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Co-derivative and derivative operators on commutative bounded integral residuated lattices

Ahmet Hamal[a]

[a]*Department of Mathematics, Ege University, 35100 Bornova, Izmir-Turkey*

Abstract. In this paper we introduce and investigate multiplicative co-derivative operators and additive derivative operators on commutative bounded integral residuated lattices. Moreover, we represent connections between multiplicative co-derivative operators and additive derivative operators on commutative bounded integral residuated lattices. Also, we indicate that multiplicative co-derivative operators are generalizations of multiplicative interior operators on this lattices. Then we describe connections between multiplicative co-derivative operators (additive derivative operators) on commutative bounded integral residuated lattice with Glivenko property and on the residuated lattices of their reguler elements. Finally, we study some properties of the commutative bounded integral residuated lattices with multiplicative co-derivative operator.

1. Introduction

The commutative residuated lattices as generalization of ideal lattices of rings were studied by M. Ward and R.P. Dilworth in [29]. The class of commutative bounded integral residuated lattices contains some class of algebras such as MV-algebras [8, 22], BL-algebras [17] and commutative Rℓ-monoids [13] as the the algebraic counterparts of many-valued and fuzzy logics. Also, Heyting algebras, which were introduced as algebric counterpart of the intuitionistic propositional logic in [3], can be considered as commutative bounded integral lattices.

Closure (interior) operators are very useful tools in both pure and applied mathematics. Many researchers have studied closure (interior) operators on different structures such as (fuzzy) topological spaces, lattices [12], Hoop-algebras [5, 28], MV-algebras [26], commutative Rℓ-monoids [27] and commutative bounded integral residuated lattices [24].

Mckinsey and Tarski in [21] said that some topological operations apart from closure operation can be treated in an algebraic way and in this way, interesting results can be obtained. Thus, the derivative algebras were introduced by Mckinsey and Tarski in [21]. Then the derivative algebras were redefined in a more general way by Esakia in [14]. In [16] derivative MV-algebras as generalizations of derivative algebras were introduced and investigated by favour of additive derivative operators.

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Email address: ahmet.hamal@ege.edu.tr (Ahmet Hamal)
ORCID iD: https://orcid.org/0000-0002-4593-7054 (Ahmet Hamal)

Modal operators were introduced and investigated on various algebras such as Heyting algebras [20] and bounded residuated ℓ-monoids [25]. Also, monotone modal operators were studied on Hoop-Algebras [28], bounded residuated lattices [23] and commutative bounded residuated lattices [19].

In the paper we introduce and study multiplicative co-derivative operators (mcd-operators) and additive derivative operators (ad-operators) on commutative bounded integral residuated lattices. We indicate that any multiplicative interior operator on commutative bounded integral residuated lattice serves as mcd-operator. However, an mcd-operator on commutative bounded integral residuated lattices may not be multiplicative interior operator. Also, any monotone modal operator on commutative bounded integral residuated lattice is an mcd-operator. However, an mcd-operator on commutative bounded integral residuated lattices may not be (monotone) modal operator. So, it is worth considiring mcd-operators on commutative bounded integral residuated lattices. After that, we investigate connections between mcd-operators and ad-operators. In section 4 we describe connections between mcd-and ad-operators on residuated lattice with Glivenko property and on the residuated lattices of their reguler elements. In section 5 we show that Boolean algebra $B(A)$ which is the set of complemented elements of A may not be co-derivative subalgebra with respect to the mcd-operator on A. In section 6 we introduce θ-filters and study on them. Moreover, we study mcd-operators on quotient commutative bounded integral residuated lattice.

2. Preliminaries

We remember that an algebra $\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called a *commutative bounded integral residuated lattice* (see, [1, 11, 18, 19, 24]) if it satisfies the following conditions:

(i) $\mathcal{A} = (A, \odot, 1)$ is a commutative monoid, i.e., $\odot$ is a commutative, associative and $x \odot 1 = x$ for all $x \in A$,

(ii) $\mathcal{A} = (A, \wedge, \vee, 0, 1)$ is a bounded lattice,

(iii) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for all $x, y, z \in A$.

We will write *residuated lattice* for short instead of commutative bounded integral residuated lattice. We define a unary operator $^-$ such that $x^- := x \rightarrow 0$ and a binary operator $\oplus$ such that $x \oplus y = (x^- \odot y^-)^-$ on any residuated lattice $\mathcal{A}$.

A residuated lattice $\mathcal{A}$ is said to be *involutive* if it satisfies the following property

(iv) $x^{--} = x$ for all $x \in A$.

A residuated lattice $\mathcal{A}$ is said to be an *Rℓ-monoid* if it satisfies the following property

(v) $(x \rightarrow y) \odot x = x \wedge y$ for all $x, y \in A$.

We have the following results.

Proposition 2.1. ([11, 18, 24]) *$\mathcal{A}$ be a residuated lattice. We have:*

(1) $x \leq y$ *implies* $y^- \leq x^-$,

(2) $x \odot y \leq x \wedge y$,

(3) $(x \rightarrow y) \odot x \leq y$,

(4) $x \leq x^{--}$,

(5) $x^{---} = x^-$,

(6) *If* $x \leq y$, *then* $z \rightarrow x \leq z \rightarrow y$, $y \rightarrow z \leq x \rightarrow z$ *and* $x \odot z \leq y \odot z$,

(7) $x^{--} \rightarrow y^{--} = x \rightarrow y^{--}$,

(8) $(x \to y^{--})^{--} = x \to y^{--}$,

(9) $(x \odot y)^{-} = y \to x^{-} = x \to y^{-} = x^{--} \to y^{-} = y^{--} \to x^{-}$,

(10) $(x \odot y)^{--} \geq x^{--} \odot y^{--}$,

for any $x, y, z \in A$.

Lemma 2.2. ([11, 19, 24]) *Suppose that* $\mathcal{A}$ *is a residuated lattice. For any* $x, y \in A$*, we have:*

(1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(2) $x \oplus y \geq x^{--} \vee y^{--} \geq x \vee y$,

(3) $x \oplus 0 = x^{--}$,

(4) $(x \oplus y)^{--} = x^{--} \oplus y^{--} = x^{--} \oplus y = x \oplus y$,

(5) $x \odot x^{-} = 0, x \oplus x^{-} = 1$.

A residuated lattice $\mathcal{A}$ is said to be *normal* if it satisfies

$$(x \odot y)^{--} = x^{--} \odot y^{--}.$$

Lemma 2.3. ([24]) *Suppose that residuated lattice* $\mathcal{A}$ *is normal . Then we have:*

(1) $(x \oplus y)^{-} = x^{-} \odot y^{-}$,

(2) $(x \odot y)^{-} = x^{-} \oplus y^{-}$.

for any $x, y \in A$.

3. Co-derivative and derivative operators on residuated lattices

Definition 3.1. Let $\mathcal{A}$ be a residuated lattice. A mapping $\theta : A \longrightarrow A$ is said to be *a multiplicative co-derivative operator (mcd-operator)* on residuated lattice $\mathcal{A}$ if it satisfies the following conditions for each $x, y \in A$:

$(T1)$ $\theta(x \odot y) = \theta(x) \odot \theta(y)$,

$(T2)$ $x \odot \theta(x) \leq \theta\theta(x)$,

$(T3)$ $\theta(1) = 1$,

$(T4)$ $x \leq y \Longrightarrow \theta(x) \leq \theta(y)$.

θ is said to be a *strong mcd-operator* on $\mathcal{A}$ if it has the below property

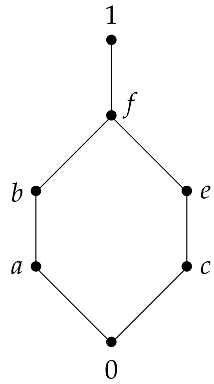
$(T5)$ $\theta(x) \leq \theta\theta(x)$

for any $x \in A$.

Theorem 3.2. *The axioms of an mcd-operator on residuated lattice* $\mathcal{A}$ *are independent.*

Proof. We have to find a model for each axiom in which the others are true while the axiom is false.

(T1) Let $A = \{0, a, b, c, e, f, 1\}$. Define $\odot$ and $\to$ as follows:



$\odot$	0	a	b	c	e	f	1
0	0	0	0	0	0	0	0
a	0	a	a	0	0	a	a
b	0	a	a	0	0	a	b
c	0	0	0	c	c	c	c
e	0	0	0	c	c	c	e
f	0	a	a	c	c	f	f
1	0	a	b	c	e	f	1

$\to$	0	a	b	c	e	f	1
0	1	1	1	1	1	1	1
a	e	1	1	e	e	1	1
b	e	f	1	e	e	1	1
c	b	b	b	1	1	1	1
e	b	b	b	f	1	1	1
f	0	b	b	e	e	1	1
1	0	a	b	c	e	f	1

Then $\mathcal{A}$ is a residuated lattice (see, [4]). We define $\theta : A \longrightarrow A$ operator as follows:

x	0	a	b	c	e	f	1
$\theta(x)$	0	b	b	c	e	f	1

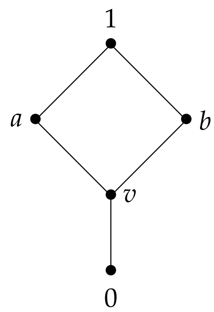
Then it is apparent that θ satisfies (T2), (T3) and (T4) axioms. We able to demonstrate that θ doesn't satisfy (T1). Really, we get $x = a$ and $y = a$. Then,

$\theta(a \odot a) = \theta(a) = b,$

$\theta(a) \odot \theta(a) = b \odot b = a,$

$\theta(a \odot a) \neq \theta(a) \odot \theta(a).$

(T2) Let $A = \{0, a, b, v, 1\}$. Define $\odot$ and $\to$ as follows:



$\odot$	0	v	a	b	1
0	0	0	0	0	0
v	0	v	v	v	v
a	0	v	a	v	a
b	0	v	v	b	b
1	0	v	a	b	1

$\rightarrow$	0	v	a	b	1
0	1	1	1	1	1
v	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	v	a	b	1

Then $\mathcal{A}$ is a residuated lattice (see, [4]). We define $\theta : A \longrightarrow A$ operator as follows:

x	0	v	a	b	1
$\theta(x)$	0	0	v	0	1

Then θ satisfies (T1), (T3) and (T4) properties. we take $x = a$. Then

$a \odot \theta(a) = a \odot v = v$

$\theta\theta(a) = \theta(\theta(a)) = \theta(v) = 0$. So, we have $a \odot \theta(a) \nleq \theta\theta(a)$.

(T3) Suppose that $\mathcal{A}$ is a residuated lattice. We describe the function $\theta : A \longrightarrow A$ by $\theta(x) = 0$ for all $x \in A$. Then it is easy to see that θ provides (T1), (T2) and (T4) axioms. However, the axiom (T3) is not satisfied.

(T4) Let $A = \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$, $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$ for any $x, y \in A$. Then $\mathcal{A} = (A, \wedge, \vee, 0, 1)$ is a bounded lattice. We define $\odot$ and $\rightarrow$ as follows:

$$x \odot y = \begin{cases} y, & x = 1 \\ x, & y = 1 \\ 0, & otherwise \end{cases}$$

and

$$x \rightarrow y = \begin{cases} 1, & x \leq y \\ y, & x = 1 \\ 0.9, & otherwise \end{cases}$$

Then $\mathcal{A}$ is a residuated lattice (see, [23]). Let $\theta : A \longrightarrow A$ be the mapping as follows:

$$\theta(x) = \begin{cases} 1, & x = 1 \\ 0, & x = 0 \\ 1 - x, & otherwise \end{cases}$$

Then it is clear that θ satisfies (T1), (T2) and (T3) axioms. Nevertheless, the mapping θ doesn't satisfy the axiom (T4). Because, we have $0.5 \leq 0.8$ but $\theta(0.5) = 0.5 \nleq \theta(0.8) = 0.2$. □

Lemma 3.3. *Suppose that $\mathcal{A}$ is an $R\ell$-monoid and $\theta : A \longrightarrow A$ is a mapping. If θ satisfies (T1)-(T3) axioms, then θ is an mcd-operator on $\mathcal{A}$.*

Proof. We have to show that θ satisfies the axiom (T4). Let $x, y \in A$ such that $x \leq y$. Then $\theta(x) = \theta(y \wedge x) = \theta((y \rightarrow x) \odot y) = \theta(y \rightarrow x) \odot \theta(y) \leq \theta(y \rightarrow x) \wedge \theta(y) \leq \theta(y)$. □

Definition 3.4. ([24]) Suppose that $\mathcal{A}$ is a residuated lattice and $f : A \longrightarrow A$ is a mapping. If f satisfies the below conditions for any $x, y \in A$:

$(I1)$ $f(x \odot y) = f(x) \odot f(y)$,

$(I2)$ $f(x) \leq x$,

$(I3)$ $ff(x) = f(x)$,

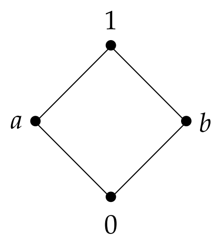
$(I4)$ $f(1) = 1$,

$(I5)$ $x \leq y \Longrightarrow f(x) \leq f(y)$,

then f is said to be *a multiplicative interior operator (mi-operator)* on $\mathcal{A}$.

Remark 3.5. Suppose that f is a mi-operator on residuated lattice $\mathcal{A}$. It is clear that f satisfies (T1), (T3), (T4) and (T5) axioms. Then f is a strong mcd-operator. However, an mcd-operator may not be a mi-operator.

Example 3.6. Let $A = \{0, a, b, 1\}$. Define $\odot$ and $\rightarrow$ as follows:



$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Then $\mathcal{A}$ is a residuated lattice. We define $\theta : A \longrightarrow A$ operator as follows:

x	0	a	b	1
$\theta(x)$	0	b	a	1

Then it is clear that θ is an mcd-operator. However, θ isn't mi-operator. Because, θ doesn't satisfy (I2) and (I3) axioms. Indeed, we take $x = a$. Then $\theta(a) = b \nleq a$, $\theta(\theta(a)) = \theta(b) = a \nleq \theta(a) = b$ and $\theta(a) = b \nleq \theta(\theta(a)) = \theta(b) = a$.

Definition 3.7. ([19, 28]) $f : A \longrightarrow A$ is called a modal operator on a residuated lattice $\mathcal{A}$ if for all $x, y \in A$, it satisfies the following conditions:

$(M1)$ $f(x \odot y) = f(x) \odot f(y)$,

$(M2)$ $x \leq f(x)$,

$(M3)$ $ff(x) = f(x)$.

A modal operator f is called monotone, if for any $x, y \in A$,

$(M4)$ $x \leq y \Longrightarrow f(x) \leq f(y)$.

Remark 3.8. Suppose that f is a monotone modal operator on residuated lattice $\mathcal{A}$. It is clear that f satisfies (T1), (T3), (T4) and (T5) axioms. Then f is a strong mcd-operator. However, an mcd-operator may not be a (monotone) modal operator.

Example 3.9. We consider the residuated lattice $\mathcal{A} = (\{0, a, b, 1\}, \wedge, \vee, \odot, \rightarrow, 0, 1)$ and the mcd-operator θ from Example 3.6. θ isn't a (monotone) modal operator. Because, θ doesn't satisfy (M2) and (M3) axioms.

Proposition 3.10. *Let θ be an mcd-operator on a residuated lattice $\mathcal{A}$. Then for any $x, y \in A$*

(1) $\theta(x \rightarrow y) \leq \theta(x) \rightarrow \theta(y)$,

(2) $x \leq \theta(x) \rightarrow \theta\theta(x)$, $\theta(x) \leq x \rightarrow \theta\theta(x)$

(3) $\theta(\theta(x) \wedge \theta(y)) \leq \theta\theta(x) \wedge \theta\theta(y)$.

Proof. **(1)** Let $x, y \in A$. Then $(x \rightarrow y) \odot x \leq y$ by Proposition 2.1 (3). We have $\theta((x \rightarrow y) \odot x) = \theta(x \rightarrow y) \odot \theta(x)$. Since θ is monotone, we must have $\theta(x \rightarrow y) \odot \theta(x) \leq \theta(y)$ and $\theta(x \rightarrow y) \leq \theta(x) \rightarrow \theta(y)$.

(2) it is clear that $x \leq \theta(x) \rightarrow \theta\theta(x)$ and $\theta(x) \leq x \rightarrow \theta\theta(x)$ by (T2).

(3) We have $\theta(x) \wedge \theta(y) \leq \theta(x)$ and $\theta(x) \wedge \theta(y) \leq \theta(y)$. Then $\theta(\theta(x) \wedge \theta(y)) \leq \theta\theta(x)$ and $\theta(\theta(x) \wedge \theta(y)) \leq \theta\theta(y)$ by (T4). So, we have $\theta(\theta(x) \wedge \theta(y)) \leq \theta\theta(x) \wedge \theta\theta(y)$. □

Let $\mathcal{A}$ be a residuated lattice and $h : A \rightarrow A$ be a mapping on $\mathcal{A}$. We define a mapping $h^- : A \rightarrow A$ by

$$h^-(x) = (h(x^-))^-.$$

Proposition 3.11. ([24]) *Suppose that $\mathcal{A}$ is a residuated lattice and h is a mapping on residuated lattice $\mathcal{A}$. If h is monotone, then the mapping h^- is monotone.*

Proposition 3.12. *Let $\theta : A \longrightarrow A$ be an mcd-operator on a normal residuated lattice $\mathcal{A}$. Then for any $x, y \in A$ we have*

(1) $\theta^-(x \oplus y) = \theta^-(x) \oplus \theta^-(y)$,

(2) $\theta^-\theta^-(x) \leq x \oplus \theta^-(x)$,

(3) $\theta^-(0) = 0$,

(4) $x \leq y \Longrightarrow \theta^-(x) \leq \theta^-(y)$.

If θ is a strong mcd-operator, then for any $x \in A$ we have

(5) $\theta^-\theta^-(x) \leq \theta^-(x)$.

Proof. **(1)** $\theta^-(x \oplus y) = (\theta((x \oplus y)^-))^- = (\theta(x^- \odot y^-))^- = (\theta(x^-) \odot \theta(y^-))^- = (\theta(x^-))^- \oplus (\theta(y^-))^- = \theta^-(x) \oplus \theta^-(y)$.

(2) $\theta^-(\theta^-(x)) = \theta^-((\theta(x^-))^-) = (\theta((\theta(x^-))^{--}))^-$. We have $\theta(x^-) \leq (\theta(x^-))^{--}$ by Proposition 2.1 (4) and $\theta(\theta(x^-)) \leq \theta((\theta(x^-))^{--})$ by (T4) axiom. Moreover, we have $x^- \odot \theta(x^-) \leq \theta(\theta(x^-)) \leq \theta((\theta(x^-))^{--})$ and also, $(\theta((\theta(x^-))^{--}))^- \leq (x^- \odot \theta(x^-))^- = x^{--} \oplus (\theta(x^-))^- = x \oplus (\theta(x^-))^-$ by Lemma 2.2 (4). Hence, $\theta^-(\theta^-(x)) \leq x \oplus \theta^-(x)$.

(3) $\theta^-(0) = (\theta(0^-))^- = (\theta(1))^- = (1)^- = 0$.

(4) It is the result of Proposition 3.11.

(5) If θ is a strong mcd-operator, then $\theta^-(\theta^-(x)) = (\theta((\theta(x^-))^{--}))^- \leq (\theta(x^-))^- = \theta^-(x)$. □

Definition 3.13. Let $\mathcal{A}$ be a residuated lattice. A mapping $\omega : A \longrightarrow A$ is called *an additive derivative operator (ad-operator)* on residuated lattice $\mathcal{A}$ if it satisfies the following conditions for each $x, y \in A$:

($D1$) $\omega(x \oplus y) = \omega(x) \oplus \omega(y)$,

($D2$) $\omega\omega(x) \leq x \oplus \omega(x)$,

($D3$) $\omega(0) = 0$,

($D4$) $x \leq y \Longrightarrow \omega(x) \leq \omega(y)$.

ω is said to be a *strong ad-operator* on $\mathcal{A}$ if it has the below property

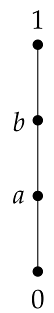
($D5$) $\omega\omega(x) \leq \omega(x)$

for any $x \in A$.

Theorem 3.14. *Let $\theta : A \longrightarrow A$ be an mcd-operator on a normal residuated lattice $\mathcal{A}$. Then the mapping $\theta^- : A \longrightarrow A$ is an ad-operator on $\mathcal{A}$. Moreover, if θ is a strong mcd-operator on $\mathcal{A}$, then θ^- is a strong ad-operator on $\mathcal{A}$.*

Proof. It follows from Proposition 3.12. □

Example 3.15. Let $A = \{0, a, b, 1\}$. Define $\odot$ and $\rightarrow$ as follows:



$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $\mathcal{A}$ is a normal residuated lattice. We define $\theta : A \longrightarrow A$ operator as follows:

x	0	a	b	1
$\theta(x)$	0	0	b	1

Then it is clear that θ is a strong mcd-operator. We have $x \oplus y = (x^- \odot y^-)^-$ and $\theta^-(x) = (\theta(x^-))^-$ for all $x, y \in A$ as follows:

$\oplus$	0	a	b	1
0	0	a	1	1
a	a	1	1	1
b	1	1	1	1
1	1	1	1	1

x	0	a	b	1
$\theta^-(x)$	0	1	1	1

Easily we can check that θ^- is a strong ad-operator.

Lemma 3.16. *Assume that $\mathcal{A}$ is a residuated lattice . If ω is an ad-operator on $\mathcal{A}$, then ω satisfies the following property*

$$\omega(x^{--}) = (\omega(x))^{--}.$$

Proof. We have $x^{--} = x \oplus 0$ for all $x \in A$ by Lemma 2.2 (3). Then $\omega(x^{--}) = \omega(x \oplus 0) = \omega(x) \oplus \omega(0) = \omega(x) \oplus 0 = (\omega(x))^{--}$. □

Proposition 3.17. *Suppose that $\mathcal{A}$ is a normal residuated lattice and $\omega : A \longrightarrow A$ is an ad-operator on $\mathcal{A}$. Then we get*

(1) $\omega^-(x \odot y) = \omega^-(x) \odot \omega^-(y)$,

(2) $x \odot \omega^-(x) \leq \omega^-\omega^-(x)$,

(3) $\omega^-(1) = 1$,

(4) $x \leq y \Longrightarrow \omega^-(x) \leq \omega^-(y)$,

for any $x, y \in A$. If ω is a strong ad-operator, then we get

(5) $\omega^-(x) \leq \omega^-\omega^-(x)$

for any $x \in A$.

Proof. **(1)** Let $x, y \in A$. Then we have
$\omega^-(x \odot y) = (\omega((x \odot y)^-))^- = (\omega(x^- \oplus y^-))^- = (\omega(x^-) \oplus \omega(y^-))^- = (\omega(x^-))^- \odot (\omega(y^-))^- = \omega^-(x) \odot \omega^-(y)$.

(2) We have $x \leq x^{--}$ for all $x \in A$ by Proposition 2.1 (4) and $x \odot \omega^-(x) \leq x^{--} \odot \omega^-(x)$ by Proposition 2.1 (6). Then $x^{--} \odot \omega^-(x) = x^{--} \odot (\omega(x^-))^- = (x^- \oplus \omega(x^-))^-$. Moreover, we have $\omega(\omega(x^-)) \leq x^- \oplus \omega(x^-)$ for any $x \in A$ by (D2) and $(x^- \oplus \omega(x^-))^- \leq (\omega(\omega(x^-)))^-$.

$(\omega(\omega(x^-)))^- = (\omega(\omega(x^-)))^{---}$ by Proposition 2.1 (5).
$= (\omega((\omega(x^-))^{--}))^-$ by Lemma 3.16
$= (\omega((\omega^-(x))^-))^- = \omega^-(\omega^-(x)) = \omega^-\omega^-(x)$.

(3) $\omega^-(1) = (\omega(1^-))^- = (\omega(0))^- = 0^- = 1$.

(4) It follows from Proposition 3.11.

(5) We assume that ω is a strong ad-operator. Then,
$\omega^-(x) = (\omega(x^-))^- \leq (\omega(\omega(x^-)))^- = (\omega(\omega(x^-)))^{---} = (\omega((\omega(x^-))^{--}))^- = (\omega((\omega^-(x))^-))^- = \omega^-(\omega^-(x))$. □

Theorem 3.18. *Let $\mathcal{A}$ be a normal residuated lattice and $\omega : A \longrightarrow A$ be an ad-operator on $\mathcal{A}$. Then the mapping $\omega^- : A \longrightarrow A$ is an mcd-operator on $\mathcal{A}$. Moreover, if ω is an strong ad-operator on $\mathcal{A}$, then ω^- is a strong mcd-operator on $\mathcal{A}$.*

Proof. It follows from Proposition 3.17. □

Example 3.19. Let $\mathcal{A}$ be any normal residuated lattice. We define $\omega(x) = 0$ for all $x \in A$. Then it is obvious that ω is an strong ad-operator on $\mathcal{A}$. We get $\omega^-(x) = 1$ for all $x \in A$. Then we can see that ω^- is a strong mcd-operator on $\mathcal{A}$.

Corollary 3.20. *Let $\mathcal{A}$ be a normal residuated lattice. If θ is an mcd-operator on $\mathcal{A}$, then θ^- is an ad-operator on $\mathcal{A}$ by Theorem 3.14. Moreover, if ω is an ad-operator on $\mathcal{A}$, then ω^- is an mcd-operator on $\mathcal{A}$ by Theorem 3.18.*

Let $\mathcal{A}$ be a normal residuated lattice. We shall denote by $mcd(\mathcal{A})$ the set of mcd-operators on $\mathcal{A}$ and by $ad(\mathcal{A})$ the set of ad-operators on $\mathcal{A}$. Assume that $mcd(\mathcal{A})$ and $ad(\mathcal{A})$ are pointwise ordered.

Theorem 3.21. *If $\mathcal{A}$ is a normal residuated lattice, then $\theta \leq \omega^-$ iff $\omega \leq \theta^-$, for any $\theta \in mcd(\mathcal{A})$ and $\omega \in ad(\mathcal{A})$.*

Proof. Let $\theta \in mcd(\mathcal{A})$ and $\omega \in ad(\mathcal{A})$ and $\theta \leq \omega^-$. Then $\theta(x) \leq \omega^-(x) = (\omega(x^-))^-$ and $(\omega(x^-))^{--} \leq (\theta(x))^-$ for any $x \in A$. So, $(\theta(x^-))^- \geq (\omega(x^{--}))^{--} \geq (\omega(x))^{--} \geq \omega(x)$ and $\theta^-(x) \geq \omega(x)$. As a result, $\theta^-(x) \geq \omega(x)$ for all $x \in A$ and $\theta^- \geq \omega$.

Conversely, let $\omega \leq \theta^-$. Then $\theta^-(x) \geq \omega(x)$ for any $x \in A$. We have $(\theta(x^-))^- \geq \omega(x)$ and $(\theta(x^-))^{--} \leq (\omega(x))^-$. So, $(\theta(x^{--}))^{--} \leq (\omega(x^-))^- = \omega^-(x)$ and $\omega^-(x) \geq (\theta(x^{--}))^{--} \geq (\theta(x))^{--} \geq \theta(x)$. Consequently, $\omega^-(x) \geq \theta(x)$ for all $x \in A$ and $\omega^- \geq \theta$. □

4. Residuated lattice with Glivenko property

Let $\mathcal{A}$ be a residuated lattice. If the following property is provided for any $x, y \in A$

$$(x \to y)^{--} = x \to y^{--}$$

then we say that a residuated lattice $\mathcal{A}$ has *Glivenko property* (see, [10]).

Proposition 4.1. ([10]) *A residuated lattice $\mathcal{A}$ has Glivenko property iff $\mathcal{A}$ satisfies the identity, for any $x \in A$,*

$$(x^{--} \to x)^{--} = 1.$$

An element x of a residuated lattice $\mathcal{A}$ is called regular if $x^{--} = x$. We shall denote by $Reg(A)$ to the set of all regular elements in A. We set $x \sqcup y := (x \vee y)^{--}$, $x \sqcap y := (x \wedge y)^{--}$, $x \rightsquigarrow y := (x \to y)^{--}$, $x \otimes y := (x \odot y)^{--}$ and $x \boxplus y := (x \oplus y)^{--}$ for $x, y \in Reg(A)$. Then $Reg(\mathcal{A}) = (Reg(A), \sqcap, \sqcup, \otimes, \rightsquigarrow, 0, 1)$ is an involutive residuated lattice. Note that $\sqcup$ and $\otimes$ are different, in general, from $\vee$ and $\odot$, respectively. Hence, $Reg(\mathcal{A})$ may not be a subalgebra of $\mathcal{A}$. We known that if $\mathcal{A}$ is a normal residuated lattice, then $x \otimes y = x \odot y$ for every $x, y \in Reg(A)$. We have $x \boxplus y = x \oplus y$ for any residuated lattice. Indeed, $x \boxplus y = (x \oplus y)^{--} = x \oplus y$ by Lemma 2.2 (4).

Theorem 4.2. ([7]) *The following conditions are equivalent for any residuated lattice $\mathcal{A}$.*

(1) *$\mathcal{A}$ has Glivenko property,*

(2) *the map $x \mapsto x^{--}$ defines a homomorphism from A onto $Reg(A)$.*

Proposition 4.3. ([24]) *A residuated lattice $\mathcal{A}$ has Glivenko property iff $(x \to y)^{--} = x^{--} \to y^{--}$, for any $x, y \in A$.*

Remark 4.4. ([24]) Every Rℓ-monoid satisfies the identity $(x \to y)^{--} = x^{--} \to y^{--}$. Because of this, it has Glivenko property .

Theorem 4.5. *Suppose that $\mathcal{A}$ is a normal residuated lattice with Glivenko property, $\theta : A \to A$ is an mcd-operator (a strong mcd-operator) on A. If $\varphi : Reg(A) \to Reg(A)$ is the mapping defined by $\varphi(x) = (\theta(x))^{--}$ for all $x \in Reg(A)$, then φ is an mcd-operator (a strong mcd-operator) on $Reg(A)$.*

Proof. **(T1)** Let $x, y \in Reg(A)$. Then $\varphi(x \otimes y) = (\theta(x \odot y))^{--} = (\theta(x) \odot (y))^{--} = (\theta(x))^{--} \odot (\theta(y))^{--} = \varphi(x) \odot \varphi(y) = \varphi(x) \otimes \varphi(y)$.

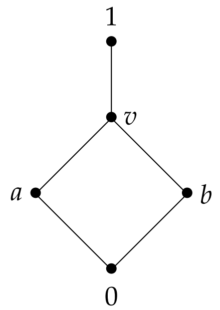
(T2) Let $x \in Reg(A)$. Then $x \otimes \varphi(x) = x \odot \varphi(x) = x \odot (\theta(x))^{--} = x^{--} \odot (\theta(x))^{--} = (x \odot \theta(x))^{--} \leq (\theta(\theta(x)))^{--} \leq (\theta((\theta(x))^{--}))^{--} = (\theta(\varphi(x)))^{--} = \varphi(\varphi(x))$.

(T3) $\varphi(1) = (\theta(1))^{--} = (1)^{--} = (1^-)^- = 0^- = 1$.

(T4) Let $x, y \in Reg(A)$ such that $x \leq y$. Then we have $(\theta(x))^{--} \leq (\theta(y))^{--}$ and $\varphi(x) \leq \varphi(y)$.

(T5) If θ is a strong mcd-operator, then for any $x \in Reg(A)$, $\varphi(x) = (\theta(x))^{--} \leq (\theta(\theta(x)))^{--} \leq (\theta((\theta(x))^{--}))^{--} = (\theta(\varphi(x)))^{--} = \varphi(\varphi(x))$. □

Example 4.6. Let $A = \{0, a, b, v, 1\}$. Define $\odot$ and $\rightarrow$ as follows:



$\odot$	0	a	b	v	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
v	0	a	b	v	v
1	0	a	b	v	1

$\rightarrow$	0	a	b	v	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
v	0	a	b	1	1
1	0	a	b	v	1

Then $\mathcal{A}$ is a normal residuated lattice (see, [4]). Moreover, $\mathcal{A}$ has Glivenko property. We define $\theta : A \longrightarrow A$ operator as follows:

$$\theta(x) = \begin{cases} 1, & x = 1 \\ v, & x \neq 1 \end{cases}$$

Then it is clear that θ is a strong mcd-operator. We have $Reg(A) = \{0, a, b, 1\}$ and $\varphi(x) = (\theta(x))^{--} = 1$ for all $x \in Reg(A)$. It is easy to ascertain that φ is a strong mcd-operator on $Reg(A)$.

Theorem 4.7. *Suppose that $\mathcal{A}$ is a residuated lattice with Glivenko property, $\omega : A \rightarrow A$ is an ad-operator (a strong ad-operator) on A. If $\delta : Reg(A) \rightarrow Reg(A)$ the mapping defined by $\delta(x) = (\omega(x))^{--}$ for all $x \in Reg(A)$, then δ is an ad-operator (a strong ad-operator) on $Reg(A)$.*

Proof. **(D1)** Let $x, y \in Reg(A)$. Then $\delta(x \boxplus y) = \delta(x \oplus y) = (\omega(x \oplus y))^{--} = (\omega(x) \oplus \omega(y))^{--} = (\omega(x))^{--} \oplus (\omega(y))^{--} = \delta(x) \oplus \delta(y) = \delta(x) \boxplus \delta(y)$.

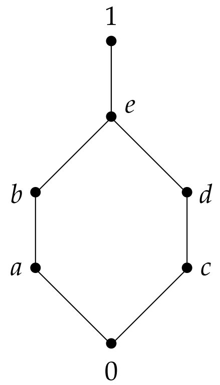
(D2) Let $x \in Reg(A)$. Then $\delta(\delta(x)) = \delta((\omega(x))^{--}) = (\omega((\omega(x))^{--}))^{--} = (\omega(\omega(x)))^{----} = (\omega(\omega(x)))^{--} \leq (x \oplus \omega(x))^{--} = x^{--} \oplus (\omega(x))^{--} = x \oplus \delta(x) = x \boxplus \delta(x)$.

(D3) $\delta(0) = (\omega(0))^{--} = (0)^{--} = (0^-)^- = 1^- = 0$.

(D4) Let $x, y \in Reg(A)$ such that $x \leq y$. Then we have $(\omega(x))^{--} \leq (\omega(y))^{--}$ and $\delta(x) \leq \delta(y)$.

(D5) If ω is a strong ad-operator, then for any $x \in Reg(A)$, $\delta(\delta(x)) = \delta((\omega(x))^{--}) = (\omega((\omega(x))^{--}))^{--} = (\omega(\omega(x)))^{----} = (\omega(\omega(x)))^{--} \leq (\omega(x))^{--} = \delta(x)$. □

Example 4.8. Let $A = \{0, a, b, c, d, e, 1\}$. Define $\odot$ and $\rightarrow$ as follows:



$\odot$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	a	0	0	a	a
b	0	a	a	0	0	a	b
c	0	0	0	c	c	c	c
d	0	0	0	c	c	c	d
e	0	a	a	c	c	e	e
1	0	a	b	c	d	e	1

$\rightarrow$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	d	1	1	d	d	1	1
b	d	e	1	d	d	1	1
c	b	b	b	1	1	1	1
d	b	b	b	e	1	1	1
e	0	b	b	d	d	1	1
1	0	a	b	c	d	e	1

Then $\mathcal{A}$ is a residuated lattice with Glivenko property. Note that $\mathcal{A}$ isn't a normal residuated lattice. We can obtain easily that $Reg(A) = \{0, b, d, 1\}$ and $\oplus$ is as follows:

$\oplus$	0	a	b	c	d	e	1
0	0	b	b	d	d	1	1
a	b	b	b	1	1	1	1
b	b	b	b	1	1	1	1
c	d	1	1	d	d	1	1
d	d	1	1	d	d	1	1
e	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1

We define $\omega : A \longrightarrow A$ operator as follows:

x	0	a	b	c	d	e	1
$\omega(x)$	0	b	b	d	d	1	1

It is easy to ascertain that ω is a strong ad-operator on A. If $\delta : Reg(A) \rightarrow Reg(A)$ the mapping defined by $\delta(x) = (\omega(x))^{--}$ for all $x \in Reg(A)$, then δ is as follows:

x	0	b	d	1
$\delta(x)$	0	b	d	1

It is easy to see that δ is a strong ad-operator on $Reg(A)$.

Theorem 4.9. *Suppose that $\mathcal{A}$ is a normal residuated lattice with Glivenko property and $\tau : Reg(A) \to Reg(A)$ is an mcd-operator (a strong mcd-operator) on the involutive residuated lattice $Reg(A)$. If $\theta : A \to A$ is the mapping defined by $\theta(x) = \tau(x^{--})$ for all $x \in A$, then θ is an mcd-operator (a strong mcd-operator) on $\mathcal{A}$.*

Proof. **(T1)** Let $x, y \in A$. Then
$\theta(x \odot y) = \tau((x \odot y)^{--}) = \tau(x^{--} \odot y^{--}) = \tau(x^{--} \otimes y^{--}) = \tau(x^{--}) \otimes \tau(y^{--}) = \tau(x^{--}) \odot \tau(y^{--}) = \theta(x) \odot \theta(y)$.

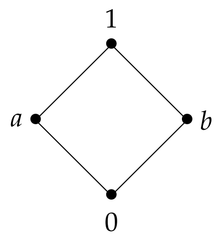
(T2) Let $x \in A$. Then $x \odot \theta(x) = x \odot \tau(x^{--}) \leq x^{--} \odot \tau(x^{--}) = x^{--} \otimes \tau(x^{--}) \leq \tau(\tau(x^{--})) \leq \tau((\tau(x^{--}))^{--}) = \tau((\theta(x))^{--}) = \theta(\theta(x))$.

(T3) $\theta(1) = \tau(1^{--}) = \tau(1) = 1$.

(T4) Let $x, y \in A$ such that $x \leq y$. Then we have $x^{--} \leq y^{--}$ and $\tau(x^{--}) \leq \tau(y^{--})$. So, we have $\theta(x) \leq \theta(y)$.

(T5) If τ is a strong mcd-operator, then for any $x \in A$, $\theta(x) = \tau(x^{--}) \leq \tau(\tau(x^{--})) \leq \tau((\tau(x^{--}))^{--}) = \tau((\theta(x))^{--}) = \theta(\theta(x))$. □

Example 4.10. We consider the residuated lattice $\mathcal{A} = (\{0, a, b, v, 1\}, \wedge, \vee, \odot, \to, 0, 1)$ from Example 4.6. Then $Reg(\mathcal{A}) = (\{0, a, b, 1\}, \sqcap, \sqcup, \otimes, \rightsquigarrow, 0, 1)$ is an involutive residuated lattice with the following operations:



$\otimes$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\rightsquigarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

We define $\tau : Reg(A) \longrightarrow Reg(A)$ operator as follows:

x	0	a	b	1
$\tau(x)$	0	a	b	1

Then it is clear that τ is a strong mcd-operator on $Reg(A)$. Define $\theta : A \to A$ by $\theta(x) = \tau(x^{--})$ for all $x \in A$. Then θ is as follows:

x	0	a	b	v	1
$\theta(x)$	0	a	b	1	1

Then it is easy to verify that θ is a strong mcd-operator on A.

Theorem 4.11. *Suppose that $\mathcal{A}$ is a residuated lattice with Glivenko property. If $\pi : Reg(A) \to Reg(A)$ is an ad-operator (a strong ad-operator) on Reg(A), then the mapping $\omega : A \to A$ such that $\omega(x) = \pi(x^{--})$ for any $x \in A$, is an ad-operator (a strong ad-operator) on $\mathcal{A}$.*

Proof. **(D1)** Let $x, y \in A$. Then $\omega(x \oplus y) = \pi((x \oplus y)^{--}) = \pi(x^{--} \oplus y^{--}) = \pi(x^{--} \boxplus y^{--}) = \pi(x^{--}) \boxplus \pi(y^{--}) = \pi(x^{--}) \oplus \pi(y^{--}) = \omega(x) \oplus \omega(y)$.

(D2) Let $x \in A$. Then $\omega(\omega(x)) = \omega(\pi(x^{--})) = \pi((\pi(x^{--}))^{--}) = \pi(\pi(x^{----})) = \pi(\pi(x^{--})) \leq x^{--} \boxplus \pi(x^{--}) = x^{--} \oplus \pi(x^{--}) = x^{--} \oplus \omega(x) = x \oplus \omega(x)$.

(D3) $\omega(0) = \pi(0^{--}) = \pi(0) = 0$.

(D4) Let $x, y \in A$ such that $x \leq y$. Then we have $x^{--} \leq y^{--}$ and $\pi(x^{--}) \leq \pi(y^{--})$. As a result, we have $\omega(x) \leq \omega(y)$.

(D5) If π is a strong ad-operator, then for any $x \in A$, $\omega(\omega(x)) = \omega(\pi(x^{--})) = \pi((\pi(x^{--}))^{--}) = \pi(\pi(x^{----})) = \pi(\pi(x^{--})) \leq \pi(x^{--}) = \omega(x)$. □

Example 4.12. *We consider the residuated lattice $\mathcal{A} = (\{0, a, b, c, d, e, 1\}, \wedge, \vee, \odot, \to, 0, 1)$ and the involutive residuated lattice $Reg(\mathcal{A}) = (\{0, b, d, 1\}, \sqcap, \sqcup, \otimes, \rightsquigarrow, 0, 1)$ from Example 4.8. We define $\pi : Reg(A) \longrightarrow Reg(A)$ operator as follows:*

x	0	b	d	1
$\pi(x)$	0	d	b	1

Then it is clear that π is an ad-operator on Reg(A). If $\omega : A \to A$ the mapping defined by $\omega(x) = \pi(x^{--})$ for any $x \in A$, then ω is as follows:

x	0	a	b	c	d	e	1
$\omega(x)$	0	d	d	b	b	1	1

It is easy to ascertain that ω is an ad-operator on A.

5. The Boolean center of the residuaded lattice

Suppose that $(A, \vee, \wedge, 0, 1)$ is a bounded lattice. We remember that an element $a \in A$ is called *complemented* if there is an element $b \in A$ such that $a \vee b = 1$ and $a \wedge b = 0$ (see, [2, 6, 15]). We shall denote $B(A)$ to the set of complemented elements of A.

Lemma 5.1. ([9]) *The following properties are true in every residuated lattice $\mathcal{A}$:*

(1) *If $a \in A$ has a complement, the complement must coincide with a^-,*

(2) $B(A) = \{a \in A : a^- \vee a = 1\}$,

(3) *If $a \in B(A)$, then for each $x \in A$, $x \odot a = x \wedge a$.*

$B(\mathcal{A}) = (B(A), \vee, \wedge, ^-, 0, 1)$ is a Boolean algebra with the operations induced by those of $\mathcal{A}$.

Remark 5.2. ([7]) $B(\mathcal{A})$ is a subalgebra both of $\mathcal{A}$ and $Reg(\mathcal{A})$. Usually, $Reg(A) \neq B(A)$. Nevertheless, $Reg(\mathcal{A})$ is a Boolean algebra iff $\mathcal{A}$ is *pseudocomplemented*, and $Reg(A) = B(A)$ iff $\mathcal{A}$ is *stonean*, i.e., the equation $x^- \vee x^{--} = 1$ hold in A.

Definition 5.3. Let $\mathcal{A}$ be a residuated lattice, C a subalgebra of $\mathcal{A}$ and $\theta : A \longrightarrow A$ ($\omega : A \longrightarrow A$) an mcd-operator (ad-operator) on A. Then, C is called a *co-derivative (derivative) subalgebra* with respect to θ (ω) if $\theta(x) \in C$ ($\omega(x) \in C$) for any $x \in C$.

Proposition 5.4. *A subalgebra C is a co-derivative (derivative) subalgebra with respect to an mcd-operator θ (an ad-operator ω) iff C is a derivative (co-derivative) subalgebra with respect to the ad-operator θ^- (mcd-operator ω^-).*

Proof. ($\Rightarrow$) Let C be a co-derivative subalgebra with respect to an mcd-operator θ and $x \in C$. Then we have $x^- \in C$, $\theta(x^-) \in C$ and $(\theta(x^-))^- \in C$. So, $\theta^-(x) \in C$ and C is derivative subalgebra of $\mathcal{A}$.

($\Leftarrow$) Let C be a derivative subalgebra with respect to an ad-operator ω and $x \in C$. Then we have $x^- \in C$, $\omega(x^-) \in C$ and $(\omega(x^-))^- \in C$. As a result, $\omega^-(x) \in C$ and C is co-derivative subalgebra of $\mathcal{A}$. □

Example 5.5. We consider the residuated lattice $\mathcal{A} = (\{0, a, b, v, 1\}, \wedge, \vee, \odot, \rightarrow, 0, 1)$ and θ operator from Example 4.6. Then it is easy to verify that $B(A) = \{0, 1\}$. Since $\theta(0) = v \notin B(A)$, $B(\mathcal{A})$ is not co-derivative subalgebra of $\mathcal{A}$ with respect to θ. However, $B(\mathcal{A})$ is co-derivative subalgebra of $Reg(\mathcal{A})$ with respect to $\varphi(x) = (\theta(x))^{--} = 1$ for all $x \in Reg(A)$.

Proposition 5.6. ([6]) *Let $\mathcal{A}$ be a residuated lattice. We consider the following statements for $a \in A$:*

(1) $a \in B(A)$,

(2) $a \odot a = a$,

(3) $a^{--} = a$,

(4) $a \odot a = a$ *and* $a^- \rightarrow a = a$,

(5) $(a \rightarrow x) \rightarrow a = a$, *for every* $x \in A$,

(6) $a \wedge a^- = 0$.

Then (1) implies (2) , (3), (4) and (5) but (2)$\not\Rightarrow$(1), (3)$\not\Rightarrow$(1), (4)$\not\Rightarrow$(1), (5)$\not\Rightarrow$(1).

Proposition 5.7. ([6]) *If residuated lattice $\mathcal{A}$ is an $R\ell$-monoid, then the statements (1), (2), (3) and (4) from Proposition 5.6 are equivalent.*

Proposition 5.8. *Let $\mathcal{A}$ be an $R\ell$-monoid and θ be an mcd-operator on A. Then $B(\mathcal{A})$ is a co-derivative subalgebra of $\mathcal{A}$ with respect to θ.*

Proof. Let $a \in B(A)$. Then $a \odot a = a$ by Proposition 5.6 (2). We have $\theta(a \odot a) = \theta(a) \odot \theta(a) = \theta(a)$. So, $\theta(a) \in B(A)$ by Proposition 5.7. □

6. Filters

Assume that $\mathcal{A}$ is a residuated lattice. A nonempty subset F of A is provided the following conditions:

(F1) $x, y \in F \Longrightarrow x \odot y \in F$,

(F2) $x \in F, y \in A, x \leq y \Longrightarrow y \in F$.

Then it is called a *filter* of A.
A subset D of A is said to be a *deductive system* of A if

(DS1) $1 \in D$,

(DS2) $x, x \rightarrow y \in D \Longrightarrow y \in D$.

It is known that a subset of A is a deductive system of A iff it is a filter (see, [24]).

Definition 6.1. Let $\mathcal{A}$ be a residuated lattice with mcd-operator θ and F be a filter of $\mathcal{A}$. Then F is called a *θ-filter* if $\theta(x) \in F$ for every $x \in F$.

Proposition 6.2. *Suppose that $\mathcal{A}$ is a residuated lattice with mcd-operator θ . Then the subset $F = \{x \in A : \theta(x) = 1\}$ of A is a θ-filter.*

Proof. **(F1)** Let $x, y \in F$. Then we have $\theta(x) = 1$ and $\theta(y) = 1$. Since $\theta(x \odot y) = \theta(x) \odot \theta(y)$ by (T1), we must have $\theta(x \odot y) = 1$. So, $x \odot y \in F$.

(F2) Let $x \in F$, $y \in A$ and $x \leq y$. Then we have $\theta(x) = 1 \leq \theta(y)$. So, $\theta(y) = 1$ and $y \in F$.

Finally, let $x \in F$. Then we get $\theta(x) = 1$. Hence, $\theta\theta(x) = \theta(\theta(x)) = \theta(1) = 1$. It means that $\theta(x) \in F$. □

Lemma 6.3. *Suppose that $\mathcal{A}$ is a normal residuated lattice with Glivenko property, θ is an mcd-operator on $\mathcal{A}$ and F is θ-filter of $\mathcal{A}$. Then, $F \cap Reg(A)$ is a φ-filter of $Reg(\mathcal{A})$ with respect to $\varphi(x) = (\theta(x))^{--}$.*

Proof. We know that $F \cap Reg(A)$ is a filter of $Reg(\mathcal{A})$ (see, [7]). Let $x \in F \cap Reg(A)$. Then $\theta(x) \in F$ by the hypothesis. Moreover, we have $\theta(x) \leq (\theta(x))^{--}$ by Proposition 2.1 (4) and $(\theta(x))^{--} \in F$. Since $(\theta(x))^{--}$ is regular, $\varphi(x) = (\theta(x))^{--} \in F \cap Reg(A)$. □

Lemma 6.4. *Suppose that $\mathcal{A}$ is a normal residuated lattice with Glivenko property. If φ is an mcd-operator on the involutive residuated lattice $Reg(A)$ and G is φ-filter of $Reg(\mathcal{A})$, then there is a θ-filter F of $\mathcal{A}$ such that $G = F \cap Reg(A)$ with respect to $\theta(x) = \varphi(x^{--})$.*

Proof. We take $F = \{x \in A : x \geq a_1 \odot \ldots \odot a_n \, for \;\; some \; n \geq 1 \;\; and \;\; a_1, \ldots, a_n \in G\}$. It is easy to see that F is a filter of $\mathcal{A}$ (see, [7]). Let $x \in F$. Then
$a_1 \odot \ldots \odot a_n \leq x$ for some $n \geq 1$ and $a_1, \ldots, a_n \in G$,
$a_1 \odot \ldots \odot a_n \leq x \leq x^{--}$ by Proposition 2.1 (4),
$\varphi(a_1 \odot \ldots \odot a_n) \leq \varphi(x^{--})$ by (T4),
$\varphi(a_1 \odot \ldots \odot a_n) = \varphi(a_1) \odot \ldots \odot \varphi(a_n) \leq \varphi(x^{--})$ by (T1).
Since G is φ-filter, $\varphi(a_i) \in G$ for $1 \leq i \leq n$. So, $\theta(x) = \varphi(x^{--}) \in F$. □

Filters of residuated lattice are in one-to-one correspondence with their congruences (see, [18]). For any filter F of residuated lattice $\mathcal{A}$, we define a relation $\sim_F$ on A as follows:

$$x \sim_F y \Longleftrightarrow x \to y, y \to x \in F.$$

Then $\sim_F$ is a congruence relation. The set of all congruence classes is denoted by A/F, i.e, $A/F := \{x/F : x \in A\}$ where $x/F = \{y \in A : x \sim_F y\}$. Define the binary operations $\wedge, \vee, \odot, \to$ on A/F as follows:

$$\begin{aligned} x/F \wedge y/F &= (x \wedge y)/F \\ x/F \vee y/F &= (x \vee y)/F \\ x/F \odot y/F &= (x \odot y)/F \\ x/F \to y/F &= (x \to y)/F \end{aligned}$$

Therefore $(A/F, \wedge, \vee, \odot, \to, 0/F, 1/F)$ is a residuated lattice which is called quotient residuated lattice with respect to F.

Theorem 6.5. *Suppose that $\mathcal{A}$ is a residuated lattice, $\theta : A \longrightarrow A$ is an mcd-operator (a strong mcd-operator) and F is a θ-filter of residuated lattice $\mathcal{A}$. Then the mapping $\vartheta : A/F \to A/F$ such that $\vartheta(x/F) = \theta(x)/F$ is an mcd-operator (a strong mcd-operator) on the quotient residuated lattice $\mathcal{A}/\mathcal{F}$.*

Proof. Let $x, y \in A$ and $x \sim_F y$. Then $x \to y, y \to x \in F$ and so $\theta(x \to y), \theta(y \to x) \in F$. Since $\theta(x \to y) \leq \theta(x) \to \theta(y)$ and $\theta(y \to x) \leq \theta(y) \to \theta(x)$ by Proposition 3.10 (1), we have $\theta(x) \to \theta(y), \theta(y) \to \theta(x) \in F$ by (F2). Thus $\theta(x) \sim_F \theta(y)$. That means the relation $\sim_F$ on A with the mcd-operator is a congruence relation.

Now we will show that ϑ satisfies (T1)-(T5) conditions. Let $x, y \in A$.

(T1) $\vartheta(x/F \odot y/F) = \vartheta((x \odot y)/F) = \theta(x \odot y)/F = (\theta(x) \odot \theta(y))/F = \theta(x)/F \odot \theta(y)/F = \vartheta(x/F) \odot \vartheta(y/F)$.

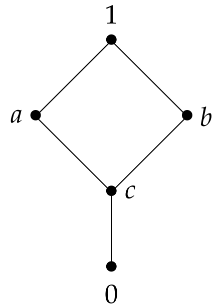
(T2) $x/F \odot \vartheta(x/F) = x/F \odot \theta(x)/F = (x \odot \theta(x))/F \leq \theta(\theta(x))/F = \vartheta(\theta(x)/F) = \vartheta(\vartheta(x/F)) = \vartheta\vartheta(x/F)$.

(T3) $\vartheta(1/F) = \theta(1)/F = 1/F$.

(T4) Let $x, y \in A$ such that $x/F \leq y/F$. Then we know that $x/F \leq y/F$ iff $x \to y \in F$. Since F is a θ-filter, we get $\theta(x \to y) \in F$. Moreover, we must have $\theta(x) \to \theta(y) \in F$ and $\theta(x)/F \leq \theta(y)/F$. So, $\vartheta(x/F) = \theta(x)/F \leq \theta(y)/F = \vartheta(y/F)$.

(T5) If θ is a strong mcd-operator, then for any $x \in A$, $\vartheta(x/F) = \theta(x)/F \leq \theta(\theta(x))/F = \vartheta(\theta(x)/F) = \vartheta(\vartheta(x/F)) = \vartheta\vartheta(x/F)$. □

Example 6.6. Let $A = \{0, a, b, c, 1\}$. Define $\odot$ and $\to$ as follows:



$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	c	c	a
b	0	c	b	c	b
c	0	c	c	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	b	1
b	0	a	1	a	1
c	0	1	1	1	1
1	0	a	b	c	1

Then $\mathcal{A}$ is a residuated lattice. We define $\theta : A \longrightarrow A$ operator as follows:

x	0	a	b	c	1
$\theta(x)$	0	c	1	c	1

Then we can check that θ be a strong mcd-operator on A. We consider $F = \{b, 1\}$. It is easy to see that F is a θ-filter of residuated lattice $\mathcal{A}$. Then $A/F = \{0/F, c/F, 1/F\}$ where $a/F = c/F$ and $b/F = 1/F$. We define $\vartheta : A/F \to A/F$ by $\vartheta(x/F) = \theta(x)/F$ for all $x/F \in A/F$. Then ϑ is as follows:

x/F	$0/F$	c/F	$1/F$
ϑ	$0/F$	c/F	$1/F$

We can check that the mapping ϑ is a strong mcd-operator on the quotient residuated lattice $\mathcal{A}/\mathcal{F}$.

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