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Some fixed point theorems in $[3, \Delta, 2]$-metric spaces

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Abstract. In this article we prove the existence and uniqueness of fixed points for self-mappings on $[3, \Delta, 2]$-metric spaces related to a nondecreasing map $\phi : [0, \infty) \to [0, \infty)$ such that $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \in (0, \infty)$.

1. Introduction

The Banach fixed point theorem [2] laid the foundation in the study of fixed points. It is a very powerful theorem with wide range of applications. It has been used in various research fields, such as physics, chemistry, certain engineering branches, economics and many areas of mathematics. This theorem has been generalized and extended by many authors in various ways and directions.

The geometric properties, their axiomatic classification and generalizations of metric spaces have been considered in lot of papers: Menger [17], Aleksandrov, Nemytskii [1], Mamuzić [18], Gähler [13], Nedev, Choban [21–23], Kopperman [15], Dhage, Mustafa, Sims [5, 19].

The notion of an (n, m, ρ)-metric, $n > m$, generalizing the usual notion of a pseudometric (the case when $n = 2, m = 1$), and the notion of an $(n + 1)$-metric (as in [17] and [13]) was introduced in [6]. Connections between some of the topologies induced by a $(3, 1, \rho)$-metric and topologies induced by a pseudo-o-metric, o-metric and symmetric (as in [22]), are given in [7]. Some other characterizations of $(3, j, \rho)$-metrizable topological spaces, $j \in \{1, 2\}$, are given in [3, 4, 9, 10].

There are many fixed point theorems in generalized metric spaces which differ by the contraction type mappings. Papers like: Gähler [13], Dhage [5], Mustafa and Sims [19] and many more, with vast number of fixed point theorems, the similarities between their axioms, motivated us to investigate fixed point theorems in the $[3, \Delta, 2]$-metric spaces.

In this paper we will consider only $[3, \Delta, 2]$-metric spaces, a subclass of the class of $(3, 2)$-metric spaces (as in [8]). Here we examine self-mappings on these spaces related to a nondecreasing map $\phi : [0, \infty) \to [0, \infty)$ such that $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \in (0, \infty)$ (as in [16]).

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2. Preliminaries

We give the definitions for $(3, 2, \rho)$-metric spaces, $(3, 2)$-metric spaces, as in [3], and $[3, \rho, 2]$-metric spaces, as in [8].

Let $M \neq \emptyset$ and $M^{(3)} = M^3/\alpha$, where α is the equivalence relation on M^3 defined by:

$$(x, y, z)\alpha(u, v, w) \Leftrightarrow \pi(u, v, w) = (x, y, z),$$

where π is a permutation. We will use the same notation (x, y, z) for the elements in $M^{(3)}$ keeping in mind that $(x, y, z) = (u, v, w)$ in $M^{(3)}$ iff (x, y, z) is a permutation of (u, v, w).

Let $d : M^{(3)} \to \mathbb{R}_0^+$. We state several conditions for such map:

$(M0)$ $d(x, x, x) = 0$, for all $x \in M$;

$(M1)$ $d(x, y, z) \leq d(x, a, b) + d(a, y, b) + d(a, b, z)$, for all $x, y, z, a, b \in M$;

$(M2)$ $d(x, y, z) \leq d(x, a, b) + d(a, y, b) + d(a, b, z)$, for all $x, y, z, a, b \in M$ with $z \neq x \neq y \neq z$;

$(M3)$ $d(x, x, y) \leq d(x, a, b) + d(y, a, b)$, for all $x, y, z, a, b \in M$;

(Ms) $d(x, x, y) = d(x, y, y)$, for all $x, y \in M$.

Let ρ be a subset of $M^{(3)}$. We state two conditions for such a set:

$(E0)$ $(x, x, x) \in \rho$, for all $x \in M$;

$(E1)$ $(x, a, b), (a, y, b), (a, b, z) \in \rho \implies (x, y, z) \in \rho$, for any $x, y, z, a, b \in M$.

Definition 2.1. *If ρ satisfies $(E0)$ and $(E1)$, we say that ρ is a $(3, 2)$-equivalence.*

It is not difficult to show the following two examples.

Example 2.2. *The set $\Delta = \{(x, x, x)|x \in M\}$ is a $(3, 2)$-equivalence on M.*

Example 2.3. *The set $\rho = \rho_d = \{(x, y, z)|(x, y, z) \in M^{(3)},\ d(x, y, z) = 0\}$ such that d satisfies $(M0)$ and $(M1)$ is a $(3, 2)$-equivalence on M.*

Definition 2.4. *Let $d : M^{(3)} \to \mathbb{R}_0^+$ and $\rho = \rho_d = \{(x, y, z) \in M^{(3)}|d(x, y, z) = 0\}$.*

- (i) *If d satisfies $(M0)$ and $(M1)$ we say that d is a $(3, 2, \rho)$-metric on M, and the pair (M, d) is a $(3, 2, \rho)$-metric space.*
- (ii) *If d satisfies $(M0)$,$(M1)$ and (Ms) we say that d is a $(3, 2, \rho)$-symmetric on M, and the pair (M, d) is a $(3, 2, \rho)$-symmetric space.*

If $\rho = \Delta = \{(x, x, x)|x \in M\}$, we write $(3, j)$ instead of $(3, j, \Delta)$.

Example 2.5. *Let M be a nonempty set. The map $d : M^{(3)} \to \mathbb{R}_0^+$ defined by:*

$$d(x, y, z) = \begin{cases} 0 & , (x, y, z) \in \Delta \\ 1 & , otherwise \end{cases},$$

is a $(3, 2)$-metric on M (the discrete 3-metric).

Example 2.6. *Let $D : M^2 \to \mathbb{R}_0^+$ be a metric on M. The map $d : M^{(3)} \to \mathbb{R}_0^+$ defined by:*

$$d(x, y, z) = \frac{D(x, y) + D(x, z) + D(y, z)}{2},$$

is a $(3, 2)$-metric on M.

Definition 2.7. *Let $d : M^{(3)} \to \mathbb{R}_0^+$ and $\rho = \rho_d = \{(x, y, z) \in M^{(3)} | d(x, y, z) = 0\}$. If d satisfies (M0), (M2) and (M3) we say that d is a $[3, \rho, 2]$-metric on M, and the pair (M, d) is a $[3, \rho, 2]$-metric space.*

Example 2.8. *Let $D : M^2 \to \mathbb{R}_0^+$ be a metric on M and $\alpha > 0$. The map $d : M^{(3)} \to \mathbb{R}_0^+$ defined by:*

$$d(x, y, z) = \alpha \max\{D(x, y), D(x, z), D(y, z)\},$$

is a $[3, \Delta, 2]$-metric on M.

Proposition 2.9. *Each $[3, \rho, 2]$-metric d on M satisfies the condition (Ms).*

Proof. *It follows directly from the condition (M3).* □

Proposition 2.10. [8] *If $d : M^{(3)} \to \mathbb{R}_0^+$ is a $[3, \rho, 2]$-metric on M, then d is also a $(3, 2, \rho)$-metric on M.*

From Proposition 2.10 it follows that the properties which are true in $(3, 2)$-metric spaces, concerning sequences and various types of convergences can be used here in $[3, \Delta, 2]$-metric spaces.

Proposition 2.11. [12] *For any $(3, 2, \rho)$-metric d and any sequence (x_n), the following conditions are equivalent.*

(C1) *$d(x_n, x_m, x_p) \to 0$, as $n, m, p \to \infty$; and*

(C2) *$d(x_n, x_m, x_m) \to 0$, as $n, m \to \infty$.*

Definition 2.12. [12] *A sequence (x_n) in a $(3, 2, \rho)$-metric space (M, d) is called $(3, 2)$-Cauchy if it satisfies* (C1) *or* (C2).

Definition 2.13. [11] *We say that a sequence (x_n) in a $(3, 2, \rho)$-metric space (M, d):*

(*i*) *1-converges to $x \in M$ if $d(x, x, x_n) \to 0$ as $n \to \infty$;*

(*ii*) *2-converges to $x \in M$ if $d(x, x_n, x_n) \to 0$ as $n \to \infty$;*

(*iii*) *3-converges to $x \in M$ if $d(x, x_n, x_m) \to 0$ as $n, m \to \infty$.*

Theorem 2.14. [11] *For any sequence (x_n) in a $(3, 2, \rho)$-metric space (M, d) the following conditions are equivalent:*

(*i*) *(x_n) 1-converges to $x \in M$;*

(*ii*) *(x_n) 2-converges to $x \in M$;*

(*iii*) *(x_n) 3-converges to $x \in M$.*

Definition 2.15. [11] *We say that a sequence (x_n) in a $(3, 2, \rho)$-metric space (M, d) is $(3, 2, \rho)$-convergent ($(3, 2, \rho)$-converges to the point x) if it satisfies any of the conditions in the previous theorem.*

3. Main results

The following type of functions was introduced by Matkowski [16].

Definition 3.1. *Let Φ denote the set of all functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:*
(F1) ϕ is nondecreasing function;
(F2) $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \in (0,\infty)$.
If $\phi \in \Phi$, then ϕ is called a Φ-map or a comparison function.

Lemma 3.2. [16] *Let ϕ be a Φ-map. Then*

(1) $\phi(t) < t$ for all $t \in (0,\infty)$; and

(2) $\phi(0) = 0$.

Proof. It follows directly from Definition 3.1. □

Let (M,d) be a $[3,\Delta,2]$-metric space such that each $(3,2)$-Cauchy sequence is $(3,2)$-convergent and ϕ be a Φ-map.

Theorem 3.3. *Let $f : M \to M$ be a map such that*

$$d(f(x), f(y), f(z)) \le \phi(d(x,y,z))$$

for all $x,y,z \in M$. Then f has a unique fixed point.

Proof. For notational simplicity in the following we will write fx instead of $f(x)$. We choose $x_0 \in M$. Let $x_n = fx_{n-1}$, $n \in \mathbb{N}$. We assume that $x_n \ne x_{n-1}$, for each $n \in \mathbb{N}$. We will prove that the sequence (x_n) is a $(3,2)$-Cauchy sequence. For $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_{n+1}, x_{n+1}) &= d(fx_{n-1}, fx_n, fx_n) \\ &\le \phi(d(x_{n-1}, x_n, x_n)) \\ &\le \phi^2(d(x_{n-2}, x_{n-1}, x_{n-1})) \\ &\vdots \\ &\le \phi^n(d(x_0, x_1, x_1)) \end{aligned}$$

Let $\epsilon > 0$. Since $\lim_{n\to\infty} \phi^n(d(x_0,x_1,x_1)) = 0$ and $\phi(\epsilon) < \epsilon$, there is an $n_0 \in \mathbb{N}$ such that

$$\phi^n(d(x_0,x_1,x_1)) < \epsilon - \phi(\epsilon)$$

for all $n \ge n_0$. So,

$$d(x_n, x_{n+1}, x_{n+1}) < \epsilon - \phi(\epsilon) \tag{1}$$

for all $n \ge n_0$.
We will prove that for $m,n \in \mathbb{N}$,

$$d(x_n, x_m, x_m) < \epsilon \tag{2}$$

for all $m > n \ge n_0$. We prove this inequality by induction on m. It holds for $m = n+1$ using the fact that $\epsilon - \phi(\epsilon) < \epsilon$ and the inequality (1). Assume that the inequality (2) holds for $m = k$. For $m = k+1$, using at first (M3), then Proposition 2.9, and the fact that ϕ is a nondecreasing function, we have

$$\begin{aligned} d(x_n, x_{k+1}, x_{k+1}) &\le d(x_n, x_{n+1}, x_{n+1}) + d(x_{k+1}, x_{n+1}, x_{n+1}) \\ &= d(x_n, x_{n+1}, x_{n+1}) + d(x_{n+1}, x_{k+1}, x_{k+1}) \\ &< \epsilon - \phi(\epsilon) + \phi(d(x_n, x_k, x_k)) \\ &< \epsilon - \phi(\epsilon) + \phi(\epsilon) = \epsilon \end{aligned}$$

So, the inequality (2) holds for all $m > n \ge n_0$. Thus, (x_n) is a $(3,2)$-Cauchy sequence. Then there is $u \in M$ such that $d(x_n, u, u) \to 0$ as $n \to \infty$. We will prove that u is a fixed point of f. If we use at first (M3), then Proposition 2.9, and finally Lemma 3.2, we obtain

$$\begin{aligned} d(u, u, fu) &\le d(u, x_{n+1}, x_{n+1}) + d(fu, x_{n+1}, x_{n+1}) \\ &\le d(u, u, x_{n+1}) + \phi(d(u, x_n, x_n)) \\ &< d(u, u, x_{n+1}) + d(u, x_n, x_n)). \end{aligned}$$

If we take $n \to \infty$ we obtain that $d(u, u, fu) = 0$. So, $fu = u$, i.e. u is a fixed point of f.
Next we will prove the uniqueness. Let v be another fixed point of f such that $v \ne u$. Using Lemma 3.2, we obtain

$$\begin{aligned} d(u, u, v) &= d(fu, fu, fv) \\ &\le \phi(d(u, u, v)) \\ &< d(u, u, v), \end{aligned}$$

which is impossible. So there is a unique fixed point of f. □

Corollary 3.4. *Let $f : M \to M$ be a map such that for some $m \in \mathbb{N}$,*

$$d(f^m x, f^m y, f^m z) \le \phi(d(x, y, z))$$

for all $x, y, z \in M$. Then f has a unique fixed point.

Proof. From the previous theorem it follows that there is a unique fixed point u of the map f^m. Since

$$fu = f(f^m u) = f^{m+1} u = f^m(fu),$$

we obtain that fu is a fixed point of f^m as well. Thus, by the uniqueness of the fixed point of f^m, we obtain that $fu = u$. □

Corollary 3.5. *Let $f : M \to M$ be a map such that*

$$d(fx, fx, fy) \le \phi(d(x, x, y))$$

for all $x, y \in M$. Then f has a unique fixed point.

Proof. It follows directly from Theorem 3.3 by setting $z = x$. □

Corollary 3.6. *Let $f : M \to M$ be a map such that*

$$d(fx, fy, fz) \le q d(x, y, z)$$

for all $x, y, z \in M$, where $q \in [0, 1)$. Then f has a unique fixed point.

Proof. We define a map $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = qt$. Obviously ϕ is a nondecreasing function and $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \in (0,\infty)$, i.e. ϕ is a Φ-map. Now from Theorem 3.3 it follows that f has a unique fixed point. □

Corollary 3.7. *Let $f : M \to M$ be a map such that*

$$d(fx, fy, fz) \leq \frac{d(x,y,z)}{1 + d(x,y,z)}$$

for all $x, y, z \in M$. Then f has a unique fixed point.

Proof. We define a map $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = \frac{t}{1+t}$. It is obvious that ϕ is a nondecreasing function and $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \in (0,\infty)$, i.e. ϕ is a Φ-map. From Theorem 3.3 it follows that f has a unique fixed point. □

Corollary 3.8. *Let $f : M \to M$ be a map such that*

$$d(fx, fy, fz) \leq \ln(1 + d(x,y,z))$$

for all $x, y, z \in M$. Then f has a unique fixed point.

Proof. We define a map $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = \ln(1+t)$. It is obvious that ϕ is a nondecreasing function and $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \in (0,\infty)$, i.e. ϕ is a Φ-map. It follows from Theorem 3.3 that f has a unique fixed point. □

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