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On τ-base and e-density of topological spaces

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Abstract. In this paper, we prove some properties of families of τ-open sets, study the properties of the space of τ-continuous mappings, as well as the properties of the e-density of topological spaces. We prove an analogue of A.V.Arkhangelskii's theorem for τ-base. Also we showed that the base of a topological space is not always a τ-base. We introduce the notion of e-density of topological spaces. As well as, we investigate some properties of this cardinal function. It is given an example of a topological space, e-density of which is not equal to its density.

1. Introduction

In recent researches an interest in the theory of cardinal invariants and their behavior under the influence of various covariant functors is increasing fast. In [3], [5], [6], [7], [11], [12], [13], [14] the authors investigated several cardinal invariants under the influence of some weakly normal and normal functors and hyperspaces.

The concept of a τ-closed subset was introduced Juhasz I. in 1980 in his book [10]. In 1987, Arkhangel'skii A.V. introduced the classes of τ-continuous and strictly τ-continuous mappings and gave examples of their discrepancy with the class of continuous mappings [2]. In 2016, in the work [15] Okunev O. introduced the concept of τ-closure of a set and presented some criteria for the τ-continuity of mappings. In 2023, Georgiou D.N., Mamadaliev N.K., Zhuraev R.M. introduced the definitions of a τ-open set and a τ-interior operator. Using new concepts, they expanded O. Okunev's theorem and introduced new criteria for τ-continuity of mappings [9]. In the work [4] some properties of ω-bounded spaces were studied. In [1] the authors introduced and investigated e-spaces and e-continuous mappings.

In this article, we proved some properties of families of τ-open sets, studied the properties of the space of τ-continuous mappings, as well as the properties of the e-density of topological spaces. Let X be a topological space and let A be a subset of X. We denote the closure of A in X by $\mathrm{cl}_X A$.

Throughout the paper all spaces are assumed to be topological spaces and τ be an infinite cardinal number.

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2. On τ-base of topological spaces

Definition 2.1. *[10] Let X be a topological space. A set $F \subset X$ is called τ-closed in X if for each $B \subset F$ such that $|B| \le \tau$, the closure of the set B in X lies in F.*

It is known that every closed subset of a topological space is τ-closed. But the opposite is not always true.

Example 2.2. *[9] On the real line we will assume that all sets whose complement is countable are open, and we will also declare the empty set is open, i.e. the set of all real numbers $\mathbb{R}$ has the following topology:*

$$\theta = \{\varnothing\} \cup \{U : U \subset \mathbb{R}, |\mathbb{R}\backslash U| \le \omega\}$$

Since every set whose cardinality of its complement does not exceed ω is open in this topological space, then an arbitrary countable set $B \subset \mathbb{R}$ is closed. Let's choose an arbitrary subset $M \subset \mathbb{R}$. Then every subset $B \subset M$, whose cardinality does not exceed ω, coincides with its closure, which means that $B \subset M$ implies that $cl_{\mathbb{R}}B \subset M$ for all $|B| \le \omega$. From the arbitrariness of the set M it follows that each subset of this space is ω-closed. In particular, the set of all irrational numbers in this space is ω-closed, but not closed.

I. Juhasz in his work [10] proved that the tightness of a topological space X does not exceed τ if and only if every τ-closed subset is closed.

Definition 2.3. *[9] Let X be a topological space. A set $U \subset X$ is called τ-open in X if its complement $X\backslash U$ is τ-closed.*

Every subset of the space defined in Example 2.2 is τ-open.

Any τ-open set containing a point $x \in X$ is called a τ-neighborhood of this point.

The τ-closure of a subset A is defined as follows:

$$[A]_\tau = \bigcup\{cl_X B : B \subset A, |B| \le \tau\}.$$

Recall that a subset A is τ-dense in X if $[A]_\tau = X$ [15].

For any subsets A and B of the space X the following relation holds: if $A \subset B$, then $[A]_\tau \subset [B]_\tau$.

Example 2.4. *On the set of real numbers with the natural topology, we choose the set of all rational numbers. Let's find its ω-closure $[\mathbb{Q}]_\omega = \bigcup\{cl_{\mathbb{R}}B : B \subset \mathbb{Q}, |B| \le \omega\}$. As a subset $B \subset \mathbb{Q}$, $|B| \le \omega$ we take the set itself $\mathbb{Q}$, the closure of which coincides with the set of real numbers. This means $[\mathbb{Q}]_\omega = \mathbb{R}$, and we can conclude that the set of rational numbers on the Euclidean line is ω-dense.*

Let Θ_τ be the family of all τ-open subsets in X. The family $\mathrm{B}_\tau \subset \Theta_\tau$ is called the τ-base of the topological T_1-space X, if every τ-open subset of $U_\tau \in \Theta_\tau$, $U_\tau \ne \varnothing$ can be represented as a union of some subfamily B_τ.

Theorem 2.5. *(An analogue of the theorem of A.V. Arkhangelsky) The family B_τ is a τ-base of the topological space X if and only if for every element x from X and for every τ-neighborhood $V \in \Theta_\tau$ of x there exists $U \in \mathrm{B}_\tau$ such that $x \in U \subset V$.*

Proof. *Necessity:* Let x be an arbitrary point in the space X and $V \in \Theta_\tau$ be its τ-neighborhood. Since the family B_τ is a τ-base by condition, then there exists $U_\alpha \in \mathrm{B}'_\tau \subset \mathrm{B}_\tau$ such that $\bigcup\{U_\alpha : U_\alpha \in \mathrm{B}'_\tau\} = V$. Then there exists $U_\alpha \in \mathrm{B}'_\tau$ such that x belongs to $U_\alpha \subset V$.

Sufficiency: Let V be an arbitrary non-empty τ-open set of X and for every x from V there exists a subset $U_x \in \mathrm{B}_\tau$ such that $x \in U_x \subset V$. Then the subfamily $\{U_x : x \in V, U_x \in \mathrm{B}_\tau\}$ covers the set V, i.e., $V = \bigcup\{U_x : x \in V, U_x \in \mathrm{B}_\tau\}$. Consequently, B_τ is a τ-base of the space X. Theorem 2.5 is proved.

Remark 2.6. *The base of a topological space is not always a τ-base.*

Example 2.7. *Consider the topological space given in Example 2.2. Since the complement of every set whose cardinality does not exceed ω is open in this topological space, then an arbitrary set $B \subset \mathbb{R}$ for which $|B| \leq \omega$ is closed. Let us choose an arbitrary subset $M \subset \mathbb{R}$. Then every subset $B \subset M$ whose cardinality does not exceed ω coincides with its closure, which means that from $B \subset M$ it follows that $cl_{\mathbb{R}}B \subset M$ for all $|B| \leq \omega$. From the arbitrariness of the choice of M it follows that every subset of this space is ω-closed, and hence every subset is ω-open. As the base of this space, we can choose a family of subsets of the form $\mathrm{B} = \{U : U \subset \mathbb{R}, |\mathbb{R}\setminus U| \leq \omega\}$. Since each element of B is an infinite set, it is impossible to represent the ω-open subset $\{x\}$, where $x \in \mathbb{R}$, as a union of some subfamily of the family B. Therefore, B is not an ω-base.*

Remark 2.8. *A τ-base is not always a base.*

Example 2.9. *The ω-base $\mathrm{B}_\omega = \{\{x\} : x \in \mathbb{R}\}$ is not a base of the space $(\mathbb{R}, \theta)$ from Example 2.4 because its elements are not open sets.*

Definition 2.10. *The family $\mathrm{B}_\tau(x)$ of τ-neighborhoods of a point x is called a τ-base of the topological space X at point x if for every τ-neighborhood V of point x there exists $U \in \mathrm{B}_\tau(x)$ such that $x \in U \subset V$.*

Definition 2.11. *Let X be a topological T_1-space and for every x from X a τ-base $\mathrm{B}_\tau(x)$ of the space X. The family $\{\mathrm{B}_\tau(x) : x \in X\}$ is called a system of τ-neighborhoods of the topological space X.*

Theorem 2.12. *Any system of τ-neighborhoods $\{\mathrm{B}_\tau(x) : x \in X\}$ has the following properties:*

($\mathrm{B}_\tau P1$) *For every element x of X we have that $\mathrm{B}_\tau(x)$ is non-empty and for every element U from $\mathrm{B}_\tau(x)$ we have that $x \in U$.*

($\mathrm{B}_\tau P2$) *If x belongs to $U \in \mathrm{B}_\tau(y)$ for some $y \in X$, then there is a set $V \in \mathrm{B}_\tau(x)$ such that $V \subset U$.*

($\mathrm{B}_\tau P3$) *For any elements U_1, U_2 of the family $\mathrm{B}_\tau(x)$ there is a set $V \in \mathrm{B}_\tau(x)$ such that $V \subset U_1 \cap U_2$.*

Proof. The property ($\mathrm{B}_\tau P1$) follows from the definition of a τ-base at a point x. The property ($\mathrm{B}_\tau P2$) follows from the fact that U is a τ-neighborhood of the point x, and therefore, by definition τ-base at a point there is a neighborhood $V \in \mathrm{B}_\tau(x)$ such that $x \in V \subset U$.

Let's prove the property ($\mathrm{B}_\tau P3$). Any elements U_1, U_2 of the family $\mathrm{B}_\tau(x)$ are τ-open sets containing x. Therefore, $U_1 \cap U_2$ is also a τ-open set and $x \in U_1 \cap U_2$. By the definition of a τ-base at a point there is an element V of $\mathrm{B}_\tau(x)$ such that $V \subset U_1 \cap U_2$. Theorem 2.12 is proved.

Proposition 2.13. *For any subset A of a topological space X, the following conditions are equivalent:*

1) A point x belongs to $[A]_\tau$;

2) For every $\mathrm{B}_\tau(x)$ and every $U \in \mathrm{B}_\tau(x)$ we have $U \cap A \neq \varnothing$;

3) There is a system of neighborhoods $\mathrm{B}_\tau(x)$ such that $U \cap A \neq \varnothing$ for every $U \in \mathrm{B}_\tau(x)$.

Proof. To prove the implication 1)⇒2) let us assume the opposite, i.e., assume that for the τ-base $\mathrm{B}_\tau(x)$ at the point x there is a neighborhood $U \in \mathrm{B}_\tau(x)$ that does not intersect the set A. Then $A \subset X\setminus U$. Since $X\setminus U$ is a τ-closed subset, then $[A]_\tau \subset X\setminus U$. Therefore, $x \notin [A]_\tau$, which contradicts 1).

Condition 3) directly follows from condition 2). Let us prove the implication 3)⇒1). Let us assume that condition 1) is not satisfied, i.e., $x \notin [A]_\tau$. Then there is a τ-closed set F containing $[A]_\tau$ such that $x \notin F$. For a τ-open set $V = X\setminus F$ we have $x \in V$ and $V \cap A = \varnothing$. Further, for every τ-base $\mathrm{B}_\tau(x)$ at the point x there is a neighborhood $U \in \mathrm{B}_\tau(x)$ such that $x \in U \subset V$. From $V \cap A = \varnothing$ it follows that $U \cap A$ is empty, which means 3) does not hold. Proposition 2.13 is proved.

Corollary 2.14. *If U is a τ-open set and A is some subset of the space X disjoint with U, then $U \cap [A]_\tau = \emptyset$. In particular, if U and V are disjoint τ-open subsets, then $U \cap [V]_\tau = [U]_\tau \cap V = \varnothing$.*

3. On τ-continuous mappings

Let X and Y be topological spaces and let τ be an infinite cardinal.

Definition 3.1. *[2]. A mapping $f : X \to Y$ is called τ-continuous if for every set $A \subset X$ such that $|A| \leq \tau$, the mapping $f|_A : A \to Y$ is continuous.*

Let $C^\tau(X, Y)$ denote the set of all τ-continuous mappings of the space X into the space Y. Note that $C^\tau(X, Y)$ does not coincide with $C(X, Y)$.

Example 3.2. *[2] Let $T(\omega_1) = \{\infty : \infty \leq \omega_1\}$ be the space of all ordinal numbers not exceeding the first uncountable ordinal ω_1 in order topology. Let us assume that $f(\infty) = 0$ for all $\infty < \omega_1$ and $f(\omega_1) = 1$. The function $f : T(\omega_1) \to \mathbb{R}$ defined in this way is ω-continuous, but not continuous.*

If $A \subset X$ and $B \subset Y$, then $\langle A, B\rangle = \{f \in C^\tau(X, Y) : f(A) \subset B\}$.

Let ξ be the family of all finite subsets of the space X. Then the family P_ξ of all sets of the form $\langle A, U\rangle$, where $A \in \xi$ and U is open set in Y the set that constitutes the prebase of some topology T_ξ is called the topology of pointwise convergence; $C^\tau(X, Y)$ together with this topology is denoted by $C^\tau_p(X, Y)$. If $Y = \mathbb{R}$, then $C^\tau_p(X, \mathbb{R})$ will be denoted by $C^\tau_p(X)$.

Proposition 3.3. *[8] Let β be the base of the space Y. Then sets of the form $W(x_1, ..., x_k, U_1, ..., U_k) = \{f \in C^\tau(X, Y) : f(x_i) \in U_i, i = 1, ..., k\}$, where $x_1, ..., x_k \in X$, $U_1, ..., U_k$ - elements of the base β and $k \in \mathbb{N}$, form the base of the space $C^\tau_p(X, Y)$.*

Theorem 3.4. *If Y is a T_i-space, then the space $C^\tau_p(X, Y)$ with the topology of pointwise convergence is also T_i-space for $i = 0, 1, 2, 3$.*

Proof. 1) Let Y be a T_0-space. Let us show that $C^\tau_p(X, Y) \in T_0$. To do this, we choose two arbitrary unequal maps f_1 and f_2 from $C^\tau_p(X, Y)$. Then there is a point x from X such that $f_1(x) \neq f_2(x)$ in the space Y. Since $Y \in T_0$, then at least for the point $f_2(x)$ there exists a neighborhood $U(f_2(x))$, which does not contain $f_1(x)$. Therefore, f_1 does not belong to $W(x, U(f_2(x)))$. This means $C^\tau_p(X, Y) \in T_0$.

2) Let $Y \in T_1$. Let us show that $C^\tau_p(X, Y) \in T_1$. Let us choose arbitrary unequal mappings f_1 and f_2 from $C^\tau_p(X, Y)$. Then there is a point x from X such that $f_1(x) \neq f_2(x)$ in the space Y. Since $Y \in T_1$, then for every points $f_1(x) \neq f_2(x)$ there is a neighborhood $U(f_2(x))$ of a point $f_2(x)$ that does not contain $f_1(x)$ and there is also a neighborhood $U(f_1(x))$ of the point $f_1(x)$ that does not contain $f_2(x)$, i.e., $f_1(x) \notin U(f_2(x))$, $f_2(x) \notin U(f_1(x))$. Therefore, $f_1 \notin W(x, U(f_2(x)))$ and $f_2 \notin W(x, U(f_1(x)))$. This means $C^\tau_p(X, Y) \in T_1$.

3) Let $Y \in T_2$. Let us show that $C^\tau_p(X, Y) \in T_2$. Let us choose arbitrary different mappings f_1 and f_2 from $C^\tau_p(X, Y)$. Then there is a point x from X such that $f_1(x) \neq f_2(x)$ in the space Y. Since $Y \in T_2$, then for every points $f_1(x) \neq f_2(x)$ there is a neighborhood $U(f_2(x))$ of the point $f_2(x)$, there is also a neighborhood $U(f_1(x))$ of the point $f_1(x)$ that do not intersect, i.e., $U(f_1(x)) \cap U(f_2(x)) = \emptyset$. Therefore, $W(x, U(f_2(x))) \cap W(x, U(f_1(x))) = \varnothing$. This means $C^\tau_p(X, Y) \in T_2$.

4) Let $Y \in T_3$. Let us choose an arbitrary f from $C^\tau_p(X, Y)$ and some arbitrary closed subset $F \subset C^\tau_p(X, Y)$ such that $f \notin F$. Then there exist $x_1, ..., x_n \in X$ and $U_i(f(x_i)) \subset Y$ such that $f \in W(x_1, ..., x_n, U_1, ..., U_n)$, where $W(x_1, ..., x_n, U_1, ..., U_n)$ is an open set in $C^\tau_p(X, Y)$ and $i = 1, ..., n$. Let $y_i = f(x_i)$. Then, since $Y \in T_3$, there are open subsets $O_i \subset Y$ and $U_i \subset Y$ such that $y_i \in O_i$, $Y \backslash U_i \subset V_i$ and $O_i \cap V_i = \varnothing$ for all $i = 1, ..., n$. The sets $G(x_1, ..., x_n, O_1, ..., O_n)$ and $H(x_1, ..., x_n, V_1, ..., V_n)$ are open in $C^\tau_p(X, Y)$ and $G(x_1, ..., x_n, O_1, ..., O_n) \cap H(x_1, ..., x_n, V_1, ..., V_n) = \varnothing$. Since $y_i \in O_i$, then $f \in G(x_1, ..., x_n, O_1, ..., O_n)$. If $g \in F$, then $g \notin W(x_1, ..., x_n, U_1, ..., U_n)$. So $g(x_i) \notin U_i$ and $g(x_i) \in V_i$. It follows from this that $g \in H$ and $F \subset H$. Since $G(x_1, ..., x_n, O_1, ..., O_n)$ and $H(x_1, ..., x_n, V_1, ..., V_n)$ do not intersect, then $C^\tau_p(X, Y) \in T_3$. Theorem 3.4 is proved.

4. On *e*-density of topological space

In this section we introduced the notion of *e*-density of topological spaces and investigated some properties of it.

A set G in a topological space X is called extremely open (briefly *e*-open) if G and its clouser $\mathrm{cl}_X G$ are open subsets of X. Recall that a subset of a topological space an *e*-closed if its complement is an *e*-open [1]. Clearly every clopen set in a topological space is an *e*-open set, but not conversely. For example $\mathbb{R}\backslash\{a\}$ is an *e*-open subset of $\mathbb{R}$ (for each $a \in \mathbb{R}$) which is not a clopen set.

The set of all *e*-open subsets of X forms a base for a topology θ_e on X. This means that θ_e is weaker topology with respect to the original topology θ. Whenever θ_e coincides with θ (i.e., $\theta = \theta_e$) the space X is called an *e-space*.

An element x is called an *e*-cluster point of A if each *e*-open subset of X containing x meets A. The set of all *e*-cluster points of A is called the *e*-closure of A and denoted by $e - \mathrm{cl}_X A$ [1]. It is easy to check that $\mathrm{cl}_X A \subset e - \mathrm{cl}_X A$ for each subset A of X. The inclusion may be proper. For example, we take the unit interval $(0, 1)$ in $\mathbb{R}$. We have $\mathrm{cl}_{\mathbb{R}}(0, 1) = [0, 1]$, but $e - \mathrm{cl}_{\mathbb{R}}(0, 1) = \mathbb{R}$. The *e*-closure of a set A in X is the intersection of all *e*-closed subsets of X containing A.

In fact, every closure of a set is closed, but the *e*-closure of a set need not be *e*-closed, in general. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ be a subspace of $\mathbb{R}$ with the standard topology and $A = \{1, \frac{1}{3}, \frac{1}{5}, \ldots\}$. Then $e - \mathrm{cl}_X A = A \cup \{0\}$ which is not *e*-closed.

Definition 4.1. *[1] Let X and Y be topological spaces and $f\colon X \to Y$ be a mapping. We say that f is e-continuous at a point $x \in X$ if for each open set V in Y containing $f(x)$ there exists an e-open set U in X containing x such that $f(U) \subset V$. A mapping $f\colon X \to Y$ is called e-continuous if it is e-continuous at each point of X.*

Clearly every *e*-continuous function is continuous, but the converse is not necessarily true in general. In fact if the mapping $id\colon \mathbb{R} \to \mathbb{R}$ is an identity, then it is continuous but not *e*-continuous.

Proposition 4.2. *[1] Let X and Y be topological spaces and $f\colon X \to Y$ be a function. Then the following statements are equivalent.*

1. *f is e-continuous.*
2. *$f^{-1}(V)$ is a union of e-open subsets of X for each open subset V of Y.*
3. *$f^{-1}(H)$ is an intersection of e-closed subsets of X for each closed subset H of Y.*
4. *$f(e - cl_X A) \subset cl_Y f(A)$ for each subset A of X.*

Definition 4.3. *Let X be a topological space. A subset A is called e-dense in X, if $e - cl_X A = X$. e-density of a topological space X is the smallest cardinal number $|A|$, where A is e-dense in X and denoted by $ed(X)$, i.e.*

$$ed(X) = \omega + \min\{|A| : A \text{ is e-dense in } X\}.$$

The density of a space X is defined as the smallest cardinal number of the form $|A|$, where A is a dense subset of X. This cardinal number is denoted by $d(X)$. Clearly for any topological space X and its density $d(X)$, we have $ed(X) \leq d(X)$ and the inequality may be proper.

Example 4.4. *Let $\mathbb{R}$ be the real line, and let $I_s = I \times \{s\}$ for every $s \in \mathbb{R}$, where $I = [0, 1]$. By letting*

$$(x, s_1)\delta(y, s_2) \text{ whenever } x = y = 0 \text{ or } x = y \text{ and } s_1 = s_2$$

we define an equivalence relation δ on the set $\bigcup_{s\in\mathbb{R}} I_s$. The formula

$$\rho([(x, s_1)], [(y, s_2)]) = \begin{cases} |x - y|, & \text{if } s_1 = s_2 \\ x + y, & \text{if } s_1 \neq s_2 \end{cases}$$

defines a metric on the set of equivalence classes of δ. The metric space thus obtained will be called the hedgehog of spininess $\mathfrak{c}$ and will be denoted by $J(\mathfrak{c})$. For every $s \in S$ the mapping f_s of the interval I to $J(\mathfrak{c})$ defined by letting

$f_s(x) = [(x,s)]$ is a homeomorphic embedding. The family of all balls with rational radii around points of the form $[(r,s)]$, where r is a rational number, is a base for $J(\mathfrak{c})$; so that $w(J(\mathfrak{c})) \leq \mathfrak{c}$. Since $D = \{[(1,s)] : s \in \mathbb{R}\}$ is a discrete subspace of $J(\mathfrak{c})$, it follows that $w(J(\mathfrak{c})) = \mathfrak{c}$. It is known that for every metrizable space X we have $w(X) = d(X)$. Therefore

$$d(J(\mathfrak{c})) = \mathfrak{c}.$$

Let s_0 be fixed in $\mathbb{R}$, and let $U_{s_0} = \{[(x,s_0)] : x \in (0,1)\}$. Since f_{s_0} is homeomorphic embedding, it follows that U_{s_0} is open subset of $J(\mathfrak{c})$. Consider a subset $\mathbb{Q}_{s_0} = \{[(x,s_0)] : x \in I \cap \mathbb{Q}\}$ of I_{s_0}. It is easy to check that $\mathbb{Q}_{s_0}$ is dense in $[I_{s_0}]$, where $[I_{s_0}] = \{[(x,s_0)] : x \in I\}$. We will prove that $\mathbb{Q}_{s_0}$ is e-dense in $J(\mathfrak{c})$. For each $s \in \mathbb{R}$ the set $[I_s]$ is connected, because $[I_s] = f_s(I)$. In this case the space $J(\mathfrak{c})$ is also connected. Let $p \in J(\mathfrak{c})$. Take an arbitrary e-open subset U of $J(\mathfrak{c})$ with $p \in U$. From the connectedness of $J(\mathfrak{c})$, it follows that $cl_{J(\mathfrak{c})}U = J(\mathfrak{c})$. In this case we have $U_{s_0} \cap U \neq \emptyset$. Since $\mathbb{Q}_{s_0}$ is dense in $[I_{s_0}]$ and the subset $U_{s_0} \cap U \neq \emptyset$ is nonempty open subset of $[I_{s_0}]$ implies $\mathbb{Q}_{s_0} \cap U \neq \emptyset$. Thus

$$ed(J(\mathfrak{c})) = |\mathbb{Q}_{s_0}| = \omega.$$

This means that $ed(J(\mathfrak{c})) < d(J(\mathfrak{c}))$.

Whenever X is an e-space then $ed(X) = d(X)$.

Proposition 4.5. *Let $f\colon X \to Y$ be an e-continuous subjective mapping. Then $ed(Y) \leq ed(X)$.*

Proof. Let $ed(X) = \tau \geq \omega$. Then there exists e-dense subset A of X such that $|A| = \tau$. Thus $e - \mathrm{cl}_X A = X$. It suffices to show that $f(A)$ is an e-dense in Y. By Proposition 4.2 we have

$$Y = f(X) = f(e - \mathrm{cl}_X A) \subset \mathrm{cl}_Y f(A) \subset e - \mathrm{cl}_Y f(A).$$

Therefore $f(A)$ is an e-dense in Y. Proposition 4.5 is proved.

Recall that topological space X is an *e-separable*, if $ed(X) = \omega$. Every separable space is e-separable.

Remark 4.6. *The e-separability of topological space is not a hereditary property.*

In Example 4.4 we consider the subset D of $J(\mathfrak{c})$. Since D is discrete subspace, it follows that $ed(D) = |D| = \mathfrak{c}$, but $ed(J(\mathfrak{c})) = \omega$.

Problem 4.7. *Let X be a topological space and Y be an e-open subspace of X. Is it true that $ed(Y) \leq ed(X)$?*

References

[1] S. Afrooz, F. Azarpanah, N. Hasan Hajee, *On e-spaces and rings of real valued e-continuous functions*, Appl. Gen. Topol. **24** (2023), 433–448.

[2] A. V. Arhangel'skii, *Functional tightness, Q-spaces, and τ-embeddings*, Comment. Math. Univ. Carolinae **24** (1983), 105–119.

[3] R. B. Beshimov, D. N. Georgiou, N. K. Mamadaliev, *On τ-bounded spaces and hyperspaces*, Filomat **36** (2022), 187–193.

[4] R. B. Beshimov, D. N. Georgiou, F. Sereti, R. M. Zhuraev, *Metric, stratifiable and uniform spaces of G-permutation degree*, Math. Slovaca **74** (2024), 235–248.

[5] R. B. Beshimov, D. N. Georgiou, R. M. Zhuraev, *Index boundedness and uniform connectedness of space of the G-permutation degree*, Appl. Gen. Topol. **22** (2021), 447–459.

[6] R. B. Beshimov, N. K. Mamadaliev, *On the functor of semiadditive τ-smooth functionals*, Topol. Appl. **221** (2017), 167–177.

[7] R. B. Beshimov, N. K. Mamadaliev, S. K. E'shtemirova, *Categorical and cardinal properties of hyperspaces with a finite number of components*, J. Math. Sci. **245** (2020), 390–397.

[8] R. B. Beshimov, N. K. Mamadaliev, R. M. Juraev, *On some properties of the space of τ-continuous mappings*, Topol. Appl., to appear.

[9] D. N. Georgiou, N. K. Mamadaliev, R. M. Zhuraev, *A note on functional tightness and minitightness of space of the G-permutation degree*, Comment. Math. Univ. Carolinae **64** (2023), 97–108.

[10] I. Juhász, *Cardinal Functions in Topology: Ten Years Later* Mathematical Center, Amsterdam, 1980.

[11] Lj. D. R. Kočinac, F. G. Mukhamadiev, A. K. Sadullaev, *Some tightness-type properties of the space of permutation degree*, Mathematics **10** (2022), Art. ID 3341.

[12] Lj. D. R. Kočinac, F. G. Mukhamadiev, A. K. Sadullaev, *Some topological and cardinal properties of the space of permutation degree*, Filomat **36** (2022), 7059–7066.

[13] Lj. D. R. Kočinac, F. G. Mukhamadiev, A. K. Sadullaev, *On the space of G-permutation degree of some classes of topological spaces*, Mathematics **11** (2023), Art. ID 4624.

[14] Lj. D. R. Kočinac, F. G. Mukhamadiev, A. K. Sadullaev, Sh. U. Meyliev, *Some network-type properties of the space of G-permutation degree*, Appl. Gen. Topol. **24** (2023), 229–237

[15] O. Okunev, *The minitightness of products*, Topol. Appl. **208** (2016), 10–16.