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On the quantitative weighted generalization of Jafari transform

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Abstract. In this paper, a quantitative weighted transform based on the Jafari transform is proposed, and the mathematical foundations of this new transform are investigated. In the first section, some information about Jafari transform and some mathematical tools are reviewed. In the second section, the quantitative weighted Jafari transform is introduced, its existence guaranteed through a theorem, and its fundamental properties are examined. Additionally, transforms of the fractional derivative and fractional integral of a function with respect to a function h and a $w-$weight are obtained. In the third section, the theoretical findings are applied to solve classical and fractional initial value problems based on a function h and $w-$weight. In the last section, the results are discussed.

1. Introduction

For about two hundred and fifty years, integral transforms have been used effectively to solve various problems in applied mathematics, physics and engineering. In the 1780s, Pierre-Simon Laplace published a book entitled "Théorie analytique des probabilités", in which he obtained some fundamental results such as the effective solution of differential and integral equations using the Laplace transform, one of the oldest and most widely used integral transforms in the mathematical literature. The Laplace transform of a function g is given as follows

$$L\left(g\left(x\right);v\right) := \int_{0}^{\infty} e^{-vx} g\left(x\right) dx = F\left(v\right), \tag{1}$$

where the function g is integrable for $x \geq 0$. Another important study on this topic is the enormous research work entitled "Théorie analytique de la chaleur" by Joseph Fourier in 1822. In this study, important

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topics such as Fourier transforms, Fourier series and the modern mathematical theory of heat transfer were mentioned. In addition, the Hartley, Hankel, Mellin and Hilbert-Stieltjes transforms are also an important part of this topic. If you want to get detailed information about the integral transforms up to this point, you can refer to the source given in [4].

So far, many transforms similar to the Laplace transform have been defined to solve certain problems such as initial value problems. Some of them are Aboodh [1], Sumudu [23], Elzaki [6], Sawi [16], Kamal [11] and Kashuri [12] transforms.
A general integral transform was defined by Jafari [8]. It is called Jafari transform, which is given by

$$T\left(g\left(x\right);v\right) := p\left(v\right)\int_{0}^{\infty} e^{-q(v)x} g\left(x\right)dx = G\left(v\right), \tag{2}$$

where $p(v) \neq 0$, g is an integrable function for $x \geq 0$ and $p, q : (0, \infty) \to (0, \infty)$ in [8]. Jafari transform is the comprehensive version of many studies and some integral transforms can be obtained according to the special cases of $p(v)$ and $q(v)$ in the equality (2):

- If $p(v) = 1$ and $q(v) = v$ in (2), then Jafari transform yields Laplace transform given in the equality (1).
- If $p(v) = 1/v$ and $q(v) = v$ in (2), then Jafari transform yields Aboodh transform defined in [1].
- If $p(v) = 1/v$ and $q(v) = 1/v$ in (2), then Jafari transform yields Sumudu transform defined in [23].
- If $p(v) = v$ and $q(v) = 1/v$ in (2), then Jafari transform yields Elzaki transform defined in [6].
- If $p(v) = 1/v^2$ and $q(v) = 1/v$ in (2), then Jafari transform yields Sawi transform defined in [16].
- If $p(v) = 1$ and $q(v) = 1/v$ in (2), then Jafari transform yields Kemal transform defined in [11].
- If $p(v) = 1/v$ and $q(v) = 1/v^2$ in (2), then Jafari transform yields Kashuri transform defined in [12].
- If $p(v) = v^2$ and $q(v) = v$ in (2), then Jafari transform yields Mohand transform defined in [15].
- If $p(v) = \frac{1}{u}$ and $q(v) = \frac{v}{u}$ in (2), then Jafari transform yields N-transform defined in [13].
- If $p(v) = 1$ and $q(v) = v^{\frac{1}{\alpha}}$ in (2) where $\alpha > 0$, then Jafari transform yields α-integral Laplace transform defined in [17].

Numerous studies have explored the Jafari transform, highlighting its versatility and applications. For instance, the double type [18] and triple type [22] of Jafari transform have been introduced and thoroughly analyzed. Moreover, in [5], this transformation has been utilized to solve differential equations relevant to medical science. Notably, the Jafari transform has been utilized to address the Ulam-Hyers stability problem, highlighting the Ulam-Hyers stability of the proposed equations as an extension of Mittag-Leffler-Ulam-Hyers stability [20].

The fractional calculus and fractional differential equations are a fascinating and increasingly influential area of mathematics, providing innovative methods and powerful tools for understanding and tackling complex phenomena in various scientific and technical fields. These branches of mathematics deal with derivatives and integrals of non-integer orders and extend the traditional notions of calculus to deal with complicated systems that defy the standard approaches of integer orders. The growing importance of fractional calculus and fractional differential equations stems from their ability to describe phenomena characterized by memory effects, long-range dependencies, and anomalous behavior that frequently arise in fields such as physics, biology, finance, and materials science. Since traditional integer integral calculus fails to capture such complicated dynamics, fractional calculus has emerged as an indispensable tool that allows researchers and scientists to gain deeper insights into the underlying mechanisms of these intricate systems. Readers interested in learning more about fractional calculus and fractional differential equations can refer to the references [14] and [19].

To begin with, we recall the basic definitions of the weighted derivative and its fractional version with respect to an increasing differentiable function h.

The m-th order w−weighted derivative of a fuction g about the function h is given by

$$D_w^m g(x) = \frac{1}{w(x)} \left(\frac{D_x}{h'(x)} \right)^m (w(x)\, g(x)) := \frac{1}{w(x)} \left(\frac{1}{h'(x)} D_x \right)^m (w(x) g(x)), \quad m = 0, 1, 2, ..., \tag{3}$$

where $D_x = \frac{d}{dx}$, $w(x) \neq 0$ and the function h is strictly increasing differentiable [2, 9]. The w-weighted Riemann- Liouville fractional integral with respect to h is recalled as

$$\left({}_{a^+}J_w^\beta g \right)(x) = \frac{1}{w(x)\,\Gamma(\beta)} \int_a^x (h(x) - h(t))^{\beta-1} w(t)\, g(t)\, h'(t)\, dt, \tag{4}$$

for $a \geq 0$ and $\beta > 0$ [2]. On the other hand, the corresponding w-weighted Riemann Liouville derivative is defined as [2]

$${}_{a^+}^{RL}D_w^\beta g(x) = D_w^m \left({}_{a^+}J_w^{m-\beta} g(x) \right), \tag{5}$$

for $a \geq 0$ and $\beta > 0$ where $m = \left[\left|\beta\right|\right] + 1$. Whilst, the associated w-weighted Caputo derivative was proposed as the following [2]

$${}_{a^+}^{C}D_w^\beta g(x) = \left({}_{a^+}J_w^{m-\beta} \left[D_w^m g(x) \right] \right), \tag{6}$$

for $a \geq 0$ and $\beta > 0$ where $m = \left[\left|\beta\right|\right] + 1$.

If $AC[a, b]$ denotes the space of absolute continuous functions on $[a, b]$. Then, the space $AC_w^m[a, b]$ can be defined as [9]

$$AC_w^m[a, b] = \left\{ g : [a, b] \to \mathbb{R} | g_{m-1} \in AC[a, b] \right\},$$

where $g_m(x) = \left(\frac{D_x}{h'(x)} \right)^m (w(x)\, g(x))$ for $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Heretofore, it has been highlighted that the Jafari transform given in (2) encompasses many well-known integral transforms for specific values of $p(v)$ and $q(v)$, and the fundamental mathematical tools to be utilized in this paper have been reviewed. In the next section, we will define a new integral transform, establish its existence, present its fundamental properties, and subsequently provide convolution-type theorem related to this transform. Furthermore, we will discuss the properties satisfied by the transform of the weighted fractional derivative and integral of a function with respect to a function h and a w weight. Additionally, we will calculate the values of the Mittag-Leffler functions with one and two parameters under this integral transform.

2. The transform ${}_p^q£_h^w$ and its properties

Now, we define an integral transform, which is the cornerstone of this paper.

Definition 2.1. *Let $w \neq 0$, $g, h, w : [a, \infty) \to \mathbb{R}$, where h is a strictly increasing differentiable function on (a, ∞) satisfying $h(x) \to \infty$ when $x \to \infty$. We define the quantitative weighted Jafari transform of the function g by*

$${}_p^q£_h^w (g(x); v) = p(v) \int_a^\infty e^{-q(v)[h(x)-h(a)]} w(x)\, g(x)\, h'(x)\, dx = R(v), \tag{7}$$

where $a \geq 0$ and the functions p and q are of form $p, q : (0, \infty) \to (0, \infty)$.

Note that some special cases of the quantitative weighted Jafari transform defined in (7) are as follows:

- If $p(v) = 1$, $q(v) = v$, $w(x) = 1$, $h(x) = x$ and $a = 0$ in (7), then transform ${}_{p}^{q}£_{h}^{w}$ yields Laplace transform given in (1).
- If $w(x) = 1$, $h(x) = x$ and $a = 0$ in (7), then transform ${}_{p}^{q}£_{h}^{w}$ yields Jafari transform given in [7]. So, the quantitative weighted Jafari transform gives many transforms such as Elzaki, Aboodh, and Sumudu transforms.
- If $p(v) = 1$ and $q(v) = v$ in (7), then transform ${}_{p}^{q}£_{h}^{w}$ yields $w-$weighted Laplace transform regard to the function h given in [9].
- If $w(x) = 1$, $p(v) = 1$ and $q(v) = v$ in (7), then transform ${}_{p}^{q}£_{h}^{w}$ yields Laplace transform regard to the function h given in [10].
- If $w(x) = 1$, $h(0) = 0$ and $a = 0$ in (7), then transform ${}_{p}^{q}£_{h}^{w}$ yields Jafari transform regard to the function h given in [8]
- If $p(v) = v$, $q(v) = v$, $w(x) = x^{n-1}$, $h(x) = x$ and $a = 0$ in (7) where $n \in \mathbb{N}$, then transform ${}_{p}^{q}£_{h}^{w}$ yields ARA transform given in [21].
- If $p(v) = 1$, $q(v) = v$, $w(x) = x^{n}$, $h(x) = x$ and $a = 0$ in (7) where $n \in \mathbb{N}$, then transform ${}_{p}^{q}£_{h}^{w}$ yields the transform given in [3].

We note that it is possible to obtain new transforms from the transformation ${}_{p}^{q}£_{h}^{w}$. For example, we introduce $w-$weighted Jafari transform the following that

$$ {}_{p}^{q}£_{e_1}^{w}(g(x);v) = p(v)\int_{0}^{\infty} w(x)\, e^{-q(v)x} g(x)\, dx, \tag{8}$$

by taking $h(x) := e_1(x) = x$ and $a = 0$ in (7).

Definition 2.2. [9] Let $w \neq 0$ and $g, w : [a, \infty) \to \mathbb{R}$. The function g is said to be of w-weighted-h-exponential order function if there are constants M, x_0 and C, where $M > 0$ and $x > x_0$ such that

$$ \left| w(x)\, g(x) \right| \leq M\, e^{C\, h(x)}. \tag{9}$$

We now demonstrate the condition for the existence of the quantitative weighted Jafari transform through the following theorem.

Theorem 2.3. *Let $w \neq 0$ and $g, w : [a, \infty) \to \mathbb{R}$ be functions such that g is of w-weighted-h-exponential order and w and g are piecewise continuous on each arbitrary subinterval of $[a, x_0)$. Then, the transforms ${}_{p}^{q}£_{h}^{w}$ of the function g exists for $q(v) > C$.*

Proof. We can write that

$$ {}_{p}^{q}£_{h}^{w}(g(x);v) = I_1 + I_2,$$

where

$$ I_1 = p(v)\int_{a}^{x_0} e^{-q(v)[h(x)-h(a)]} w(x)\, g(x)\, h'(x)\, dx \quad \text{and} \quad I_2 = p(v)\int_{x_0}^{\infty} e^{-q(v)[h(x)-h(a)]} w(x)\, g(x)\, h'(x)\, dx. \tag{10}$$

Accordingly, since the product wg is piecewise continuous, the integral I_1 has an existing real value. So, if we prove that the integral I_2 converges, we obtain the existence of the transform ${}_p^q\pounds_h^w$ of function. With a w-weighted-h-exponential order function g and a strictly increasing function h, we obtain that

$$\begin{aligned} I_2 &\leq \left| p(v) \int_{x_0}^{\infty} e^{-q(v)[h(x)-h(a)]} w(x)\, g(x)\, h'(x)\, dx \right| \\ &\leq \left| p(v) \right| \int_{x_0}^{\infty} e^{-q(v)[h(x)-h(a)]} \left| w(x)\, g(x) \right| h'(x)\, dx \\ &\leq M \left| p(v) \right| e^{Ch(a)} \int_{x_0}^{\infty} e^{-[q(v)-C][h(x)-h(a)]} h'(x)\, dx \\ &\leq M \left| p(v) \right| e^{C\,h(a)} \frac{e^{-q(v)[h(x_0)-h(a)]}}{q(v)-C}, \quad q(v) > C. \end{aligned}$$

Finally, the integral of I_2 converges due to the above inequality and thus the proof is complete. □

Theorem 2.4. *Let the transforms ${}_p^q\pounds_h^w$ of functions $f : [a, \infty) \to \mathbb{R}$ and $g : [a, \infty) \to \mathbb{R}$ be available for $q(v) > C_1$ and $q(v) > C_2$, respectively exist. Then,*

$${}_p^q\pounds_h^w (Af(x) + Bg(x); v) = A\, {}_p^q\pounds_h^w (f(x); v) + B\, {}_p^q\pounds_h^w (g(x); v),$$

for $q(v) > \max\{C_1, C_2\}$ where A, B, C_1, C_2 are real constants. Thus the transform ${}_p^q\pounds_h^w$ is linear.

If we take $u = h(x) - h(a)$ in the definition of ${}_p^q\pounds_h^w$ given in (7), we have the following result.

Theorem 2.5. *The following relations are provided between the Jafari transform given in (2) with the new integral transform given in (7)*

i. $${}_p^q\pounds_h^w (f(x); v) = T\left(w\left(h^{-1}(x + h(a))\right) f\left(h^{-1}(x + h(a))\right); v\right)$$

ii. $${}_p^q\pounds_h^w \left(\frac{f(h(x) - h(a))}{w(x)}; v\right) = T(f(x); v)$$

where T is Jafari transform given in (2) and the function h^{-1} is inverse of h.

Proposition 2.6. *The transforms ${}_p^q\pounds_h^w$ of some basic functions are as follows*

i. $${}_p^q\pounds_h^w \left(\frac{1}{w(x)}; v\right) = \frac{p(v)}{q(v)}, \qquad q(v) > 0.$$

ii. $${}_p^q\pounds_h^w \left(\frac{e^{\beta(h(x)-h(a))}}{w(x)}; v\right) = \frac{p(v)}{q(v) - \beta}, \qquad q(v) > \beta.$$

iii. $${}_p^q\pounds_h^w \left(\frac{(h(x) - h(a))^{\beta-1}}{w(x)}; v\right) = \frac{\Gamma(\beta)\, p(v)}{[q(v)]^{\beta}}, \qquad q(v) > 0.$$

iv. $${}_p^q\pounds_h^w \left(\frac{\sin[\beta(h(x) - h(a))]}{w(x)}; v\right) = \frac{\beta p(v)}{[q(v)]^2 + \beta^2}, \qquad q(v) > 0.$$

v. $${}_p^q\pounds_h^w \left(\frac{\cos[\beta(h(x) - h(a))]}{w(x)}; v\right) = \frac{p(v)\, q(v)}{[q(v)]^2 + \beta^2}, \qquad q(v) > 0.$$

vi. $_{p}^{q}£_{h}^{w}\left(\frac{\sinh\left[\beta\left(h\left(x\right)-h\left(a\right)\right)\right]}{w\left(x\right)};v\right)=\frac{\beta p\left(v\right)}{\left[q\left(v\right)\right]^{2}-\beta^{2}},\quad q\left(v\right)>\left|\beta\right|.$

vii. $_{p}^{q}£_{h}^{w}\left(\frac{\cosh\left[\beta\left(h\left(x\right)-h\left(a\right)\right)\right]}{w\left(x\right)};v\right)=\frac{p\left(v\right)q\left(v\right)}{\left[q\left(v\right)\right]^{2}-\beta^{2}},\quad q\left(v\right)>\left|\beta\right|,$

where β is a real constant.

Definition 2.7. *[9] The $w-$weighted convolution respect to a function h of the functions f and g is given by*

$$\left(f*_{h}^{w}g\right)(x)=\frac{1}{w\left(x\right)}\int_{a}^{x}w\left(h^{-1}\left[h\left(x\right)+h\left(a\right)-h\left(t\right)\right]\right)f\left(h^{-1}\left[h\left(x\right)+h\left(a\right)-h\left(t\right)\right]\right)w\left(t\right)g\left(t\right)h'\left(t\right)dt \tag{11}$$

where h^{-1} is the inverse of the function h.

It can be seen that $f*_{h}^{w}g=g*_{h}^{w}f$ by using definition given in (11) [9].

Theorem 2.8. *If the transforms $_{p}^{q}£_{h}^{w}$ of the functions f and g exist for $q\left(v\right)>M_1$ and $q\left(v\right)>M_2$, respectively, then it is that*

$$_{p}^{q}£_{h}^{w}\left[\left(f*_{h}^{w}g\right)(x);v\right]=\frac{1}{p\left(v\right)}\left\{_{p}^{q}£_{h}^{w}\left(f\left(x\right);v\right)\right\}\left\{_{p}^{q}£_{h}^{w}\left(g\left(x\right);v\right)\right\} \tag{12}$$

for $q\left(v\right)>\max\left\{M_1,M_2\right\}$ where M_1 and M_2 are real constants.

Proof. *We can write that*

$$\frac{1}{p\left(v\right)}\left\{_{p}^{q}£_{h}^{w}\left(f\left(x\right);v\right)\right\}\left\{_{p}^{q}£_{h}^{w}\left(g\left(x\right);v\right)\right\}=p\left(v\right)\int_{a}^{\infty}\int_{a}^{\infty}e^{-q(v)[h(x)+h(t)-2h(a)]}w\left(x\right)f\left(x\right)h'\left(x\right)w\left(t\right)g\left(t\right)h'\left(t\right)dxdt. \tag{13}$$

By taking $h\left(z\right)=h\left(x\right)+h\left(t\right)-h\left(a\right)$ in the integral in (13), we get the following equality

$$\begin{aligned}&\frac{1}{p\left(v\right)}\left\{_{p}^{q}£_{h}^{w}\left(f\left(x\right);v\right)\right\}\left\{_{p}^{q}£_{h}^{w}\left(g\left(x\right);v\right)\right\}\\&=p\left(v\right)\int_{a}^{\infty}\int_{t}^{\infty}e^{-q(v)[h(z)-h(a)]}w\left(h^{-1}\left(h\left(z\right)+h\left(a\right)-h\left(t\right)\right)\right)f\left(h^{-1}\left(h\left(z\right)+h\left(a\right)-h\left(t\right)\right)\right)h'\left(z\right)w\left(t\right)g\left(t\right)h'\left(t\right)dzdt.\end{aligned} \tag{14}$$

We obtain

$$\begin{aligned}&\frac{1}{p\left(v\right)}\left\{_{p}^{q}£_{h}^{w}\left(f\left(x\right);v\right)\right\}\left\{_{p}^{q}£_{h}^{w}\left(g\left(x\right);v\right)\right\}\\&=p\left(v\right)\int_{a}^{\infty}\int_{a}^{z}e^{-q(v)[h(z)-h(a)]}w\left(h^{-1}\left(h\left(z\right)+h\left(a\right)-h\left(t\right)\right)\right)f\left(h^{-1}\left(h\left(z\right)+h\left(a\right)-h\left(t\right)\right)\right)h'\left(z\right)w\left(t\right)g\left(t\right)h'\left(t\right)dtdz\\&=p\left(v\right)\int_{a}^{\infty}e^{-q(v)[h(z)-h(a)]}\left(f*_{h}^{w}g\right)(z)w\left(z\right)h'\left(z\right)dz\\&=\left\{_{p}^{q}£_{h}^{w}\left(\left(f*_{h}^{w}g\right)(x);v\right)\right\}\end{aligned}$$

by changing the order of integration in (14) with the help of Fubini's theorem. □

Now, let's examine the properties of the the quantitative weighted Jafari transform regard to the weighted derivative and weighted fractional derivative.

Theorem 2.9. *Let the function $g \in AC_w[a, x_0)$ be w-weighted-h-exponential order function. If the function $D_w g$ is piecewise continuous on every subinterval of $[a, x_0)$, then the transform ${}_p^q£_h^w$ of the $D_w g$ function exists and*

$$ {}_p^q£_h^w (D_w g(x); v) = q(v) \left\{ {}_p^q£_h^w (g(x); v) \right\} - p(v) w(a) g(a). \tag{15} $$

Proof. Using the definition of transform ${}_p^q£_h^w$ given in (7), we get

$$ \begin{aligned} {}_p^q£_h^w (D_w g(x); v) &= p(v) \int_a^\infty e^{-q(v)[h(x)-h(a)]} w(x) [D_w g(x)] h'(x) \, dx \\ &= p(v) \int_a^\infty e^{-q(v)[h(x)-h(a)]} (w(x) g(x))' \, dx. \end{aligned} \tag{16} $$

Since g is a w-weighted-h-exponential order function, we can clearly write that

$$ \left| e^{-q(v)[h(x)-h(a)]} w(x) g(x) \right| \to 0 \text{ such that } x \to \infty. \tag{17} $$

By applying partial integration in the integral on the right side of equation (16) for $u = e^{-q(v)[h(x)-h(a)]}$ and $ds = (w(x) g(x))' \, dx$ and using the limit in (17), it is verified that

$$ \begin{aligned} {}_p^q£_h^w (D_w g(x); v) &= p(v) \left[e^{-q(v)[h(x)-h(a)]} w(x) g(x) \right]_{x=a}^\infty + q(v) p(v) \int_a^\infty e^{-q(v)[h(x)-h(a)]} w(x) g(x) h'(x) \, dx. \\ &= q(v) \left\{ {}_p^q£_h^w (g(x); v) \right\} - p(v) w(a) g(a). \end{aligned} $$

□

Theorem 2.10. *Let the functions $g \in AC_w^{m-1}[a, x_0)$ and g_k be a w-weighted-h-exponential order functions for $k = 1, 2, \ldots, m-1$. If the function $D_w^m g$ is piecewise continuous on every subinterval of $[a, x_0)$, then the transform ${}_p^q£_h^w$ of the $D_w^m g$ function exists and*

$$ {}_p^q£_h^w (D_w^m g(x); v) = [q(v)]^m \left\{ {}_p^q£_h^w (g(x); v) \right\} - p(v) \sum_{k=0}^{m-1} [q(v)]^{m-1-k} g_k(a) \tag{18} $$

for $m \in \mathbb{N}$ where $g_k(x) = \left(\frac{D_x}{h'(x)} \right)^k (w(x) g(x))$ for $k \in \mathbb{N}_0$.

Proof. We are going to prove this theorem by induction method.

i. If it is $m = 1$, the equality in (18) is correct in view of the equality in (15).

ii. For $m - 1$, consider that the equality in (18) be correct. Then, we have

$$ {}_p^q£_h^w \left(D_w^{m-1} g(x); v \right) = [q(v)]^{m-1} \left\{ {}_p^q£_h^w (g(x); v) \right\} - p(v) \sum_{k=0}^{m-2} [q(v)]^{m-2-k} g_k(a). \tag{19} $$

Using the definition of transform ${}_p^q£_h^w$ given in (7), we get

$$ {}_p^q£_h^w (D_w^m g(x); v) = p(v) \int_a^\infty e^{-q(v)[h(x)-h(a)]} \left[\left(\frac{D_x}{h'(x)} \right)^m (w(x) g(x)) \right] h'(x) \, dx $$

$$= p(v) \int_a^\infty e^{-q(v)[h(x)-h(a)]} D_x \left[\left(\frac{D_x}{h'(x)} \right)^{m-1} (w(x) g(x)) \right] dx$$

$$= p(v) \int_a^\infty e^{-q(v)[h(x)-h(a)]} g'_{m-1}(x)\, dx.$$

By applying integration by parts in the integral on the right side of the above equality for $u = e^{-q(v)[h(x)-h(a)]}$ *and* $ds = g'_{m-1}(x)\, dx$ *and using the limit given as follow*

$$\left| e^{-q(v)[h(x)-h(a)]} g_{m-1}(x) \right| \to 0 \text{ such that } (x \to \infty)$$

for $m \in \mathbb{N}$*, it is verified that*

$$\begin{aligned} {}^q_p\text{£}^w_h \left(D^m_w g(x); v \right) &= p(v) \left[e^{-q(v)[h(x)-h(a)]} g_{m-1}(x) \right]_{x=a}^\infty + q(v) p(v) \int_a^\infty e^{-q(v)[h(x)-h(a)]} g_{m-1}(x)\, dx \\ &= q(v) \left\{ {}^q_p\text{£}^w_h \left(D^{m-1}_w g(x); v \right) \right\} - p(v) g_{m-1}(a) \\ &= q(v) \left[[q(v)]^{m-1} \left\{ {}^q_p\text{£}^w_h (g(x); v) \right\} - p(v) \sum_{k=0}^{m-2} [q(v)]^{m-2-k} g_k(a) \right] - p(v) g_{m-1}(a) \\ &= [q(v)]^m \left\{ {}^q_p\text{£}^w_h (g(x); v) \right\} - p(v) \sum_{k=0}^{m-1} [q(v)]^{m-1-k} g_k(a). \end{aligned}$$

From the above equality, the equality in (18) is correct for m. So, the proof is done. □

Theorem 2.11. *Let the function g be a w-weighted-h-exponential order and piecewise continuous on each subinterval of* $[a, x_0)$*. Then, we have*

$$ {}^q_p\text{£}^w_h \left({}_{a^+}J^\beta_w g(x); v \right) = \frac{{}^q_p\text{£}^w_g (g(x); v)}{[q(v)]^\beta}. \tag{20}$$

Proof. With the help of equality (11), we can write

$$\begin{aligned} \frac{(h(x)-h(a))^{\beta-1}}{w(x)} *^w_h g(x) &= \frac{1}{w(x)} \int_a^x w\left(h^{-1}[h(x)+h(a)-h(t)] \right) \\ &\quad \times \frac{(h(x)+h(a)-h(t)-h(a))^{\beta-1}}{w(h^{-1}[h(x)+h(a)-h(t)])} w(t) g(t) h'(t)\, dt \\ &= \frac{1}{w(x)} \int_a^x (h(x)-h(t))^{\beta-1} w(t) g(t) h'(t)\, dt. \end{aligned} \tag{21}$$

So, we have

$$\begin{aligned} {}^q_p\text{£}^w_h \left({}_{a^+}J^\beta_w g(x); v \right) &= {}^q_p\text{£}^w_g \left(\frac{1}{w(x)\Gamma(\beta)} \int_a^x (h(x)-h(t))^{\beta-1} w(t) g(t) h'(t)\, dt; v \right) \\ &= \frac{1}{\Gamma(\beta)} {}^q_p\text{£}^w_g \left(\frac{(h(x)-h(a))^{\beta-1}}{w(x)} *^w_h g(x); v \right) \end{aligned}$$

$$= \frac{1}{\Gamma(\beta)} \left\{ {}_{p}^{q}\pounds_{g}^{w} \left(\frac{(h(x)-h(a))^{\beta-1}}{w(x)}; v \right) \right\} \left\{ {}_{p}^{q}\pounds_{g}^{w} \left(g(x); v \right) \right\}$$
$$= \frac{{}_{p}^{q}\pounds_{g}^{w} (g(x); v)}{[q(v)]^{\beta}}$$

by using Proposition 2.6, Theorem 2.8 and the equality (21). □

Taking Theorems 2.10 and 2.11 together, the following two results can be given without proof.

Corollary 2.12. *Let $\beta > 0$, $g \in AC_w^m[a,b]$, $h \in C^m[a,b]$, $h'(x) > 0$ and $\left({}_{a^+}J_w^{m-\beta} g\right)_k$ be $w-$weighted $h-$exponential order for $k = 0,1,2,\ldots,m-1$. So, we have*

$$ {}_{p}^{q}\pounds_{h}^{w} \left({}_{a^+}^{RL}D_w^{\beta} g(x); v \right) = [q(v)]^{\beta} \left\{ {}_{p}^{q}\pounds_{g}^{w} (g(x); v) \right\} - p(v) \sum_{k=0}^{m-1} [q(v)]^{m-1-k} g_k^{RL}(a^+) \tag{22}$$

for $m = [|\beta|] + 1$ where $g_k^{RL}(x) = \left(\frac{D_x}{h'(x)}\right)^k \left(w(x)\left({}_{a^+}J_w^{m-\beta} g(x)\right)\right)$, $k \in \mathbb{N}_0$.

Corollary 2.13. *Let $\beta > 0$, $g \in AC_w^m[a,b]$, $h \in C^m[a,b]$, $h'(x) > 0$ and g_k be $w-$weighted $h-$exponential order function for $k = 0,1,2,\ldots,m-1$. So, the following equality is satisfied*

$$ {}_{p}^{q}\pounds_{h}^{w} \left({}_{a^+}^{C}D_w^{\beta} g(x); v \right) = [q(v)]^{\beta} \left\{ {}_{p}^{q}\pounds_{g}^{w} (g(x); v) \right\} - p(v) \sum_{k=0}^{m-1} [q(v)]^{\beta-1-k} g_k(a^+) \tag{23}$$

for $m = [|\beta|] + 1$ where $g_k(x) = \left(\frac{D_x}{h'(x)}\right)^k (w(x) g(x))$, $k \in \mathbb{N}_0$.

Now let's recollect the definition of the Mittag-Leffler function with one and two parameters, and then we will give a property related to the transform ${}_{p}^{q}\pounds_{h}^{w}$.

The one and two-parameter Mittag-Leffler functions are defined, respectively as follows

$$E_{\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\beta k + 1)}, \tag{24}$$

$$E_{\beta,\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\beta k + \alpha)} \tag{25}$$

where $\alpha, \beta, x \in \mathbb{R}$, $\alpha > 0$ and $\beta > 0$, respectively [19].

Theorem 2.14. *The quantitative weighted Jafari transform of the Mittag–Leffler functions with one and two parameters are given by*

i. $${}_{p}^{q}\pounds_{h}^{w} \left(\frac{(h(x)-h(a))^{\alpha-1}}{w(x)} E_{\beta,\alpha}\left(C(h(x)-h(a))^{\beta}\right); v \right) = \frac{p(v)[q(v)]^{\beta-\alpha}}{[q(v)]^{\beta} - C},$$

ii. $${}_{p}^{q}\pounds_{h}^{w} \left(\frac{1}{w(x)} E_{\beta}\left(C(h(x)-h(a))^{\beta}\right); v \right) = \frac{p(v)[q(v)]^{\beta-1}}{[q(v)]^{\beta} - C}$$

for $\left|\frac{C}{[q(v)]^{\beta}}\right| < 1$ where α, β and C are real constants independent of x.

Proof. *i. We get*

$$
\begin{aligned}
&{}_{p}^{q}\pounds_{h}^{w}\left(\frac{(h(x)-h(a))^{\alpha-1}}{w(x)}E_{\beta,\alpha}\left(C(h(x)-h(a))^{\beta}\right);v\right)\\
&= {}_{p}^{q}\pounds_{h}^{w}\left(\sum_{k=0}^{\infty}\frac{1}{\Gamma(\beta k+\alpha)}\frac{C^{k}(h(x)-h(a))^{\beta k+\alpha-1}}{w(x)};v\right)\\
&= \sum_{k=0}^{\infty}\frac{C^{k}}{\Gamma(\beta k+\alpha)}\,{}_{p}^{q}\pounds_{h}^{w}\left(\frac{(h(x)-h(a))^{\beta k+\alpha-1}}{w(x)};v\right)\\
&= \sum_{k=0}^{\infty}\frac{C^{k}}{\Gamma(\beta k+\alpha)}\,\frac{\Gamma(\beta k+\alpha)\,p(v)}{[q(v)]^{\beta k+\alpha}}\\
&= \frac{p(v)}{[q(v)]^{\alpha}}\sum_{k=0}^{\infty}\left(\frac{C}{[q(v)]^{\beta}}\right)^{k} = \frac{p(v)\,[q(v)]^{\beta-\alpha}}{[q(v)]^{\beta}-C}
\end{aligned}
$$

by using Proposition 2.6 and the linearity of the quantitative weighted Jafari transform.

ii. Since $E_{\beta,1}=E_{\beta}$, the proof is clear. □

In the next section, using the results obtained, we will solve certain w-weighted classical and fractional initial value problems with respect to an h function by utilizing the quantitative weighted Jafari transform.

3. Applications of the transform ${}_{p}^{q}\pounds_{h}^{w}$

Now, we are going to solve the following weighted initial value problems about the function h with the help of Proposition 2.6 and Theorem 2.10.

Example 3.1. *Solve the following initial value problem (IVP)*

$$D_{w}y(x)=Ky(x),\ w(a)\,y(a)=C$$

where $K,C\in\mathbb{R}$.

Solution.

$$
\begin{aligned}
&{}_{p}^{q}\pounds_{h}^{w}(D_{w}y(x);v)=K\,{}_{p}^{q}\pounds_{h}^{w}(y(x);v)\\
\Rightarrow\ &{}_{p}^{q}\pounds_{h}^{w}(y(x);v)=C\frac{p(v)}{q(v)-K}\\
\Rightarrow\ &y(x)=\left({}_{p}^{q}\pounds_{h}^{w}\right)^{-1}\left(C\frac{p(v)}{q(v)-K};x\right)=C\frac{e^{K(h(x)-h(a))}}{w(x)}.
\end{aligned}
$$

Example 3.2. *Solve the next IVP*

$$\left(D_{w}^{2}-3D_{w}+2\right)y(x)=0,\ y_{1}(a)=A,\ y_{0}(a)=B$$

where $A,B\in\mathbb{R}$ and

$$y_{k}(a)=\left[\left(\frac{D_{x}}{h'(x)}\right)^{k}(w(x)\,y(x))\right]_{x=a}$$

for $k=0,1$.

Solution.

$$
\begin{aligned}
& {}_{p}^{q}\pounds_{h}^{w}\left(D_{w}^{2}y(x);v\right)-3\,{}_{p}^{q}\pounds_{h}^{w}\left(D_{w}y(x);v\right)+2\,{}_{p}^{q}\pounds_{h}^{w}\left(y(x);v\right)=0\\
\Rightarrow\ & {}_{p}^{q}\pounds_{h}^{w}\left(y(x);v\right)=p(v)\,\frac{Bq(v)+A-3B}{[q(v)]^{2}-3q(v)+2}=p(v)\left(\frac{A-B}{q(v)-2}-\frac{A-2B}{q(v)-1}\right)\\
\Rightarrow\ & y(x)=\left({}_{p}^{q}\pounds_{h}^{w}\right)^{-1}\left(p(v)\left(\frac{A-B}{q(v)-2}-\frac{A-2B}{q(v)-1}\right);x\right)\\
\Rightarrow\ & y(x)=(A-B)\,\frac{e^{2(h(x)-h(a))}}{w(x)}+(2B-A)\,\frac{e^{h(x)-h(a)}}{w(x)}.
\end{aligned}
$$

Finally, we are going to solve a w-weighted fractional initial value problem regard to the function h by considering the features obtained in the previous section.

Example 3.3. *Consider Caputo type IVP*

$$
{}_{a^{+}}^{C}D_{w}^{\beta}y(x)+Ky(x)=\frac{f(h(x)-h(a))}{w(x)} \tag{26}
$$

with the initial conditions:

$$
m-1<\beta<m \quad \textit{and} \quad y_{k}(a^{+})=\left[\left(\frac{D_{x}}{h'(x)}\right)^{k}(w(x)\,y(x))\right]_{x=a^{+}}=A_{k} \tag{27}
$$

for $k=0,1,...,m-1$ where $K\in\mathbb{R}$, $A_k\in\mathbb{R}$, $m\in\mathbb{N}$ and $f(x)$ is verify that

$$
f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^{k} \tag{28}
$$

for $x\in\mathbb{R}$. Prove that solution of this problem can be given by

$$
\begin{aligned}
y(x)=&\sum_{n=0}^{m-1}A_{n}\frac{(h(x)-h(a))^{n}}{w(x)}E_{\beta,n+1}\left(-K(h(x)-h(a))^{\beta}\right)\\
&+\sum_{k=0}^{\infty}f^{(k)}(0)\,\frac{(h(x)-h(a))^{\beta+k}}{w(x)}E_{\beta,\beta+k+1}\left(-K(h(x)-h(a))^{\beta}\right).
\end{aligned}
$$

Solution.
We firstly calculate the transform ${}_{p}^{q}\pounds_{h}^{w}$ for the function f in (28) the following that

$$
\begin{aligned}
{}_{p}^{q}\pounds_{h}^{w}\left(\frac{f(h(x)-h(a))}{w(x)};v\right)&={}_{p}^{q}\pounds_{h}^{w}\left(\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}\frac{(h(x)-h(a))^{k}}{w(x)};v\right)\\
&=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}\,{}_{p}^{q}\pounds_{h}^{w}\left(\frac{(h(x)-h(a))^{k}}{w(x)};v\right)\\
&=\sum_{k=0}^{\infty}f^{(k)}(0)\,\frac{p(v)}{[q(v)]^{k+1}},
\end{aligned}
$$

by using Proposition 2.6 Then, if we apply the transform ${}_{p}^{q}\pounds_{h}^{w}$ *to both sides of the equality (26) and make the necessary adjustments, we get*

$$ {}_{p}^{q}\pounds_{h}^{w}\left(y\left(x\right);v\right)=\sum_{n=0}^{m-1}A_{n}\frac{p\left(v\right)\left[q\left(v\right)\right]^{\beta-n-1}}{\left[q\left(v\right)\right]^{\beta}+K}+\sum_{k=0}^{\infty}f^{(k)}\left(0\right)\frac{p\left(v\right)\left[q\left(v\right)\right]^{-k-1}}{\left[q\left(v\right)\right]^{\beta}+K}, $$

from Corollary 2.13. Consequently, if we take the inverse of transform ${}_{p}^{q}\pounds_{h}^{w}$ *of both sides of the above equation with the help of Theorem 2.14, we acquire the desired result.*

Example 3.4. *Take account of Riemann Liouville type IVP*

$$ {}_{a^{+}}^{RL}D_{w}^{\beta}y\left(x\right)+Ky\left(x\right)=\frac{f\left(h\left(x\right)-h\left(a\right)\right)}{w\left(x\right)} \tag{29} $$

with the following conditions:

$$ m-1<\beta<m \quad \text{and} \quad y_{k}^{RL}\left(a^{+}\right)=B_{k} \tag{30} $$

for $k=0,1,...,m-1$ *where* $K\in\mathbb{R}$, $B_k\in\mathbb{R}$, $m\in\mathbb{N}$,

$$ y_{k}^{RL}\left(a^{+}\right)=\left[\left(\frac{D_{x}}{h'\left(x\right)}\right)^{k}\left(w\left(x\right)\left({}_{a^{+}}J_{w}^{m-\beta}y\left(x\right)\right)\right)\right]_{x=a^{+}} $$

and $f\left(x\right)$ *is satisfy that*

$$ f\left(x\right)=\sum_{k=0}^{\infty}\frac{f^{(k)}\left(0\right)}{k!}x^{k} \tag{31} $$

for $x\in\mathbb{R}$. *Demonstrate that this problem has the solution given by*

$$ \begin{aligned} y\left(x\right)=&\sum_{n=0}^{m-1}B_{n}\frac{\left(h\left(x\right)-h\left(a\right)\right)^{n}}{w\left(x\right)}E_{m,n+1}\left(-K\left(h\left(x\right)-h\left(a\right)\right)^{m}\right)\\ &+\sum_{k=0}^{\infty}f^{(k)}\left(0\right)\frac{\left(h\left(x\right)-h\left(a\right)\right)^{\beta+k}}{w\left(x\right)}E_{\beta,\beta+k+1}\left(-K\left(h\left(x\right)-h\left(a\right)\right)^{\beta}\right). \end{aligned} $$

Solution.
With the help of Theorem 2.14 and Corollary 2.12, the desired result can be reached with operations similar to the solution of the previous example.

4. Conclusion

In this study, a new perspective on the Jafari transform has been introduced by proposing the quantitative weighted Jafari transform, and its mathematical foundations have been thoroughly examined. Within this framework, the transform of the fractional derivative and fractional integral of a function with respect to an h function and a w−weight has been derived, and the fundamental properties of this transform have been analyzed. Furthermore, the transform values of Mittag-Leffler functions with one and two parameters under the proposed transform have been investigated. The theoretical findings have been applied to solve classical and fractional initial value problems based on a h function and w−weight, showing that the quantitative weighted Jafari transform is an effective tool for solving analytical problems. This study establishes that the proposed transform provides a significant contribution to both theoretical mathematics and applied sciences. In the future, we anticipate that this transform will provide innovative perspectives and solutions for a wide range of renowned equations. Finally, as an open problem for researchers, we propose investigating the solution of the Ulam–Hyers stability problem using the quantitative weighted Jafari transform.

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