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Rings with uniformly *S-SFT*

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Abstract. In this article, we examine the notion of uniformly S-SFT and study its properties. Let R be a commutative ring and S a multiplicative subset of R. A ring R is said to be uniformly S-SFT if there exists an element s in S such that for every ideal I of R, there exist a finitely generated sub-ideal J of I and a positive integer n with the property that $sa^n \in J$ for all a in I. Our investigation includes proving Cohen's Theorem for uniformly S-SFT rings and analyzing the behavior of uniformly S-SFT property under various ring operations like Nagata's idealization and amalgamation of algebras.

1. Introduction

Throughout this article, R is always a commutative ring with identity. Recall from [4] that R is called an *SFT ring* if for any ideal I of R, there exist a finitely generated sub-ideal J of I and a positive integer n such that $a^n \in J$ for any $a \in I$. In [4], Arnold showed that if R is not an SFT-ring, then $dim(R[[X]]) = \infty$.

A subset S of ring R is a multiplicative subset if $1 \in S$, $0 \notin S$, and for any $s, t \in S$, the product st is also in S. In the first part of this paper, we introduce the concept of uniformly S-SFT ring and study its basic properties. Let R be a commutative ring. We say that R is a *uniformly S-SFT* ring if there exists an $s \in S$ such that for any ideal I of R, there exist a finitely generated sub-ideal J of I and a positive integer n such that $sa^n \in J$ for all $a \in I$. It is clear that if R is a SFT ring, then R is an uniformly S-SFT ring. However, this implication is not reversible. Some counterexamples are given in Example 2.2 and Example 2.25. An increasing sequence $(I_k)_{k\in\mathbb{N}}$ of ideals of R is called *S-root* if there exist two positive integers n, m and an $s \in S$ such that for each $k \geq n$ if $x \in I_k$, then $sx^m \in I_n$. Now, let $s \in S$. We say that every increasing sequence of ideals of R is S-root with respect to s if for every increasing sequence $(I_k)_{k\in\mathbb{N}}$ of ideals of R there exist two positive integers n, m such that for each $k \geq n$ and for every $x \in I_k$, $sx^m \in I_n$. We show that, if S is a multiplicative subset of R, then R satisfies the uniformly S-SFT property if and only if there exists an $s \in S$ such that every increasing sequence of ideals of R is S-root with respect to s. (Theorem 2.6). Cohen's type theorem is of importance in the analysis of Noetherian rings. In the 1950s, Cohen made a groundbreaking discovery that states that a ring R is Noetherian if and only if each prime ideal of R can be generated by a finite number of elements (see [6]). This result has since then been extensively used in the field. More recently, in [4], J.T. Arnold expanded upon Cohen's work by demonstrating that a similar statement holds true for SFT rings. Specifically, a ring R is considered SFT if it satisfies the following condition: for

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every prime ideal P of R, there exist a finitely generated sub-ideal Q of P and a positive integer n such that for any element $a \in P$, $a^n \in Q$. In our research, we aim to build upon these findings by providing a more comprehensive understanding of Cohen's theorem and its applications to uniformly S-SFT rings. First, recall that a multiplicative set S of a commutative ring R is called *anti-Archimedean* if for each $s \in S$, $S \cap (\cap_{n\geq 1} s^n R) \neq \emptyset$, (see [1]). Let R be a ring and S an anti-Archimedean multiplicative subset of R, then R is a uniformly S-SFT ring, if and only if there exists an $s \in S$ such that for every prime ideal P of R there exist a finitely generated sub-ideal Q of P and a positive integer n such that $sa^n \in Q$ for all $a \in P$. We also give a necessary and sufficient condition for a product of rings $\prod_{i\in\Lambda} R_i$ to be uniformly S-SFT, where $S = \prod_{i\in\Lambda} S_i$. We demonstrate that the following assertions are equivalent:

1. R is a uniformly S-SFT ring.
2. Λ is finite and for each $i \in \Lambda$, R_i is a uniformly S_i-SFT ring.

Finally, we consider the uniformly S-SFT property over some ring constructions, specifically, Nagata's idealization ring $R(+)M$ and the amalgamated algebras along an ideal $A \bowtie^f J$ (the concepts of the Nagata's idealization ring and amalgamated algebras along an ideal will be reviewed in Section 3). We prove that if $f : A \longrightarrow B$ is a ring homomorphism, J an ideal of B, S an anti-Archimedean multiplicative subset of A and $S' = \{(s, f(s)) \mid s \in S\}$, then $A \bowtie^f J$ is a uniformly S'-SFT ring if and only if A is a uniformly S-SFT ring and $f(A) + J$ is a uniformly $f(S)$-SFT ring (Theorem 3.3). Additionally, we show that if M is a unitary R-module, N an R-submodule of M and S an anti-Archimedean multiplicative subset of R, then R is a uniformly S-SFT ring if and only if $R(+)M$ is a uniformly $(S(+)N)$-SFT ring (Theorem 3.7).

2. Uniformly *S-SFT* Rings

We start this section by introducing the following definition in order to generalize some known results about rings satisfying the SFT property.

Definition 2.1. *Let R be a commutative ring, S a multiplicative subset of R, and s an element of S. We say that an ideal I of R is of strong finite type with respect to s if there exist a positive integer n and a finitely generated sub-ideal J of I such that for any $a \in I$, $sa^n \in J$.*

We also define R to satisfies the uniformly S-SFT property if there exists an $s \in S$ such that each ideal of R is of strong finite type with respect to s.

Example 2.2. *Let F be a field, $R = F[X_1, X_2, ...]/(X_iX_j, i \neq j)$ and $S = \{\overline{X_1}^i \mid i \in \mathbb{N}\}$. Assume that R is an SFT ring. Let $I = (\overline{X_1}, \overline{X_2}, ...)$ be an ideal of R. There exist a positive integer n and a finitely generated sub-ideal J of I such that for any $a \in I$, $a^n \in J$. Assume that $J = (\overline{X_1}, \overline{X_2}, ..., \overline{X_k})$ for some $k \geq 1$. Since $\overline{X_{k+1}} \in I$, $\overline{X_{k+1}}^n \in J$, a contradiction.*

We show that R is uniformly S-SFT. Let P be an ideal of R. Then by [15, Example 3.1], $\overline{X_1}P$ is a principal ideal. Thus for any $a \in P$, $\overline{X_1}a \in \overline{X_1}P \subseteq P$, and hence R is a uniformly S-SFT ring.

Example 2.3. *Let p be a prime integer, $R = \prod_{n\in\mathbb{N}^*} \mathbb{Z}/p^n\mathbb{Z}$. Then R is not an SFT ring. Indeed, let $I = ((e_i), i \in \mathbb{N})$ with $e_i = (0, ..., 1, 0, ...)$. Assume that I is an SFT ideal, there exist a positive integer n and a finitely generated sub-ideal J of I such that $x^n \in J$ for all $x \in I$. Assume that $J = (e_1, ..., e_k)$ for some $k \geq 1$. Thus $e_{k+1}^n \in J$ which is a contradiction. Now, let $s = (\overline{1}, \overline{p}, \overline{p}, \overline{0}, ...)$. Note that $s^2 = (\overline{1}, \overline{0}, \overline{p}^2, \overline{0}, ...)$, $s^3 = (\overline{1}, \overline{0}, \overline{0}, \overline{0}, ...)$ and $s^k = s^3$ for all $k \geq 3$. Let $S = \{1, s, s^2, s^3\}$. Then S is a multiplicative subset of R. Let I be an ideal of R and $a \in I$. Then $sa \in sI$. An element of sI is of the form $(\overline{a_1}, \overline{a_2}, \overline{a_3}, \overline{0}, ...)$ with $\overline{a_i} \in \mathbb{Z}/p^i\mathbb{Z}$. Then sI is a finitely generated ideal of R. It is also contained in I which implies that $sa \in sI \subseteq I$. Hence, R is uniformly S-SFT ring.*

Let R be a commutative ring and S a multiplicative subset of R. We define R to be *S-strongly finite type ring* (in short *S-SFT ring*) if for each ideal I of R there exist an $s \in S$, a finitely generated sub-ideal J of I and positive integer m such that $sa^m \in J$ for any $a \in I$ [10].

Remark 2.4. *Let R be a ring and S a finite multiplicative subset of R. Then R is a uniformly S-SFT ring if and only if R is an S-SFT ring. Indeed, it is clear that if R is a uniformly S-SFT ring, then R is an S-SFT ring. Conversely, let $S = \{s_1, ..., s_r\}$ and put $s := s_1 \cdots s_r$. Assume that for any ideal I of R there exist a finitely generated sub-ideal J of I and an positive integer n such that $s_i a^n \in J$ for some $s_i \in S$ $i \in \{1, ..., r\}$. Then $sa^n = s_1 \cdots s_r a^n \in s_1 \cdots s_{i-1} s_{i+1} \cdots s_r J \subseteq J$. This implies that R is a uniformly S-SFT ring.*

A ring extension $A \subseteq B$ is called a *root extension* if for each element $b \in B$, there exists a positive integer n (depending on b) such that $b^n \in A$. ([2]). Expanding on this notion, we introduce the following new definition to ideals:

Definition 2.5. *Let R be a commutative ring, S a multiplicative subset of R and $(I_k)_{k\in\mathbb{N}}$ an increasing sequence of ideals of R.*

1. *$(I_k)_{k\in\mathbb{N}}$ is called S-*root *if there exist an $s \in S$ (depending on $(I_k)_{k\in\mathbb{N}}$) and two positive integers n, m such that for each k greater than or equal to n and for every $x \in I_k$, sx^m belongs to I_n.*
2. *Let $s \in S$. It is said that every increasing sequence of ideals of R is S-root with respect to s if for every increasing sequence $(I_k)_{k\in\mathbb{N}}$ of ideals of R there exist two positive integers n, m such that for each $k \geq n$ and for every $x \in I_k$, $sx^m \in I_n$.*
3. *In the specific case where $S = \{1\}$, the sequence $(I_k)_{k\in\mathbb{N}}$ is termed a "root" sequence if there exist two positive integers n and m, such that for all $k \geq n$, and for all $x \in I_k$, x^m is an element of I_n.*

Theorem 2.6. *Let R be a commutative ring and S a multiplicative subset of R. The following statements are equivalent.*

1. R satisfies the uniformly S-SFT property.
2. There exists an $s \in S$ such that every increasing sequence of ideals of R is S-root with respect to s.

Proof. (1)"⇒"(2). Assume that R satisfies the uniformly S-SFT property. There exists an $s \in S$ such that for any ideal I of R there exist a finitely generated sub-ideal J of I and a positive integer m such that $sx^m \in J$ for all $x \in I$. Let $(I_n)_{n\in\mathbb{N}}$ be an increasing sequence of ideals of R. We prove that this increasing sequence is S-root with respect to s. Put $I = \bigcup_{n\in\mathbb{N}} I_n$. Then I is an ideal of R. Moreover by hypothesis there exist a finitely generated sub-ideal J of I and a positive integer m such that $sx^m \in J$ for all $x \in I$. Put $J = a_1R + \cdots + a_nR$ for some $a_1, ..., a_n \in I$. Note that for $1 \leq i \leq n$, there exists an $n_i \in \mathbb{N}$ such that $a_i \in I_{n_i}$. Let $n_0 = max\{n_i, 1 \leq i \leq n\}$. Then $J \subseteq I_{n_0}$. This implies that for all $k \geq n_0$, for any $x \in I_k \subseteq I$, $sx^m \in J \subseteq I_{n_0}$. Hence the sequence $(I_n)_{n\in\mathbb{N}}$ is S-root with respect to s.

(2)"⇒"(1). Let $s \in S$ in (2). Assume that R is not uniformly S-SFT with respect to s. There exists an ideal I of R such that for each finitely generated sub-ideal J of I and every positive integer m, there exists an $a_0 \in I$ such that $sa_0^m \notin J$. Let $a \in I$ and define $I_0 = aR$. For $n = 1$, there exists an $a_{1I_0} \in I$ such that $sa_{1I_0} \notin I_0$. Define $I_1 = aR + a_{1I_0}R$. For $n = 2$, there exists $a_{2I_1} \in I$ such that $sa_{2I_1}^2 \notin I_1$. By induction, define $I_{n-1} = aR + a_{1I_0}R + \cdots + a_{n-1I_{n-2}}R$. Since I is not of strong finite type ideal with respect to s, for any $n = m$ there exists an $a_{mI_{m-1}} \in I$ such that $sa_{mI_{m-1}}^m \notin I_{m-1}$. Thus, we construct an increasing sequence of ideals (I_n) of R. Therefore, the sequence $(I_n)_n$ is S-root with respect to s. There exist $n, m \in \mathbb{N}$ such that for all $k \geq n$, $sx^m \in I_n$ for all $x \in I_k$. Choose $k > \max\{n, m\}$. Then, $sa_{kI_{k-1}}^k = sa_{kI_{k-1}}^m a_{kI_{k-1}}^{k-m} \in I_n \subseteq I_{k-1}$, a contradiction. □

In the particular case when $S = \{1\}$, we find the following corollary.

Corollary 2.7. *Let R be a commutative ring. Then the following statements are equivalent.*

1. *R satisfies the SFT property.*
2. *Every increasing sequence of ideals of R is root.*

Example 2.8. *Let $R = (\mathbb{Z}/4\mathbb{Z})[X_1, X_2, \cdots]$ and $S = \{\overline{3}^n, n \in \mathbb{N}\}$. Then R is not uniformly S-SFT ring. Indeed, let $I_1 \subseteq I_2 \subseteq \cdots$ an ascending chain of ideals of R with $I_k = \langle 2X_1, 2X_2, ..., 2X_k \rangle$. Assume that there exist an $s \in S$ and two positive integers m, k such that for every $n \geq k$ if $x \in I_n$, then $sx^m \in I_k$. Thus $s2X_{k+1}^m \in I_k$, which is a contradiction.*

Proposition 2.9. *Let R be a commutative ring and S be an at most countable multiplicative subset of R. Then the following statements are equivalent.*

1. *R satisfies the S-SFT property.*
2. *Every increasing sequence of ideals of R is S-root.*

Proof. (1)"⇒"(2). Assume that for any ideal I of R there exist a finitely generated sub-ideal J of I and a positive integer m such that for all $x \in I$, $sx^m \in J$ for some $s \in S$. Let $(I_n)_{n\in\mathbb{N}}$ be an increasing sequence of ideals of R. We prove that this increasing sequence $(I_n)_{n\in\mathbb{N}}$ is S-root. Put $I = \bigcup_{n\in\mathbb{N}} I_n$. Then I is an ideal of R. Moreover by hypotheses there exist an $s \in S$, a finitely generated sub-ideal J of I and an positive integer m such that $sx^m \in J$ for all $x \in I$. Put $J = a_1R + \cdots + a_nR$. Note that for $1 \leq i \leq n$, there exist an $n_i \in \mathbb{N}$ such that $a_i \in I_{n_i}$. Let $n_0 = max\{n_i, 1 \leq i \leq n\}$. Then $J \subseteq I_{n_0}$. This implies that for all $k \geq n_0$, for any $x \in I_k \subseteq I$, $sx^m \in J \subseteq I_{n_0}$. Hence the sequence $(I_n)_{n\in\mathbb{N}}$ is S-root.

(2)"⇒"(1). * Suppose that $S = \{s_1, ..., s_n\}$ is finite and let $s = s_1 \cdots s_n$. Then by Remark 2.4, R is uniformly S-SFT if and only if R is an S-SFT ring.

*Assume that $S = (s_n)_{n\geq 0}$ is a countable multiplicative subset of R. Suppose that R is not an S-SFT ring. Then there exists an ideal I of R such that for every $s \in S$, every positive integers m and every finitely generated sub-ideal J of I, there exists an element $a \in I$ such that $sa^m \notin J$. Let $x \in I$ and define $J_0 = xR$, which is a finitely generated sub-ideal of I. For $n = 1$ and $s = s_1 \in S$, there exists an element $x_{s_1 1 J_0} \in I$ such that $s_1 x_{s_1 1 J_0} \notin J_0$. Define $J_1 = xR + x_{s_1 1 J_0}R$, which is again a finitely generated sub-ideal of I. For $n = 2$ and $s = s_1 \in S$, there exists $x_{s_1 2 J_1} \in I$ such that $s_1 x^2_{s_1 2 J_1} \notin J_1$. Similarly, for $n = 2$ and $s = s_2 \in S$, there exists $x_{s_2 2 J_1} \in I$ such that $s_2 x^2_{s_2 2 J_1} \notin J_1$. By induction, assume $J_{n-1} = xR + x_{s_1 1 J_0}R + x_{s_1 2 J_1}R + x_{s_2 2 J_1}R + \cdots + x_{s_1 n-1 J_{n-2}}R + \cdots + x_{s_{n-1} n-1 J_{n-2}}R$. For each $s = s_i \in S$ and $n = m$, there exists $x_{s_i n J_{n-1}} \in I$ such that $s_i x^n_{s_i n J_{n-1}} \notin J_{n-1}$. Thus, we construct an increasing sequence of ideals (J_n) of R, where each J_n is finitely generated and $J_{n-1} \subseteq J_n$. So J_n is S-root. There exist $s_r \in S$ and positive integers n, m such that for all $k \geq n$ and $x \in J_k$, $s_r x^m \in J_n$. Choose $k > \max\{r, n, m\}$. Then, $x_{s_r k J_{k-1}} \in I_k$ and hence $s_r x^m_{s_r k J_{k-1}} \in J_n$. Therefore,

$$s_r x^k_{s_r k J_{k-1}} = s_r x^m_{s_r k J_{k-1}} x^{k-m}_{s_r k J_{k-1}} \in J_n \subseteq J_{k-1},$$

which is a contradiction. □

Let R be a commutative ring with identity and S a multiplicative subset of R. We say that S is saturated if for every $a, b \in R$, if $ab \in S$, then both a and b are in S. Additionally, the set $S' = \{x \in R \mid x \text{ divides } s \text{ for some } s \in S\}$ is a saturated multiplicative subset of R called the saturation of S which includes S.

Theorem 2.10. *Let R be a ring and S a multiplicative subset of R.*

1. *Let T be a multiplicative subset of R such that $S \subseteq T$. If R is a uniformly S-SFT ring, then R is a uniformly T-SFT ring.*
2. *Let S' be the saturation of S in R. Then R is a uniformly S-SFT ring if and only if R is a uniformly S'-SFT ring.*
3. *Let $f : R \to R'$ be a surjective ring homomorphism and S a multiplicative subset of R such that $f(S)$ does not contain 0. If R is a uniformly S-SFT ring, then R' is a uniformly $f(S)$-SFT ring.*

Proof. (1). Obvious.

(2). If R is a uniformly S-SFT ring, then by (1), R is a uniformly S'-SFT ring. Conversely, assume that R is a uniformly S'-SFT ring. There exists an $s \in S'$ such that for any ideal I of R there exist a finitely generated sub-ideal J of I and a positive integer n such that for any $a \in I$, $sa^n \in J$. Let $t \in S$ such that $t = sr$ where $r \in R$. $ta^n = sra^n \in rJ \subseteq J$, and hence R is a uniformly S-SFT ring.

(3). Assume that R is a uniformly S-SFT ring. There exists an $s \in S$ such that each ideal of R is of strong finite type with respect to s. Let J be an ideal of R'. Since f is a surjective homomorphism, $J = f(I)$ for some ideal I of R. Thus I is of strong finite type with respect to s. Let $b \in J$, then $b = f(a)$ for some $a \in I$. So $sa^n \in K$ for some finitely generated sub-ideal K of I and some positive integer n. This implies that $f(sa^n) \in f(K)$, thus $f(s)b^n = f(sa^n) \in f(K)$. Note that $f(K)$ is a finitely generated sub-ideal of $f(I) = J$. Hence R' is a uniformly $f(S)$-SFT ring. □

Remark 2.11. *(1) Consider the multiplicative set S in Example 2.2, and let $T = \{1\}$. Then R is uniformly S-SFT. Clearly $T \subseteq S$ and R is not uniformly T-SFT.*

(2) Note that the condition "f is surjective" in Theorem 2.10 (3) is necessary. Indeed, let $R = K[X_1, X_2, ...]$ be the polynomial ring in countably infinite variables over a field K and $S = U(K) = K - \{0\}$ (a multiplicative subset of K). Let $\Psi : K \to R$ defined by $\Psi(a) = a$. Then $\Psi(S) = S$. It is clear that Ψ is not surjective and K is a uniformly S-SFT ring. But R is not uniformly S-SFT.

Let R be a ring, S a multiplicative subset of R, and I an ideal of R disjoint with S. Let $s \in S$, we denote by $\bar{s}$ the equivalence class of s in R/I. Let $\overline{S} = \{\bar{s} \mid s \in S\}$, then $\overline{S}$ is a multiplicative subset of R/I.

Corollary 2.12. *Let R be a ring, S a multiplicative subset of R and I an ideal of R disjoint with S. If R satisfies the uniformly S-SFT property, then R/I satisfies the uniformly $\overline{S}$-SFT property.*

Let R be a ring and S a multiplicative subset of R. For any non-nilpotent element $s \in S$, consider the multiplicative subset $\langle s \rangle := \{1, s, s^2, ...\}$ of S. We denote by R_s the localization of R at $\langle s \rangle$.

We next study the Cohen's type theorem for uniformly S-SFT rings. To do this, we need the following results.

Lemma 2.13. *Let R be a ring, S a multiplicative subset of R and I an ideal of R. Let s a non-nilpotent element of S. If I is of strong finite type with respect to s, then I_s is an SFT ideal of R_s.*

Proof. Suppose that I is of strong finite type with respect to s. There exist a finitely generated sub-ideal J of I and positive integer n such that for any $x \in I$, $sx^n \in J$. Let $b \in I_s$. Then $b = \dfrac{a}{s^r}$ for some $a \in I$ and some positive integer r. This implies that $b^n = \dfrac{a^n}{s^{rn}} = \dfrac{sa^n}{s^{rn+1}} \in J_s$. Note that J_s is a finitely generated sub-ideal of I_s. Thus I_s is an SFT ideal of R_s. □

Let R be a ring and S a multiplicative subset of R. Recall that S is called *anti-Archimedean* if for each $s \in S$, $S \bigcap (\bigcap_{n \geq 1} s^n R) \neq \emptyset$, see [1]. In [13], the authors showed that, a finite multiplicative set is an anti-Archimedean set. For example, let $R = \mathbb{Z}/12\mathbb{Z}$ and $S = \{\bar{1}, \bar{2}, \bar{4}, \bar{8}\} \nsubseteq U(R)$ is an anti-Archimedean multiplicative set of R. It is clear that if R is a uniformly S-SFT ring, then for any ideal I of R, there exist an $s \in S$, a finitely generated sub-ideal of I and a positive integer n such that $sa^n \in J$ for any $a \in I$. Our next example show that the converse of this implication is not true in general. First, we need the following proposition.

Proposition 2.14. *Let R be a ring and S a multiplicative subset of R disjoint from $Nil(R)$.*

1. *If R is a uniformly S-SFT ring, then there exists an $s \in S$ such that R_s is an SFT ring.*
2. *If S is an anti-Archimedean multiplicative subset of R and R_s an SFT ring for some $s \in S$, then R is a uniformly S-SFT ring.*

Proof. (1). Assume that R is a uniformly S-SFT ring. There exists an $s \in S$ such that any ideal I of R is strongly of finite type with respect to s. We will show that R_s is an SFT ring. Let F be an ideal of R_s, then $F = I_s$ for some ideal I of R. Thus by Lemma 2.13, $F = I_s$ is an SFT ideal of R_s, hence R_s is an SFT ring.

(2). Suppose that R_s is an SFT ring for some $s \in S$. Take $t \in S \bigcap (\bigcap_{n \geq 1} s^n R)$. We will show that R is uniformly S-SFT with respect to t. Let I be an ideal of R. Then I_s is an SFT ideal of R_s, so there exist an $n \in \mathbb{N}$ and a finitely generated sub-ideal J of I such that $x^n \in J_s$ for all $x \in I_s$. We will show that for all $a \in I$, $ta^n \in J$. Let $a \in I$. Then $\dfrac{a^n}{1} \in J_s$, thus $\dfrac{a^n}{1} = \dfrac{\alpha}{s^r}$ for some positive integer r and $\alpha \in J$. There exists a positive integer r' such that $s^{r'} a^n \in J$. As $t \in S \cap_{n \in \mathbb{N}} s^n R$, $t = s^{r'} a_{r'}$ for some $a_{r'} \in R$. This implies that $ta^n = s^{r'} a^n a_{r'} \in J$. □

Example 2.15. *Let $R = K[X_1, X_2, ...]$ be the polynomial ring in countably infinite variables over a field K. Set $S := R - \{0\}$. It is clear that for any ideal I of R, there exist an $s \in S$, a finitely generated sub-ideal J of I and a positive integer n such that $sa^n \in J$ for any $a \in I$. But R is not uniformly S-SFT. Indeed, let $s \in S$. Assume that R_s is an SFT ring. Let n be the minimal integer such that X_m does not divide any monomial of s for any $m \geqslant n$. Then $s \in K[X_1, X_2, ..., X_{n-1}]$. Let φ be the following mapping*

$$\begin{array}{rcll} \varphi : & R_s & \longrightarrow & K[X_1, X_2, ..., X_{n-1}]_s[X_n, X_{n+1}, ...] \\ & P = \dfrac{1}{s^k}\displaystyle\sum_i f_i(X_1, ..., X_{n-1})h_i(X_n, X_{n+1}, ...) & \longrightarrow & \displaystyle\sum_i \dfrac{f_i(X_1, ..., X_{n-1})}{s^k} h_i(X_n, X_{n+1}, ...) \end{array}$$

is an isomorphism.

Assume that $K[X_1, X_2, ..., X_{n-1}]_s[X_n, X_{n+1}, ...]$ is an SFT ring. Then for any ideal I of $K[X_1, X_2, ..., X_{n-1}]_s[X_n, X_{n+1}, ...]$, there exist a finitely generated sub-ideal J of I and positive integer r such that for any $a \in I$, $a^r \in J$. Let $I = (X_n, X_{n+1}, ...)$ the ideal of $K[X_1, X_2, ..., X_{n-1}]_s[X_n, X_{n+1}, ...]$. There exist a finitely generated sub-ideal J of I and positive integer n such that for any $a \in I$, $a^n \in J$. Assume that $J = (X_n, X_{n+1}, ..., X_k)$ for some $k > n$. Since $X_{k+1} \in I$, then $X_{k+1}^r \in J$, which is a contradiction. This implies that R_s is not SFT. Thus by Proposition 2.14, R is not uniformly S-SFT.

We are now ready to give the Cohen type theorem for uniformly S-SFT rings.

Theorem 2.16. *Let R be a commutative ring and S an anti-Archimedean multiplicative subset of R such that $S \cap Nil(R) = \emptyset$. The following statements are equivalent.*

1. *R is a uniformly S-SFT ring.*
2. *There exists an $s \in S$ such that every radical ideal of R is of strong finite type with respect to s.*
3. *There exists an $s \in S$ such that every prime ideal of R is of strong finite type with respect to s.*

Proof. (1)⇒(2). Obvious.

(2)⇒(3). Follows from the fact that every prime ideal of R is a radical ideal of R.

(3)⇒(1). Suppose that R is not a uniformly S-SFT ring. By Proposition 2.14, for all $s \in S$, R_s is not an SFT ring. Let s be such that there exists an $s \in S$ such that every prime ideal of R is of strong finite type with respect to s. Since R_s is not an SFT ring, there exists an ideal I_s of R_s which is not of strong finite type. Let $\mathcal{F} = \{Q$ ideal of R such that Q_s is not an SFT ideal of $R_s\}$. We have $\mathcal{F} \neq \emptyset$, since $I \in \mathcal{F}$. Let $(I_\lambda)_{\lambda\in\Lambda}$ be a chain in $\mathcal{F}$ and $L = \bigcup_{\lambda\in\Lambda} I_\lambda$. Now, we will show that $L \in \mathcal{F}$. Assume that $L \notin \mathcal{F}$. There exist a finitely generated sub-ideal J of L and a positive integer n such that for any $a \in L_s$, $a^n \in J_s$. Since J is finitely generated, there exists a $\lambda_0 \in \Lambda$ such that $J \subseteq I_{\lambda_0}$. Let $a \in (I_{\lambda_0})_s \subseteq L_s$. Then $a^n \in J_s \subseteq (I_{\lambda_0})_s$, a contradiction. Hence, by Zorn's Lemma, there is a maximal element P of $\mathcal{F}$. We prove that the maximal element P of $\mathcal{F}$ is a prime ideal of R. Suppose that P is not prime. There exist $a, b \in R \setminus P$ such that $ab \in P$. We put $I := P + aR$ and $J := P + bR$. Then $IJ = P^2 + (aR)P + (bR)P + (ab)R \subseteq P$. Since $P \subsetneq I$ and $P \subsetneq J$, by maximality of P there exist a finitely generated sub-ideal I' (respectively, J') of I (respectively, of J) and $n, m \in \mathbb{N}$ such that for any $a \in I_s$, and $b \in J_s$, we get $a^n \in I'_s$ and $b^m \in J'_s$. Let $x \in P_s$. Then $x \in I_s$ and $x \in J_s$, thus $x^n \in I'_s$ and $x^m \in J'_s$. So $x^{n+m} \in I'_sJ'_s$ which implies that $x^{n+m} \in (I'J')_s \subseteq P_s$. Then P_s is an SFT ideal. Thus $P \notin \mathcal{F}$, a contradiction. Hence P is a prime ideal of R. Note that P_s is not an SFT ideal. Then by Lemma 2.13, P is not of strong finite type with respect to s, a contradiction. □

According to [17], a commutative ring R is called *uniformly S-Noetherian* if there exists an $s \in S$ such that for any ideal I of R, there exists a finitely generated sub-ideal J of I such that $sI \subseteq J$. In [18], the authors demonstrated that if S is an anti-Archimedean multiplicative subset of the ring R and T is a ring extension of R such that T is an S-finite R-module, then R is a uniformly S-Noetherian ring if and only if there exists an $s \in S$ such that for every prime ideal P of R, PT is an S-finite ideal of T with respect to s. By analogy, in Proposition 2.14 and Lemma 2.13, it is easy to show that if S is regular anti-Archimedean, then R_s is a Noetherian ring for some $s \in S$ implies that R is a uniformly S-Noetherian ring and if there exists an $s \in S$ such that I is finitely generated with respect to s, then I_s is a finitely generated ideal of R_s. Next remark, provides another proof of Cohen's theorem.

Remark 2.17. *Let R be a commutative ring and S be a regular anti-Archimedean multiplicative subset of R. Then the following statements are equivalent.*

1. *R is a uniformly S-Noetherian ring.*
2. *There exists an $s \in S$ such that every prime ideal of R is finitely generated with respect to s.*

Proof. (2)⇒(1). Suppose that R is not uniformly S-Noetherian ring. Let s be such that every prime ideal of R is finitely generated with respect to s. As R_s is not Noetherian, thus there exists an ideal I_s of R_s which is not finitely generated. Let $\mathcal{F} = \{Q$ ideal of R such that Q_s is not finitely generated in $R_s\}$. We have $\mathcal{F} \neq \emptyset$, since $I \in \mathcal{F}$. Let $(I_\lambda)_{\lambda\in\Lambda}$ be a chain in $\mathcal{F}$ and $L = \bigcup_{\lambda\in\Lambda} I_\lambda$. We show that $L \in \mathcal{F}$. Assume that $L \notin \mathcal{F}$. There exists a finitely generated sub-ideal J of L such that $L_s = J_s$. Since J is finitely generated, there exists $\lambda_0 \in \Lambda$ such that $J \subseteq I_{\lambda_0}$. Then $(I_{\lambda_0})_s \subseteq L_s = J_s \subseteq (I_{\lambda_0})_s$, a contradiction. Hence, by Zorn's Lemma, there is a maximal element P of $\mathcal{F}$. We prove that the maximal element P of $\mathcal{F}$ is a prime ideal of R. Suppose that P is not a prime ideal of R. There exist $a, b \in R \setminus P$ such that $ab \in P$. Since $P \subsetneq P + aR$, by maximality of P, $(P + aR)_s$ is finitely generated in R_s, then there exist $p_1, ..., p_n \in P$, $r_1, ..., r_n \in R$ and a positive integer q such that $(P + aR)_s = (\frac{p_1+ar_1}{s^q}, ..., \frac{p_n+ar_n}{s^q})$. Let $x \in P_s \subseteq (P + aR)_s$. Then $x = \frac{p_1+ar_1}{s^q}\frac{\alpha_1}{s^k} + \cdots + \frac{p_n+ar_n}{s^q}\frac{\alpha_n}{s^k}$ for some $\alpha_1, ..., \alpha_n \in R$ and a positive integer k, thus $\frac{a}{1}(\frac{r_1}{s^q}\frac{\alpha_1}{s^k} + \cdots + \frac{r_n}{s^q}\frac{\alpha_n}{s^k}) = x - \frac{p_1}{s^q}\frac{\alpha_1}{s^k} - \cdots - \frac{p_n}{s^q}\frac{\alpha_n}{s^k} \in P_s$. Since $P \subset (P : a)$, again by maximality of P, $(P : a)_s$ is a finitely generated ideal of R_s. So there exist $\gamma_1, ..., \gamma_l \in (P : a)_s$ such that $(P : a)_s = (\gamma_1, ..., \gamma_l)R_s$. Put $y := \frac{r_1}{s^q}\frac{\alpha_1}{s^k} + \cdots + \frac{r_n}{s^q}\frac{\alpha_n}{s^k}$. Then $y \in (P_s : \frac{a}{1}) \subseteq (P : a)_s$. Then there exist a positive integer t and $\beta_1, ..., \beta_l \in R$, such that $y = \gamma_1\frac{\beta_1}{s^t} + \cdots + \gamma_l\frac{\beta_l}{s^t}$. This implies that

$$x = \frac{p_1}{s^q}\frac{\alpha_1}{s^k} + \cdots + \frac{p_n}{s^q}\frac{\alpha_n}{s^k} + \gamma_1\frac{a\beta_1}{s^t} + \cdots + \gamma_l\frac{a\beta_l}{s^t}.$$

Thus $x \in (\frac{p_1}{s^q}, ..., \frac{p_n}{s^q}, \gamma_1\frac{a}{s^t}, ..., \gamma_l\frac{a}{s^t}) \subseteq P_s$. So $P_s \subseteq (\frac{p_1}{s^q}, ..., \frac{p_n}{s^q}, \gamma_1\frac{a}{s^t}, ..., \gamma_l\frac{a}{s^t}) \subseteq P_s$, a contradiction. Hence P is a prime ideal of R such that P_s is not finitely generated ideal of R_s. So P is not a finitely generated with respect to s, which is a contradiction. □

Let R be a ring and S a multiplicative subset of R. Then R is called *of uniformly S-Noetherian spectrum* if there exists an $s \in S$ such that for any ideal I of R, $sI \subseteq \sqrt{J}$ for some finitely generated sub-ideal J of I (see [12]).

Remark 2.18. *Let R be a ring and S a multiplicative subset of R. If R is uniformly S-SFT, then R is of uniformly S-Noetherian spectrum. Indeed, as R is uniformly S-SFT, there exists an $s \in S$ such that for any ideal I of R there exist a finitely generated sub-ideal J of I and a positive integer n such that for any $a \in I$, $sa^n \in J$. We show that R is of uniformly S-Noetherian spectrum with respect to s. Let K be an ideal of R and x an element of K, then $sx^n \in J$ for some finitely generated sub-ideal J of K and some positive integer n. Thus $s^nx^n \in J$; so $sx \in \sqrt{J}$. Hence $sI \subseteq \sqrt{J}$.*

Example 2.19. *Let F be a field, $X = \{X_1, X_2, ...\}$ a countably set of indeterminates over F, $J = \langle X_n^n, n \geq 1\rangle F[X]$, $R = F[X]/J$ and $S = F \setminus \{0\}$. If P is a prime ideal of R, then there exists a prime ideal Q of $F[X]$ such that $J \subseteq Q$; so for all $n \in \mathbb{N}^*$, $X_n^n \in Q$ which implies that $X_n \in Q$ for all $n \in \mathbb{N}^*$. Thus $\langle X_n, n \geq 1\rangle \subseteq Q$. Since $\langle X_n, n \geq 1\rangle$ is a maximal ideal of $F[X]$, $\langle X_n, n \geq 1\rangle = Q$, hence $P = \langle \overline{X_n}, n \geq 1\rangle$. So $P = Nil(R)$. Then the only prime ideal of R is $Nil(R)$ which implies that R is of uniformly S-Noetherian spectrum. On the other hand, R is not uniformly S-SFT because for all $s \in S$, $I = (\overline{X_1}, \overline{X_2}, ...)$ is not a strongly finite type ideal with respect to s. Indeed, if not, there exist $s \in S$ and two positive integers n, k such that for any $x \in I$, $sx^n \in (\overline{X_1}, \overline{X_2}, ..., \overline{X_k})$. Thus $s\overline{X}_{n+k+1}^n \in (\overline{X_1}, \overline{X_2}, ..., \overline{X_k})$ a contradiction.*

Proposition 2.20. *Let $R_1 \subseteq R_2$ be a ring extension such that for each finitely generated ideal I of R_1, $IR_2 \cap R_1 = I$ and S a multiplicative subset of R_1. If R_2 is uniformly S-SFT, then R_1 is uniformly S-SFT.*

Proof. Let I be a ideal of R_1. Since the ring R_2 is uniformly S-SFT, there exists an $s \in S$ such that for any ideal J of R_2, there exist a finitely generated sub-ideal K of J and a positive integer n such that for any $x \in J$, $sx^n \in K$. Since IR_2 an ideal of R_2, there exist $k \geq 1$ and a finitely generated ideal $K \subseteq IR_2$ of R_2 such that $sx^k \in K$ for every $x \in IR_2$. Let $F \subseteq I$ be a finitely generated ideal of R_1 such that $K \subseteq FR_2$ and $a \in I$. Hence $sa^k \subseteq K \cap R_1 \subseteq FR_2 \cap R_1 = F$ which implies that the ring R_1 is uniformly S-SFT. □

Let R be a commutative ring and P a prime ideal of R. Then $R \setminus P$ is a multiplicative subset of R. We say that R is a uniformly P-SFT ring if R is a uniformly $(R\backslash P)$-SFT ring.

Theorem 2.21. *The following assertions are equivalent for a commutative ring R.*

1. *R is an SFT ring.*
2. *R is a uniformly P-SFT ring for any $P \in Spec(R)$.*
3. *R is a uniformly M-SFT ring for any $M \in Max(R)$.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3). These implications are trivial.

(3) $\Rightarrow$ (1). By hypothesis, for any $M \in \mathrm{Max}(R)$, there exists an $s_M \in R\backslash M$ such that for any ideal I of R, there exist a positive integer r and a finitely generated sub-ideal F_M of I such that for any $x \in I$, $s_M x^r \in F_M$. Let J be the ideal of R generated by the set $\{s_M \mid M \in \mathrm{Max}(R)\}$. If $J \neq R$, then $J \subseteq M_0$ for some $M_0 \in \mathrm{Max}(R)$. So $s_M \in M_0$, a contradiction. Thus $J = R$. Hence $1 = s_{M_1}\alpha_1 + \cdots + s_{M_n}\alpha_n$ for some $\alpha_1, ..., \alpha_n \in R$. Now, let I be an ideal of R. For each $1 \leqslant i \leqslant n$, there exists a finitely generated sub-ideal F_i of I and a positive integer r_i such that for every $x \in I$, $s_{M_i}x^{r_i} \in F_{M_i}$. Put $r := \prod_{i=1}^{n} r_i$ and $F := \sum_{i=1}^{n} F_{M_i}$. Then F is a finitely generated sub-ideal of I. Moreover, for any $x \in I$,

$$x^r = 1 \cdot x^r = (s_{M_1}\alpha_1 + \cdots + s_{M_n}\alpha_n)x^r \subseteq s_{M_1}x^r + \cdots + s_{M_n}x^r \subseteq \sum_{i=1}^{n} F_{M_i} = F.$$

Hence R is an SFT ring. □

Proposition 2.22. *Let R be a commutative ring with identity and $T \subseteq R$ a multiplicative subset of R consisting of non-zero-divisors. Let S be another multiplicative subset of R. If R satisfies the uniformly S-SFT property, then $T^{-1}R$ satisfies the uniformly S'-SFT property where $S' = \{\frac{s}{1}, s \in S\}$.*

Proof. Since R is a uniformly S-SFT ring, there exists an $s \in S$ such that each ideal of R is of strong finite type with respect to s. Let $J = T^{-1}I$ be an ideal of $T^{-1}R$. There exist a finitely generated sub-ideal K of I and a positive integer n such that for any $x \in I$, $sx^n \in K$. Let $y \in J$, then $y = \dfrac{a}{t}$ for some $a \in I$ and $t \in T$, thus $sa^n \in K$. So $\dfrac{s}{1}y^n = \dfrac{sa^n}{1t^n} \in T^{-1}K$. Since $K \subseteq I$, $T^{-1}K \subseteq T^{-1}I = J$. This shows that J is of strong finite type with respect to $\frac{s}{1}$. So $T^{-1}R$ is a uniformly S'-SFT ring. □

The next Theorem give a necessary and sufficient condition for a product of rings $\prod_{i\in\Lambda} R_i$ to be uniformly S-SFT, where $S = \prod_{i\in\Lambda} S_i$.

Theorem 2.23. *Let $\Lambda \subseteq \mathbb{N}$ and $(R_i)_{i\in\Lambda}$ be a family of commutative rings. For each $i \in \Lambda$, let S_i be a multiplicative subset of R_i. Let $R = \prod_{i\in\Lambda} R_i$ and $S = \prod_{i\in\Lambda} S_i$. Then the following assertions are equivalent:*

1. *R is a uniformly S-SFT ring.*
2. *Λ is finite and for each $i \in \Lambda$, R_i is a uniformly S_i-SFT ring.*

Proof. (1)$\Rightarrow$(2). Suppose that Λ is infinite. Since R is a uniformly S-SFT ring, there exists an $s = (s_1, s_2, ...) \in S$ such that for any ideal J of R, there exist a finitely generated sub-ideal F of J and a positive integer r such that for any $a \in J$, $sa^r \in F$. Let $J = (e_i \mid i \in I)$, with $e_i = (1, 1, ..., \underbrace{1}_{i-place}, 0, ...)$. So there exists a finitely generated sub-ideal F of J and a positive integer r such that for any $a \in J$, $sa^r \in F$. Put $F := (e_i \mid 1 \leq i \leq n)$. Since $e_{n+1} \in J$, then $se_{n+1}^r \in F$. Hence $s_{n+1} = 0$, a contradiction.

Now, let φ_k be the k^{th} projection mapping, that is, $\varphi_k : R \to R_k$; $\varphi(x_1, ..., x_k, ...) = x_k$. Then φ_k is a surjective homomorphism of rings. Since $\varphi_k(S) = S_k$, by Theorem 2.10 (3), R_k is a uniformly S_k-SFT ring.

(2) ⇒ (1). To prove this implication, it is sufficient to show it in the case $n = 2$ and conclude by induction on n. Let $R = R_1 \times R_2$, and $S = S_1 \times S_2$ be such that R_1 (respectively, R_2) is a uniformly S_1-SFT ring (respectively, uniformly S_2-SFT). Let $s_1 \in S_1$ (respectively, $s_2 \in S_2$) such that each ideal of R_1 (respectively, R_2), is of strong finite type with respect to s_1 (respectively, s_2). Now, let $I = I_1 \times I_2$ be an ideal of R. For each $1 \leq i \leq 2$, there exists a finitely generated sub-ideal J_i of I_i and positive integers r_1, r_2 such that for any $a_1 \in I_1, a_2 \in I_2$, $s_i a_i^{r_i} \in J_i$. Let $y \in I$, then $y = (a_1, a_2)$ for some $a_1 \in I_1$ and $a_2 \in I_2$. We have

$$(s_1, s_2)(a_1, a_2)^{r_1 r_2} = (s_1 a_1^{r_1 r_2}, s_2 a_2^{r_1 r_2}) \in J_1 \times J_2.$$

Hence R is a uniformly S-SFT ring. □

In the particular case when $S = \{1\}$, we find this result.

Corollary 2.24. *Let $\Lambda \subseteq \mathbb{N}$ and $(R_i)_{i\in\Lambda}$ be a family of commutative rings. Let $R = \prod_{i\in\Lambda} R_i$. Then the following assertions are equivalent:*

1. *R is a SFT ring.*
2. *Λ is finite and for each $i \in \Lambda$, R_i is a SFT ring.*

Example 2.25. *Let R_1 be a non SFT ring and R_2 be a uniformly S_2-SFT ring, where S_2 is a multiplicative subset of R_2. We consider $R = R_1 \times R_2$ and $S := (S_1 \cup \{0\} \times S_2)$, where S_1 is a multiplicative subset of R_1. Then S is a multiplicative subset of R. Since R_2 is a uniformly S_2-SFT ring, there exists $s_2 \in S_2$ such that for any ideal K of R_2 there exists a finitely generated sub-ideal J of K and a positive integer n such that $sx^n \in J$ for any $x \in K$. Let $I := I_1 \times I_2$ be an ideal of R, where I_1 is an ideal of R_1 and I_2 is an ideal of R_2. Take $s := (0, s_2) \in S$. Then for any $a = (a_1, a_2) \in I$, $sa^n = (0, s_2)(a_1, a_2)^n = (0, s_2 a_2^n) \in \{0\} \times J$ for some sub-ideal J of I_2 and some positive integer n. Note that $\{0\} \times J$ is a finitely generated sub-ideal of I. This implies that R is uniformly S-SFT. However, as R_1 is not an SFT ring by Corollary 2.24, R is not an SFT ring.*

3. Uniformly S-SFT properties on amalgamated algebras

In this section, we give a necessary and sufficient condition for the amalgamated algebra along an ideal to be uniformly S-SFT. To do this, we first recall the definition of the amalgamated algebra introduced in [7].

Definition 3.1. *Let A and B be commutative rings with identity, $f : A \to B$ a ring homomorphism and J an ideal of B. Then the sub-ring $A \bowtie^f J$ of $A \times B$ is defined as follows:*

$$A \bowtie^f J = \{(a, f(a) + j) \mid a \in A \text{ and } j \in J\}.$$

The ring $A \bowtie^f J$ is called the amalgamation of A with B along J with respect to f.

Let A and B be commutative rings with identity, $f : A \to B$ a ring homomorphism and J an ideal of B. Then $f(A) + J$ is a sub-ring of B. For a multiplicative subset S of A, let $S' = \{(s, f(s)) \mid s \in S\}$. Then it is easy to see that S' is a multiplicative subset of $A \bowtie^f J$ and $f(S)$ is a multiplicative subset of $f(A) + J$. For prime ideals P and Q of A and B, respectively, we put

$$P \bowtie^f J := \{(p, f(p) + j) \mid p \in P \text{ and } j \in J\}; \text{ and}$$
$$\overline{Q}^f := \{(a, f(a) + j) \mid a \in A, j \in J \text{ and } f(a) + j \in Q\}.$$

Then the prime ideals of $A \bowtie^f J$ are exactly of the type $P \bowtie^f J$ or $\overline{Q}^f$ for some prime ideals P of A and Q of B which do not contain J. (See [8, Proposition 2.6(3)] or [11, Theorem 1.4]). Our next result give a necessary and sufficient condition for the amalgamated algebra $A \bowtie^f J$ to be uniformly S'-SFT. First, we need the following Remark.

Remark 3.2. *Let A and B be commutative rings, $f : A \longrightarrow B$ a ring homomorphism, J an ideal of B and S an anti-Archimedean multiplicative subset of A. Then $S' = \{(s, f(s)) \mid s \in S\}$ is an anti-Archimedean multiplicative subset of $A \bowtie^f J$. Indeed, let $s \in S$ and $t \in S \bigcap(\bigcap_{n\in\mathbb{N}} s^n A)$. Then for all $n \in \mathbb{N}$, $t = s^n a_n$ for some $a_n \in A$. Thus*

$$(t, f(t)) = (s^n a_n, f(s^n a_n)) = (s^n a_n, f(s^n)f(a_n)) = (s, f(s))^n (a_n, f(a_n)))$$

for all positive integers n. So $(t, f(t)) \in S' \bigcap(\bigcap_{n\in\mathbb{N}}(s, f(s))^n A \bowtie^f J)$.

It is clear that $S \cap Nil(R) = \emptyset$ if and only if $S' \cap Nil(A \bowtie^f J) = \emptyset$.

Theorem 3.3. *Let A and B be commutative rings, $f : A \longrightarrow B$ a ring homomorphism, J an ideal of B and S an anti-Archimedean multiplicative subset of A such that $S \cap Nil(R) = \emptyset$ and $f(S) \cap J = \emptyset$. Then the following statements are equivalent.*

1. *$A \bowtie^f J$ is a uniformly S'-SFT ring.*
2. *A is a uniformly S-SFT ring and $f(A) + J$ is a uniformly $f(S)$-SFT ring.*

Proof. (1) ⇒ (2). Let $P_A : A \bowtie^f J \rightarrow A$ and $P_B : A \bowtie^f J \rightarrow f(A) + J$ be the canonical epimorphisms.

Suppose that $A \bowtie^f J$ is a uniformly S'-SFT ring. Note that $P_A(A \bowtie^f J) = A$, $P_A(S') = S$, $P_B(A \bowtie^f J) = f(A) + J$ and $P_B(S') = f(S)$. Thus by Theorem 2.10 (3), A is a uniformly S-SFT ring and $f(A) + J$ is a uniformly $f(S)$-SFT ring.

(2) ⇒ (1) Suppose that A is a uniformly S-SFT ring and $f(A) + J$ is a uniformly $f(S)$-SFT ring. There exist $s_1, s_2 \in S$ such that for any ideal I of A and for any ideal F of $f(A) + J$, there exist $a_1, ..., a_n \in I$ and $b_1, ..., b_r \in F$ such that for any $a \in I, b \in F$, $s_1 a^{k_1} \in (a_1, ..., a_n)$ and $f(s_2)b^{k_2} \in (b_1, ..., b_r)$ for some positive integers k_1, k_2.

Since $f(A) + J$ is an uniformly $f(S)$-SFT ring, $(f(A) + J)/J$ is a uniformly $\overline{f(S)}$-SFT ring by Corollary 2.12; so there exists an $s_3 \in S$ such that for any ideal P of $(f(A) + J)/J$, there exist a finitely generated sub-ideal P' of P and a positive integer k_0 such that for any $p \in P$, $\overline{f(s_3)}p^{k_0} \in P'$.

Now, let $s = s_1 s_2 s_3 \in S$. Since S is anti-Archimedean, $S \bigcap(\bigcap_{n\geq 1} s^n A) \neq \emptyset$. Let $t \in S \bigcap(\bigcap_{n\geq 1} s^n A)$. According to Theorem 2.16, it suffices to show that for each prime ideal $\mathcal{P}$ of $A \bowtie^f J$, there exist a finitely generated sub-ideal L of $\mathcal{P}$ and a positive integer m such that $(t, f(t))(a, f(a) + b)^{n_0} \in L$ for any $(a, f(a) + b) \in \mathcal{P}$.

Case 1: $\mathcal{P} = \overline{Q}^f = \langle (a, f(a) + j), a \in A, j \in J \text{ and } f(a) + j \in Q\rangle$ where Q is a prime ideal of B. Let $Q_0 = \overline{(Q + J) \cap f(A) + J}$. Then Q_0 is an ideal of $(f(A) + J)/J$; so there exist a finitely generated sub-ideal $Q'_0 := \{\overline{f(a_i) + b_i} \mid i = 1, ..., n\}$ of Q_0 and a positive integer k_0 such that for any $p \in Q_0$, $\overline{f(s_3)}p^{k_0} \in Q'_0$. Let L_0 be the ideal of $A \bowtie^f J$ generated by the set $\{(a_i, f(a_i) + b_i) \mid i = 1, ..., n\}$.

Note that $I := f^{-1}(J) \cap P_A(\overline{Q}^f)$ is an ideal of A. There exist a finitely generated sub-ideal $I_0 := (\alpha_1, ..., \alpha_m)$ of I and a positive integer k_1 such that for any $a \in I$, $s_1 a^{k_1} \in I_0$. For each $i \in \{1, ..., m\}$, take any element $\beta_i \in J$ such that $f(\alpha_i) + \beta_i \in Q$. Let L_1 be the ideal of $A \bowtie^f J$ generated by the set $\{(\alpha_i, f(\alpha_i) + \beta_i) \mid i = 1, ..., m\}$.

Note that $Q_1 := Q \cap J$ is an ideal of $f(A) + J$. There exist a finitely generated sub-ideal $Q'_1 := (\gamma_1, ..., \gamma_p)$ of Q_1 and a positive integer k_2 such that for any $b \in Q_1$, $f(s_2)b^{k_2} \in Q'_1$. Let L_2 be the ideal of $A \bowtie^f J$ generated by the set $\{(0, \gamma_i) \mid i = 1, ..., p\}$.

Now, Let $(a, f(a) + b)$ be an element of $\overline{Q}^f$. Then $f(a) + b \in (Q + J) \cap (f(A) + J)$ which implies that $\overline{f(a) + b} \in \overline{(Q + J) \cap (f(A) + J)} = Q_0$. Then

$$\overline{f(s_3)}\,\overline{(f(a) + b)}^{k_0} = \sum_{i=1}^{n} \overline{(f(a_i) + b_i)(f(c_i) + d_i)}$$

for some $c_1, ..., c_n \in A$ and $d_1, ..., d_n \in J$. Let

$$X = f(s_3)(f(a) + b)^{k_0} - \sum_{i=1}^{n}(f(a_i) + b_i)(f(c_i) + d_i).$$

Then $X \in J$. We have

$$\begin{aligned} X &= f(s_3)(f(a)+b)^{k_0} - \sum_{i=1}^{n}(f(a_i)+b_i)(f(c_i)+d_i) \\ &= f((s_3a^{k_0}) - \sum_{i=1}^{n} a_ic_i) + \sum_{i=0}^{k_0-1} C_{k_0}^{i} f(s_3a^{i})b^{k_0-i} - \sum_{i=1}^{n} b_if(c_i) + d_if(a_i) + b_id_i. \end{aligned}$$

This implies that $f((s_3a^{k_0}) - \sum_{i=1}^{n} a_ic_i) \in J$. Let

$$Y = s_3a^{k_0} - \sum_{i=1}^{n} a_ic_i.$$

Then $Y \in f^{-1}(J)$, hence we obtain

$$\begin{aligned} (s_3, f(s_3))(a, f(a)+b)^{k_0} &= (s_3a^{k_0}, f(s_3)(f(a)+b)^{k_0}) \\ &= (Y + \sum_{i=1}^{n} a_ic_i, X + \sum_{i=1}^{n}(f(a_i)+b_i)(f(c_i)+d_i)) \\ &= (Y, X) + \sum_{i=1}^{n}(a_i, f(a_i)+b_i)(c_i, f(c_i)+d_i). \end{aligned}$$

Since for all $i \in \{1, ..., n\}$, $f(a_i) + b_i \in Q$,

$$Z_1 = \sum_{i=1}^{n}(c_i, f(c_i)+d_i)(a_i, f(a_i)+b_i) \in \overline{Q}^{f}.$$

Thus $(Y, X) \in \overline{Q}^{f}$. Let $e = f(s_3)\sum_{i=1}^{k_0} C_{k_0}^{i} f(a)^{k_0-i}b^{i} - \sum_{i=1}^{n}(f(a_i)d_i + b_if(c_i) + b_id_i)$; so $(Y, X) = (Y, f(Y) + e)$. Since $Y \in f^{-1}(J)$ and $(Y, X) \in \overline{Q}^{f}$, $Y \in P_A(\overline{Q}^{f})$, and so $Y \in I = f^{-1}(J) \cap P_A(\overline{Q}^{f})$. Therefore $s_1Y^{k_1} = \sum_{i=1}^{m} \alpha_ir_i$ for some $r_1, ..., r_m \in A$. Hence we obtain

$$\begin{aligned} (s_1, f(s_1))(Y, X)^{k_1} &= (s_1, f(s_1))(Y, f(Y)+e)^{k_1} \\ &= (s_1Y^{k_1}, f(s_1)(f(Y)+e)^{k_1}) \\ &= (s_1Y^{k_1}, f(s_1)f(Y)^{k_1} + j_1) \\ &= (\sum_{i=1}^{m} \alpha_ir_i, \sum_{i=1}^{m}(f(\alpha_i)+\beta_i)f(r_i) + j_2) \\ &= \sum_{i=1}^{m}(\alpha_i, f(\alpha_i)+\beta_i)(r_i, f(r_i)) + (0, j_2), \end{aligned}$$

where $j_1 = f(s_1)\sum_{i=1}^{k_1} C_{k_1}^{i} f(Y)^{k_1-i}e^{i}$ and $j_2 = j_1 - \sum_{i=1}^{m} \beta_if(r_i)$.

Let

$$Z_2 = \sum_{i=1}^{m}(r_i, f(r_i))(\alpha_i, f(\alpha_i)+\beta_i) \in L_1.$$

Since $(\alpha_i, f(\alpha_i)+\beta_i) \in \overline{Q}^f$ for all $i \in \{1, ..., m\}$, $Z_2 \in \overline{Q}^f$, so $(0, j_2) \in \overline{Q}^f$. Therefore we can find a positive integer k_2 such that $f(s_2)j_2^{k_2} = \sum_{i=1}^{p} \gamma_i(f(x_i) + y_i)$ for some $x_1, ..., x_p \in A$ and $y_1, ..., y_p \in J$. Then

$$(s_2, f(s_2))(0, j_2)^{k_2} = \sum_{i=1}^{p}(x_i, f(x_i) + y_i)(0, \gamma_i) \in L_2.$$

Put $k := k_0k_1k_2$ and $t := s^k\zeta_k = (s_1s_2s_3)^k\zeta_k$.

$$\begin{aligned}
(t, f(t))(a, f(a) + b)^k &= ((s_1s_2s_3)^k\zeta_k, f((s_1s_2s_3)^k\zeta_k)) \left[(a, f(a) + b)^{k_0})\right]^{k_1k_2} \\
&= ((s_1s_2)^k s_3^{k-k_1k_2}\zeta_k, f((s_1s_2)^k s_3^{k-k_1k_2}\zeta_k)) \left[(s_3, f(s_3))(a, f(a) + b)^{k_0}\right]^{k_1k_2} \\
&= ((s_1s_2)^k s_3^{k-k_1k_2}\zeta_k, f((s_1s_2)^k s_3^{k-k_1k_2}\zeta_k)) \left[(Y, X) + Z_1\right]^{k_1k_2} \\
&= ((s_1s_2)^k s_3^{k-k_1k_2}\zeta_k, f((s_1s_2)^k s_3^{k-k_1k_2}\zeta_k)) \left[((Y, X) + Z_1)^{k_1}\right]^{k_2} \\
&= (s_1^{k-k_2}s_2^k s_3^{k-k_1k_2}\zeta_k, f(s_1^{k-k_2}s_2^k s_3^{k-k_1k_2}\zeta_k)) \left[(s_1, f(s_1)) \sum_{i=0}^{k_1} C_{k_1}^i Z_1^i (Y, X)^{k_1-i}\right]^{k_2} \\
&= (s_1^{k-k_2}s_2^k s_3^{k-k_1k_2}\zeta_k, f(s_1^{k-k_2}s_2^k s_3^{k-k_1k_2}\zeta_k)) \left[(s_1, f(s_1))(Y, X)^{k_1} + Z_3\right]^{k_2} \\
&= (s_1^{k-k_2}s_2^k s_3^{k-k_1k_2}\zeta_k, f(s_1^{k-k_2}s_2^k s_3^{k-k_1k_2}\zeta_k)) \left[Z_2 + (0, j_2) + Z_3\right]^{k_2} \\
&= (s_1^{k-k_2}s_2^k s_3^{k-k_1k_2}\zeta_k, f(s_1^{k-k_2}s_2^k s_3^{k-k_1k_2}\zeta_k)) \sum_{i=1}^{k_2} C_{k_2}^i (Z_2 + Z_3)^i (0, j_2)^{k_2-i} \\
&\quad + (s_1^{k-k_2} s_3^{k-k_1k_2}\zeta_k, f(s_1^{k-k_2} s_3^{k-k_1k_2}\zeta_k))(s_2, f(s_2))^k (0, j_2)^{k_2}.
\end{aligned}$$

with $Z_3 = (s_1, f(s_1)) \sum_{i=1}^{k_1} C_{k_1}^i Z_1^i (Y, X)^{k_1-i} \in L_0$. Note that $Z_2 \in L_1$ and $Z_3 \in L_0$; so $\sum_{i=1}^{k_2} C_{k_2}^i (Z_2 + Z_3)^i (0, j_2)^{k_2-i} \in L_0 + L_1$. As $(s_2, f(s_2))(0, j_2)^{k_2} \in L_2$, $(s_2, f(s_2))^k(0, j_2)^{k_2} \in L_2$. Therefore

$$(t, f(t))(a, f(a) + b)^k \in (L_0 + L_1 + L_2).$$

Note that $L_0 + L_1 + L_2$ is a finitely generated sub-ideal of $\overline{Q}^f$.

Case 2: $\mathcal{P} = P \bowtie^f J$ for some prime ideal P of A.

There exist $p_1, ..., p_n \in P$, $b_1, ..., b_r \in J$ such that for any $a \in P, b \in J$, $s_1a^m \in (p_1, ..., p_n)$ and $f(s_2)b^q \in (b_1, ..., b_r)$ for some positive integers m, q. Put $t = s^k\zeta_k$ in (Case 1). Let $a \in P$ and $j \in J$.

$$\begin{aligned}
(t, f(t))(a, f(a) + j)^{m+q} &= (t, f(t))((a, f(a)) + (0, j))^{m+q} \\
&= (t, f(t)) \sum_{i=0}^{m+q} C_{m+q}^i (a, f(a))^i (0, j)^{m+q-i} \\
&= (t, f(t))(\sum_{i=0}^{m} C_{m+q}^i (a, f(a))^i (0, j)^{m+q-i} + \sum_{i=m+1}^{m+q} C_{m+q}^i (a, f(a))^i (0, j)^{m+q-i}) \\
&= \sum_{i=0}^{m} C_{m+q}^i (t, f(t))(a, f(a))^i (0, j)^q (0, j)^{m-i} + \sum_{i=m+1}^{m+q} C_{m+q}^i (t, f(t))(a, f(a))^m (a, f(a))^{i-m} (0, j)^{m+q-i} \\
&= \sum_{i=0}^{m} C_{m+q}^i (a, f(a))^i (0, j)^{m-i} \sum_{\alpha=1}^{r} (0, b_\alpha(f(x_\alpha) + y_\alpha))
\end{aligned}$$

$$+\sum_{i=m+1}^{m+q} C^i_{m+q}(a, f(a))^{i-m}(0, j)^{m+q-i}\sum_{\beta=1}^{n}(p_\beta r_\beta, f(p_\beta r_\beta))$$
$$\in \quad ((0, b_\alpha), (p_\beta, f(p_\beta)), 1 \le \alpha \le r, 1 \le \beta \le n).$$

So $(t, f(t))(a, f(a) + j)^{m+q} \in ((0, b_\alpha), (p_\beta, f(p_\beta)), 1 \le \alpha \le r, 1 \le \beta \le n)$. Since $a_\beta \in P$ for all $1 \le \beta \le n$ and $b_\alpha \in J$ for all $1 \le \alpha \le r$, then $((0, b_\alpha), (p_\beta, f(p_\beta)), 1 \le \alpha \le r, 1 \le \beta \le n) \subseteq \mathcal{P}$. □

The *amalgamated duplication of a ring R along an ideal I* is a ring that is defined as the following sub-ring of $R \times R$ (as a particular case of the amalgamation) [9]:

$$R \bowtie I = \{(r, r + i) | r \in R, i \in I\}.$$

Let $S' = \{(s, s) \mid s \in S\}$, where S is an anti-Archimedean multiplicative subset of R. Then S' is an anti-Archimedean multiplicative subset of $R \bowtie I$. Combining Theorem 3.3 and Theorem 2.23, we obtain the following Corollaries.

Corollary 3.4. *The following statements are equivalent for a commutative ring R.*

1. *R is a uniformly S-SFT ring.*
2. *$R \bowtie I$ is a uniformly S'-SFT ring.*
3. *$R \times R$ is a uniformly $S \times S$-SFT ring.*

Corollary 3.5. *Let R be a ring, I an ideal of R, $s : R \mapsto R/I$ be the canonical homomorphism, and J an ideal of R/I. Then $R \bowtie^s J$ is a uniformly S-SFT-ring if and only if R is a uniformly S-SFT-ring.*

Proof. We have $s(R) + J = R/I + J = R/I$. By Theorem 2.10(3), if R is an uniformly S-SFT-ring, so is R/I. □

Let R be a commutative ring with identity and M a unitary R-module. Then the *Nagata's idealization* of M in R (or *trivial extension* of R by M) is the commutative ring

$$R(+)M := \{(r, m) \,|\, r \in R \text{ and } m \in M\}$$

Endowed with the usual addition and the multiplication defined by $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ for all $(r_1, m_1), (r_2, m_2) \in R(+)M$. It is clear that $(1, 0)$ is the identity of $R(+)M$. It was shown that if Q is a prime ideal of $R(+)M$, then $Q = P(+)M$ for some prime ideal P of R. Conversely if P is a prime ideal of R, then $P(+)M$ is a prime ideal of $R(+)M$ [14, Theorem 25.1(3)] (or [3, Theorem 3.2(2)]).

It is clear that if S is a multiplicative subset of R and N a submodule of M, then $S(+)N$ is a multiplicative subset of $R(+)M$. Our next result give a necessary and sufficient condition for the Nagata's idealization $R(+)M$ to be uniformly $(S(+)N)$-SFT ring. First, we need the following Remark.

Remark 3.6. *Let R be a commutative ring with identity and M a unitary R-module. If S is an anti-Archimedean multiplicative subset of R, then $(S(+)\{0\})$ is an anti-Archimedean multiplicative subset of $R(+)M$. Indeed, let $s \in S$ and $t \in S \bigcap(\bigcap_{n\in\mathbb{N}} s^nR)$. Then for all $n \in \mathbb{N}$, $t = s^na_n$ for some $a_n \in R$. Thus for all $n \in \mathbb{N}$,*

$$(t, 0) = (s^na_n, 0) = (s, 0)^n(a_n, 0).$$

So $(t, 0) \in S(+)\{0\} \bigcap(\bigcap(s, 0)^nR(+)M)$.

It is clear that $S \cap Nil(R) = \emptyset$ if and only if $(S(+)\{0\}) \cap Nil(R(+)M) = \emptyset$.

Theorem 3.7. *Let R be a commutative ring with identity, S an anti-Archimedean multiplicative subset of R disjoint from $Nil(R)$ and M a unitary R-module. Then the following statements are equivalent.*

1. *R is a uniformly S-SFT ring.*
2. *$R(+)M$ is an $(S(+)\{0\})$-SFT ring.*
3. *$R(+)M$ is an $(S(+)N)$-SFT ring.*

Proof. (1) ⇒ (2). Suppose that R is a uniformly S-SFT ring. There exists an $s \in S$ such that any ideal I of R is strong finite type with respect to s. Let $\mathcal{P} = P(+)M$ be a prime ideal of $R(+)M$, where P is a prime ideal of R. Then P is of strong finite type with respect to s. There exist a finitely generated sub-ideal J of P and positive integer r such that for any $a \in P$, $sa^r \in J$.

Let $(a, m) \in \mathcal{P}$. We show that $(s, 0)(a, m)^{r+1} \in J(+)JM$. Since $a \in P$, $(sa)^r \in J$. Then

$$(s, 0)(a, m)^{r+1} = (s, 0)(a^{r+1}, (r + 1)a^r m) = (\underbrace{sa^{r+1}}_{\in J}, \underbrace{(r + 1)sa^r m}_{\in JM}) \in J(+)JM.$$

Note that $J(+)JM = (J \times \{0\})R(+)M$. As J is a finitely generated sub-ideal of I, there exist $j_1, ..., j_t \in I$ such that $J = (j_1, ..., j_t)R$. So $J(+)JM = (J \times \{0\})R(+)M = ((j_1, 0), ..., (j_t, 0))R(+)M$. This implies that $J(+)JM$ is a finitely generated sub-ideal of $\mathcal{P}$ and $(s, 0) \in S(+)\{0\}$. Thus $\mathcal{P}$ is of strong finite type ideal of $R(+)M$, with respect to $(s, 0)$, and hence $R(+)M$ is a uniformly $(S(+)\{0\})$-SFT ring.

(2)⇒(3). As $S(+)\{0\} \subseteq S(+)N$, by Theorem 2.10(1), $R(+)M$ is a uniformly $(S(+)N)$-SFT ring.

(3)⇒(1). Follows from Theorem 2.10 (3) and the fact that the naturel mapping $\Phi : R(+)M \to R$ defined by $\Phi(r, m) = r$ is a surjective ring homomorphism with $\Phi(S(+)N) = S$. □

Let R be a commutative ring and S a multiplicative subset of R. If R is uniformly S-Noetherian, then R is uniformly S-SFT. This implication follows from the fact that if R is uniformly S-Noetherian, then there exists an $s \in S$ such that for every ideal I there exists a finitely generated sub-ideal of I such that $sI \subseteq J \subseteq I$. So for every $x \in I$, $sx \in sI \subseteq J \subseteq I$ which implies that I is of strong finite type ideal with respect to s. The converse is not necessarily true, which means that there exist rings that are uniformly S-SFT but not uniformly S-Noetherian.

Example 3.8. *Let $R = \mathbb{Z}(+)\mathbb{Z}[X]$ and $S = \{1\}$ is an anti-archimedean multiplicative set. Then $S(+)\mathbb{Z}[X]$ is a multiplicative subset of $\mathbb{Z}(+)\mathbb{Z}[X]$. Since $\mathbb{Z}$ is a uniformly S-SFT ring, by Theorem 3.7, $\mathbb{Z}(+)\mathbb{Z}[X]$ is a uniformly $(S(+)\mathbb{Z}[X])$-SFT ring. Now, by [17, Proposition 3.1], if $\mathbb{Z}(+)\mathbb{Z}[X]$ is a uniformly $(S(+)\mathbb{Z}[X])$-Noetherian ring, then $\mathbb{Z}$ is a uniformly S-Noetherian ring and $\mathbb{Z}[X]$ is a uniformly S-Noetherian $\mathbb{Z}$-module. This implies that $\mathbb{Z}[X]$ is an S-finite $\mathbb{Z}$-module a contradiction. So $\mathbb{Z}(+)\mathbb{Z}[X]$ is a uniformly $(S(+)\mathbb{Z}[X])$-SFT ring which is not uniformly $(S(+)\mathbb{Z}[X])$-Noetherian.*

Example 3.9. *Let $R = \mathbb{Z}/6\mathbb{Z}$, $M = \mathbb{Z}/6\mathbb{Z}[X]$ and $S = \{\overline{1}, \overline{3}\}$. By [13], S is an anti-archimedean multiplicative subset of R. Then $S(+)M$ is a multiplicative subset of $R(+)M$. Since R is a uniformly S-SFT ring, by Theorem 3.7, $R(+)M$ is a uniformly $(S(+)M)$-SFT ring. Now, by [17, Proposition 3.1], if $R(+)M$ is a uniformly $(S(+)M)$-Noetherian ring, then M is an S-finite R-module a contradiction.*

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