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A new class of Finsler metrics: douglas curvature and its generalizations

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Abstract. This paper introduces new classes of Finsler metrics, namely $\tilde{D}$-stretch metrics, isotropic $\tilde{D}$-stretch metrics, and relatively isotropic $\tilde{D}$-metrics, by exploring the Douglas curvature in Finsler geometry. The class of relatively isotropic $\tilde{D}$-metrics encompasses two additional classes: $\tilde{D}$-stretch metrics and isotropic $\tilde{D}$-stretch metrics. The study delves into the properties of relatively isotropic $\tilde{D}$-metrics, elucidating their geometric characteristics and situating them within the broader context of Finsler metrics dependent on Douglas curvature, such as Douglas or GDW-metrics. Additionally, the research investigates the interplay between relatively isotropic $\tilde{D}$-metrics and other key curvatures, including $\bar{E}$-curvature and S-curvature, building upon prior studies on the relationships between Douglas curvature and these curvatures. Furthermore, examples of Finsler metrics are provided to elucidate the distinguishing criteria for the class of relatively isotropic $\tilde{D}$-metrics in comparison to well-known classes of Finsler metrics like Douglas, Weyl, and GDW-metrics.

1. Introduction

Recent years have seen important progress in the study of Finsler geometry, with a special emphasis on exploring the behaviors of different curvatures present in Finsler spaces. One of the most significant contributions to the study of Finsler geometry involves the development and exploration of Douglas curvature, first introduced by Douglas in his pioneering work [14]. Douglas curvature plays a crucial role in understanding the geometric properties of Finsler spaces due to its projective invariance property. In this research, we focus on the development of new classes of Finsler metrics based on an in-depth analysis of Douglas curvature, introducing $\tilde{D}$-stretch metrics, isotropic $\tilde{D}$-stretch metrics, and relatively isotropic $\tilde{D}$-metrics.

The behavior of various curvatures in Douglas spaces unveils intriguing characteristics of the underlying geometry. Additionally, our study investigates their relationships with other curvatures and provides illustrative examples to emphasize their distinctions from established classes associated with Douglas spaces, such as Douglas and GDW-metrics. The innovations of this study are detailed in Sections 3 to 5. The Douglas curvature of projectively related Finsler metrics is the same, with projectively flat Finsler metrics having a Douglas curvature of zero. These projectively flat metrics are part of the Finsler metrics

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with scalar curvature category. The S-curvature is crucial, and extensive research has focused on Finsler metrics with scalar flag curvature and isotropic S-curvature [10], [8], [7], [9]. The relationship between the S-curvature and Douglas curvature in Finsler geometry is a fascinating and significant topic in this field.

Even though Douglas curvature remains a projectively invariant tensor in Finsler geometry, $\bar{E}$-curvature emerges as a significant non-Riemannian quantity within this field. The paper [15] demonstrates that when a projectively flat Finsler metric F has non-zero flag curvature, it is Riemannian if and only if $\bar{E} = 0$.

Although these two curvatures seem to be distinct, they share some common ground in describing the geometric properties of Finsler spaces. For instance, the behavior of $\bar{E}$-curvature in compact Douglas manifolds are considered in some researches. In particular, it has been established in [4] that a Douglas metric with vanishing stretch curvature is R-quadratic if and only if its $\bar{E}$-curvature vanishes. Moreover, in [27], it has been shown that any compact Douglas space with zero $\bar{E}$-curvature simplifies to a Berwald metric. The relation between Douglas and stretch curvature in the context of Douglas-Randers manifolds with vanishing stretch tensor is discussed in [28]. The paper proves that every Douglas-Randers metric with vanishing stretch curvature is a Berwald metric. These various research efforts are contributing significantly to our knowledge of the relationships between diverse curvatures in Finsler geometry, with a specific focus on Douglas curvature along with other relevant curvatures.

This paper delves into the study of Douglas curvature in Finsler geometry to introduce new classes of Finsler metrics known as $\tilde{D}$-stretch metrics, isotropic $\tilde{D}$-stretch metrics, and relatively isotropic $\tilde{D}$-metric. The approach used to define these new classes bears some similarities to the approach taken in [1] for defining $\tilde{B}$-metrics or (isotropic) $\tilde{B}$-stretch metrics. The class of relatively isotropic $\tilde{D}$-metrics contains two other introduced classes: $\tilde{D}$-stretch metrics and isotropic $\tilde{D}$-stretch metrics. This research presents several theorems, lemmas, and examples that characterize the properties of a new class of Finsler metrics, the relatively isotropic $\tilde{D}$-metrics. The study situates this new class of Finsler metrics within the broader context of Finsler metrics that depend on Douglas curvature, such as Douglas or GDW-metrics. Furthermore, the research explores the relationship between the relatively isotropic $\tilde{D}$-metrics and other important curvatures, including $\bar{E}$-curvature and S-curvature, in light of previous studies on the relationship between Douglas curvature and these curvatures.

The paper is structured as follows.

- The first section introduces the research and its significance.

- In Section 2, we provide the essential preliminaries needed for this study.
  In Section 3, the focus is on comprehending these novel Finsler metrics through the establishment of efficient theorem that illuminate their properties and delineate the complete characterization of all relatively isotropic $\tilde{D}$-metrics.
  While these metrics generalize Douglas metrics, it is noteworthy that not all metrics within this class are GDW-metrics. Instead, they exhibit an intersection with Weyl or W-quadratic Finsler metrics without entirely containing them. Characterizing the intersection of relatively isotropic $\tilde{D}$-metrics with Weyl metrics is of great significance. This characterization enables us to discover Finsler metrics with scalar flag curvature that are not of relatively isotropic $\tilde{D}$-metric. This is important because the class of relatively isotropic $\tilde{D}$-metrics only intersects with the class of GDW-metrics, rather than being fully contained within it.

- In Section 4, we provide a clear image of this new class of Finsler metrics within the class of Finsler metrics with scalar flag curvature. We have discovered the distinct class of Finsler metrics, characterized by their scalar flag curvature.

- In order to make this new class of Finsler metrics more discernible, we restrict our focus to a specific class of Finsler metrics, namely those with constant flag curvature. By concentrating on this specific class, we aim to recognize well-known Finsler metrics within the relatively isotropic $\tilde{D}$-metrics. This approach allows us to gain a deeper understanding of the properties and characteristics of these Finsler metrics, and to establish connections with existing Finsler metrics that are already well-studied in the literature as discussed in Section 5.

- Discovering Finsler metrics that display relatively isotropic $\tilde{D}$-metric characteristics is incredibly significant. The introduction of a standby condition aids in the effective identification and characterization of these metrics, simplifying the search process for these specialized geometric structures. This paper employs some theorems in Section 5.1 to exemplify this approach. Then, we provide examples of Finsler metrics that help to clarify the criteria for the class of relatively isotropic $\tilde{D}$-metrics and differentiate it from other well-known classes of Finsler metrics, such as Douglas, Weyl, and GDW-metrics.

Throughout this article, the notations "$_{.}$" and "$_{|}$" represent the vertical and horizontal derivatives associated with the Berwald connection, respectively.

Additionally, the subscript "$_0$" denotes the contraction by y^m indicated by the subscript "$_m$", and the symbol "$_{;m}$" denotes the differential with respect to x^m.

2. Preliminaries

A Finsler metric on a manifold M is a nonnegative function F on TM with the following properties

1. F is C^∞ on $TM \setminus \{0\}$;
2. $F(\lambda y) = \lambda F(y), \forall \lambda > 0,\ y \in TM$;
3. For each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\Big[F^2(y + su + tv)\Big]\,|_{s,t=0}, \qquad u, v \in T_xM. \tag{1}$$

At each point $x \in M$, $F_x := F\,|_{T_xM}$, is an Euclidean norm, if and only if $\mathbf{g}_y$ is independent of $y \in T_xM \setminus \{0\}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to R$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]\,|_{t=0}, \qquad u, v, w \in T_xM. \tag{2}$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM\setminus\{0\}}$ is called the *Cartan torsion*. A curve $c(t)$ is called a *geodesic* if it satisfies

$$\frac{d^2c^i}{dt^2} + 2G^i(c(t), \dot{c}(t)) = 0, \tag{3}$$

where $G^i(x, y)$ are local functions on TM given by

$$G^i(x, y) := \frac{1}{4}g^{il}(x, y)\{\frac{\partial^2 F^2}{\partial x^k \partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\}, \quad y \in T_xM, \tag{4}$$

and called the spray coefficients of $F = F(x, y)$. Here,

$$G = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

denotes the associated spray to (M, F). The projection of an integral curve of G is called a geodesic in M.

The Riemann curvature $R_y = R^i{}_k \frac{\partial}{\partial x^i} \otimes dx^k$ of F is given by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^m \partial y^k}y^m + 2G^m\frac{\partial^2 G^i}{\partial y^m \partial y^k} - \frac{\partial G^i}{\partial y^m}\frac{\partial G^m}{\partial y^k}.$$

For the Riemann curvature of Finsler metric F one has [22]

$$R^i{}_{kl} = \frac{1}{3}(R^i{}_{k.l} - R^i{}_{l.k}), \quad and \quad R_j{}^i{}_{kl} = R^i{}_{kl.j}. \tag{5}$$

Here, "$_{.k}$" denotes the differential with respect to y^k.

F is called a Berwald metric if $G^i(y)$ are quadratic in $y \in T_xM$ for all $x \in M$. Define

$$B_y : T_xM \times T_xM \times T_xM \to T_xM$$

$$B_y(u, v, w) = B_j{}^i{}_{kl} u^j v^k w^l \frac{\partial}{\partial x^i},$$

where, $B^i_{jkl} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}$, and

$$E_y : T_xM \times T_xM \to R$$

$$E_y(u, v) = E_{jk} u^j v^k,$$

where, $E_{jk} = \frac{1}{2} B_j{}^m{}_{km}$, $u = u^i \frac{\partial}{\partial x^i}$, $v = v^i \frac{\partial}{\partial x^i}$ and $w = w^i \frac{\partial}{\partial x^i}$. B and E are called the Berwald curvature and the mean Berwald curvature, respectively. F is called a Berwald metric and weakly Berwald (WB) metric if $B = 0$ and $E = 0$, respectively [22]. The connection between the Berwald curvature and the Riemann curvature is articulated through the following Ricci identity [22].

$$B_j{}^i{}_{kl|m} - B_j{}^i{}_{km|l} = R_j{}^i{}_{lm.k}. \tag{6}$$

By means of E-curvature, we can define $\bar{E}$-curvature as follows

$$\bar{E}_y : T_xM \times T_xM \times T_xM \longrightarrow R$$

$$\bar{E}_y(u, v, w) := \bar{E}_{jkl}(y) u^i v^j w^k = E_{jk|l} u^j v^k w^l.$$

It is worth noting that $\bar{E}_{ijk}$ is not completely symmetric with respect to all three indices. To define the H-curvature, we take the covariant derivative of E along geodesics. Specifically, $H_{ij} = E_{ij|m} y^m$,

$$H_y : T_xM \times T_xM \longrightarrow R$$

$$H_y(u, v) := H_{ij} u^i v^j$$

Define

$$D_j{}^i{}_{kl} = B_j{}^i{}_{kl} - \frac{1}{n+1} \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left(\frac{\partial G^m}{\partial y^m} y^i\right). \tag{7}$$

The tensor $D := D_j{}^i{}_{kl} dx^j \otimes \frac{\partial}{\partial x^i} \otimes dx^k \otimes dx^l$ is a well-defined tensor on the slit tangent bundle TM_0, and is called the Douglas tensor. The Douglas tensor D is a non-Riemannian projective invariant, meaning that if two Finsler metrics F and $\bar{F}$ are projectively equivalent, i.e., if $G^i = \bar{G}^i + P y^i$ where the projective factor $P = P(x, y)$ is positively y-homogeneous of degree one, then the Douglas tensor of F is the same as that of $\bar{F}$ [12], [22]. One could easily show that

$$D_j{}^i{}_{kl} = B_j{}^i{}_{kl} - \frac{2}{n+1} \{E_{jk} \delta^i_l + E_{jl} \delta^i_k + E_{kl} \delta^i_j + E_{jk.l} y^i\}. \tag{8}$$

The Douglas curvature, denoted by $D_j{}^i{}_{kl}$, is a projective invariant that is constructed from the Berwald curvature. Finsler metrics with $D_j{}^i{}_{kl} = 0$ are called Douglas metrics. Additionally, metrics satisfying the following condition are called GDW-metrics, which are also projective invariants

$$D_j{}^i{}_{kl|m} y^m = T_{jkl} y^i,$$

for some tensors $T_{jkl} = T_{jkl}(x, y)$.

Z. Shen proposed a non-Riemannian quantity $\tilde{B}$, derived from the Berwald curvature B, through covariant horizontal differentiation along Finslerian geodesics [22]. Extending the concept further, we define a metric based on the expanded notion of Douglas curvature, termed $\tilde{D}$-metric. Given a vector $y \in T_xM$, define

$$\tilde{D}_y : T_xM \times T_xM \times T_xM \longrightarrow T_xM$$

$$\tilde{D}_y(u, v, w) = \tilde{D}_j{}^i{}_{kl} u^j v^k w^l \frac{\partial}{\partial x^i},$$

where $\tilde{D}_j{}^i{}_{kl} = D_j{}^i{}_{kl|0} = D_j{}^i{}_{kl|m} y^m$. For a vector $y \in T_xM$, we define

$$\mathfrak{D}_y : T_xM \times T_xM \times T_xM \times T_xM \longrightarrow T_xM$$

$$\mathfrak{D}_y(u, v, w, z) = \mathfrak{D}_j{}^i{}_{klm} u^j v^k w^l z^m \frac{\partial}{\partial x^i},$$

where $\mathfrak{D}_j{}^i{}_{klm} = 2(\tilde{D}_j{}^i{}_{kl|m} - \tilde{D}_j{}^i{}_{km|l})$.

A Finsler metric (M, F) is called $\tilde{D}$-stretch if

$$\mathfrak{D}_j{}^i{}_{klm} = 0.$$

Additionally, if the metric satisfies the extra requirement below, it becomes an isotropic $\tilde{D}$-stretch metric

$$\mathfrak{D}_j{}^i{}_{klm} = \lambda F(D_j{}^i{}_{kl|m} - D_j{}^i{}_{km|l}),$$

where $\lambda = \lambda(x, y)$ is a scalar function on TM.

The new class of Finsler metrics which is introduced in the following, includes the previously mentioned classes, is termed relatively isotropic $\tilde{D}$-metric. The method of defining these new classes mirrors that of relatively isotropic (mean) Landsberg metrics.

A Finsler metric is called relatively isotropic $\tilde{D}$-metric if it satisfies the given equation

$$\tilde{D}_j{}^i{}_{kl|0} + \lambda F \tilde{D}_j{}^i{}_{kl} = 0,$$

where $\lambda = \lambda(x, y)$ is scalar function on TM.

The Douglas curvature is identical for projectively related Finsler metrics, and projectively flat Finsler metrics that have a Douglas curvature of zero. Two Finsler metrics F and $\bar{F}$ are said to be projectively related if their geodesic coefficients G^i and $\bar{G}^i$ are related as follows [22],

$$G^i = \bar{G}^i + P y^i,$$

where P is a homogeneous function of degree 1. A Finsler metric F is considered projectively flat if its geodesic coefficients satisfy the condition

$$G^i = P y^i,$$

for a homogeneous function P of degree 1. It is known that every projective Finsler metric is of scalar curvature, namely, there is a scalar function $K(P, y) = \lambda(x, y)$, where

$$R^i{}_k = \lambda(x, y)[\delta^i{}_k F^2 - y_k y^i].$$

There exists another significant quantity closely linked to flag curvature known as the S-curvature $S = S(x, y)$ [23], [21]. The isotropic nature of S-curvature is characterized by $S = (n + 1)cF$, where $c = c(x)$ is a scalar function on M. It has been demonstrated that for a Finsler metric F with scalar flag curvature $\lambda = \lambda(x, y)$ and isotropic S-curvature $S = (n + 1)cF$, the flag curvature takes the form

$$\lambda = \frac{3c_{;0}}{F} + \sigma, \tag{9}$$

where $\sigma = \sigma(x)$ and $c_{;0} = \frac{\partial c}{\partial x^m} y^m$. This relationship illustrates the close connection between flag curvature and S-curvature. For further advancements, refer to [19].

The class of (α, β)-metrics is a significant and well-studied class in Finsler geometry. These metrics are defined by a Riemannian metric α and a 1-form β, and they have been shown to play a crucial role in understanding various categories within Finsler spaces. They are expressed in the form $F = \alpha\varphi(s)$, $s = \frac{\beta}{\alpha}$ where $\alpha(y) = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta(y) = b_i(x)y^i$, with $\| \beta_x \|_\alpha < b_0$, are a Riemannian metric and 1-form on manifold M and $\varphi(s)$ is a C^∞ positive function on $(-b_0, b_0)$. It is known that $F = \alpha\varphi(s)$ is a (positive definite) Finsler metric for any α and β, with $\| \beta_x \|_\alpha < b_0$, if and only if φ satisfies the following condition [24],

$$\varphi(s) - s\varphi'(s) + (b^2 - s^2)\varphi''(s) > 0, \qquad | s | \leq b < b_0.$$

Such a metric is called an (α, β)-metric. Clearly, Finsler metrics of Randers type are special (α, β)-metrics. For an (α, β)-metric $F = \alpha\varphi(s)$, we define

$$b_{i|j}\theta^j = db_i - b_j\theta^j{}_i,$$

where $\theta^i = dx^i$ and $\theta^j{}_i = \tilde{\Gamma}^j{}_{ik}dx^k$ denote the Levi-Civita connection form of α. Put

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}), \quad s^i{}_j = a^{ih}s_{hj},$$

$$s_j = b_i s^i{}_j, \quad r_j = b_i r^i{}_j, \quad e_{ij} = r_{ij} + b_i s_j + b_j s_i.$$

For more details, one could refer to [11], [22].

3. Relatively Isotropic $\tilde{D}$-Metrics

In [30], Weyl introduces a projective invariant for Riemannian metrics. Then Douglas extends Weyl's projective invariant to Finsler metrics [14]. Finsler metrics with vanishing projective Weyl curvature are called Weyl metrics or W-metrics. In [25], Szabó proves that Weyl metrics are exactly Finsler metrics of scalar flag curvature.

In this section, we aim to deepen the understanding of these new Finsler metrics by establishing key theorems that shed light on their properties and relationships within the broader context of Finsler geometry. While the relatively isotropic $\tilde{D}$-metrics extend beyond Douglas metrics, it is important to note that not all metrics in this category are GDW-metrics. They intersect with Weyl or W-quadratic Finsler metrics without encompassing them entirely. Characterizing the intersection of relatively isotropic $\tilde{D}$-metrics with Weyl metrics holds significant importance. This characterization unveils Finsler metrics with scalar flag curvature that do not fall under the category of relatively isotropic $\tilde{D}$-metrics. Our next task is to demonstrate the truth of the following Theorem, which characterizes all relatively isotropic $\tilde{D}$-metrics.

Theorem 3.1. *[Characterization of Relatively Isotropic $\tilde{D}$-Metric]*
A Finsler metric F is of relatively isotropic $\tilde{D}$-metric if and only if it satisfies the following equation,

$$\left(W_j{}^i{}_{ml.k|0} + \mu F W_j{}^i{}_{ml.k}\right)y^m + \frac{1}{n+1}\left(\theta_{jkl|0} + \mu F\theta_{jkl}\right)y^i = 0, \tag{10}$$

for some scalar function $\mu = \mu(x, y)$ on TM. $W_j{}^i{}_{ml.k}$ and θ_{jkl} are defined as in

$$W_j{}^i{}_{ml.k} = \frac{1}{3}(W^i{}_{m.l} - W^i{}_{l.m})_{.j.k}, \tag{11}$$

and

$$\theta_{jkl} = 2E_{jk|l} - (R^m{}_{lm} - R_{.l})_{.j.k}, \tag{12}$$

respectively, in terms of the E-curvature E, Weyl curvature W, and Riemann curvature R of the Finsler metric F.

Proof of Theorem 3.1

Proof. The Weyl curvature of a Finsler metric (M, F) is defined as [22]

$$W^i{}_k = A^i{}_k - \frac{1}{n+1}A^m{}_{k.m}y^i,$$

where $A^i{}_k = R^i{}_k - R\delta^i{}_k$ and $R = \frac{1}{n-1}R^m{}_m$. From this definition, we can express the Riemann curvature tensor as follows.

$$R^i{}_k = W^i{}_k + R\delta^i{}_k + \frac{1}{n+1}A^m{}_{k.m}y^i.$$

Substituting this expression into (5), we obtain

$$\begin{aligned}3R_j{}^i{}_{ml} &= (W^i{}_{m.l} - W^i{}_{l.m})_{.j} + (\frac{1}{n+1}A^s{}_{m.s} - R_{.m})_{.j}\delta^i{}_l - (\frac{1}{n+1}A^s{}_{l.s}\\ &-R_{.l})_{.j}\delta^i{}_m + \frac{1}{n+1}(A^s{}_{m.l} - A^s_{l.m})_{.s}\delta^i{}_j + \frac{1}{n+1}(A^s{}_{m.l} - A^s{}_{l.m})_{.s.j}y^i.\end{aligned} \tag{13}$$

However, per the definition of $A^i{}_k$, we can see that

$$\frac{1}{n+1}A^s{}_{k.s} - R_{.k} = \frac{1}{n+1}(R^s{}_{k.s} - (n+2)R_{.k}),$$

and

$$A^s{}_{k.l} - A^s{}_{l.k} = 3R^s{}_{kl} - (R_{.l}\delta^s{}_k - R_{.k}\delta^s{}_l).$$

To compute $R_j{}^i{}_{ml.k}$, we first differentiate (13) with respect to y^k. Then, by substituting the resulting equations into $R_j{}^i{}_{ml.k}$, we obtain the following expression.

$$\begin{aligned}3R_j{}^i{}_{ml.k} &= 3W_j{}^i{}_{ml.k} + \frac{1}{n+1}\Big[(R^s{}_{m.s} - (n+2)R_{.m})_{.j.k}\delta^i{}_l - (R^s{}_{l.s}\\ &-(n+2)R_{.l})_{.j.k}\delta^i{}_m + 3R_s{}^s{}_{ml.k}\delta^i{}_j + 3R_s{}^s{}_{ml.j}\delta^i{}_k + 3R_s{}^s{}_{ml.j.k}y^i\Big],\end{aligned} \tag{14}$$

where $W_j{}^i{}_{kl} = \frac{1}{3}(W^i{}_{k.l} - W^i{}_{l.k})_{.j}$. By utilizing the Ricci identity (6), we can express the relations as follows.

$$R_j{}^i{}_{ml.k}y^m = B_j{}^i{}_{kl|0}, \qquad R_s{}^s{}_{ml.k} = 2(E_{kl|m} - E_{km|l}). \tag{15}$$

Combining equations (5) and (6), and taking into account the previous equation, we derive

$$\begin{aligned}R_s{}^s{}_{ml.k}y^m &= \frac{1}{3}R^s{}_{m.s.l.k}y^m = 2H_{kl}.\\ R_s{}^s{}_{ml.j.k}y^m &= 2H_{jl.k} - R_s{}^s{}_{kl.j} = 2H_{jl.k} - 2(E_{jl|k} - E_{jk|l})\\ &= 2E_{jl|p.k}y^p + 2E_{jl|k} - 2(E_{jl|k} - E_{jk|l}) = 2(E_{jlk|0} + E_{jk|l}).\end{aligned} \tag{16}$$

Substituting equations (15) and (16) into the contracted form of (14) by y^m, we obtain

$$\begin{aligned}W_j{}^i{}_{ml.k}y^m = R_j{}^i{}_{ml.k}y^m - \frac{1}{n+1}\Big[&2H_{jk}\delta^i{}_l + 2H_{kl}\delta^i{}_j + 2H_{jk}\delta^i{}_k\\ &+2E_{jl.k|0}y^i + (2E_{jk|l} - \frac{1}{3}(R^s{}_{l.s} - (n+2)R_{.l})_{.j.k})y^i\Big]\end{aligned} \tag{17}$$

Utilizing the above equation, (15) and (8) in the equation (17), one gets

$$W_j{}^i{}_{ml.k}y^m = D_j{}^i{}_{kl|0} - \frac{1}{n+1}\theta_{jkl}y^i, \tag{18}$$

where $\theta_{jkl} = 2E_{jk|l} - \frac{1}{3}(R^s{}_{l.s} - (n+2)R_{.l})_{.j.k}$. Considering (5) alongside the relationship $R = \frac{1}{n-1}R^m{}_m$, a new expression for θ_{jkl} might be derived.

$$\theta_{jkl} = 2E_{jk|l} - (R^m{}_{lm} - R_{.l})_{.j.k}.$$

Based on the previous equation, a Finsler metric is a relatively isotropic $\tilde{D}$-metric if and only if there exists a scalar function $\mu = \mu(x, y)$ on TM such that

$$0 = D_j{}^i{}_{kl|0|0} + \mu F D_j{}^i{}_{kl|0} = \Big(W_j{}^i{}_{ml.k|0} + \mu F W_j{}^i{}_{ml.k}\Big)y^m + \frac{1}{n+1}\Big(\theta_{jkl|0} + \mu F \theta_{jkl}\Big)y^i.$$

□

According to (18), we can derive the following corollaries

Corollary 3.2. *A Finsler metric (M, F) is GDW-metric if and only if*

$$W_j{}^i{}_{ml.k}y^m = w_{jkl}y^i,$$

for tensor $w_{jkl} = w_{jkl}(x, y)$ on TM.

Corollary 3.3. *Every Finsler metric of scalar curvature is a GDW-metric.*

Corollary 3.4. *Every Finsler metric with quadratic Weyl curvature is a GDW-metric.*

4. Relatively Isotropic $\tilde{D}$-Metrics: A Class of Finsler Metrics with Scalar Flag Curvature

Upon examining the broader scope of relatively isotropic $\tilde{D}$-metrics, we discover a distinct class of Finsler metrics characterized by scalar flag curvature. This class goes beyond the conventional Douglas metrics, demonstrating an intersection with Weyl or W-quadratic Finsler metrics. By investigating this intersection, we uncover Finsler metrics with scalar flag curvature that do not conform to the traits of relatively isotropic $\tilde{D}$-metrics. This analysis illuminates the complex relationships and properties of this class of Finsler metrics within the context of an essential class of Finsler metrics, namely scalar flag curvatures, and provides a discernible image of this new class of Finsler metrics.

Theorem 4.1. *[Characterization of Finsler Metrics with Scalar Flag Curvature and Relatively Isotropic $\tilde{D}$-Metrics]*
A Finsler metric F of scalar flag curvature $\lambda = \lambda(x, y)$ is a relatively isotropic $\tilde{D}$-metric if and only if it satisfies the following equation.

$$\frac{2}{n+1}(E_{jk|l|0} + \mu F E_{jk|l}) + (t_{l.j.k|0} + \mu F t_{l.j.k}) = 0, \tag{19}$$

where $\mu = \mu(x, y)$ is a scalar function on TM and t_l is defined as in

$$t_l = \frac{F^2}{3}\lambda_{.l} + \lambda y_l, \tag{20}$$

and the other notations remain consistent with Theorem 3.1.

Proof of Theorem 4.1

Proof. Assume that the Finsler metric F has scalar flag curvature. According to its definition in [22], the Riemann curvature tensor is given by

$$R^i{}_k = \lambda(F^2\delta^i{}_k - y_k y^i).$$

Then we could find

$$\Big(R^s{}_{k.s} - (n+2)R_{.k}\Big) = -(n+1)\Big(F^2\lambda_{.k} + 3\lambda y_k\Big). \tag{21}$$

On the other hand, the Weyl curvature of every Finsler metric with scalar flag curvature vanishes. Using this fact in (18), we obtain

$$D_j{}^i{}_{kl|0} = \frac{1}{n+1}\theta_{jkl}y^i. \tag{22}$$

Using (22) and (21) in (18), we obtain

$$D_j{}^i{}_{kl|0} = \frac{2}{n+1}E_{jk|l} + \Big(\frac{F^2}{3}\lambda_{.l} + \lambda y_l\Big)_{.j.k}.$$

By defining $t_l = \frac{F^2}{3}\lambda_{.l} + \lambda y_l$ and applying the definition of a relatively isotropic $\tilde{D}$-metric, it follows that the Finsler metric F with scalar curvature $\lambda = \lambda(x, y)$ is a relatively isotropic $\tilde{D}$-metric if and only if there is scalar function $\mu = \mu(x, y)$ on TM such that

$$\frac{2}{n+1}(E_{jk|l|0} + \mu F E_{jk|l}) + (t_{l.j.k|0} + \mu F t_{l.j.k}) = 0.$$

□

The study of the relationship between flag curvature and the S-curvature has been the subject of numerous investigations in Finsler geometry. For a Finsler metric of scalar flag curvature $\lambda = \lambda(x, y)$ and isotropic S-curvature, $S = (n + 1)cF$, for $c = c(x)$ the flag curvature must be in a specific form that involves the S-curvature, as expressed in (9). This relationship has been used to characterize Finsler metrics of scalar curvature and isotropic S-curvature. In the following proposition, we aim to further explore this relationship in the context of relatively isotropic $\tilde{D}$-metrics.

Proposition 4.2. *Let (M, F) be an n-dimensional Finsler manifold of scalar curvature with flag curvature $\lambda = \lambda(x, y)$. Suppose that the S-curvature is isotropic, $S = (n + 1)cF$, then F is relatively isotropic $\tilde{D}$-metric if and only if the following equation holds for $\mu = \mu(x, y)$ on TM and $c = c(x)$ on M,*

$$cL_{jkl|0} - \omega L_{jkl} - F\eta C_{jkl} = \bar{c}_l h_{jk} + \bar{c}_j h_{lk} + \bar{c}_k h_{jl} + \bar{c}_0 F F_{.j.l.k}. \tag{23}$$

Here, $\bar{c}_l$, ω and η are defined as $\bar{c}_l = c_{;l|0} + \mu F c_{;l}$, $\omega = \lambda F - (2c_{;0} + \mu F c)$, and $\eta = \lambda_{|0} + \mu F\lambda - \frac{3}{F}\bar{c}_0$, respectively.

Proof. Assume that (M, F) be an manifold of scalar curvature with flag curvature $\lambda = \lambda(x, y)$ which is of isotropic S-curvature, $S = (n + 1)cF(x, y)$, with the scalar function $c = c(x)$ on M. Then, according to (9),there is a scalar function $\sigma = \sigma(x)$ on M such that

$$\lambda = 3\frac{c_{;0}}{F} + \sigma. \tag{24}$$

Here, $c_{;m} = \frac{\partial c}{\partial x^m}$. Using the facts $E_{jk} = \frac{1}{2}S_{.j.k}$ and $F_{.j.k|l} = \frac{1}{F}(g_{jk} - l_j l_k)_{|l} = -\frac{2}{F}L_{jkl}$, we have

$$\frac{2}{n+1}E_{jk|l} = (cF_{.j.k})_{|l} = -\frac{2c}{F}L_{jkl} + c_{;l}F_{.j.k}. \tag{25}$$

Therefore

$$\frac{2}{n+1}(E_{jk|l|0} + \mu F E_{jk|l}) = -\frac{2}{F}cL_{jkl|0} - \frac{2}{F}(c_{;0} + \mu F c)L_{jkl} + (c_{;l|0} + \mu F c_{;l})F_{jk}.$$

Based on (24), we have

$$\frac{F^2}{3}\lambda_{.l} = c_{;l}F - c_{;0}F_{.l}, \quad \lambda y_l = (3c_{;0} + \sigma F)F_{.l}, \quad \lambda_{.j}y_l = 3c_{;j}F_{.l} - 3\frac{c_{;0}}{F}F_{.j}F_{.l}.$$

By using the equations mentioned earlier and (20), we can derive the following expressions for t_l, $t_{l.j}$, and $t_{l.j.k}$

$$t_l = \frac{F^2}{3}\lambda_{.l} + \lambda y_l = c_{;l}F + (2c_{;0} + \sigma F)F_{.l},$$

$$t_{l.j} = c_{;l}F_{.j} + (2c_{;j} + \sigma F_{.j})F_{.l} + (2c_{;0} + \sigma F)F_{.l.j},$$

$$t_{l.j.k} = c_{;l}F_{.j.k} + 2c_{;j}F_{.l.k} + 2c_{;k}F_{.j.l} + 2c_{;0}F_{.j.l.k}$$
$$+\sigma(F_{.l}F_{.j.k} + F_{.j}F_{.l.k} + F_{.k}F_{.j.l} + FF_{.j.l.k}).$$

Therefore, using the definition of C_{jkl}, simplifying $t_{l.j.k}$ and equation (25), we can express $T_{ljk} = \frac{2}{n+1}E_{jk|l} + t_{l.j.k}$ as

$$T_{ljk} = \frac{2}{n+1}E_{jk|l} + t_{l.j.k} = -\frac{2}{F}cL_{jkl} + 2\sigma C_{jkl} + 2(c_{;l}F_{.j.k} + c_{;j}F_{.l.k} + c_{;k}F_{.j.l} + c_{;0}F_{.j.l.k}). \tag{26}$$

Then using the facts

$$F_{.j.k|0} = \frac{1}{F}(g_{jk} - l_j l_k)_{|0} = 0,$$

and

$$F_{.j.k.l|0} = F_{.j.k|p.l}y^p = F_{.j.k|0.l} - F_{.j.k|l} = \frac{2}{F}L_{jkl},$$

we obtain

$$T_{ljk|0} = -\frac{2}{F}cL_{jkl|0} + \frac{2}{F}(\sigma F + c_{;0})L_{jkl} + 2\sigma_{;0}C_{jkl} + 2(c_{;l|0}F_{.j.k} + c_{;j|0}F_{.l.k} + c_{;k|0}F_{.j.l} + c_{;0|0}F_{.j.l.k}).$$

Applying the above equation and (26) in the equation (19) in the Theorem 4.1 we find that F is a relatively isotropic $\tilde{D}$-metric if and only if

$$T_{ljk|0} + \mu F T_{ljk} = -\frac{2}{F}cL_{jkl|0} + \frac{2}{F}(\sigma F + c_{;0} - \mu F c)L_{jkl}$$
$$+2(\sigma_{;0} + \mu F\sigma)C_{jkl} + 2(\bar{c}_l F_{.j.k} + \bar{c}_j F_{.l.k} + \bar{c}_k F_{.j.l} + \bar{c}_0 F_{.j.l.k}) = 0, \tag{27}$$

where $\bar{c}_l = c_{;l|0} + \mu F c_{;l}$. Based on (24), we have

$$\sigma F + c_{;0} - \mu F c = \lambda F - (2c_{;0} + \mu F c),$$

$$\sigma_{;0} + \mu F\sigma = \sigma_{|0} + \mu F\sigma = \lambda_{|0} + \mu F\lambda - \frac{3}{F}\bar{c}_0,$$

which upon merging with (27) and the fact, $h_{jk} = FF_{.j.k}$, we derive (23). □

According to a theorem in [9], every non-Randers type regular (α, β)-metric on an n-dimensional manifold M ($n \geq 3$) is a Finsler metric with scalar flag curvature λ and vanishing S-curvature if and only if the flag curvature λ is identically zero. In this case, the metric is also a Berwald metric. Therefore, by applying the Proposition 4.2 we have the following Corollary.

Corollary 4.3. *Every non-Randers type regular (α, β)-metric on an n-dimensional manifold M ($n \geq 3$) of scalar flag curvature λ and vanishing S-curvature is a relatively isotropic $\tilde{D}$-metric.*

For Randers metrics that meet the criteria outlined in the previous corollary, we have the following result.

Corollary 4.4. *Let (M, F) be an n-dimensional Randers manifold of scalar curvature with flag curvature $\lambda = \lambda(x, y)$. Suppose that the S-curvature is isotropic, $S = (n+1)cF$, then F is relatively isotropic $\tilde{D}$-metric if and only if the following equation holds for $\mu = \mu(x, y)$ on TM and $c = c(x)$ on M.*

$$\alpha A_k - A_0 \alpha_{.k} + \omega s_{k0} = 0, \tag{28}$$

where

$$A_k = (\lambda_{|0} + \mu(\lambda + c^2)F + 2cc_{;0})b_k - c(\lambda F - 2c_{;0} - \mu cF)s_k + 2(\bar{\tau}_k + cF\tau_k),$$

$$\bar{\tau}_k = \bar{c}_k - \bar{c}_0 \frac{F_{.k}}{F}, \quad \tau_k = c_{;k} - c_{;0}\frac{F_{.k}}{F},$$

$$\omega = \lambda F - 2c_{;0} - \mu cF,$$

for some scalar function $\mu = \mu(x, y)$ on TM and $A_0 = A_k y^k$.

Proof. Assuming F is a Randers metric with scalar curvature and flag curvature $\lambda = \lambda(x, y)$, and isotropic S-curvature, $S = (n+1)cF$, it is a known fact that every Randers metric is C-reducible, which implies

$$C_{jkl} = \frac{1}{n+1}(I_j h_{kl} + I_k h_{jl} + I_l h_{jk}).$$

Subsequently, we have

$$L_{jkl} = \frac{1}{n+1}(J_j h_{kl} + J_k h_{jl} + J_l h_{jk}),$$

and

$$L_{jkl|0} = \frac{1}{n+1}(J_{j|0} h_{kl} + J_{k|0} h_{jl} + J_{l|0} h_{jk}),$$

and utilizing the relationship $2C_{jkl} = \frac{F_{.j}}{F}h_{kl} + \frac{F_{.k}}{F}h_{jl} + \frac{F_{.l}}{F}h_{jk} + FF_{.j.k.l})$, we derive

$$FF_{.j.k.l} = (\frac{2}{n+1}I_j - \frac{F_{.j}}{F})h_{kl} + (\frac{2}{n+1}I_k - \frac{F_{.k}}{F})h_{jl} + (\frac{2}{n+1}I_l - \frac{F_{.l}}{F})h_{jk}.$$

By incorporating the above expressions into equation (23) from Proposition 4.2, we arrive at

$$\rho_j h_{kl} + \rho_k h_{jl} + \rho_l h_{jk} = 0,$$

where $\rho_k = cJ_{k|0} - \omega J_k - (\eta F + 2\bar{c}_0)I_k - (n+1)\bar{\tau}_k$, $\bar{\tau}_k = \bar{c}_k - \bar{c}_0 \frac{F_{.k}}{F}$ and the remaining symbols retain the same definitions as presented in Proposition 4.2. The proceeding equation yields

$$cJ_{k|0} = \omega J_k + (\eta F + 2\bar{c}_0)I_k + (n+1)\bar{\tau}_k \tag{29}$$

However, we have the following Bianchi identity for every Finsler metric with scalar flag curvature K, [13]

$$J_{k|0} + KF^2 I_k = -\frac{n+1}{3}F^2 K_{.k}.$$

On the other hand, based on (9), we have

$$\frac{F}{3}\lambda_{.k} = c_{;k} - c_{;0}\frac{F_{.k}}{F} := \tau_k.$$

By combining the two aforemention equations with (29), we get

$$\omega J_k + \Omega F I_k + (n+1)(\bar{\tau}_k + cF\tau_k) = 0, \tag{30}$$

where $\Omega = \lambda_{|0} + (\mu + c)\lambda F - \frac{\bar{c}_0}{F}$.

According to equation (38) of Lemma 4.1 in [11], for Randers metric with isotropic S-curvature, $S = (n+1)cF$, we have

$$J_k = -cFI_k + \frac{n+1}{2\alpha^2}\Big(\alpha s_{k0} - c(\alpha^2 s_k - s_0 y_k)\Big), \tag{31}$$

where

$$I_k F = \frac{n+1}{2\alpha^2}(\alpha^2 b_k - \beta y_k).$$

Applying these two equations in (30), we arrive at

$$[(\Omega - c\omega)b_k - c\omega s_k + 2(\bar{\tau}_k + cF\tau_k)] - \frac{1}{\alpha^2}[(\Omega - c\omega)\beta - c\omega s_0]y_k + \frac{\omega}{\alpha}s_{k0} = 0.$$

Putting $A_k = (\Omega - c\omega)b_k - c\omega s_k + 2(\bar{\tau}_k + cF\tau_k)$ in the above equation we reach (28). □

5. Relatively Isotropic $\tilde{D}$-Metrics and Constant Flag Curvature in (α, β)-Metrics

This section conducts a comparative analysis of relatively isotropic $\tilde{D}$-metrics with constant flag curvature and examines the criteria that (α, β)-metrics must satisfy to belong to this class. The study of Finsler metrics with constant flag curvature remains an active area of research in Finsler geometry, with numerous examples of Finsler metrics that have been presented in the literature. Understanding the conditions under which these metrics are relatively isotropic $\tilde{D}$-metrics is crucial for identifying a wide range of Finsler metrics that fall within this new class. By studying the conditions for Finsler metrics of constant flag curvature to be relatively isotropic $\tilde{D}$-metrics, we can uncover a wealth of examples of Finsler metrics that belong to this new class. This, in turn, can help us to develop a more intelligible image of this new class of Finsler metrics, deepening our understanding of their properties and relationships with other Finsler metrics. The class of (α, β)-metrics plays a significant role in Finsler geometry, as they aid in the development and understanding of various categories within Finsler spaces. By determining the conditions under which (α, β)-metrics can be classified as relatively isotropic $\tilde{D}$-metrics, we can gain a more comprehensive and clearer picture of this new class. In this section, we will consider (α, β)-metrics as examples to illustrate our approach and further elucidate the properties of this class. To begin, we establish the following Lemma as a direct consequence of Theorem 4.1.

Lemma 5.1. *A Finsler metric F of constant flag curvature λ_0 is a relatively isotropic $\tilde{D}$-metric if and only if it satisfies the following equation.*

$$(E_{jk|l|0} + \mu F E_{jk|l}) + (n+1)\lambda_0(L_{jkl} + \mu F C_{jkl}) = 0, \tag{32}$$

As mentioned in [5], when a Randers metric possesses constant flag curvature, it also exhibits constant S-curvature. Combining this with the Corollary 4.4 yields.

Corollary 5.2. *A n-dimensional Randers metric F, $(n \geq 3)$, of constant flag curvature λ_0 is a relatively isotropic $\tilde{D}$-metric if and only if it satisfies the following equation.*

$$\mu(\kappa + c^2)(\alpha^2 b_k - \beta y_k) + (\kappa - \mu c)(\alpha s_{k0} - c\alpha^2 s_k + c s_0 y_k), \tag{33}$$

where $\mu = \mu(x, y)$ is a scalar function on TM and c_0 is a constant such that $S = (n+1)c_0 F$.

Proof. Supposing that the Randers metric F exhibits a constant flag curvature κ, then it must also exhibit isotropic S-curvature of c, with $S = (n+1)cF$ as reported in [5], while the scalar quantity $\bar{c}$ stated in Theorem 4.2 remains at zero. Based on Corollary 4.4, calculate the following indexes in the corollary.

$$A_k = \mu(\kappa + c^2)b_k - c(\kappa - c\mu)s_k, \quad \bar{\tau}_k = 0, \quad \tau_k = 0.$$

Putting (28) yields (33). □

5.1. Relatively Isotropic $\tilde{D}$-Metrics: A Study of Constant Flag Curvature and Vanishing $\bar{E}$-Curvature

Douglas curvature and $\bar{E}$-curvature are both important quantities in Finsler geometry, with the former being projectively invariant and the latter emerging as a significant non-Riemannian quantity. While they may seem to be distinct quantities, they both play crucial roles in understanding the geometry of Finsler spaces. Despite their differences, these two curvatures share some common ground in describing the geometric properties of Finsler spaces, which have been considered in some research studies. In the subsequent discussion, leveraging this significant connection and referencing equation (32), we will explore Finsler metrics characterized by constant flag curvature and exhibiting vanishing $\bar{E}$-curvature.

Theorem 5.3. *Let F be a Finsler metric of constant flag curvature λ_0 with $\bar{E} = 0$ on a manifold M. Then*

1. *If $\lambda_0 = 0$, then F is a relatively isotropic $\tilde{D}$-metric.*
2. *If $\lambda_0 \neq 0$, then F is relatively isotropic $\tilde{D}$-metric if and only if it is general relatively Landsberg metric, as described by*

$$L_{jkl} + \mu F C_{jkl} = 0, \tag{34}$$

for scalar function $\mu = \mu(x, y)$ on TM.

Proof of Theorem 5.3

Proof. Let F be a Finsler metric of constant flag curvature λ_0. Then one has

$$R^i{}_k = \lambda_0 F^2 h^i{}_k.$$

According to equation (32) and the condition $\bar{E} = 0$, this Finsler metric F is a relatively isotropic $\tilde{D}$-metric if and only if

$$\lambda_0 (L_{jkl} + \mu F C_{jkl}) = 0,$$

for $\mu = \mu(x, y)$. If $\lambda_0 = 0$, the above equation is satisfied, indicating that F is a relatively isotropic $\tilde{D}$-metric. Now, in the case where $\lambda_0 \neq 0$, we observe that F is a relatively isotropic $\tilde{D}$-metric if and only if

$$L_{jkl} + \mu F C_{jkl} = 0.$$

□

Finding the scalar functions μ for Finsler manifolds that satisfy the specific equation (34) is crucial. In the subsequent theorem, the solution to this equation reveals the general form of μ for relatively isotropic $\tilde{D}$-metrics with constant flag curvature and $\bar{E} = 0$. We will now establish proof for the Theorem.

Theorem 5.4. *Let F be a non-Riemannian Finsler metric of constant flag curvature $\lambda_0 \neq 0$ with $\bar{E} = 0$ on a manifold M. If F be a relatively isotropic $\tilde{D}$-metric, $\tilde{D}_j{}^i{}_{kl|0} + \mu F \tilde{D}_j{}^i{}_{kl} = 0$, then*

1. *If $\mu_{|0} = 0$, then $\lambda_0 \leq 0$.*
2. *If $\mu_{|0} \neq 0$, we have the equation $\lambda_0 + \mu^2 = \frac{\xi_0}{||C||^2}$,*

with $||C||$ representing the norm of Cartan torsion of F as $||C||^2 = C^{jkl} C_{jkl}$, and where $\xi_0 = \xi_0(x, y)$ serves as a scalar function on TM satisfying $\xi_{0|0} = 0$.

Proof of Theorem 5.4

Proof. Given a metric F with constant flag curvature and $\bar{E} = 0$, suppose it is a relatively isotropic $\tilde{D}$-metric. Consequently, based on the earlier Theorem, we obtain equation (34), $L_{jkl} + \mu F C_{jkl} = 0$. On the other hand, with F having constant flag curvature λ_0, we have

$$R^i{}_k = \lambda_0 F^2 h^i{}_k.$$

Based on (5) and (6) we have

$$B_j{}^i{}_{kl|m} - B_j{}^i{}_{km|l} = R_j{}^i{}_{ml.k} = 2\lambda_0 (C_{jkl}\delta^i{}_m - C_{jkm}\delta^i{}_l),$$

which yields

$$B_j{}^i{}_{kl|0} = 2\lambda_0 C_{jkl} y^i, \quad E_{jk|0} = H_{jk} = 0. \tag{35}$$

Through contracting it by y_i, we can derive

$$L_{jkl|0} + \lambda_0 F^2 C_{jkl} = 0. \tag{36}$$

By referring to the equations denoted by (34) and (36), it becomes evident that

$$C(t)\Big(\mu'(t) - (\lambda_0 + \mu^2(t))F\Big) = 0. \tag{37}$$

Here, $\gamma(t)$ is the geodesic parameterized by the arc length on M with the start point $\gamma(0) = p$ and the tangent vector $\gamma'(0) = y$, $U = U(t)$, $V = V(t)$ and $W(t)$ are parallel vector fields along $\gamma = \gamma(t)$ with $U(0) = u$, $V(0) = v$ and $W(0) = w$, and then

$$C(t) = C_{\gamma'(t)}(U(t), V(t), W(t)) = C_{jk}(\gamma(t), \gamma'(t))U^i V^j W^k.$$

and $\mu(t) = \mu(\gamma(t), \gamma'(t))$. Now, F is non-Riemannian, then $C(t) \neq 0$ and by (37), one has $\mu'(t) = (\lambda_0 + \mu^2(t))F$.

In the case where $\mu'(t) = 0$, the value of $\lambda_0 = -\mu^2(t)$, indicating that λ_0 will not have a positive value.

However, when $\mu'(t) \neq 0$, we find that $\mu' = (\lambda_0 + \mu^2)F$ which is equivalent to

$$(\lambda_0 + \mu^2)' = 2\mu F(\lambda_0 + \mu^2). \tag{38}$$

On the other hand, by setting $Y = Y(t) = ||C||^2 = C_{jkl}C^{jkl}$ and implementing equation (34), it can be determined that

$$Y' = 2L_{jkl}C^{jkl} = -2\mu F Y. \tag{39}$$

The combination of the equations (38) and (39) results in

$$\lambda_0 + \mu^2 = \frac{\xi_0}{||C||^2},$$

where $\xi_0 = \xi_0(x, y)$ is a scalar function on TM which $\xi_0' = 0$. □

In this study, it is crucial to explore and identify Finsler metrics that demonstrate properties of the relatively isotropic $\tilde{D}$-metric introduced earlier. By incorporating a standby condition into the analysis, the process of recognizing and describing these metrics is significantly improved, making the search for these unique geometric structures more efficient. In the following, the evidence supporting Theorem 5.5 showcases a structured method for uncovering Finsler metrics belonging to this category.

Theorem 5.5. *Every (α, β)-metric of constant flag curvature κ with $\bar{E} = 0$ on manifold M of dimension $n \geq 3$ is a relatively isotropic $\tilde{D}$-metric if and only if one of the following conditions is true.*

1. $\kappa = 0$,

2. $\mathcal{P}_{|0} = 0$, *where*

$$\mathcal{P} = \frac{n+1}{\mathfrak{a}\mathcal{A}}\Big(s\varphi\varphi'' - \varphi'(\varphi - s\varphi')\Big), \tag{40}$$

and $\mathfrak{a} = \mathfrak{a}(s)$ *and* $\mathcal{A} = \mathcal{A}(s)$ *are given by*

$$\mathfrak{a} = \varphi(\varphi - s\varphi'), \tag{41}$$

$$\mathcal{A} = \frac{3s\varphi'' - (b^2 - s^2)\varphi'''}{\varphi - s\varphi' + (b^2 - s^2)\varphi''} + (n-2)\frac{s\varphi''}{\varphi - s\varphi'} - (n+1)\frac{\varphi'}{\varphi}. \tag{42}$$

Proof of Theorem 5.5

Proof. Assume that $F = \alpha\varphi(s)$, $s = \frac{\beta}{\alpha}$ be a non-Riemannian (α, β)-metric on a manifold M of dimension $n \geq 3$, then [17]

$$C_{jkl} = \frac{\mathcal{P}}{n+1}(I_j h_{kl} + I_k h_{jk} + I_l h_{jk}) + \frac{1-\mathcal{P}}{||I||^2} I_j I_k I_l, \tag{43}$$

where $\mathcal{P}$ can be represented as (40) as demonstrated in the study by [18] on the semi C-reducibility of these metrics. By taking horizontal covariant derivative with respect to Finsler metric F we have

$$\begin{aligned} L_{jkl} = \frac{\mathcal{P}}{n+1}(J_j h_{kl} + J_k h_{jk} + J_l h_{jk}) + \frac{1-\mathcal{P}}{||I||^2}(J_j I_k I_l + I_j J_k I_l + I_j I_k J_l) \\ + \frac{\mathcal{P}'}{\mathcal{P}} C_{jkl} - \frac{1}{||I||^2}\Big(\frac{\mathcal{P}'}{\mathcal{P}} + 2(1-\mathcal{P})\frac{||I||'}{||I||}\Big) I_j I_k I_l. \end{aligned} \tag{44}$$

In this scenario, $||I||'$ is equivalent to the derivative of $||I||$ with respect to t, where $||I||^2(t) = I_i(t)I^i(t)$ and $I(t)$ is equal to $I_{\gamma(t)}(U(t)) = I_j(\gamma(t), \gamma'(t))U^j$. Here, $\gamma(t)$ stands for the geodesic which is defined with arc length as its parameter on the manifold M starting at $\gamma(0) = p$ and having a tangent vector of $\gamma'(0) = y$, whereas $U = U(t)$ acts as a parallel vector fields along $\gamma = \gamma(t)$ whose initial value is set to $U(0) = u$. The same scenario applies to $\mathcal{P}'$.

On the other hand, the mean Cartan torsion of a (α, β)-metric can be identified by [18]

$$I_i = -\frac{1}{2}\mathcal{A}_{.s} s_{.i},$$

where $\mathcal{A}_{.s} = \frac{\partial \mathcal{A}}{\partial s}$, $s_{.i} = \frac{\partial s}{\partial y^i}$ and $\mathcal{A}$ is denoted by (42). With a contraction of the equation (44) by I^j, I^k and I^l and applying the equation $J_k I^k = I'_k I^k = \frac{1}{2}(I_k I^k)' = ||I||||I||'$ we get

$$L_{jkl}I^j I^k I^l = \Big(1 - \frac{n-2}{n+1}\mathcal{P}\Big)||I||'||I||^3 + \frac{\mathcal{P}'}{\mathcal{P}}\Big(C_{jkl}I^j I^k I^l - ||I||^4\Big).$$

Nevertheless, as per (43), one easily obtains

$$C_{jkl}I^j I^k I^l = \Big(1 - \frac{n-2}{n+1}\mathcal{P}\Big)||I||^4. \tag{45}$$

When these two equations are merged together, we get

$$L_{jkl}I^j I^k I^l = \Big(1 - \frac{n-2}{n+1}\mathcal{P}\Big)||I||'||I||^3 - \frac{n-2}{n+1}\mathcal{P}'||I||^4. \tag{46}$$

Now, according to Theorem 5.3, the (α, β)-metric F of constant flag curvature $\kappa \neq 0$ with $\bar{E} = 0$ is of relatively isotropic $\tilde{D}$-metric if and only if there is a function $\mu = \mu(x, y)$ on TM such that $L_{jkl} + \mu F C_{jkl} = 0$.

Then assume that the condition

$$L_{jkl} + \mu F C_{jkl} = 0, \tag{47}$$

holds for the (α, β)-metric F. Then based on (45) we have

$$L_{jkl} I^j I^k I^l = -\mu F\Big(1 - \frac{n-2}{n+1}\mathcal{P}\Big)||I||^4. \tag{48}$$

When we contract equation (47) with g^{jl} and I^k, and also use $J_k I^k = ||I|| ||I||'$, the result is

$$-\mu F = \frac{||I||'}{||I||}.$$

Plugging the aforementioned formula into (48) gives us

$$L_{jkl} I^j I^k I^l = (1 - \frac{n-2}{n+1}\mathcal{P}\Big)||I||' ||I||^3,$$

which by combing with (46) one finds $\mathcal{P}' = 0$. □

When it comes to Randers metric, the value $\mathcal{P} = 1$, therefore, as per the Theorem mentioned above, one can effortlessly arrive at the following Corollary.

Corollary 5.6. *Every Randers metric of constant flag curvature with $\bar{E} = 0$ ($n \geq 3$) is a relatively isotropic $\tilde{D}$-metric.*

Given corollary and Corollary 4.3, it is straightforward to derive the following result.

Corollary 5.7. *Every (α, β)-metric with vanishing either flag curvature, $\lambda = 0$ and S-curvature, $S = 0$ ($n \geq 3$) is a relatively isotropic $\tilde{D}$-metric.*

According to the work in [16], projectively flat (α, β)-metrics with constant flag curvature have been categorized. The findings indicate that Finsler metrics which are not of Randers type can be considered as essentially square metrics, given by $F = \frac{(\alpha+\beta)^2}{\alpha}$, where α is a Riemannian metric and β is a 1-form on M. It was later demonstrated by [29] that any square metric with constant flag curvature must exhibit local projective flatness. Then one easily finds that

Corollary 5.8. *There does not exist a non-trivial relatively isotropic $\tilde{D}$-metric ($n \geq 3$) in the form of a square metric having constant flag curvature $\kappa \neq 0$.*

When examining (α, β)-metrics with the requirement of relatively isotropic $\tilde{D}$-metric, it is observed in [7] that a regular (α, β)-metric possessing a constant length Killing 1-form β and constant flag curvature must either be Riemannian metric or locally Minkowskian. Following this, we come to the subsequent Corollary.

Corollary 5.9. *There does not exist a non-trivial relatively isotropic $\tilde{D}$-metric in the form of a regular (α, β)-metric possessing a constant length Killing 1-form β and constant flag curvature $\kappa \neq 0$ ($n \geq 3$).*

The following example demonstrates the existence of Non-Douglas relatively isotropic $\tilde{D}$-metric.

Example 5.1. *[6]*

The family of Randers metrics on S^3 constructed by Bao-Shen are weakly Berwald which are not Berwaldian. Denote generic tangent vectors on S^3 as

$$u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$

The Finsler function for Bao-Shen's Randers space is given by

$$F(x, y, z; u, v, w) = \alpha(x, y, z; u, v, w) + \beta(x, y, z; u, v, w),$$

with

$$\alpha = \frac{\sqrt{\lambda(cu - zv + yw)^2 + (zu + cv - xw)^2 + (-yu + xv + cw)^2}}{1 + x^2 + y^2 + z^2},$$

$$\beta = \frac{\pm\sqrt{\lambda - 1}(cu - zv + yw)}{1 + x^2 + y^2 + z^2},$$

where $\lambda > 1$ is a real constant. The above Randers metric has vanishing S-curvature and with positive constant flag curvature 1. The metric has constant flag curvature, it will also have Weyl curvature and be classified as a GDW-metric. Despite not fitting the criteria for a Douglas type metric, according to (5.6)*, the Randers metric F falls under the category of a relatively isotropic $\tilde{D}$-metric based on Theorem 5.5.*

According to Corollary 5.7, the following example is a non-Douglas relatively isotropic $\tilde{D}$-metric.

Example 5.2. *The Shen's fish tank metric F is a Finsler metric with vanishing S-curvature and flag curvature $\lambda = 0$, while it is not a Berwald metric. It is defined as follows*

$$F(p, y) = \frac{\sqrt{(-yu + xv)^2 + (u^2 + v^2 + w^2)(1 - x^2 - y^2)}}{1 - x^2 - y^2} + \frac{-yu + xv}{1 - x^2 - y^2},$$

where $p = (x, y, z)$ and $y = (u, v, w)$ are elements of the tangent space $T\Omega$ of the $\Omega = \{(x, y, z) \mid x^2 + y^2 = 1\}$.

The following example, is a Randers metric of Weyl curvature (non-constant flag curvature) and is consequently a *GDW*-metric, but it is not of relatively isotropic Douglas metric.

Example 5.3. *[24]*

Let us consider the Randers metric $F = \alpha + \beta$ which is given by

$$\alpha = \frac{\sqrt{(1- \mid a \mid^2 \mid x \mid^2) \mid y \mid^2 + (\mid x \mid^2 < a, y > -2 < a, x >< x, y >)^2}}{1- \mid a \mid^2 \mid x \mid^2},$$

and

$$\beta = -\frac{\mid x \mid^2 < a, y > -2 < a, x >< x, y >}{1- \mid a \mid^2 \mid x \mid^2}.$$

F is of isotropic S-curvature, $S = (n + 1)cF$, with $c =< a, x >$, and of scalar flag curvature λ as

$$\lambda = 3\frac{c_{;0}}{F} + 3c^2 - 2 \mid a \mid^2 \mid x \mid^2 .$$

However, we have

$$\begin{aligned} a_{jk} &= \frac{\delta_{jk}}{\Delta} + b_j b_k, \\ b_j &= 2\frac{c}{\Delta} x_k - \frac{\mid x \mid^2}{\Delta} c_{;k}, \end{aligned} \tag{49}$$

where $\Delta = 1- \mid a \mid^2 \mid x \mid^2$. Using Maple for the computation, which has been done in [24], we have

$$\begin{aligned} s_{jk} &= \frac{2}{\Delta^2}(c_{;k} x_j - c_{;j} x_k), \\ s_k &= 2\frac{\mid a \mid^2 \mid x \mid^2}{\Delta} x_k + 2\frac{c}{\Delta} c_{;k} \end{aligned} \tag{50}$$

and

$$G^i = {}^{\alpha}G^i + Py^i + \alpha s^i{}_0, \tag{51}$$

where $P = c(\alpha - \beta) - s_0$.

Now, let us assume $n = 3$, *constant vector* $a = (-1, 0, 0)$, $X = (x, y, z)$ *and* $Y = (u, v, w)$. *Then we have* $c = -x$ *and* $c_{;k} = -\delta_{1k}$. *Hence, we have* $s_{jk} = \frac{2}{\Delta^2}(\delta_{1k}x_j - \delta_{1j}x_k)$, *which indicates that* β *is not closed. Consequently,* F *is not a Douglas metric, even though it has scalar flag curvature. Therefore, it is a Weyl metric and, as a result, a GDW-metric. Moreover, based on the notations in Corollary 4.4we have* $\bar{c}_k = \delta_{1r}N^r{}_k - (\mu + c)F\delta_{1k}$. *For* $k \neq 1$ *we have*

$$2(\bar{\tau}_l + cF\tau_k) = 2(\bar{c}_k + cFc_{;k}) - \frac{2}{F}(\bar{c}_0 + cFc_{;0})F_{.k} = 2\delta_{1r}N^r{}_k - \frac{4}{F}\delta_{1r}G^rF_{.k},$$

and

$$A_k = Ab_k - c\omega s_k + 2\delta_{1r}N^r{}_k - \frac{4}{F}\delta_{1r}G^rF_{.k}.$$

where $A = \lambda_{|0} + \mu(\lambda + c^2)F + 2cc_{;0}$ *and* $\omega = \lambda F - 2c_{;0} - \mu cF$.

Using the above equations and the fact $F_{.k} = \alpha_{.k} + b_k$ *in* (28), *the metric* F *is relatively isotropic* $\tilde{D}$*-metric if and only if*

$$(\alpha A + A_0)b_k + \omega(s_{k0} - c\alpha s_k) + 2\delta_{1r}N^r{}_k - (A_0 + \frac{2}{F}\delta_{1r}G^r)F_{.k} = 0,$$

where $A_0 = A_k y^k$. *The equation obtained implies that all coefficients of* y_i, *and the coefficients of* $y_1 = u$, *are zero. Based on the formulas of* (49) *and* (50), *the term* $(\alpha A + A_0)b_k + \omega(s_{k0} - c\alpha s_k)$ *cannot be a coefficient of every* y_i *and the coefficient of* u. *Furthermore, since* $k \neq 0$, *the term* $(A_0 + \frac{2}{F}\delta_{1r}G^r)F_{.k}$ *is not a coefficient of* u *either. Therefore, only certain terms in* $\delta_{1r}N^r{}_k$ *can be the coefficient of* u, *and they are presented below. Based on* (51), *we have*

$$\delta_{1r}N^r{}_k = \delta_{1r}\{{}^{\alpha}N^r{}_k + \alpha s^1{}_k - s^1{}_0 b_k + s^1{}_0 F_{.k} + P_{.k}u\}.$$

Using the expression

$${}^{\alpha}N^r{}_k = \tilde{N}^r k + \gamma^r kl(x)u,$$

where there is no multiple of u *in* $\bar{N}^r k$, *we find that the coefficient of* u *is equal to*

$$0 = \delta_{1r}\gamma^r{}_{k1}(x) + P_{.k} = \delta_{1r}\gamma^r{}_{k1}(x) + xb_k - \frac{x}{\alpha}y_k,$$

which is a contradiction. Therefore F *is not a relatively isotropic* $\tilde{D}$*-metric.*

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