Filomat 39:10 (2025), 3343–3352
https://doi.org/10.2298/FIL2510343R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Probabilistic extension of a general fixed point theorem

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Abstract. As a probabilistic extension of the theorem obtained by Pant et al. [R.P. Pant, V. Rakočević, D. Gopal, A. Pant, M. Ram, A General Fixed Point Theorem, Filomat 35(12) (2021), 4061–4072] in this paper, we prove existence and uniqueness of a fixed point for a wide class of self-mapping defined on complete Menger PM-space. Our theorem generalizes well-known fixed point theorems proved for Menger PM spaces. Also, this theorem characterizes probabilistic metric completeness. Some examples and comments are provided based on the obtained results.

1. Introduction

In 1971, Ćirić [5] introduced the notion of orbital continuity.

Definition 1.1 ([5]). *If f is a self-mapping of a metric space (X,d) then the set $O(x,f) = \{f^n x : n = 0,1,2,\ldots\}$ is called the orbit of f at x and f is called orbitally continuous if $u = \lim_{i\to\infty} f^{m_i}x$ implies $fu = \lim_{i\to\infty} ff^{m_i}x$.*

Every continuous self-mapping is orbitally continuous, but not conversely. Pant and Pant [17] introduced another, weaker form of continuity.

Definition 1.2 ([17]). *A self-mapping f of a metric space X is called a k-continuous, $k = 1,2,3,\ldots$, if $f^k x_n \to ft$ whenever $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X such that $f^{k-1}x_n \to t$.*

A k-continuous mapping is obviously orbitally continuous. Recently, Pant et al. [18] introduced a weaker form of these definitions.

Definition 1.3 ([18]). *A self-mapping f of a metric space (X,d) will be called a weakly orbitally continuous if the set $\{y \in X : \lim_i f^{m_i}y = u \Rightarrow \lim_i ff^{m_i}y = fu\}$ is nonempty whenever the set $\{x \in X : \lim_i f^{m_i}x = u\}$ is nonempty.*

2020 *Mathematics Subject Classification*. Primary 47H10.
Keywords. Menger PM-spaces, Fixed point, Weak orbital continuity, Contractive mapping.
Received: 09 September 2024; Revised: 26 January 2025; Accepted: 28 January 2025
Communicated by Ljubiša D. R. Kočinac
For the third author, this paper is supported by the Ministry of Defence, Republic of Serbia, Grant no. VA-TT/2/20-25.
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Using the notion of weak orbital continuity, Pant et al. [18] generalized Caristi's fixed point theorem [3] omitting the assumption that the function ϕ is a lower semi-continuous. Pant et al. [16] extended the previous theorem to the framework of Menger PM-spaces (see also Hadžić and Ovcin [9] (Theorem 9, p. 207) and Shisheng et al. [23] (Theorem 3, p. 221)).

Theorem 1.4 ([16]). *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space. Let f be a weakly orbitally continuous or f^k is continuous or f is k-continuous mapping, for some $k \geq 1$, satisfying*

$$F_{x,fx}(t) \geq \varepsilon_0\left(t - \left(\phi(x) - \phi(fx)\right)\right), \tag{1}$$

for every $x \in X$ and $t > 0$, where $\phi : X \mapsto [0, \infty)$. Then f has a fixed point.

Remark 1.5. *The condition (1) is equivalent to the following condition (see [9])*

$$(\forall \alpha \in (0,1]) \ \ \sup\left\{t : F_{x,fx}(t) \leq 1 - \alpha\right\} \leq \phi(x) - \phi(fx). \tag{2}$$

Recently, Pant et al. [19] established a theorem that guarantees the existence and uniqueness of a fixed point and can be applied to mappings that meet contractive type conditions as well as mappings that do not. This result is independent of Caristi's fixed point theorem [3]. Furthermore, Banach's fixed point theorem [1], Kannan's fixed point theorem [11, 12], Chatterjea's fixed point theorem [4] and Ćirić's fixed point theorem [6] are particular cases of Theorem 2.1 proved in [19]. Further generalizations and extensions in this direction can be found in Pant and Bisht [15], Mitrović et al. [14], Bisht [2].

As a probabilistic extension of the theorem obtained by Pant et al. [19], in this paper we prove the existence and uniqueness of a fixed point for a wide class of self-mappings defined on complete Menger PM-space. Our theorem generalizes well-known fixed point theorems proved for Menger PM-spaces (e.g. results obtained by Sehgal and Bharucha-Reid [22] and Ćirić [7]). Also, this theorem characterizes probabilistic metric completeness. Some examples and comments are given according to the obtained results.

2. Preliminaries

Menger [13] introduced the concept of a probabilistic metric space (briefly, PM-space) using distribution functions instead of non-negative real numbers as the values of the metric. Schweizer and Sklar [20, 21] studied the properties of spaces introduced by Menger and have developed their theory in depth. The first result from the fixed point theory for probabilistic metric spaces was obtained by Sehgal and Bharucha–Reid [22] as a generalization of the classical Banach Contraction Mapping Principle.

Let Δ^+ be the set of all distribution functions $F : \mathbb{R} \to [0,1]$ such that F is a non-decreasing, left-continuous mapping, which satisfies $F(0) = 0$ and $\sup_{x \in \mathbb{R}} F(x) = 1$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

Definition 2.1 ([21]). *A binary operation $T : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $([0,1], T(\cdot,\cdot))$ is a topological monoid with unit 1 such that $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$, for all $a,b,c,d \in [0,1]$.*

Some examples of t-norm are minimum $T(a,b) = \min\{a,b\}$ (briefly T_{min}), the product $T(a,b) = a \cdot b$ and *Łukasievicz* t-norm $\mathcal{L}(a,b) = \max\{a+b-1, 0\}$. For such t-norms, it holds $\min\{a,b\} \geq a \cdot b \geq \mathcal{L}(a,b)$.

Definition 2.2. *A Menger probabilistic metric space (briefly, Menger PM-space) is a triple $(X, \mathcal{F}, T)$ where X is a nonempty set, T is a continuous t-norm, and $\mathcal{F}$ is a mapping from $X \times X$ into D^+ such that, if $F_{x,y}$ denotes the value of $\mathcal{F}$ at the pair (x,y), the following conditions hold:*

(PM1) $F_{x,y}(t) = \varepsilon_0(t)$ *if and only if* $x = y$;

(PM2) $F_{x,y}(t) = F_{y,x}(t)$;

(PM3) $F_{x,z}(t+s) \geq T\Big(F_{x,y}(t), F_{y,z}(s)\Big)$ *for all* $x, y, z \in X$ *and* $s, t \geq 0$.

Remark 2.3 ([22]). *Every metric space is a PM-space. Let* (X, d) *be a metric space and let* $T(a, b) = \min\{a, b\}$ *is a continuous t-norm. Define* $F_{x,y}(t) = \varepsilon_0(t - d(x, y))$ *for all* $x, y \in X$ *and every* $t > 0$. *The triple* $(X, \mathcal{F}, T)$ *is a PM-space induced by the metric* d.

Definition 2.4. *Let* $(X, \mathcal{F}, T)$ *be a Menger PM-space.*

(1) A sequence $\{x_n\}_{n\in\mathbb{N}}$ *in* X *is said to be convergent to* x *in* X *if, for every* $\varepsilon > 0$ *and* $\lambda > 0$ *there exists positive integer* N *such that* $F_{x_n,x}(\varepsilon) > 1 - \lambda$ *whenever* $n \geq N$.

(2) A sequence $\{x_n\}_{n\in\mathbb{N}}$ *in* X *is called Cauchy sequence if, for every* $\varepsilon > 0$ *and* $\lambda > 0$ *there exists positive integer* N *such that* $F_{x_n,x_m}(\varepsilon) > 1 - \lambda$ *whenever* $n, m \geq N$.

(3) A Menger PM-space is said to be complete if every Cauchy sequence in X *converges to a point in* X.

Lemma 2.5 ([10]). *Let* $(X, \mathcal{F}, T)$ *be a Menger PM-space and let* $\varphi\colon (0, \infty) \to (0, \infty)$ *be a continuous, non-decreasing function which satisfies* $\varphi(t) < t$ *for every* $t > 0$. *Then the following statement holds:*

If for $x, y \in X$ *we have* $F_{x,y}\Big(\varphi(t)\Big) \geq F_{x,y}(t)$ *for every* $t > 0$ *then* $x = y$.

3. Main results

Theorem 3.1. *Let* f *be a self-mapping of a complete Menger PM-space* $(X, \mathcal{F}, T)$ *and* $\phi : X \to [0, \infty)$ *satisfying*

$$F_{fx,fy}(t) \geq \varepsilon_0\Big(t - \Big(\phi(x) - \phi(fx) + \phi(y) - \phi(fy)\Big)\Big), \tag{3}$$

for every $x, y \in X$ *and* $t > 0$. *If* f *is a weakly orbitally continuous, or* f *is a orbitally continuous, or* f *is k-continuous mapping (for some* $k \in \mathbb{N}$*), then* f *has a unique fixed point.*

Proof. Let $x_0 \in X$. Define a sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_1 = fx_0$, $x_2 = fx_1, \ldots, x_n = fx_{n-1}$, i.e. $x_n = f^n x_0, n \in \mathbb{N}$. Then, we obtain that

$$F_{fx_0,fx_1}(t) \geq \varepsilon_0\Big(t - \Big(\phi(x_0) - \phi(fx_0) + \phi(x_1) - \phi(fx_1)\Big)\Big) = \varepsilon_0\Big(t - \Big(\phi(x_0) - \phi(x_2)\Big)\Big).$$

is satisfied for every $t > 0$. Similarly, we obtain that the following inequalities

$$\begin{aligned} F_{fx_1,fx_2}(t) &\geq \varepsilon_0\Big(t - \Big(\phi(x_1) - \phi(fx_1) + \phi(x_2) - \phi(fx_2)\Big)\Big) = \varepsilon_0\Big(t - \Big(\phi(x_1) - \phi(x_3)\Big)\Big) \\ F_{fx_2,fx_3}(t) &\geq \varepsilon_0\Big(t - \Big(\phi(x_2) - \phi(fx_2) + \phi(x_3) - \phi(fx_3)\Big)\Big) = \varepsilon_0\Big(t - \Big(\phi(x_2) - \phi(x_4)\Big)\Big) \\ &\vdots \end{aligned}$$

hold for every $t > 0$. If there exists a point x_i, for some $i \in \mathbb{N} \cup \{0\}$, such that $\phi(x_i) - \phi(fx_{i+2}) \leq 0$, then using condition (3) we obtain that $x_{i+1} = fx_{i+1}$, i.e. x_{i+1} is a fixed point of f. Hence, we will suppose that

$$\phi(x_i) - \phi(x_{i+2}) > 0, \tag{4}$$

holds for every $i \in \mathbb{N} \cup \{0\}$. From inequality (4) it follows that $\{\phi(x_{2n})\}_{n\in\mathbb{N}}$ and $\{\phi(x_{2n-1})\}_{n\in\mathbb{N}}$ are a strictly decreasing sequence. Then, there exist $s_1, s_2 \geq 0$ such that

$$\lim_{n\to\infty} \phi(x_{2n}) = s_1, \lim_{n\to\infty} \phi(x_{2n-1}) = s_2.$$

Hence, for any given $\varepsilon, \lambda > 0$ such that $\frac{\varepsilon}{2} > \lambda$, it follows that

$$s_1 \leq \phi(x_{2n}) < s_1 + \lambda, s_2 \leq \phi(x_{2n-1}) < s_2 + \lambda, \tag{5}$$

is satisfied for sufficiently large $n \in \mathbb{N}$. Now, we will show that

$$F_{x_n,x_m}(\varepsilon) > 1 - \lambda, \tag{6}$$

holds for sufficiently large $n, m \in \mathbb{N}$. Hence, let us consider the following four cases.

Case 1. Let us assume that $n = 2p+1, (p \in \mathbb{N}), m = 2q, (q \in \mathbb{N})$ and $p < q$. From (5) we get that $\phi(x_{2p})-\phi(x_{2q}) < \lambda$ and $\phi(x_{2p+1}) - \phi(x_{2q-1}) < \lambda$ hold. Then, we obtain that

$$\phi(x_{2p}) - \phi(x_{2q}) - \Big(\phi(x_{2p+1}) - \phi(x_{2q-1})\Big) = \phi(x_{2p}) - \phi(x_{2p+1}) + \phi(x_{2q-1}) - \phi(x_{2q}) < \lambda.$$

Now, having in mind inequality (3), it follows that

$$F_{x_n,x_m}(\varepsilon) = F_{x_{2p+1},x_{2q}}(\varepsilon) \geq \varepsilon_0\Big(\varepsilon - \Big(\phi(x_{2p}) - \phi(x_{2p+1}) + \phi(x_{2q-1}) - \phi(x_{2q})\Big)\Big) \geq \varepsilon_0(\varepsilon - \lambda) = 1 > 1 - \lambda,$$

is satisfied for sufficiently large $n, m \in \mathbb{N}$.

Case 2. If we assume that $n = 2p, (p \in \mathbb{N})$ and $m = 2q + 1, (q \in \mathbb{N})$, then Case 2 reduces to Case 1.

Case 3. Let us assume that $n = 2p, (p \in \mathbb{N}), m = 2q, (q \in \mathbb{N})$ and $p < q$. Hence, $n + 1$ is odd and m is even and from previous cases we get that

$$F_{x_{n+1},x_m}\left(\frac{\varepsilon}{2}\right) \geq \varepsilon_0\left(\frac{\varepsilon}{2} - \lambda\right),$$

holds for sufficiently large $n, m \in \mathbb{N}$. On the other hand, it follows that

$$F_{x_n,x_{n+1}}\left(\frac{\varepsilon}{2}\right) = F_{fx_{n-1},fx_n}\left(\frac{\varepsilon}{2}\right) \geq \varepsilon_0\left(\frac{\varepsilon}{2} - \Big(\phi(x_{n-1}) - \phi(x_{n+1})\Big)\right) \geq \varepsilon_0\left(\frac{\varepsilon}{2} - \lambda\right),$$

holds, for sufficiently large $n \in \mathbb{N}$. Finally, we obtain that

$$F_{x_n,x_m}(\varepsilon) \geq T\left(F_{x_n,x_{n+1}}\left(\frac{\varepsilon}{2}\right), F_{x_{n+1},x_m}\left(\frac{\varepsilon}{2}\right)\right) \geq T\left(\varepsilon_0\left(\frac{\varepsilon}{2} - \lambda\right), \varepsilon_0\left(\frac{\varepsilon}{2} - \lambda\right)\right) = T(1,1) > 1 - \lambda,$$

holds for sufficiently large $n, m \in \mathbb{N}$.

Case 4. If we assume that $n = 2p + 1, (p \in \mathbb{N})$ and $m = 2q + 1, (q \in \mathbb{N})$, then Case 4 reduces to Case 3.

From all cases we can conclude that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is complete, there exists a point z in X such that $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} f^p x_n = z$, for every $p \geq 1$. Suppose that f is a weakly orbitally continuous mapping. Since $\{f^n x_0\}_{n\in\mathbb{N}}$ converges for every x_0 in X, weak orbital continuity implies that there exists $y_0 \in X$ such that $f^n y_0 \to u$, and $f^{n+1} y_0 \to fu$ for some u in X. This implies $u = fu$, that is u is a fixed point of f. If f is k-continuous for some $k \geq 1$, or orbitally continuous, then f is a weakly orbitally continuous.

Finally, we will prove the uniqueness of fixed point. If we assume that u and v are two fixed points of self-mapping f, then it follows that

$$F_{fu,fv}(t) \geq \varepsilon_0\Big(t - \Big(\phi(u) - \phi(fu) + \phi(v) - \phi(fv)\Big)\Big) = \varepsilon_0(t) = 1,$$

holds for every $t > 0$, i.e. $fu = fv$, i.e. $u = v$. This completes the proof. □

Remark 3.2. *The condition* (3) *is equivalent to the following condition*

$$\Big(\forall \alpha \in (0, 1]\Big) \ \sup\Big\{t : F_{fx,fy}(t) \leq 1 - \alpha\Big\} \leq \phi(x) - \phi(fx) + \phi(y) - \phi(fy). \tag{7}$$

Example 3.3. *In accordance with Remark 2.3 we have that the triple $(X, \mathcal{F}, T_{min})$ is a complete Menger PM-space, for $X \subseteq \mathbb{R}$. Hence, let $X = [2,4]$ equipped with the Euclidean metric. Define $f : X \mapsto X$ by*

$$fx = \begin{cases} \frac{1}{2}x + 1, & \text{if } 2 \le x \le 3, \\ \frac{9}{4}, & \text{if } 3 < x \le 4. \end{cases}$$

It is easy to see that the mapping f is weakly orbitally continuous. Indeed, we have that $f^n x \to 2$ and $f f^n x \to 2 = f2$ hold for every $x \in [2,3]$. Let us define $\phi : X \to [0,\infty)$ by $\phi(x) = 5x$.

Now, let us observe next three cases:

1. *if $x, y \in [2,3]$, then it follows that*

$$\begin{aligned} F_{fx,fy}(t) &= \varepsilon_0(t - |fx - fy|) = \varepsilon_0\left(t - \frac{1}{2}|x-y|\right) \\ &\ge \varepsilon_0\left(t - \frac{1}{2}\right) \ge \varepsilon_0\left(t - \frac{1}{2}(5x + 5y - 10)\right) \\ &= \varepsilon_0\Big(t - \big(\phi(x) - \phi(fx) + \phi(y) - \phi(fy)\big)\Big), \end{aligned}$$

   holds for every $t > 0$;

2. *if $x, y \in (3,4]$, then we have that*

$$\begin{aligned} F_{fx,fy}(t) &= \varepsilon_0\big(t - |fx - fy|\big) = \varepsilon_0(t) \ge \varepsilon_0\left(t - \left(5x - 5\cdot\frac{9}{4} + 5y - 5\cdot\frac{9}{4}\right)\right) \\ &= \varepsilon_0\Big(t - \big(\phi(x) - \phi(fx) + \phi(y) - \phi(fy)\big)\Big), \end{aligned}$$

   holds for every $t > 0$;

3. *if $x \in [2,3]$ and $y \in (3,4]$, then we have that*

$$\begin{aligned} F_{fx,fy}(t) &= \varepsilon_0\big(t - |fx - fy|\big) = \left(t - \left|\frac{1}{2}x + 1 - \frac{9}{4}\right|\right) \ge \varepsilon_0\left(t - \frac{1}{4}\right) \\ &\ge \varepsilon_0\big(t - (\phi(y) - \phi(fy))\big) \ge \varepsilon_0\Big(t - \big(\phi(x) - \phi(fx) + \phi(y) - \phi(fy)\big)\Big), \end{aligned}$$

   holds for every $t > 0$.

For all three cases, condition (3) is satisfied for all $x, y \in X$ and every $t > 0$. Hence, all the conditions of Theorem 3.1 are satisfied and mapping f has a fixed point $x = 2$.

The first fixed point theorem in probabilistic metric spaces was proved by Sehgal and Bharucha-Reid [22].

Theorem 3.4 ([22]). *If a self-mapping f of a complete Menger PM-space $(X, \mathcal{F}, T_{min})$ satisfies contractive condition*

$$F_{fx,fy}(t) \ge F_{x,y}\left(\frac{t}{q}\right), \tag{8}$$

for all $x, y \in X$, every $t > 0$ and $q \in (0,1)$, then there exists a unique fixed point of the mapping f.

In next theorem we show that Theorem 3.4 is particular case of Theorem 3.1.

Theorem 3.5. *If a self-mapping f of a complete Menger PM-space $(X, \mathcal{F}, T_{min})$ satisfies condition (8), then the mapping f also satisfies conditions of Theorem 3.1 and has a unique fixed point.*

Proof. From condition (8) and taking that $y = fx$, it follows that

$$F_{fx,f^2x}(t) \geq F_{x,fx}\left(\frac{t}{q}\right), \tag{9}$$

holds for every $x \in X$, $t > 0$ and $q \in (0,1)$. Furthermore, from condition (9) we obtain that

$$\frac{1}{q}\sup\left\{t : F_{fx,f^2x}(t) \leq 1-\alpha\right\} \leq \sup\left\{t : F_{x,fx}(t) \leq 1-\alpha\right\}, \tag{10}$$

is satified for every $\alpha \in (0,1]$. Since t-norm $T = T_{min}$, using condition (8) and condition (PM3) from Definition 2.2 we have that

$$F_{fx,fy}(t) \geq F_{x,y}\left(\frac{t}{q}\right) \geq T\Big(F_{x,fy}(at), F_{fy,y}(bt)\Big) = \min\left\{F_{x,fy}(at), F_{fy,y}(bt)\right\},$$

holds for every $t > 0$ and $q \in (0,1)$, whereby we choose $a > 1$ and $b \in (0,1)$ such that $a + b = \frac{1}{q}$. Furthermore, we have that

$$\begin{aligned} F_{fx,fy}(t) &\geq \min\left\{F_{x,fy}(at), F_{fy,y}(bt)\right\} \\ &\geq \min\left\{\min\left\{F_{x,fx}(ct), F_{fx,fy}(dt)\right\}, F_{fy,y}(bt)\right\} \\ &= \min\left\{F_{x,fx}(ct), F_{fx,fy}(dt), F_{fy,y}(bt)\right\}, \end{aligned} \tag{11}$$

is satisfied for every $t > 0$, for such chosen $a > 1$ and $b \in (0,1)$, whereby we choose $d > 1$, and $c \in (0,1)$ such that $c + d = a$. From condition (11) it follows that

$$F_{fx,fy}(t) \geq F_{fx,fy}(dt) \tag{12}$$

holds, for every $t > 0$, and for such chosen $d > 1$, or

$$F_{fx,fy}(t) \geq \min\left\{F_{x,fx}(ct), F_{fy,y}(bt)\right\} \tag{13}$$

holds, for every $t > 0$, and for such chosen $b, c \in (0,1)$.

If condition (12) holds, having in mind Lemma 2.5, then we obtain that $fx = fy$, i.e. $F_{fx,fy}(dt) = 1$. Hence, from (11) we have that $1 = F_{fx,fy}(dt) \leq F_{x,fx}(ct)$ and $1 = F_{fx,fy}(dt) \leq F_{y,fy}(bt)$ i.e. $F_{x,fx}(ct) = F_{y,fy}(bt) = 1$, hold for every $t > 0$, and for such chosen $b, c \in (0,1)$. Finally, we obtain $x = fx = fy = y$, i.e. $\phi(x) = \phi(fx) = \phi(fy) = \phi(y)$, for an arbitrary function $\phi : X \to [0,\infty)$. Finally, it follows that

$$F_{fx,fy}(t) = \varepsilon_0(t) = \varepsilon_0\Big(t - \big(\phi(x) - \phi(fx) + \phi(y) - \phi(fy)\big)\Big)$$

holds for every $t > 0$, and contractive condition (3) is trivially satisfied.

Now we will assume that condition (13) is satisfied. Without loss of generality, let us assume that $b > c$. Now, from condition (13) we obtain that

$$F_{fx,fy}(t) \geq \min\left\{F_{x,fx}(ct), F_{y,fy}(ct)\right\}$$

is satisfied for every $t > 0$, and for the chosen $c \in (0,1)$. Now, we get that $F_{fx,fy}(t) \geq F_{x,fx}(ct)$ or $F_{fx,fy}(t) \geq F_{y,fy}(ct)$, i.e. $\sup\left\{t : F_{fx,fy}(t) \leq 1-\alpha_0\right\} \leq \frac{1}{c}\sup\left\{t : F_{x,fx}(t) \leq 1-\alpha_0\right\}$ or $\sup\left\{t : F_{fx,fy}(t) \leq 1-\alpha_0\right\} \leq \frac{1}{c}\sup\big\{t :$

$F_{y,fy}(t) \leq 1-\alpha_0\Big\}$ for arbitrary fixed $\alpha_0 \in (0,1]$. Finally, we get that

$$\begin{aligned}\sup\Big\{t : F_{fx,fy}(t) \leq 1-\alpha_0\Big\} &\leq \frac{1}{c}\sup\Big\{t : F_{x,fx}(t) \leq 1-\alpha_0\Big\} + \frac{1}{c}\sup\Big\{t : F_{y,fy}(t) \leq 1-\alpha_0\Big\} \\ &\leq \frac{1}{c(1-q)}\sup\Big\{t : F_{x,fx}(t) \leq 1-\alpha_0\Big\} - \frac{q}{c(1-q)}\sup\Big\{t : F_{x,fx}(t) \leq 1-\alpha_0\Big\} \\ &\quad + \frac{1}{c(1-q)}\sup\Big\{t : F_{y,fy}(t) \leq 1-\alpha_0\Big\} - \frac{q}{c(1-q)}\sup\Big\{t : F_{y,fy}(t) \leq 1-\alpha_0\Big\} \\ &\leq \frac{1}{c(1-q)}\sup\Big\{t : F_{x,fx}(t) \leq 1-\alpha_0\Big\} - \frac{q}{c(1-q)}\cdot\frac{1}{q}\sup\Big\{t : F_{fx,f^2x}(t) \leq 1-\alpha_0\Big\} \quad (14) \\ &\quad + \frac{1}{c(1-q)}\sup\Big\{t : F_{y,fy}(t) \leq 1-\alpha_0\Big\} - \frac{q}{c(1-q)}\cdot\frac{1}{q}\sup\Big\{t : F_{fy,f^2y}(t) \leq 1-\alpha_0\Big\} \\ &\leq \frac{1}{c(1-q)}\sup\Big\{t : F_{x,fx}(t) \leq 1-\alpha_0\Big\} - \frac{1}{c(1-q)}\sup\Big\{t : F_{fx,f^2x}(t) \leq 1-\alpha_0\Big\} \\ &\quad + \frac{1}{c(1-q)}\sup\Big\{t : F_{y,fy}(t) \leq 1-\alpha_0\Big\} - \frac{1}{c(1-q)}\sup\Big\{t : F_{fy,f^2y}(t) \leq 1-\alpha_0\Big\},\end{aligned}$$

holds for arbitrary fixed $\alpha_0 \in (0,1]$, every $t > 0$ and $q \in (0,1)$, and chosen $c \in (0,1)$. If we define a function $\phi : X \to [0,\infty)$ by $\phi(x) = \frac{1}{c(1-q)}\sup\Big\{t : F_{x,fx}(t) \leq 1-\alpha_0\Big\}$, for $q \in (0,1)$, and chosen $c \in (0,1)$ we obtain

$$\sup\Big\{t : F_{fx,fy}(t) \leq 1-\alpha_0\Big\} \leq \phi(x) - \phi(fx) + \phi(y) - \phi(fy).$$

Keeping in mind condition (2) it is obvious that $\phi(fx) \leq \phi(x)$ holds. Thus, f satisfies the conditions of Theorem 3.1 and, hence, has a unique fixed point. This proves that the Theorem 3.4 is a particular case of Theorem 3.1. □

Ćirić introduced the notion of a generalized contraction in PM-spaces in [7].

Definition 3.6 ([7]). *A mapping f will be called a generalized contraction if there exists a constant $q \in (0,1)$, such that for all $x, y \in X$*

$$F_{fx,fy}(qt) \geq \min\Big\{F_{x,y}(t), F_{x,fx}(t), F_{y,fy}(t), F_{x,fy}(2t), F_{y,fx}(2t)\Big\}. \quad (15)$$

Theorem 3.7. *If a self-mapping f of a complete Menger PM-space $(X, \mathcal{F}, T_{min})$ satisfies the condition* (15)*, then the mapping f also satisfies conditions of Theorem 3.1 and has a unique fixed point.*

Proof. Since $T = T_{min}$, using condition (PM3) from Definition 2.2 we get $F_{x,fy}(2t) \geq \min\Big\{F_{x,fx}(t), F_{fx,fy}(t)\Big\}$ and $F_{y,fx}(2t) \geq \min\Big\{F_{y,fy}(t), F_{fx,fy}(t)\Big\}$. Then condition (15) becomes

$$F_{fx,fy}(qt) \geq \min\Big\{F_{x,y}(t), F_{x,fx}(t), F_{y,fy}(t), F_{fx,fy}(t)\Big\},$$

for every $t > 0$ and $q \in (0,1)$. From previous inequality it follows

$$F_{fx,fy}(qt) \geq F_{x,y}(t),$$

i.e.

$$F_{fx,fy}(t) \geq F_{x,y}\left(\frac{t}{q}\right), \quad (16)$$

or

$$F_{fx,fy}(qt) \geq \min\Big\{F_{x,fx}(t), F_{y,fy}(t), F_{fx,fy}(t)\Big\},$$

i.e.

$$F_{fx,fy}(t) \geq \min\left\{F_{x,fx}\left(\frac{t}{q}\right), F_{y,fy}\left(\frac{t}{q}\right), F_{fx,fy}\left(\frac{t}{q}\right)\right\}, \tag{17}$$

is satisfied for every $t > 0$, and $q \in (0, 1)$. If condition (16) is satisfied, then this case reduces to the proof of Theorem 3.5. Hence, we will consider only condition (17). If in condition (17) holds that

$$\min\left\{F_{x,fx}\left(\frac{t}{q}\right), F_{y,fy}\left(\frac{t}{q}\right), F_{fx,fy}\left(\frac{t}{q}\right)\right\} = F_{fx,fy}\left(\frac{t}{q}\right),$$

is satisfied for every $t > 0$, and $q \in (0, 1)$, then having in mind Lemma 2.5 we obtain that $fx = fy$, and analogously to Theorem 3.5 we get that condition (3) holds. Hence, in the sequel, we will consider the case when condition

$$F_{fx,fy}(t) \geq \min\left\{F_{x,fx}\left(\frac{t}{q}\right), F_{y,fy}\left(\frac{t}{q}\right)\right\}, \tag{18}$$

holds for every $t > 0$, and $q \in (0, 1)$. From previous, it follows that $F_{fx,fy}(t) \geq F_{x,fx}\left(\frac{t}{q}\right)$ or $F_{fx,fy}(t) \geq F_{y,fy}\left(\frac{t}{q}\right)$, i.e. $\frac{1}{q}\sup\left\{t : F_{fx,fy}(t) \leq 1-\alpha_0\right\} \leq \sup\left\{t : F_{x,fx}(t) \leq 1-\alpha_0\right\}$ or $\frac{1}{q}\sup\left\{t : F_{fx,fy}(t) \leq 1-\alpha_0\right\} \leq \sup\left\{t : F_{y,fy}(t) \leq 1-\alpha_0\right\}$ for arbitrary fixed $\alpha_0 \in (0, 1]$.

On the other hand, from inequality (18), for $y = fx$ and every $x \in X$, it follows

$$F_{fx,f^2x}(t) \geq \min\left\{F_{x,fx}\left(\frac{t}{q}\right), F_{fx,f^2x}\left(\frac{t}{q}\right)\right\}, \tag{19}$$

for every $t > 0$ and $q \in (0, 1)$. Again, if we assume that

$$\min\left\{F_{x,fx}\left(\frac{t}{q}\right), F_{fx,f^2x}\left(\frac{t}{q}\right)\right\} = F_{fx,f^2x}\left(\frac{t}{q}\right)$$

is satisfied for every $t > 0$ and $q \in (0, 1)$, applying Lemma 2.5 we obtain that $fx = f^2x$, i.e. $y = fy$ is a fixed point and condition (3) is satisfied. Finally, we will consider only the case when condition

$$F_{fx,f^2x}(t) \geq F_{x,fx}\left(\frac{t}{q}\right), \tag{20}$$

holds for every $t > 0$ and $q \in (0, 1)$, i.e. $\frac{1}{q}\sup\left\{t : F_{fx,f^2x}(t) \leq 1-\alpha\right\} \leq \sup\left\{t : F_{x,fx}(t) \leq 1-\alpha\right\}$. The rest of the proof is analogous to the proof of Theorem 3.5. □

We now show that Theorem 3.1 characterizes probabilistic metric completeness.

Theorem 3.8. *Let $(X, \mathcal{F}, T)$ be a Menger PM-space. If every k-continuous or weakly orbitally continuous self-mappings of X satisfying the condition* (3) *of Theorem 3.1 has a fixed point, then X is complete.*

Proof. Let us assume the opposite, i.e that X is not complete. Then there exists a Cauchy sequence in X, say $\{a_n\}_{n\in\mathbb{N}}$, consisting of distinct points that does not converge. Let $x \in X$ be given. Then, since x is not a limit point of the Cauchy sequence $\{a_n\}_{n\in\mathbb{N}}$, there exists a least positive integer $N(x)$ such that $x \neq a_{N(x)}$ and

$$F_{a_{N(x)},a_m}\left(\frac{t}{2}\right) \geq F_{x,a_{N(x)}}(t), \tag{21}$$

is satisfied for every $m \geq N(x)$ and $t > 0$.

Let us define a mapping $f : X \to X$ by $f(x) = a_{N(x)}$. Then, $f(x) \neq x$, for every x in X. Using inequality (21) it follows

$$F_{fx,fy}\left(\frac{t}{2}\right) = F_{a_{N(x)},a_{N(y)}}\left(\frac{t}{2}\right) \geq F_{x,a_{N(x)}}(t) = F_{x,fx}(t),$$

if $N(x) \leq N(y)$, or

$$F_{fx,fy}\left(\frac{t}{2}\right) = F_{a_{N(x)},a_{N(y)}}\left(\frac{t}{2}\right) \geq F_{y,a_{N(y)}}(t) = F_{y,fy}(t),$$

if $N(x) > N(y)$, for all x, y in X and every $t > 0$. From the previous two inequalities we get that $F_{fx,fy}\left(\frac{t}{2}\right) \geq F_{x,fx}(t)$ or $F_{fx,fy}\left(\frac{t}{2}\right) \geq F_{y,fy}(t)$, i.e. $\frac{1}{2}\sup\left\{t : F_{x,fx}(t) \leq 1-\alpha_0\right\} \geq \sup\left\{t : F_{fx,fy}(t) \leq 1-\alpha_0\right\}$ or $\frac{1}{2}\sup\left\{t : F_{y,fy}(t) \leq 1-\alpha_0\right\} \geq \sup\left\{t : F_{fx,fy}(t) \leq 1-\alpha_0\right\}$, for arbitrary fixed $\alpha_0 \in (0,1]$.

On the other hand, taking $y = fx$ in (21) we get $N(fx) > N(x)$ and

$$F_{fx,f^2x}\left(\frac{t}{2}\right) = F_{a_{N(x)},a_{N(fx)}}\left(\frac{t}{2}\right) \geq F_{x,a_{N(x)}}(t) = F_{x,fx}(t),$$

for every $t > 0$. From the previous inequality we have that

$$\sup\left\{t : F_{fx,f^2x}(t) \leq 1-\alpha\right\} \leq \frac{1}{2}\sup\left\{t : F_{x,fx}(t) \leq 1-\alpha\right\}, \tag{22}$$

holds for every $\alpha \in (0,1]$. Analogous to the previous one, we have that

$$F_{fy,f^2y}\left(\frac{t}{2}\right) \geq F_{y,fy}(t),$$

for every $t > 0$, and

$$\sup\left\{t : F_{fy,f^2y}(t) \leq 1-\alpha\right\} \leq \frac{1}{2}\sup\left\{t : F_{y,fy}(t) \leq 1-\alpha\right\}, \tag{23}$$

holds for every $\alpha \in (0,1]$.

Now, let us define a function $\phi : X \to [0,\infty)$ by $\phi(x) = 2\sup\left\{t : F_{x,fx}(t) \leq 1-\alpha_0\right\}$, for arbitrary fixed $\alpha_0 \in (0,1]$. Then, using (22) and (23) we get that

$$\begin{aligned}\phi(x) - \phi(fx) + \phi(y) - \phi(fy) &= 2\sup\left\{t : F_{x,fx}(t) \leq 1-\alpha_0\right\} - 2\sup\left\{t : F_{fx,f^2x}(t) \leq 1-\alpha_0\right\}\\ &\quad + 2\sup\left\{t : F_{y,fy}(t) \leq 1-\alpha_0\right\} - 2\sup\left\{t : F_{fy,f^2y}(t) \leq 1-\alpha_0\right\}\\ &\geq 2\sup\left\{t : F_{x,fx}(t) \leq 1-\alpha_0\right\} - \sup\left\{t : F_{x,fx}(t) \leq 1-\alpha_0\right\}\\ &\quad + 2\sup\left\{t : F_{y,fy}(t) \leq 1-\alpha_0\right\} - \sup\left\{t : F_{y,fy}(t) \leq 1-\alpha_0\right\}\\ &= \sup\left\{t : F_{x,fx}(t) \leq 1-\alpha_0\right\} + \sup\left\{t : F_{y,fy}(t) \leq 1-\alpha_0\right\}\\ &\geq \sup\left\{t : F_{fx,fy}(t) \leq 1-\alpha_0\right\},\end{aligned} \tag{24}$$

holds for arbitrary fixed $\alpha_0 \in (0,1]$, and every $t > 0$. From (24) and Remark 3.2 it is clear that f satisfies condition (3) of Theorem 3.1.

Since the range of f is contained in the non-convergent Cauchy sequence $\{a_n\}_{n\in\mathbb{N}}$, there exists no sequence $\{x_n\}_{n\in\mathbb{N}}$ in X for which $\{fx_n\}_{n\in\mathbb{N}}$ converges and the condition $fx_n \to t \Rightarrow f^2x_n \to ft$ is violated. Therefore, f is a 2-continuous mapping. In a similar manner, it can be proved that f is a weakly orbitally continuous. Thus, f is a 2-continuous as well as weak orbitally continuous self-mapping of X which satisfy the condition (3) of Theorem 3.1, but does not possess a fixed point. This contradicts the hypothesis of the theorem. Hence X is complete. This completes the proof. □

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