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On the behavior of the modulus of m-th derivatives of algebraic polynomials in the whole complex plane without recurrence formula in the weighted Lebesgue space

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Abstract. In this paper we study the growth of $m-th$ derivatives of an arbitrary algebraic polynomial in weighted Lebesgue spaces in bounded and unbounded regions of the complex plane, ignoring the recurrence formula. We also provide estimates in the whole complex plane.

1. Introduction

Let $\mathbb{C}$ be a complex plane; $G \subset \mathbb{C}$ be a finite region bounded by Jordan curve $L := \partial G$ (without loss of generality, we will assume that $0 \in G$); $\Omega := \overline{\mathbb{C}} \setminus \overline{G} = extL$, where $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is the extended complex plane. For $t \in \mathbb{C}$ and $\delta > 0$, let $\Delta(t,\delta) := \{w \in \mathbb{C} : |w-t| > \delta\}$; $\Delta := \Delta(0,1)$. Let $\Phi : \Omega \to \Delta$ be the univalent conformal mapping normalized by $\Phi(\infty) = \infty$ and $\lim_{z\to\infty} \frac{\Phi(z)}{z} > 0$; $\Psi := \Phi^{-1}$. For $t \geq 1$, the us sets L_t, G_t and Ω_t are defined as follows:

$$L_t := \{z : \ |\Phi(z)| = t\}, \ L_1 \equiv L, \ G_t := intL_t, \ \Omega_t := extL_t.$$

For $z \in \mathbb{C}$ and some set $S \subset \mathbb{C}$, let

$$d(z,S) := dist(z,S) = \inf\{|\zeta - z| : \ \zeta \in S\}.$$

The class of all algebraic polynomials $P_n(z)$, $degP_n \leq n$, $n \in \mathbb{N}$ denote by $\wp_n$. Let $\left\{z_j\right\}_{j=1}^{l} \in L$ be a fixed system of distinct points located on L sequentially, without loss of generality, in the positive direction. For some fixed R_0, $1 < R_0 < \infty$, consider the generalized Jacobi weight function

$$h(z) := \begin{cases} h_0(z) \prod\limits_{j=1}^{l} \left|z - z_j\right|^{\gamma_j}, & z \in \overline{G}_{R_0}, \\ 0, & z \in \mathbb{C} \setminus \overline{G}_{R_0}, \end{cases} \tag{1}$$

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where $\gamma_j > -1$, for all $j = 1, 2, ..., l$ and for a measurable function h_0 the inequality $h_0(z) \geq c_0(L, h) > 0, z \in G_{R_0}$, holds for some constant $c_0(L, h) > 0$, depending only on L and h.

For each $0 < p \leq \infty$ and rectifiable Jordan curve $L = \partial G$, we introduce:

$$\|P_n\|_p := \|P_n\|_{\mathcal{L}_p(h,L)} := \left(\int_L h(z) \left|P_n(z)\right|^p |dz| \right)^{1/p} < \infty, 0 < p < \infty, \tag{2}$$

$$\|P_n\|_\infty := \|P_n\|_{\mathcal{L}_\infty(1,L)} := \max_{z \in L} |P_n(z)|, \; p = \infty; \; \mathcal{L}_p(1, L) =: \mathcal{L}_p(L).$$

In many problems of the theory of approximations of functions in the complex plane when studying the growth of polynomials with the expansion of a given region, the following, so-called Bernstein-Walsh inequality, is often used [43]:

$$\|P_n\|_{C(\overline{G}_R)} \leq R^n \|P_n\|_{C(\overline{G})}, \; \forall P_n \in \wp_n, \tag{3}$$

which means that the value $\|P_n\|_\infty$ has the same order of growth in n when the region $\overline{G}$ is expanded to $\overline{G}_{1+cn^{-1}}$ for all constants $c := c(G) > 0$.

In [29], the "symmetric" analogue of inequality (3) in the space $\mathcal{L}_p(L)$ was given as:

$$\|P_n\|_{\mathcal{L}_p(L_R)} \leq R^{n+\frac{1}{p}} \|P_n\|_{\mathcal{L}_p(L)}, \; \forall P_n \in \wp_n, \; p > 0.$$

Further, in [8, Lemma 2.4] this estimate was generalized to the space $\mathcal{L}_p(h, L)$ with the weight function defined as in (1), as follows:

$$\|P_n\|_{\mathcal{L}_p(h,L_R)} \leq R^{n+\frac{1+\gamma^*}{p}} \|P_n\|_{\mathcal{L}_p(h,L)}, \; \gamma^* = \max\left\{0; \gamma_j : 1 \leq j \leq l\right\}. \tag{4}$$

Along with (2), for the arbitrary Jordan region G, weight function h and $P_n \in \wp_n$, we also introduce:

$$\|P_n\|_{A_p(h,G)} := \left(\iint_G h(z) \left|P_n(z)\right|^p d\sigma_z \right)^{1/p}, \; 0 < p < \infty,$$

$$\|P_n\|_{A_\infty(1,G)} := \max_{z \in \overline{G}} |P_n(z)|, \; p = \infty, \; A_p(1, G) \equiv A_p(G),$$

where σ be the two-dimensional Lebesgue measure.

To give an inequality similar to inequalities (3) and (4) for the space $A_p(h, G)$, we need to introduce the corresponding notations and definition. For any $\delta > 0$ and arbitrary $t, w \in \mathbb{C}$ let $B(w, \delta) := \{t : |t - w| < \delta\}$ and $\varphi : G \to B := B(0, 1)$ be a conformal and univalent map which is normalized by $\varphi(0) = 0$ and $\varphi'(0) > 0$; $\psi := \varphi^{-1}$.

Definition 1.1. ([36, p.286]). A bounded Jordan region G is called a κ-quasidisk, $0 \leq \kappa < 1$, if any conformal mapping ψ can be extended to a K-quasiconformal, $K = \frac{1+\kappa}{1-\kappa}$, homeomorphism of the plane $\overline{\mathbb{C}}$ on $\overline{\mathbb{C}}$. In that case the curve $L := \partial G$ is called a κ-quasicircle. The region G (curve L) is called a quasidisk (quasicircle), if it is κ-quasidisk (κ-quasicircle) with some $0 \leq \kappa < 1$.

For an arbitrary quasidisk G and a weighted function $h(z)$ defined as in (1) with $\gamma_j > -2$, for all $j = 1, 2, ..., l$, the analog of the estimates (3) and (4) for the $\|P_n\|_{A_p(h,G)}$, was given in [2] as follows:

$$\|P_n\|_{A_p(h,G_R)} \leq c_1 \, R^{*^{n+\frac{1}{p}}} \|P_n\|_{A_p(h,G)}, \; R > 1, \; p > 0, \tag{5}$$

where $R^* := 1 + c_2(R-1)$, $c_2 > 0$ and $c_1 := c_1(G, p, c_2) > 0$ constants, independent of n and R. In [4, Theorem1.1], estimate (5) was generalized to the case of an arbitrary Jordan region G, $h(z) \equiv 1$, as follows:

$$\|P_n\|_{A_p(G_R)} \leq c_3 R^{n+\frac{2}{p}} \|P_n\|_{A_p(G_{R_1})}, \ R > R_1 = 1 + \frac{1}{n}, \ p > 0,$$

where $c_3 = \left(\frac{2}{e^p - 1}\right)^{\frac{1}{p}} \left[1 + O(\frac{1}{n})\right]$, $n \to \infty$, is asymptotically sharp constant.

N. Stylianopoulos [39] replaced the norm $\|P_n\|_{C(\overline{G})}$ with norm $\|P_n\|_{A_2(G)}$ on the right-hand side of (3) and found a new version of the inequality(3) for the rectifiable quasicircle L and arbitrary polynomials $P_n \in \wp_n$ as follows:

$$|P_n(z)| \leq C \frac{\sqrt{n}}{d(z, L)} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}, \ \ z \in \Omega,$$

where a constant $C = C(L) > 0$ depending only on L.

In this paper, for κ-quasidisks G, $0 \leq \kappa < 1$, and also for quasidisks satisfying an additional more general condition, we study for the derivative $\left|P_n^{(m)}(z)\right|$, $m = 0, 1, 2, ...$, pointwise estimates in the unbounded region $\Omega_{1+cn^{-1}} = \overline{\mathbb{C}} \backslash \overline{G}_{1+cn^{-1}}$ for arbitrary constant $c > 0$ independet of n, in the following form:

$$\left|P_n^{(m)}(z)\right| \leq \eta_n(L, h, p, m, z) \|P_n\|_p, \ z \in \Omega_{1+cn^{-1}} \tag{6}$$

where $\eta_n := \eta_n(L, h, p, m, z)$, $\eta_n(\cdot) \to \infty$, as $n \to \infty$, depending on the properties of the G and h.

Subsequently, estimates of the type (6) for $z \in \Omega$, $m = 0$ and various weight functions h were objects of study in [5]-[9], [28], [27, p.418-428], [41] and others. For the $m \geq 1$ derivatives estimates of the (6) type were investigated in [13], [14], [24], [25] and others using a recurrence formula, i.e. the inequality for each derivative is obtained using estimates for the previous derivative. And this leads to some cumbersome calculations. In this study, estimates of (6)-type will be obtained for each $m \geq 1$ independently of the estimates for the previous derivative and without using the recurrence formula.

On the other hand, using inequalities of the Bernstein-Markoff-Nikolskii type estimate $\left|P_n^{(m)}(z)\right|$, $z \in \overline{G}$, of the following type:

$$\|P_n^{(m)}\|_\infty \leq \lambda_n \|P_n\|_p, m = 1, 2, ..., \tag{7}$$

where $\lambda_n := \lambda_n(L, h, p, m) > 0$, $\lambda_n \to \infty$, $n \to \infty$, is a constant, depending on the geometrical properties of the curve L and the weight function h in general, and combining it with inequality (6), we eventually find the growth of the $m-th$ derivative of the polynomial $\left|P_n^{(m)}(z)\right|$, $m = 1, 2, ...$, on the whole complex plane in the following form:

$$\left|P_n^{(m)}(z)\right| \leq c_4 \|P_n\|_p \begin{cases} \lambda_n & z \in \overline{G}_{1+cn^{-1}}, \\ \eta_n & z \in \Omega_{1+cn^{-1}}, \end{cases} \tag{8}$$

where $c_4 = c_4(L, p) > 0$ is a constant independent of n, h, P_n, and $\lambda_n \to \infty$, $\eta_n \to \infty$, as $n \to \infty$, depending on the properties of L and h.

The study inequalities of type (7) began with works [20], [21], [40]. Similar studies were then carried out in numerous papers. In recent years, such inequalities for $m \geq 0$ and various spaces have been studied by [27, pp. 418-428], [32]-[35, pp.122-133], [38], [26] (see also the references cited therein) and continued to be studied in [7], [8], [10]-[12], [15] and other, for various general regions in the complex plane.

2. The class of curves

Throughout this paper, $c, c_0, c_1, c_2, ...$ are positive and $\varepsilon_0, \varepsilon_1, \varepsilon_2, ...$ are sufficiently small positive constants (generally, different in different relations), which depends on L in general and, on parameters inessential for the argument, otherwise, the dependence will be explicitly stated. The notation $i = \overline{k,m}$ denotes $i = k, k+1, ..., m$ for all $k \geq 0$ and $m > k$.

Let z_1, z_2 be an arbitrary points on L and $L(z_1, z_2)$ denotes the subarc of L of shorter diameter with endpoints z_1 and z_2. The curve L is a *quasicircle* if and only if the quantity

$$\frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|} \tag{9}$$

is bounded for all $z_1, z_2 \in L$ and $z \in L(z_1, z_2)$ ([30, p.100]-three point property). Lesley [31, p.341] said that the curve L is "$c-$*quasiconformal*", if there exists the positive constant c, independent from points z_1, z_2 and z such that

$$\frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|} \leq c.$$

The Jordan curve L is called *asymptotically conformal* [23], [37], if

$$\max_{z \in L(z_1,z_2)} \frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|} \to 1, \ \ |z_1 - z_2| \to 0.$$

According to the geometric criteria of quasicircles ([16, p.81], [37, p.107]), every asymptotically conformal curve is a quasicircle. Every smooth curve without corners, is asymptotically conformal. Moreover, asymptotically conformal curves may not be rectifiable.

Following [36, p.163], we say that a bounded Jordan curve L is $\lambda-$*quasismooth or Lavrentiev curve*, if for every pair z_1, $z_2 \in L$, there exists a constant $\lambda := \lambda(L) \geq 1$, such that

$$|L(z_1, z_2)| \leq \lambda \, |z_1 - z_2| \, , \ z_1, \ z_2 \in L,$$

where $|L(z_1, z_2)|$ is the linear measure (length) of $L(z_1, z_2)$.

Let S be rectifiable Jordan curve or arc and let $z = z(s)$, $s \in [0, |S|]$, $|S| := mes\, S$, be the natural parametrization of S.

A Jordan curve or arc $S \in C_\theta$, if S has a continuous tangent $\theta(z) := \theta(z(s))$ at every point $z(s)$. We will write $G \in C_\theta$, if $\partial G \in C_\theta$.

Following [36, p.48], we say that a Jordan curve S called *Dini-smooth*, if it has a parametrization $z = z(s)$, $0 \leq s \leq |S| := mes\, S$ such that $z'(s) \neq 0$, $0 \leq s \leq |S|$ and $\left|z'(s_2) - z'(s_1)\right| < g(s_2 - s_1)$, $s_1 < s_2$, where g is an increasing function for which

$$\int_0^1 \frac{g(x)}{x} dx < \infty.$$

A Jordan curve $L := \partial G$ called piecewise Dini-smooth, if L consists of the union of finite Dini-smooth arcs L_j, $j = \overline{1,m}$, such that they have exterior (with respect to $\overline{G}$) angles $\lambda_j \pi$, $0 < \lambda_j < 2$, at the corner points $\left\{z_j\right\}$, $j = \overline{1,m}$, where two arcs meet.

According to the "three-point" criterion [30, p.100], every piecewise $C_\theta-$curve and Dini-smooth curve (without cusps) is quasiconformal.

We give the definition of κ-quasicircles in Definition 1.1. Denote by $Q(\kappa)$, $0 \leq \kappa < 1$, class of κ-quasicircles and say that $G \in Q(\kappa)$, if $L = \partial G \in Q(\kappa)$, $0 \leq \kappa < 1$. Further, we denote that $L \in Q$, $(G \in Q)$ if $L \in Q(\kappa)$ $(G \in Q(\kappa))$ for some $0 \leq \kappa < 1$. Quasicircles can be non-rectifiable [22]. Since the object of study will be integrals along a curve, then we will say that $L \in \widetilde{Q}(\kappa)$, $0 \leq \kappa < 1$, if $L \in Q(\kappa)$ and $L := \partial G$ is rectifiable. Correspondingly, $L \in \widetilde{Q}$ $(G \in \widetilde{Q})$, if $L \in \widetilde{Q}(\kappa)$ $(G \in \widetilde{Q}(\kappa))$ for some $0 \leq \kappa < 1$.

In this work, firstly, the above (6) problem will be solved for the class $Q(\kappa)$, $0 \leq \kappa < 1$. Secondly, we will try to get the result for more general curves, also including the above class of curves.

For this, we need to give the following definitions of the class of quasicircles with some general functional conditions. This will allow us to unite in one class all the curves defined above.

Definition 2.1. We say that the Jordan curve $L=\partial G=\partial\Omega\in Q_{\alpha}$, if L is a quasicircle and $\Phi\in H^{\alpha}(\overline{\Omega})$ for some $0<\alpha\leq 1$, i.e. $|\Phi(z)-\Phi(\zeta)|\leq M_L\,|z-\zeta|^{\alpha}$, $0<\alpha\leq 1$, for all $z,\zeta\in\overline{\Omega}$, where $M_L>0$ is a constant depending only on L. Additionally, say that $L\in\widetilde{Q}_{\alpha}$, $0<\alpha\leq 1$, if L is rectifiable and $L\in Q_{\alpha}$.

The class Q_{α} is sufficiently large. We can find more detailed information about the elements of this class from [31], [37], [42] (also references therein). Here are just a few examples:

a) A piecewise Dini-smooth curve L having largest exterior angle opening $\alpha\pi$, $0<\alpha\leq 1$, belong to the class $\widetilde{Q}_{\alpha}$ [37, p.52].

b) A smooth curve having continuous tangent line belong to the class $\widetilde{Q}_{\alpha}$ for all $0<\alpha<1$.

c) If G is "L–shaped" region, then $L=\partial G\in\widetilde{Q}_{\frac{2}{3}}$.

d) A Lavrentiev curve $L\in\widetilde{Q}_{\alpha}$ for $\alpha=\frac{1}{2}(1-\frac{1}{\pi}\arcsin\frac{1}{c})^{-1}$ and $c>1$ [42].

e) A "c-quasiconformal" curve $L\in Q_{\alpha}$ for $\alpha=\frac{\pi}{2(\pi-\arcsin\frac{1}{c})}$;

f) An asymptotic conformal curve $L\in Q_{\alpha}$ for all $0<\alpha<1$ [31].

3. Main results

We are already beginning to formulate the new results. Firstly, we present estimate for $\left|P_n^{(m)}(z)\right|$, $m\geq 1$, $z\in\Omega$, for the classes $\widetilde{Q}(\kappa)$ and $\widetilde{Q}_{\alpha}$.

Theorem 3.1. *Let $p\geq 1$; $L\in\widetilde{Q}(\kappa)$, for some $0\leq\kappa<1$ and $h(z)$ be defined by (1). Then, for any $P_n\in\wp_n$, $n\in\mathbb{N}$, and every $m=1,2,...$, we have:*

$$\left|P_n^{(m)}(z)\right|\leq c_1\left|\Phi^{n-m+1}(z)\right|\frac{\|P_n\|_p}{d(z,L)}A^1_{n,p}(m),\ z\in\Omega_{1+\frac{\varepsilon_0}{n}},\tag{10}$$

where $c_1=c_1(L,\gamma,m,p)>0$ is a constant independent of n and z;

$$A^1_{n,p}(m):=\begin{cases} n^{\left(\frac{\gamma+1}{p}+m-1\right)(1+\kappa)}, & \begin{cases} p<1+\frac{1+\kappa}{2+\kappa}\gamma, & \gamma>0,\\ p=1+\frac{1+\kappa}{2+\kappa}\gamma, & \gamma>\frac{2+\kappa}{1+\kappa},\\ p>1+\frac{1+\kappa}{2+\kappa}\gamma, & \gamma\geq\frac{2+\kappa}{1+\kappa},\end{cases}\\ n^{m(1+\kappa)+1-\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p=1+\frac{1+\kappa}{2+\kappa}\gamma,\ 0<\gamma\leq\frac{2+\kappa}{1+\kappa},\\ n^{m(1+\kappa)+1-\frac{1}{p}}, & p>1+\frac{1+\kappa}{2+\kappa}\gamma,\ 0<\gamma<\frac{2+\kappa}{1+\kappa},\\ n^{m(1+\kappa)}, & p\geq 1,\ -1<\gamma\leq 0.\end{cases}$$

Theorem 3.2. *Let $p\geq 1$; $L\in\widetilde{Q}_{\alpha}$ for some $\frac{1}{2}\leq\alpha\leq 1$ and $h(z)$ be defined by (1). Then, for any $P_n\in\wp_n$, $n\in\mathbb{N}$, and every $m=1,2,...$, we have:*

$$\left|P_n^{(m)}(z)\right|\leq c_2\left|\Phi^{n-m+1}(z)\right|\frac{\|P_n\|_p}{d(z,L)}A^2_{n,p}(m),\ z\in\Omega_{1+\frac{\varepsilon_0}{n}},\tag{11}$$

where $c_2=c_2(L,\gamma,m,p)>0$ is a constant independent of n and z;

$$A^2_{n,p}(m):=\begin{cases} n^{\left(\frac{\gamma+1}{p}+m-1\right)\frac{1}{\alpha}}, & \begin{cases} p<1+\frac{\gamma}{1+\alpha}, & \gamma>0,\\ p=1+\frac{\gamma}{1+\alpha}, & \gamma>1+\alpha,\\ p>1+\frac{\gamma}{1+\alpha}, & \gamma\geq 1+\alpha,\end{cases}\\ n^{\frac{m}{\alpha}+1-\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p=1+\frac{\gamma}{1+\alpha}, 0<\gamma\leq 1+\alpha,\\ n^{\frac{m}{\alpha}+1-\frac{1}{p}}, & p>1+\frac{\gamma}{1+\alpha}, 0<\gamma<1+\alpha,\\ n^{\frac{m}{\alpha}}, & p\geq 1,\ -1<\gamma\leq 0,\end{cases}$$

Remark 3.3. a) The Theorems 3.1 and 3.2 gives the estimates for $\left|P_n^{(m)}(z)\right|$, $m \geq 1$, regardless of estimate for $\left|P_n^{(m-1)}(z)\right|$. They allow us to find the growth of $\left|P_n^{(m)}(z)\right|$, for any given $m \geq 1$.

b) Comparing (11) with the corresponding result [13, Theorems 2.5; 2.7], we see that the growth rate of the value $|\Phi(z)|$ has significantly decreased, which makes it possible to improve the growth rate of $\left|P_n^{(m)}(z)\right|$.

c) For the $p \geq 2$, $0 < \gamma < 1 + \alpha$ and $p > 1 + \frac{\gamma}{1+\alpha}$, $\gamma \geq 1 + \alpha$, Theorem 3.2 gives better estimates in the sense of n than the corresponding estimate in [13, Theorem 2.7].

d) We added also case of $p = 1$.

4. Estimates $\left|P_n^{(m)}(z)\right|$, $m \geq 1$, for $z \in \overline{G}$

In order to formulate estimates for $\left|P_n^{(m)}(z)\right|$, $m \geq 1$, in the whole complex plane, we need estimates for the $\left|P_n^{(m)}(z)\right|$, $m \geq 1$, in bounded regions $G \in \widetilde{Q}(\kappa)$ or $\widetilde{Q}_\alpha$. We present them.

Theorem 4.1. ([15, Th.2.5]) *Let $p > 0$; $L \in \widetilde{Q}(\kappa)$, for some $0 \leq \kappa < 1$ and $h(z)$ be defined by (1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and every $m \geq 0$, we have:*

$$\left\|P_n^{(m)}\right\|_\infty \leq c_3 n^{\left(\frac{\gamma^*+1}{p}+m\right)(1+\kappa)} \|P_n\|_p\,, \tag{12}$$

where a constant $c_3 = c_3(L, \gamma, m, p) > 0$ independent of n and z;

$$\gamma^* := \max\{0; \gamma_j,\ j = \overline{1, l}\}.$$

Theorem 4.2. ([15, Th.2.10]) *Let $p > 0$; $L \in \widetilde{Q}_\alpha$ for some $\frac{1}{2} \leq \alpha \leq 1$ and $h(z)$ be defined by (1). Then for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and every $m \geq 0$, we have:*

$$\|P_n^{(m)}\|_\infty \leq c_4 n^{\frac{1}{\alpha}\left(\frac{\gamma^*+1}{p}+m\right)} \|P_n\|_p, \tag{13}$$

where a constant $c_4 = c_4(L, \gamma, m, p) > 0$ independent of n and z; γ^* is defined as in (12).

Remark 4.3. a) The sharpness of inequality (12) and (13) was given in [15, Th.2.10].

b) From (12) at $n \to \infty$ we find

$$\|P_n^{(m)}\|_\infty \leq c_3 n^{m(1+k)} \|P_n\|_\infty.$$

This result was obtained in [17] for $m = 1$ and is exact in the sense of order n for any $m \geq 1$ and $0 \leq \kappa < 1$.

5. Estimates for $\left|P_n^{(m)}(z)\right|$ in whole complex plane

First of all, we note that estimates (12) and (13) are valid for the points $z \in \overline{G}_{R_1}$, $R_1 = 1 + \frac{\varepsilon_0}{n}$, with a different constant, in accordance with (3) (applied to the polynomial $P_n^{(m)}(z)$). Therefore, combining these estimations with (10), (11) and considering that $\mathbb{C} = \overline{G}_{R_1} \cup \Omega_{R_1}$, we will obtain estimation on the growth for $|P_n^{(m)}(z)|$, for any $m \geq 1$ (regardless of the assessment for $|P_n^{(m-1)}(z)|$), in the whole complex plane:

Theorem 5.1. *Let $p \geq 1$; $G \in \widetilde{Q}(\kappa)$, for some $0 \leq \kappa < 1$ and $h(z)$ be defined by (1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and every $m = 1, 2, ...$, we have:*

$$\left|P_n^{(m)}(z)\right| \leq c_5 \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma^*+1}{p}+m\right)(1+\kappa)}, & z \in \overline{G}_{R_1}, \\ \frac{\left|\Phi^{n-m+1)}(z)\right|}{d(z,L)} A^1_{n,p}(m), & z \in \Omega_{R_1}, \end{cases}$$

where $c_5 = c_5(L, \gamma, p) > 0$ is a constant independent of n and z; γ^* is defined as in (12); $A^1_{n,p}(m)$ defined as in Theorem 3.1 for all $z \in \Omega_{R_1}$.

Theorem 5.2. *Let $p \geq 1$; $L \in \widetilde{Q}_\alpha$ for some $\frac{1}{2} \leq \alpha \leq 1$ and $h(z)$ be defined by (1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and every $m = 1, 2, ...$, we have:*

$$\left|P_n^{(m)}(z)\right| \leq c_6 \|P_n\|_p \begin{cases} n^{\frac{1}{\alpha}\left(\frac{\gamma^*+1}{p}+m\right)}, & z \in \overline{G}_{R_1}, \\ \frac{\left|\Phi^{n-m+1)}(z)\right|}{d(z,L)} A^2_{n,p}(m), & z \in \Omega_{R_1}, \end{cases}$$

where $c_6 = c_6(L, \gamma, p) > 0$ is a constant independent of n and z; γ^* is defined as in (12); $A^2_{n,p}(m)$ defined as in Theorem 3.2 for all $z \in \Omega_{R_1}$.

6. Some auxiliary results

Throughout this paper we denote "$a \preceq b$" and "$a \asymp b$" are equivalent to $a \leq cb$ and $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2, respectively.

Lemma 6.1. ([1]) *Let G be a quasidisk, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \leq d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. Then*

a) The statements $|z_1 - z_2| \preceq |z_1 - z_3|$ and $|w_1 - w_2| \preceq |w_1 - w_3|$ are equivalent. Therefore, $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$ also are equivalent.

b) If $|z_1 - z_2| \preceq |z_1 - z_3|$, then

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c_1} \preceq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preceq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c_2},$$

where $0 < r_0 < 1$ a constant, depending on G.

Corollary 6.2. *Under the conditions of Lemma 6.1, we have:*

$$|w_1 - w_2|^{c_1} \preceq |z_1 - z_2| \preceq |w_1 - w_2|^{\varepsilon},$$

where $\varepsilon = \varepsilon(G) < 1$.

Lemma 6.3. *Let $L \in Q_\alpha$ for some $\frac{1}{2} \leq \alpha \leq 1$. Then, for all $w_1, w_2 : |w_1| \geq 1, |w_2| \geq 1$, we have:*

$$|\Psi(w_1) - \Psi(w_2)| \succeq |w_1 - w_2|^{\frac{1}{\alpha}}.$$

Lemma 6.4. *$L \in Q(\kappa)$, for some $0 \leq \kappa < 1$. Then, for all $w_1, w_2 : |w_1| \geq 1, |w_2| \geq 1$, we have:*

$$|\Psi(w_1) - \Psi(w_2)| \succeq |w_1 - w_2|^{1+\kappa}.$$

This fact follows from an appropriate result for the mapping $f \in \sum(\kappa)$[36, p.287] and estimation for Ψ'[18, Th.2.8]:

$$d(\Psi(\tau), L) \asymp |\Psi'(\tau)| (|\tau| - 1). \tag{14}$$

Lemma 6.5. ([3]) *Let $L = G$ be a rectifiable Jordan curve and $P_n(z)$, $\deg P_n \leq n, n = 1, 2, ...,$ be arbitrary polynomial and weight function $h(z)$ satisfies the condition (1). Then for any $R > 1$, $p > 0$ and $n = 1, 2, ...$*

$$\|P_n\|_{\mathcal{L}_p(h,L_R)} \preceq R^{n+\frac{1+\gamma^*}{p}} \|P_n\|_{\mathcal{L}_p(h,L)}, \ \gamma^* = \max\left\{0; \gamma_j : j = \overline{1, l}\right\}.$$

7. Proofs of theorems

Proof. **[Proofs of Theorems 3.1 and 3.2]** The proofs of Theorem 3.1 and 3.2 will be simultaneously.

Let $L \in \widetilde{Q}(\kappa)$, $0<\kappa<1$ ($L \in \widetilde{Q}_{\alpha}$, $\frac{1}{2} \leq \alpha \leq 1$) and let $R = 1+\frac{\varepsilon_0}{n}$, $R_1 := 1+\frac{R-1}{2}$. For $z \in \Omega$ and $1 \leq m < n$, let us define $H_{n,m}(z) := \frac{P_n^{(m)}(z)}{\Phi^{n-m+1}(z)}$. Since the function $H_{n,m}(z)$ is analytic in Ω, continuous on $\overline{\Omega}$ and $H_{n,m}(\infty)=0$, then using the Cauchy integral representation we have:

$$H_{n,m}(z) = -\frac{1}{2\pi i}\int\limits_{L_{R_1}} H_{n,m}(\zeta)\frac{d\zeta}{\zeta - z},\ z \in \Omega_{R_1}.$$

Then,

$$\left|\frac{P_n^{(m)}(z)}{\Phi^{n-m+1}(z)}\right| \leq \frac{1}{2\pi}\int\limits_{L_{R_1}}\left|\frac{P_n^{(m)}(\zeta)}{\Phi^{n-m+1}(\zeta)}\right|\frac{|d\zeta|}{|\zeta - z|} \leq \frac{1}{2\pi d(z,L_{R_1})}\int\limits_{L_{R_1}}\left|P_n^{(m)}(\zeta)\right||d\zeta|, \tag{15}$$

since $\left|\Phi^{n-m+1}(\zeta)\right| = R_1^{n-m+1} > 1$, for all $\zeta \in L_{R_1}$, and consequently,

$$\left|P_n^{(m)}(z)\right| \leq \frac{\left|\Phi^{n-m+1}(z)\right|}{d(z,L_{R_1})}\int\limits_{L_{R_1}}\left|P_n^{(m)}(\zeta)\right||d\zeta|. \tag{16}$$

Let us write out the Cauchy integral representation for $P_n^{(m)}(\zeta)$:

$$P_n^{(m)}(\zeta) = \frac{1}{2\pi i}\int\limits_{L_R} P_n(t)\frac{dt}{(t-\zeta)^{m+1}},\ \zeta \in G_R.$$

Taking $\zeta \in L_{R_1}$ and substituting this formula into (16), we find:

$$\begin{aligned}\left|P_n^{(m)}(z)\right| &\leq \frac{\left|\Phi^{n-m+1}(z)\right|}{d(z,L_{R_1})}\int\limits_{L_{R_1}}\left|\frac{1}{2\pi i}\int\limits_{L_R} P_n(t)\frac{dt}{(t-\zeta)^{m+1}}\right||d\zeta| \\ &\leq \frac{\left|\Phi^{n-m+1}(z)\right|}{d(z,L_{R_1})}\int\limits_{L_{R_1}}\left(\int\limits_{L_R}|P_n(t)|\frac{|dt|}{|t-\zeta|^{m+1}}\right)|d\zeta|\end{aligned} \tag{17}$$

$$\left|P_n^{(m)}(z)\right| \leq \frac{\left|\Phi^{n-m+1}(z)\right|}{d(z,L_{R_1})}\sup_{t\in L_R}\left(\int\limits_{L_{R_1}}\frac{|d\zeta|}{|t-\zeta|^{m+1}}\right)\cdot\int\limits_{L_R}|P_n(t)||dt|. \tag{18}$$

Denote by

$$A_{n,m}(t) := \int\limits_{L_{R_1}}\frac{|d\zeta|}{|t-\zeta|^{m+1}};\quad B_n := \int\limits_{L_R}|P_n(t)||dt|,$$

and estimate this integrals. For $A_{n,m}(t)$, after replacing the variable $\tau = \Phi(\zeta)$, $\zeta \in L_{R_1}$; $w = \Phi(t)$, $t \in L_R$, and applying (14) and Lemma 6.1, we get:

$$\begin{aligned}A_{n,m}(t) &= \int\limits_{L_{R_1}}\frac{|d\zeta|}{|\zeta - t|^{m+1}} = \int\limits_{|\tau|=R_1}\frac{|\Psi'(\tau)||d\tau|}{|\Psi(\tau)-\Psi(w)|^{m+1}} \\ &\leq n\int\limits_{|\tau|=R_1}\frac{d(\Psi(\tau),L)|d\tau|}{|\Psi(\tau)-\Psi(w)|^{m+1}} \leq n\int\limits_{|\tau|=R_1}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{m}}.\end{aligned}$$

Therefore, applying Lemma 6.1 and Lemma 6.4, we get:

$$A_{n,m}(t) \le n \int\limits_{|\tau|=R_1} \frac{|d\tau|}{|\tau - w|^{m(1+\kappa)}} \le n^{m(1+\kappa)}, \text{ for all } t \in L_R, \tag{19}$$

for the case $L \in \widetilde{Q}(\kappa)$ and Lemma 6.3,

$$A_{n,m}(t) \le n \int\limits_{|\tau|=R_1} \frac{|d\tau|}{|\tau - w|^{\frac{m}{\alpha}}} \le n^{\frac{m}{\alpha}}, \text{ for all } t \in L_R, \tag{20}$$

for the case $L \in \widetilde{Q}_\alpha$.

For the estimate B_n, we give some notations.

For $0 < \delta_j < \delta_0 := \frac{1}{4}\min\left\{\left|z_i - z_j\right| : i, j = 1, 2, ..., l,\ i \ne j\right\}$, let $\Omega(z_j,\ \delta_j) := \Omega \cap \left\{z : \left|z - z_j\right| \le \delta_j\right\};\ \delta := \min\limits_{1 \le j \le l} \delta_j$. Let $w_j := \Phi(z_j)$, $\varphi_j := \arg w_j$. Without loss of generality, we will assume that $\varphi_l < 2\pi$. For $\eta := \min\left\{\eta_j, j = \overline{1, l}\right\}$ with $\eta_j = \min\limits_{\varpi \in \partial\Phi(\Omega(z_j,\ \delta_j))} \left|\varpi - w_j\right| > 0$, let us set:

$$\Delta_j(\eta_j) := \left\{\varpi : \left|\varpi - w_j\right| \le \eta_j\right\} \subset \Phi(\Omega(z_j,\ \delta_j)),$$

$$\Delta(\eta) := \bigcup_{j=1}^{l} \Delta_j(\eta),\ \widehat{\Delta}_j = \Delta \backslash \Delta(\eta_j); \widehat{\Delta}(\eta) := \Delta \backslash \Delta(\eta);\ \Delta_1^{'} := \Delta_1^{'}(1),$$

$$\Delta_1^{'}(\rho) := \left\{\omega = Re^{i\theta} : R \ge \rho > 1,\ \frac{\varphi_0 + \varphi_1}{2} \le \theta < \frac{\varphi_1 + \varphi_2}{2}\right\},$$

$$\Delta_j^{'} := \Delta_j^{'}(1),\ \Delta_j^{'}(\rho) := \left\{\omega = Re^{i\theta} : R \ge \rho > 1,\ \frac{\varphi_{j-1} + \varphi_j}{2} \le \theta < \frac{\varphi_j + \varphi_0}{2}\right\},\ j = 2, 3..., l,$$

where $\varphi_0 = 2\pi - \varphi_l$; $\Omega_j := \Psi(\Delta_j^{'})$, $L_{R_1}^j := L_{R_1} \cap \Omega_j$; $\Omega = \bigcup\limits_{j=1}^{l} \Omega_j$.

We can limit our consideration to one point on the boundary for computational ease. Consequently, we set $\gamma_1 = \gamma$ and let the weight function h be defined as in (1) for $l = 1$. After multiplying the integrands numerator and denominator by $h^{\frac{1}{p}}(\zeta)$ and using the Hölder inequality, we may estimate B_n as follows:

$$B_n = \int\limits_{L_R} |P_n(t)|\,|dt| \le \left(\int\limits_{L_R} h(t)\,|P_n(t)|^p\,|dt|\right)^{\frac{1}{p}} \times \left(\int\limits_{L_R} \frac{|dt|}{h^{\frac{q}{p}}(t)}\right)^{\frac{1}{q}}$$

$$= \|P_n\|_{\mathcal{L}_p(h, L_R)} \left(\int\limits_{L_R} \frac{|dt|}{h^{q-1}(t)}\right)^{\frac{1}{q}},\ \frac{1}{p} + \frac{1}{q} = 1.$$

Applying Lemma 6.5 and passing to the variable $\tau = \Phi(t)$, we obtain:

$$B_n \le \|P_n\|_p \times \left(\int\limits_{|\tau|=R} \frac{|\Psi'(w)|\,|dw|}{|\Psi(w) - \Psi(w_1)|^{\gamma(q-1)}}\right)^{\frac{1}{q}}.$$

To estimate the last integral, we put:

$$E_R^{11} := \left\{w : w \in F_R^1,\ |w - w_1| < c_1(R-1)\right\}, \tag{21}$$

$$E_R^{12} := \left\{w : w \in F_R^1,\ c_1(R-1) \le |w - w_1| < \eta\right\},$$

$$E_R^{13} := \left\{w : w \in \Phi(L_R),\ |w - w_1| \ge \eta\right\},$$

where $F_R^1 := \Phi(L_R^1) = \Delta_1^{'} \cap \{w : |w| = R\}$, $F_R^2 := \Phi(L_R) \backslash F_R^1$ and $0 < c_1 < \eta$ chosen so that

$$\{w : |w - w_1| < c_1(R-1)\} \cap \Delta \neq \varnothing \text{ and } \Phi(L_R) = \bigcup_{k=1}^{3} E_R^{1k}.$$

Then, taking into account (21), we have:

$$B_n \leq \|P_n\|_p \times \sum_{k=1}^{3} J_n^k, \tag{22}$$

$$J_n^k := \left(\int_{E_R^{1k}} \frac{|\Psi'(w)|\,|dw|}{|\Psi(w) - \Psi(w_1)|^{\gamma(q-1)}} \right)^{\frac{1}{q}}, \; k = 1, 2, 3.$$

For any $k = 1, 2$, denote by

$$\left(I(E_R^{1k})\right)^q := \begin{cases} \int\limits_{E_R^{1k}} \frac{|\Psi'(w)||dw|}{|\Psi(w)-\Psi(w_1)|^{\gamma(q-1)}}, & \text{if } \gamma > 0, \\ \int\limits_{E_R^{1k}} |\Psi(w) - \Psi(w_1)|^{(-\gamma)(q-1)} |\Psi'(w)|\,|dw|, & \text{if } \gamma \leq 0. \end{cases} \tag{23}$$

We will estimate integrals $\left(I(E_R^{1k})\right)^q$ separately for any $k = 1, 2, 3$.

1. Let $L \in \widetilde{Q}(\kappa)$, for some $0 < \kappa < 1$ ($L \in \widetilde{Q}_\alpha$ for some $\frac{1}{2} \leq \alpha \leq 1$).

1.1. Let $\gamma > 0$. Applying Lemma 6.4 (6.3) to (23), we get:

$$\begin{aligned} \left(I(E_R^{11})\right)^q &\leq \int\limits_{E_R^{11}} \frac{d(\Psi(w), L)\,|dw|}{(|w|-1)\,|\Psi(w) - \Psi(\tau)|^{\gamma(q-1)}} \\ &\leq n \int\limits_{E_R^{11}} \frac{|dw|}{|w-\tau|^{[\gamma(q-1)-1](1+\kappa)}} \leq n^{1+[\gamma(q-1)-1](1+\kappa)} mes E_R^{11} \leq n^{[\gamma(q-1)-1](1+\kappa)}; \end{aligned} \tag{24}$$

$$\begin{aligned} \left(I(E_R^{11})\right)^q &\leq \int\limits_{E_R^{11}} \frac{d(\Psi(w), L)\,|dw|}{(|w|-1)\,|\Psi(w) - \Psi(\tau)|^{\gamma(q-1)}} \\ &\leq n \int\limits_{E_R^{11}} \frac{|dw|}{|w-\tau|^{\frac{\gamma(q-1)-1}{\alpha}}} \leq n^{1+\frac{\gamma(q-1)-1}{\alpha}} mes E_R^{11} \leq n^{\frac{\gamma(q-1)-1}{\alpha}}. \end{aligned}$$

Then, for the $I(E_R^{11})$ we obtain:

$$I(E_R^{11}) \leq \begin{cases} n^{\left(\frac{\gamma+1}{p}-1\right)(1+\kappa)}, & \text{for the case } L \in \widetilde{Q}(\kappa), \\ n^{\left(\frac{\gamma+1}{p}-1\right)\frac{1}{\alpha}}, & \text{for the case } L \in \widetilde{Q}_\alpha. \end{cases} \tag{25}$$

Analogously, for the $I(E_R^{12})$ we get:

$$\left(I(E_R^{12})\right)^q \leq n \int\limits_{E_R^{12}} \frac{|dw|}{|w-\tau|^{[\gamma(q-1)-1](1+\kappa)}} \leq \begin{cases} n^{[\gamma(q-1)-1](1+\kappa)}, & [\gamma(q-1)-1](1+\kappa) > 1, \\ n \ln n, & [\gamma(q-1)-1](1+\kappa) = 1, \\ n, & [\gamma(q-1)-1](1+\kappa) < 1, \end{cases}$$

$$\left(I(E_R^{12})\right)^q \leq n \int\limits_{E_R^{12}} \frac{|dw|}{|w-\tau|^{\frac{\gamma(q-1)-1}{\alpha}}} \leq \begin{cases} n^{\frac{\gamma(q-1)-1}{\alpha}}, & \gamma(q-1)-1 > \alpha, \\ n\ln n, & \gamma(q-1)-1 = \alpha, \\ n, & \gamma(q-1)-1 < \alpha. \end{cases}$$

Therefore,

$$I(E_R^{12}) \leq \begin{cases} n^{\left(\frac{\gamma+1}{p}-1\right)(1+\kappa)}, & p < 1 + \frac{1+\kappa}{2+\kappa}\gamma, \\ (n\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{1+\kappa}{2+\kappa}\gamma, \\ n^{1-\frac{1}{p}}, & p > 1 + \frac{1+\kappa}{2+\kappa}\gamma, \end{cases} \tag{26}$$

for the case $L \in \widetilde{Q}(\kappa)$ and

$$I(E_R^{12}) \leq \begin{cases} n^{\left(\frac{\gamma+1}{p}-1\right)\frac{1}{\alpha}}, & p < 1 + \frac{\gamma}{1+\alpha}, \\ (n\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma}{1+\alpha}, \\ n^{1-\frac{1}{p}}, & p > 1 + \frac{\gamma}{1+\alpha}, \end{cases}$$

for the case $L \in \widetilde{Q}_\alpha$.

For $w \in E_R^{13}$, we have $\eta < |w - w_1| < 2\pi R$. Then, $|\Psi(w) - \Psi(w_1)| \geq 1$, by Lemma 6.1. Applying (14), for $w \in \Delta(w_1, \eta)$, we get:

$$\left(J_2^3\right)^q \leq \int\limits_{E_R^{13}} |\Psi'(w)|\,|dw| \leq mesE_R^{13} \leq 1, \text{ for the case } L \in \widetilde{Q}(\kappa);$$

$$\left(J_2^3\right)^q \leq \int\limits_{E_R^{13}} |\Psi'(w)|\,|dw| \leq mesE_R^{13} \leq 1, \text{ for the case } L \in \widetilde{Q}_\alpha,$$

and, consequently,

$$J_2^3 \leq 1, \text{for the cases } L \in \widetilde{Q}(\kappa) \text{ and } L \in \widetilde{Q}_\alpha. \tag{27}$$

Combining (22-27), for $p > 1$ and $\gamma > 0$, we get:

$$\sum_{k=1}^{3} J_n^k \leq n^{\left(\frac{\gamma+1}{p}-1\right)(1+\kappa)} + \begin{cases} n^{\left(\frac{\gamma+1}{p}-1\right)(1+\kappa)}, & p < 1 + \frac{1+\kappa}{2+\kappa}\gamma, \\ (n\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{1+\kappa}{2+\kappa}\gamma, \\ n^{1-\frac{1}{p}}, & p > 1 + \frac{1+\kappa}{2+\kappa}\gamma, \end{cases} + 1 \tag{28}$$

$$= \begin{cases} n^{\left(\frac{\gamma+1}{p}-1\right)(1+\kappa)}, & \begin{cases} p < 1 + \frac{1+\kappa}{2+\kappa}\gamma, & \gamma > 0, \\ p = 1 + \frac{1+\kappa}{2+\kappa}\gamma, & \gamma > \frac{2+\kappa}{1+\kappa}, \\ p > 1 + \frac{1+\kappa}{2+\kappa}\gamma, & \gamma \geq \frac{2+\kappa}{1+\kappa}, \end{cases} \\ (n\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{1+\kappa}{2+\kappa}\gamma, 0 < \gamma \leq \frac{2+\kappa}{1+\kappa}, \\ n^{1-\frac{1}{p}}, & p > 1 + \frac{1+\kappa}{2+\kappa}\gamma, 0 < \gamma < \frac{2+\kappa}{1+\kappa}, \end{cases}$$

for the case $L \in \widetilde{Q}(\kappa)$, and

$$\sum_{k=1}^{3} J_n^k \leq n^{\left(\frac{\gamma+1}{p}-1\right)\frac{1}{\alpha}} + \begin{cases} n^{\left(\frac{\gamma+1}{p}-1\right)\frac{1}{\alpha}}, & p < 1 + \frac{\gamma}{1+\alpha}, \\ (n\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma}{1+\alpha}, \\ n^{1-\frac{1}{p}}, & p > 1 + \frac{\gamma}{1+\alpha}, \end{cases} + 1 \tag{29}$$

$$= \begin{cases} n^{\left(\frac{\gamma+1}{p}-1\right)\frac{1}{\alpha}}, & \begin{cases} p < 1 + \frac{\gamma}{1+\alpha}, & \gamma > 0, \\ p = 1 + \frac{\gamma}{1+\alpha}, & \gamma > 1+\alpha, \\ p > 1 + \frac{\gamma}{1+\alpha}, & \gamma \geq 1+\alpha, \end{cases} \\ (n\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma}{1+\alpha}, 0 < \gamma \leq 1+\alpha, \\ n^{1-\frac{1}{p}}, & p > 1 + \frac{\gamma}{1+\alpha}, 0 < \gamma < 1+\alpha, \end{cases}$$

for the case $L \in \widetilde{Q}_\alpha$, and consequently, combining (22)- (29), we obtain:

$$B_n \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+1}{p}-1\right)(1+\kappa)}, & \begin{cases} p < 1 + \frac{1+\kappa}{2+\kappa}\gamma, & \gamma > 0, \\ p = 1 + \frac{1+\kappa}{2+\kappa}\gamma, & \gamma > \frac{2+\kappa}{1+\kappa}, \\ p > 1 + \frac{1+\kappa}{2+\kappa}\gamma, & \gamma \geq \frac{2+\kappa}{1+\kappa}, \end{cases} \\ (n\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{1+\kappa}{2+\kappa}\gamma, 0 < \gamma \leq \frac{2+\kappa}{1+\kappa}, \\ n^{1-\frac{1}{p}}, & p > 1 + \frac{1+\kappa}{2+\kappa}\gamma, 0 < \gamma < \frac{2+\kappa}{1+\kappa}, \end{cases} \tag{30}$$

for the case $L \in \widetilde{Q}(\kappa)$, and

$$B_n \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+1}{p}-1\right)\frac{1}{\alpha}}, & \begin{cases} p < 1 + \frac{\gamma}{1+\alpha}, & \gamma > 0, \\ p = 1 + \frac{\gamma}{1+\alpha}, & \gamma > 1+\alpha, \\ p > 1 + \frac{\gamma}{1+\alpha}, & \gamma \geq 1+\alpha, \end{cases} \\ (n\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma}{1+\alpha}, 0 < \gamma \leq 1+\alpha, \\ n^{1-\frac{1}{p}}, & p > 1 + \frac{\gamma}{1+\alpha}, 0 < \gamma < 1+\alpha, \end{cases}$$

for the case $L \in \widetilde{Q}_\alpha$.

1.2. If $\gamma \leq 0$, for $w \in \Delta(w_1, \eta)$, according Lemma 6.1, for the case of $L \in \widetilde{Q}(\kappa)$, we have:

$$\left(I(E_R^{11})\right)^q \preceq \int\limits_{E_R^{11}} \frac{d(\Psi(w), L)\,|\Psi(w) - \Psi(w_1)|^{(-\gamma)(q-1)}\,|dw|}{|w| - 1} \tag{31}$$

$$\preceq n \int\limits_{E_R^{11}} |\Psi(w) - \Psi(w_1)|^{(-\gamma)(q-1)+1}\,|dw| \preceq n^{1+[\gamma(q-1)-1](1-\kappa)} mes E_R^{11} \preceq n^{[\gamma(q-1)-1](1-\kappa)};$$

$$I(E_R^{11}) \preceq n^{(\frac{\gamma+1}{p}-1)(1-\kappa)} \preceq 1.$$

In a completely similar way for the case of $L \in \widetilde{Q}_\alpha$, we find:

$$I(E_R^{11}) \preceq 1. \tag{32}$$

For $w \in E_R^{12}$ we have $|w - w_1| < \eta$ and, so, $|\Psi(w) - \Psi(w_1)| \preceq 1$, from Lemma 6.1. Then, for $\tau \in \Delta(w_1, \eta)$, applying Lemma 6.4, we get:

$$\left(I(E_R^{12})\right)^q \preceq \int\limits_{E_R^{12}} |\Psi'(w)|\,|dw| \preceq mes E_R^{12} \preceq 1; \quad I(E_R^{12}) \preceq 1; \tag{33}$$

$$\left(I(E_R^{12})\right)^q \preceq n \int\limits_{E_R^{12}} |\Psi'(w)|\,|dw| \preceq 1; \;\; I(E_R^{12}) \preceq 1;$$

Then,

$$I(E_R^{12}) \preceq 1, \text{ for both cases } L \in \widetilde{Q}(\kappa) \text{ and } L \in \widetilde{Q}_\alpha.$$

For $w \in E_R^{13}$ and each $w \in \Delta(w_1, \eta)$ we have $\eta < |w - w_1| < 2\pi R$. Therefore, from Lemma 6.1 and applying (14), we get:

$$\left(I(E_R^{13})\right)^q \asymp \int\limits_{E_R^{13}} |\Psi'(w)|\,|dw| \preceq 1; \; I(E_R^{13}) \preceq 1. \tag{34}$$

$$\left(I(E_R^{13})\right)^q \asymp \int\limits_{E_R^{13}} |\Psi'(w)|\,|dw| \preceq 1;\ I(E_R^{13}) \preceq 1. \tag{35}$$

Further, combining (31)-(35) in case of $\gamma \leq 0$, we have:

$$\sum_{k=1}^{3} J_n^k \preceq 1, \text{ for both cases } L \in \widetilde{Q}(\kappa) \text{ and } L \in \widetilde{Q}_\alpha.$$

Then, for $-1 < \gamma \leq 0$, from (22), we have:

$$B_n \preceq \|P_n\|_p\,, \text{ for both cases } L \in \widetilde{Q}(\kappa) \text{ and } L \in \widetilde{Q}_\alpha. \tag{36}$$

Let us $p = 1$. After multiplying the numerator and denominator in the inner integral of the integrand by h and applying Lemma 6.5, we obtain:

$$\begin{aligned} \left|P_n^{(m)}(z)\right| &\preceq \frac{\left|\Phi^{n-m+1}(z)\right|}{d(z, L_{R_1})} \int\limits_{L_{R_1}} \left(\int\limits_{L_R} h(t)\,|P_n(t)| \frac{|dt|}{h(t)\,|t-\zeta|^{m+1}} \right) |d\zeta| \\ &\preceq \frac{\left|\Phi^{n-m+1}(z)\right|}{d(z, L_{R_1})} \sup_{t\in L_R} \left(\int\limits_{L_{R_1}} \frac{|d\zeta|}{h(t)\,|t-\zeta|^{m+1}} \right) \cdot \left(\int\limits_{L_R} h(t)\,|P_n(t)|\,|dt| \right) \\ &\preceq \frac{\left|\Phi^{n-m+1}(z)\right|}{d(z, L_{R_1})} \|P_n\|_1 \cdot \sup_{t\in L_R} \left(\int\limits_{L_{R_1}} \frac{|d\zeta|}{h(t)\,|t-\zeta|^{m+1}} \right). \end{aligned} \tag{37}$$

Denote by

$$D_{n,m} := \sup_{t\in L_R} \left(\int\limits_{L_{R_1}} \frac{|d\zeta|}{h(t)\,|t-\zeta|^{m+1}} \right)$$

and estimate this integral. According to Lemmas 6.3, 6.4 and using (19), (20), taking into account our assumption at the end of page 7, we have:

$$D_{n,m} \preceq \sup_{t\in L_R} \left(\int\limits_{L_{R_1}} \frac{|d\zeta|}{|t-z_1|^\gamma\,|t-\zeta|^{m+1}} \right) \preceq \frac{1}{d^{\gamma^*}(z_1, L_R)} A_{n,m} \preceq \begin{cases} n^{(m+\gamma^*)(1+\kappa)}, & \text{if } L \in \widetilde{Q}(\kappa), \\ n^{\frac{m+\gamma^*}{\alpha}}, & \text{if } L \in \widetilde{Q}_\alpha, \end{cases} \tag{38}$$

where γ^* is defined as in (12).

Combining (17)-(22), (30) (36), (37) and (38), for any $\gamma > -1$, $p \geq 1$, $m \geq 1$, we get:

$$\left|P_n^{(m)}(z)\right| \leq \frac{\left|\Phi^{n-m+1}(z)\right|}{d(z,L_{R_1})} \|P_n\|_p \cdot n^{m(1+\kappa)} \begin{cases} n^{\left(\frac{\gamma+1}{p}-1\right)(1+\kappa)}, & \begin{cases} p < 1+\frac{1+\kappa}{2+\kappa}\gamma, & \gamma > 0, \\ p = 1+\frac{1+\kappa}{2+\kappa}\gamma, & \gamma > \frac{2+\kappa}{1+\kappa}, \\ p > 1+\frac{1+\kappa}{2+\kappa}\gamma, & \gamma \geq \frac{2+\kappa}{1+\kappa}, \end{cases} \\ (n\ln n)^{1-\frac{1}{p}}, & p = 1+\frac{1+\kappa}{2+\kappa}\gamma, 0 < \gamma \leq \frac{2+\kappa}{1+\kappa}, \\ n^{1-\frac{1}{p}}, & p > 1+\frac{1+\kappa}{2+\kappa}\gamma, 0 < \gamma < \frac{2+\kappa}{1+\kappa}, \\ 1, & p \geq 1, \ -1 < \gamma \leq 0, \end{cases}$$

$$= \frac{\left|\Phi^{n-m+1}(z)\right|}{d(z,L_{R_1})} \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+1}{p}+m-1\right)(1+\kappa)}, & \begin{cases} p < 1+\frac{1+\kappa}{2+\kappa}\gamma, & \gamma > 0, \\ p = 1+\frac{1+\kappa}{2+\kappa}\gamma, & \gamma > \frac{2+\kappa}{1+\kappa}, \\ p > 1+\frac{1+\kappa}{2+\kappa}\gamma, & \gamma \geq \frac{2+\kappa}{1+\kappa}, \end{cases} \\ n^{m(1+\kappa)+1-\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p = 1+\frac{1+\kappa}{2+\kappa}\gamma, 0 < \gamma \leq \frac{2+\kappa}{1+\kappa}, \\ n^{m(1+\kappa)+1-\frac{1}{p}}, & p > 1+\frac{1+\kappa}{2+\kappa}\gamma, 0 < \gamma < \frac{2+\kappa}{1+\kappa}, \\ n^{m(1+\kappa)}, & p \geq 1, \ -1 < \gamma \leq 0, \end{cases}$$

if $L \in \widetilde{Q}(\kappa)$ and

$$\left|P_n^{(m)}(z)\right| \leq \frac{\left|\Phi^{n-m+1}(z)\right|}{d(z,L_{R_1})} \|P_n\|_p \cdot n^{\frac{m}{\alpha}} \begin{cases} n^{\left(\frac{\gamma+1}{p}-1\right)\frac{1}{\alpha}}, & \begin{cases} p < 1+\frac{\gamma}{1+\alpha}, & \gamma > 0, \\ p = 1+\frac{\gamma}{1+\alpha}, & \gamma > 1+\alpha, \\ p > 1+\frac{\gamma}{1+\alpha}, & \gamma \geq 1+\alpha, \end{cases} \\ (n\ln n)^{1-\frac{1}{p}}, & p = 1+\frac{\gamma}{1+\alpha}, 0 < \gamma \leq 1+\alpha, \\ n^{1-\frac{1}{p}}, & p > 1+\frac{\gamma}{1+\alpha}, 0 < \gamma < 1+\alpha, \\ 1, & p \geq 1, \ -1 < \gamma \leq 0, \end{cases}$$

$$\leq \frac{\left|\Phi^{n-m+1}(z)\right|}{d(z,L_{R_1})} \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+1}{p}+m-1\right)\frac{1}{\alpha}}, & \begin{cases} p < 1+\frac{\gamma}{1+\alpha}, & \gamma > 0, \\ p = 1+\frac{\gamma}{1+\alpha}, & \gamma > 1+\alpha, \\ p > 1+\frac{\gamma}{1+\alpha}, & \gamma \geq 1+\alpha, \end{cases} \\ n^{\frac{m}{\alpha}+1-\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p = 1+\frac{\gamma}{1+\alpha}, 0 < \gamma \leq 1+\alpha, \\ n^{\frac{m}{\alpha}+1-\frac{1}{p}}, & p > 1+\frac{\gamma}{1+\alpha}, 0 < \gamma < 1+\alpha, \\ n^{\frac{m}{\alpha}}, & p \geq 1, \ -1 < \gamma \leq 0, \end{cases}$$

if $L \in \widetilde{Q}_\alpha$, Thus, Theorems 3.1 and 3.2 have been proved.

Finally, we note that $d(z, L_{R_1})$ appears throughout the proofs. We show that for all $z \in \Omega_R$, $d(z, L_{R_1}) \geq d(z, L)$ holds.

For $L = \partial G$ and $\delta > 0$, we set: $U_\infty(L,\delta) := \bigcup_{\zeta\in L} U(\zeta, \ \delta)$ - infinite open cover of the curve L; $U_N(L,\delta) := \bigcup_{j=1}^{N} U_j(L,\delta) \subset U_\infty(L,\delta)$−finite open cover of the curve L; For any $R \geq 1$, let us set: $\Omega_R(\delta) := \Omega(L_R,\delta) := \Omega_R \cap U_N(L_R,\delta)$, $\Omega(\delta) := \Omega_1(\delta)$. For the points $z \notin \Omega(L_{R_1}, d(L_{R_1}, L_R))$, we have: $d(z, L_{R_1}) \geq \delta \geq d(z,L)$. Now, let $z \in \Omega(L_{R_1}, d(L_{R_1}, L_R))$. Denote by $\xi_1 \in L_{R_1}$ the point such that $d(z, L_{R_1}) = |z-\xi_1|$, and point $\xi_2 \in L$, such that $d(z,L) = |z-\xi_2|$, and for $w = \Phi(z)$, $t_1 = \Phi(\xi_1), t_2 = \Phi(\xi_2)$, we have: $|w-w_1| \geq \left||w-w_2| - |w_2-w_1|\right| \geq \left||w-w_2| - \frac{1}{2}|w-w_2|\right| \geq \frac{1}{2}|w-w_2|$. Then, according to Lemma 6.1, we obtain $d(z,L_{R_1}) \geq d(z,L)$. □

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