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Some novel fractional Milne-type inequalities for twice differentiable $s-$convex functions in the second sense

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Abstract. In this paper, a special identity for twice differentiable functions, which is available in the literature, is utilized. By combining this identity with Riemann-Liouville fractional integrals, various fractional Milne-type inequalities are obtained for functions whose second derivatives in absolute value are $s-$convex in the second sense. Moreover, Hölder and Young inequalities are used to prove the different and original results. These approaches contribute to the analysis of $s-$convex functions and offer new perspectives in the field of fractional analysis.

1. Introduction

Numerical integration, as one of the cornerstones of mathematical calculations, is especially important when analytical integration of complex functions is not possible. Using numerical integration techniques, many mathematicians and researchers aim to improve the accuracy of calculated integration values and to determine error upper bounds more precisely. Studies in this context have questioned the effectiveness of different numerical methods and have included in-depth analyses to understand the error behavior of each method. Investigating the error bounds of numerical integration requires a detailed study of mathematical inequalities for different classes of functions, such as convex, bounded and Lipschitzian functions. These inequalities provide important insights into how to optimize error calculations according to the structural properties of functions.

In particular, the study of functions whose derivatives or second derivatives satisfy the convexity condition allows a more precise estimation of the error upper bounds of numerical integration. Such functions have specific mathematical structures and geometric properties, and thus play an important role in error analysis. These analyses pave the way for a continuous development process aimed at improving the effectiveness of numerical integration methods at the theoretical and practical level. Conventional

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and modern numerical integration methods each set different error upper bounds and provide various conclusions about the accuracy and efficiency of the solution.

In this context, a more detailed consideration of numerical integration methods and their associated error upper bounds will make improvements that can be made to increase the accuracy of the integration process more apparent. In particular, this paper aims to provide a comprehensive review in order to understand the different techniques of digital integration and their associated error bounds. Let us now develop a broader view of the basics of numerical integration methods and the general characteristics of the upper error bounds associated with these methods:

1. The following expression represents Simpson's quadrature formula and is commonly known as Simpson's $\frac{1}{3}$ rule:

$$\int_{\kappa_1}^{\kappa_2} F(\epsilon)d\epsilon \approx \frac{\kappa_2-\kappa_1}{6}[F(\kappa_1)+4F\Big(\frac{\kappa_1+\kappa_2}{2}\Big)+F(\kappa_2)] \tag{1}$$

2. The definition of Simpson's second formula, known as the Newton-Cotes quadratic formula or more commonly as Simpson's $\frac{3}{8}$ rule (See [15]), is expressed as follows

$$\int_{\kappa_1}^{\kappa_2} F(\epsilon)d\epsilon \approx \frac{\kappa_2-\kappa_1}{8}[F(\kappa_1)+3F\Big(\frac{2\kappa_1+\kappa_2}{3}\Big)+3F\Big(\frac{\kappa_1+2\kappa_2}{3}\Big)+F(\kappa_2)] \tag{2}$$

Equations (1) and (2) hold for any function F whose fourth derivative exists continuously on the interval $[\kappa_1,\kappa_2]$.

The standard form of Simpson's inequality is given as follows:

Theorem 1.1. *When considering $F:[\kappa_1,\kappa_2]\to\mathbb{R}$, a function with four continuous derivatives within the interval (κ_1,κ_2), and $\left\|F^{(4)}\right\|_\infty=\sup_{\epsilon\in(\kappa_1,\kappa_2)}|F^{(4)}(\epsilon)|<\infty$, the subsequent inequality holds:*

$$\left|\frac{1}{6}\left[F(\kappa_1)+4F\Big(\frac{\kappa_1+\kappa_2}{2}\Big)+F(\kappa_2)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(\epsilon)d\epsilon\right|\le\frac{1}{2880}\left\|F^{(4)}\right\|_\infty(\kappa_2-\kappa_1)^4.$$

Some new Simpson type inequalities were proved by Sarıkaya et al. in [29] using convex functions. In the field of Riemann-Liouville fractional integrals, there are three different types of Simpson-type inequalities classified according to their fractional integral representations. Some novel different inequalities have been developed and elaborated in [3]-[20]. These works contributed to the extension of Simpson's inequality in the context of fractional analysis and demonstrated the applicability of these inequalities for various types of fractional integrals. Also, in [27]-[35], special attention has been paid to Simpson-type inequalities applicable to twice differentiable functions. In these works, the use of Simpson's inequality on doubly differentiable functions is studied in depth and the special cases of the inequality for this class of functions are detailed. Thus, the scope of Simpson-type inequalities has been extended, contributing to obtain more precise and specific results for certain classes of functions.

The classical Newton inequality is of fundamental importance in mathematical analysis and is defined as follows:

Theorem 1.2 (See [15]). *If $F:[\kappa_1,\kappa_2]\to\mathbb{R}$ represents a function with a continuous fourth derivative defined over (κ_1,κ_2), and $\left\|F^{(4)}\right\|_\infty=\sup_{\epsilon\in(\kappa_1,\kappa_2)}|F^{(4)}(\epsilon)|<\infty$, then the inequality presented below is valid:*

$$\begin{aligned}&\left|\frac{1}{8}\left[F(\kappa_1)+3F\Big(\frac{2\kappa_1+\kappa_2}{3}\Big)+3F\Big(\frac{\kappa_1+2\kappa_2}{3}\Big)+F(\kappa_2)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(\epsilon)d\epsilon\right|\\ \le\;&\frac{1}{6480}\left\|F^{(4)}\right\|_\infty(\kappa_2-\kappa_1)^4.\end{aligned}$$

The works [22]-[23] establish Newton-type inequalities using convex functions in local fractional integrals. These works provide new perspectives by extending the application of Newton's inequality in fractional analysis. In [34], the first proofs of Newton-type inequalities for Riemann-Liouville fractional integrals were presented and fundamental results in this field were established. This work is considered to be an important step in fractional calculus and a fundamental reference for further research. Subsequently, several studies on Riemann-Liouville fractional integrals have focused especially on the derivation of Newton-type inequalities and in-depth investigations on the validity of these inequalities for different cases [21, 36]. These papers have contributed significantly to the development of fractional integral theory and filled the gaps in the literature on the subject.

The classical Milne inequality is a mathematical relationship that has an important place in analysis and is used in various applications and is formulated as follows:

Theorem 1.3 (See [32]). *Let $F : [\kappa_1,\kappa_2] \to \mathbb{R}$ be a function with a continuous fourth derivative over (κ_1, κ_2), and $\left\|F^{(4)}\right\|_\infty = \sup_{\epsilon\in(\kappa_1,\kappa_2)} |F^{(4)}(\epsilon)| < \infty$, In such a case, the subsequent inequality is valid:*

$$\left|\frac{1}{3}\left[2F(\kappa_1) - F\Big(\frac{\kappa_1+\kappa_2}{2}\Big) + 2F(\kappa_2)\right] - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\epsilon)d\epsilon\right| \le \frac{7(\kappa_2-\kappa_1)^4}{23040}\left\|F^{(4)}\right\|_\infty$$

In [17], Djenaoui and Meftah developed Milne-type inequalities using the concept of convexity for the first time. This work constitutes an important step in the treatment of Milne inequalities within the framework of convex analysis and allows for a reinterpretation of the inequalities from an extended perspective. In [6], Budak and his colleagues extended the applicability of these inequalities and discussed them in the context of Riemann-Liouville fractional integrals. This research has contributed to the strengthening of the theoretical background by deepening the use of Milne-type inequalities in fractional analysis and has given a new perspective to the adaptation of inequalities to different fields of analysis. In particular, recent works in [1, 5] introduced new fractional variations of Milne-type inequalities and treated them in terms of separable convex functions. These works also extend the applicability of Milne-type inequalities by detailed investigations on different classes of functions such as bounded functions, Lipschitz functions and bounded variational functions. To learn more about Milne-type inequalities and to deepen the research in this area, references [16]-[33] provide valuable information. These references provide a detailed overview of the different variations of Milne-type inequalities, their theoretical foundations and applications.

Definition 1.4. *(See [26]) Let $\Psi \in L_1[\epsilon_1, \epsilon_2]$. The RL-integrals $J^{\alpha}_{\epsilon_1^+}\Psi$ and $J^{\alpha}_{\epsilon_2^-}\Psi$ of order $\alpha > 0$ with $\epsilon_1 \ge 0$ are defined by*

$$J^{\alpha}_{\epsilon_1^+}\Psi(u_1) = \frac{1}{\Gamma(\alpha)}\int_{\epsilon_1}^{u_1}(u_1-\zeta)^{\alpha-1}\Psi(\zeta)d\zeta, \quad u_1 > \epsilon_1$$

and

$$J^{\alpha}_{\epsilon_2^-}\Psi(u_1) = \frac{1}{\Gamma(\alpha)}\int_{u_1}^{\epsilon_2}(\zeta-u_1)^{\alpha-1}\Psi(\zeta)d\zeta, \quad u_1 < \epsilon_2$$

where $\Gamma(\alpha) = \int_0^\infty e^{-\zeta}u^{\alpha-1}du$, here is $J^{0}_{\epsilon_1^+}\Psi(u_1) = J^{0}_{\epsilon_2^-}\Psi(u_1) = \Psi(u_1)$.

In the above definition, if we set $\alpha = 1$, the definition overlaps with the classical integral. Features of the fractional integral operator can be found in the references [9]-[26].

Let us continue our study by giving the definition of convexity and s-convexity in the second sense.

Definition 1.5. *[28] Let I be on interval in $\mathbb{R}$. Then $f : I \to \mathbb{R}$ is said to be convex, if*

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

In [4], W.W. Breckner defined the $s-$convex in the second sense class as follows

Definition 1.6. *A function $F:\mathbb{R}^{+}\rightarrow\mathbb{R}$, is said to be $s-$convex in the second sense if*

$$F(\beta_1 u_1+\beta_2 u_2)\leq \beta_1^s F(u_1)+\beta_2^s F(u_2)$$

for all $\beta_1,\beta_2\geq 0,\ u_1,u_2\geq 0$ with $\beta_1+\beta_2=1$ and for some fixed $s\in(0,1]$. This class of functions is denoted by K_s^2.

Lemma 1.7. *[2] If $F:[\kappa_1,\kappa_2]\longrightarrow R$ is absolutely continuous over (κ_1,κ_2) and $F''\in L_1([\kappa_1,\kappa_2])$, then the following holds:*

$$\frac{\Gamma(\zeta+1)}{2(\kappa_2-\kappa_1)^{\zeta}}[\mathfrak{I}^{\zeta}_{\kappa_1^+}F(\kappa_2)+\mathfrak{I}^{\zeta}_{\kappa_2^-}F(\kappa_1)]-\frac{1}{3}[2F(\kappa_1)-F\Big(\frac{\kappa_1+\kappa_2}{2}\Big)+2F(\kappa_2)]=\frac{(\kappa_2-\kappa_1)^2}{2(\zeta+1)}\sum_{k=1}^{4}I_k, \tag{3}$$

where

$$I_1=\int_0^{\frac{1}{2}}\Big(\epsilon^{\zeta+1}-\frac{\zeta+4}{3}\epsilon\Big)F''(\epsilon\kappa_2+(1-\epsilon)\kappa_1)d\epsilon,$$
$$I_2=\int_0^{\frac{1}{2}}\Big(\epsilon^{\zeta+1}-\frac{\zeta+4}{3}\epsilon\Big)F''(\epsilon\kappa_1+(1-\epsilon)\kappa_2)d\epsilon,$$
$$I_3=\int_{\frac{1}{2}}^{1}\Big(\epsilon^{\zeta+1}-\epsilon\Big)F''(\epsilon\kappa_2+(1-\epsilon)\kappa_1)d\epsilon,$$
$$I_4=\int_{\frac{1}{2}}^{1}\Big(\epsilon^{\zeta+1}-\epsilon\Big)F''(\epsilon\kappa_1+(1-\epsilon)\kappa_2)d\epsilon.$$

The aim of this paper is to derive fractional Milne-type inequalities for functions whose second derivatives are $s-$convex. To achieve this goal, we first establish a basic framework by introducing Riemann-Liouville fractional integrals, convexity and $s-$convexity in the second sense. Then, based on these concepts, a new formulation of fractional Milne inequalities with convexity condition will be introduced. This paper aims to contribute to a broader understanding of Milne-type inequalities in fractional analysis.

2. Main Results

Theorem 2.1. *Assume the conditions of Lemma 1.7 are satisfied. Furthermore, if $\left|F''\right|$ exhibits s-convex in the second sense over $[\kappa_1,\kappa_2]$, then :*

$$\begin{aligned}&\left|\frac{\Gamma(\zeta+1)}{2(\kappa_2-\kappa_1)^{\zeta}}[\mathfrak{I}^{\zeta}_{\kappa_1^+}F(\kappa_2)+\mathfrak{I}^{\zeta}_{\kappa_2^-}F(\kappa_1)]-\frac{1}{3}[2F(\kappa_1)-F\Big(\frac{\kappa_1+\kappa_2}{2}\Big)+2F(\kappa_2)]\right|\\ \leq\ &\frac{(\kappa_2-\kappa_1)^2}{2(\zeta+1)}\left[\frac{\zeta+4}{3}\left(\frac{\left(2^{s+1}-1\right)2^{s+2}}{2^{s+1}2^{s+2}(s+1)(s+2)}\right)+\frac{2^{s+2}\left(\left(2^{s+1}-1\right)(s+1)+(s+2)\right)}{2^{s+1}2^{s+2}(s+1)(s+2)}\right.\\ &\left.-\frac{1}{s+\zeta+2}-\beta(\zeta+2,s+1)\right]\left[\left|F''(\kappa_1)\right|+\left|F''(\kappa_2)\right|\right]\end{aligned}$$

for $s\in(0,1],\zeta>0$.

Proof. On applying the modulus operation to Lemma 1.7, we obtain:

$$\left|\frac{\Gamma(\zeta+1)}{2(\kappa_2-\kappa_1)^{\zeta}}[\mathfrak{I}^{\zeta}_{\kappa_1^+}F(\kappa_2)+\mathfrak{I}^{\zeta}_{\kappa_2^-}F(\kappa_1)]-\frac{1}{3}[2F(\kappa_1)-F\Big(\frac{\kappa_1+\kappa_2}{2}\Big)+2F(\kappa_2)]\right|$$

$$\begin{aligned}
\leq\quad & \frac{(\kappa_2-\kappa_1)^2}{2(\zeta+1)}\left[\int_0^{\frac{1}{2}}\left|\epsilon^{\zeta+1}-\frac{\zeta+4}{3}\epsilon\right|\left|F''(\epsilon\kappa_2+(1-\epsilon)\kappa_1)\right|d\epsilon\right. \\
& +\int_0^{\frac{1}{2}}\left|\epsilon^{\zeta+1}-\frac{\zeta+4}{3}\epsilon\right|\left|F''(\epsilon\kappa_1+(1-\epsilon)\kappa_2)\right|d\epsilon \\
& +\int_{\frac{1}{2}}^{1}\left|\epsilon^{\zeta+1}-\epsilon\right|\left|F''(\epsilon\kappa_2+(1-\epsilon)\kappa_1)\right|d\epsilon \\
& \left.+\int_{\frac{1}{2}}^{1}\left|\epsilon^{\zeta+1}-\epsilon\right|\left|F''(\epsilon\kappa_1+(1-\epsilon)\kappa_2)\right|d\epsilon\right].
\end{aligned}$$

Leveraging the $s-$convex in the second sense property of $\left|F''\right|$, we derive

$$\begin{aligned}
& \left|\frac{\Gamma(\zeta+1)}{2(\kappa_2-\kappa_1)^{\zeta}}[\Im^{\zeta}_{\kappa_1^+}F(\kappa_2)+\Im^{\zeta}_{\kappa_2^-}F(\kappa_1)]-\frac{1}{3}[2F(\kappa_1)-F\Big(\frac{\kappa_1+\kappa_2}{2}\Big)+2F(\kappa_2)]\right| \\
\leq\quad & \frac{(\kappa_2-\kappa_1)^2}{2(\zeta+1)}\left[\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\epsilon-\epsilon^{\zeta+1}\right)\left[\epsilon^s\left|F''(\kappa_2)\right|+(1-\epsilon)^s\left|F''(\kappa_1)\right|\right]d\epsilon\right. \\
& +\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\epsilon-\epsilon^{\zeta+1}\right)\left[\epsilon^s\left|F''(\kappa_1)\right|+(1-\epsilon)^s\left|F''(\kappa_2)\right|\right]d\epsilon \\
& +\int_{\frac{1}{2}}^{1}\left(\epsilon-\epsilon^{\zeta+1}\right)\left[\epsilon^s\left|F''(\kappa_2)\right|+(1-\epsilon)^s\left|F''(\kappa_1)\right|\right]d\epsilon \\
& \left.+\int_{\frac{1}{2}}^{1}\left(\epsilon-\epsilon^{\zeta+1}\right)\left[\epsilon^s\left|F''(\kappa_1)\right|+(1-\epsilon)^s\left|F''(\kappa_2)\right|\right]d\epsilon\right]. \\
= & \frac{(\kappa_2-\kappa_1)^2}{2(\zeta+1)}\left[\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\epsilon-\epsilon^{\zeta+1}\right)+\int_{\frac{1}{2}}^{1}\left(\epsilon-\epsilon^{\zeta+1}\right)\right](\epsilon^s+(1-\epsilon)^s)\left(\left|F''(\kappa_2)\right|+\left|F''(\kappa_1)\right|\right)d\epsilon \\
= & \frac{(\kappa_2-\kappa_1)^2}{2(\zeta+1)}\left[\frac{\zeta+4}{3}\left(\frac{\left(2^{s+1}-1\right)2^{s+2}}{2^{s+1}2^{s+2}(s+1)(s+2)}\right)+\frac{2^{s+2}\left(\left(2^{s+1}-1\right)(s+1)+(s+2)\right)}{2^{s+1}2^{s+2}(s+1)(s+2)}\right. \\
& \left.-\frac{1}{s+\zeta+2}-\beta(\zeta+2,s+1)\right]\left[\left|F''(\kappa_1)\right|+\left|F''(\kappa_2)\right|\right].
\end{aligned}$$

The proof is completed.

Remark 2.2. *(See [2]) If $s=1$ in Theorem 2.1, the following inequality is obtained:*

$$\begin{aligned}
& \left|\frac{\Gamma(\zeta+1)}{2(\kappa_2-\kappa_1)^{\zeta}}[\Im^{\zeta}_{\kappa_1^+}F(\kappa_2)+\Im^{\zeta}_{\kappa_2^-}F(\kappa_1)]-\frac{1}{3}[2F(\kappa_1)-F(\frac{\kappa_1+\kappa_2}{2})+2F(\kappa_2)]\right| \\
\leq\quad & \frac{(\kappa_2-\kappa_1)^2}{48}\left(\frac{\zeta^2+15\zeta+2}{(\zeta+1)(\zeta+2)}\right)\left[\left|F''(\kappa_1)\right|+\left|F''(\kappa_2)\right|\right]
\end{aligned}$$

Remark 2.3. *(See [16]) If $s=1$ and $\zeta=1$ in Theorem 2.1, the following inequality is obtained:*

$$\left|\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}F(\epsilon)d\epsilon-\frac{1}{3}[2F(\kappa_1)-F(\frac{\kappa_1+\kappa_2}{2})+2F(\kappa_2)]\right|\leq\frac{(\kappa_2-\kappa_1)^2}{16}\left[\left|F''(\kappa_1)\right|+\left|F''(\kappa_2)\right|\right]$$

□

Theorem 2.4. *Assume the conditions of Lemma 1.7 are met and, additionally, if $\left|F''\right|^q$, where $q>1$, exhibits $s-$convex in the second sense over the interval $[\kappa_1,\kappa_2]$, then:*

$$\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\kappa_2-\kappa_1)^\zeta}[\mathfrak{I}^\zeta_{\kappa_1^+}F(\kappa_2)+\mathfrak{I}^\zeta_{\kappa_2^-}F(\kappa_1)]-\frac{1}{3}[2F(\kappa_1)-F\Big(\frac{\kappa_1+\kappa_2}{2}\Big)+2F(\kappa_2)]\right|\\
\leq\ & \frac{(\kappa_2-\kappa_1)^2}{2(\zeta+1)}\left[\left(\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)^{\frac{1}{p}}+\left(\frac{1}{\zeta}\mathcal{B}\left(p+1,\frac{p+1}{\zeta},1-\left(\frac{1}{2}\right)^\zeta\right)\right)^{\frac{1}{p}}\right]\\
&\times\left[\left(\frac{\left(2^{s+1}-1\right)\left|F''(\kappa_2)\right|^q+\left|F''(\kappa_1)\right|^q}{2^{s+1}(s+1)}\right)^{\frac{1}{q}}+\left(\frac{\left(2^{s+1}-1\right)\left|F''(\kappa_1)\right|^q+\left|F''(\kappa_2)\right|^q}{2^{s+1}(s+1)}\right)^{\frac{1}{q}}\right]
\end{aligned}$$

for $s\in(0,1]$, $\zeta>0$ where $\frac{1}{p}+\frac{1}{q}=1$ and $\mathcal{B}$ represents the incomplete beta function, defined as:

$$\mathcal{B}(k,y,r)=\int_0^r\epsilon^{k-1}(1-\epsilon)^{y-1}d\epsilon.$$

Proof. On applying the modulus operation to Lemma 1.7, we obtain:

$$\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\kappa_2-\kappa_1)^\zeta}[\mathfrak{I}^\zeta_{\kappa_1^+}F(\kappa_2)+\mathfrak{I}^\zeta_{\kappa_2^-}F(\kappa_1)]-\frac{1}{3}[2F(\kappa_1)-F\Big(\frac{\kappa_1+\kappa_2}{2}\Big)+2F(\kappa_2)]\right|\\
\leq\ & \frac{(\kappa_2-\kappa_1)^2}{2(\zeta+1)}\left[\int_0^{\frac{1}{2}}\left|\epsilon^{\zeta+1}-\frac{\zeta+4}{3}\epsilon\right|\left|F''(\epsilon\kappa_2+(1-\epsilon)\kappa_1)\right|d\epsilon\right.\\
&+\int_0^{\frac{1}{2}}\left|\epsilon^{\zeta+1}-\frac{\zeta+4}{3}\epsilon\right|\left|F''(\epsilon\kappa_1+(1-\epsilon)\kappa_2)\right|d\epsilon\\
&+\int_{\frac{1}{2}}^1\left|\epsilon^{\zeta+1}-\epsilon\right|\left|F''(\epsilon\kappa_2+(1-\epsilon)\kappa_1)\right|d\epsilon\\
&\left.+\int_{\frac{1}{2}}^1\left|\epsilon^{\zeta+1}-\epsilon\right|\left|F''(\epsilon\kappa_1+(1-\epsilon)\kappa_2)\right|d\epsilon\right].
\end{aligned}$$

When we apply Hölder's inequality to the above inequality, we obtain the following result:

$$\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\kappa_2-\kappa_1)^\zeta}[\mathfrak{I}^\zeta_{\kappa_1^+}F(\kappa_2)+\mathfrak{I}^\zeta_{\kappa_2^-}F(\kappa_1)]-\frac{1}{3}[2F(\kappa_1)-F\Big(\frac{\kappa_1+\kappa_2}{2}\Big)+2F(\kappa_2)]\right|\\
\leq\ & \frac{(\kappa_2-\kappa_1)^2}{2(\zeta+1)}\left[\left(\int_0^{\frac{1}{2}}\left|\epsilon^{\zeta+1}-\frac{\zeta+4}{3}\epsilon\right|^p d\epsilon\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|F''(\epsilon\kappa_2+(1-\epsilon)\kappa_1)\right|^q d\epsilon\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^{\frac{1}{2}}\left|\epsilon^{\zeta+1}-\frac{\zeta+4}{3}\epsilon\right|^p d\epsilon\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|F''(\epsilon\kappa_1+(1-\epsilon)\kappa_2)\right|^q d\epsilon\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^1\left|\epsilon^{\zeta+1}-\epsilon\right|^p d\epsilon\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|F''(\epsilon\kappa_2+(1-\epsilon)\kappa_1)\right|^q d\epsilon\right)^{\frac{1}{q}}\\
&\left.+\left(\int_{\frac{1}{2}}^1\left|\epsilon^{\zeta+1}-\epsilon\right|^p d\epsilon\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|F''(\epsilon\kappa_1+(1-\epsilon)\kappa_2)\right|^q d\epsilon\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Utilizing the $s-$convex in the second sense of $\left|F''\right|^q$, we obtain

$$
\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\kappa_2-\kappa_1)^\zeta}[\mathfrak{I}^\zeta_{\kappa_1^+}F(\kappa_2)+\mathfrak{I}^\zeta_{\kappa_2^-}F(\kappa_1)]-\frac{1}{3}[2F(\kappa_1)-F\Big(\frac{\kappa_1+\kappa_2}{2}\Big)+2F(\kappa_2)]\right|\\
\le\ &\frac{(\kappa_2-\kappa_1)^2}{2(\zeta+1)}\left[\left(\int_0^{\frac12}\left(\frac{\zeta+4}{3}\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)^{\frac1p}\left(\int_0^{\frac12}\epsilon^s\left|F''(\kappa_2)\right|^q+(1-\epsilon)^s\left|F''(\kappa_1)\right|^q d\epsilon\right)^{\frac1q}\right.\\
&+\left(\int_0^{\frac12}\left(\frac{\zeta+4}{3}\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)^{\frac1p}\left(\int_0^{\frac12}\epsilon^s\left|F''(\kappa_1)\right|^q+(1-\epsilon)^s\left|F''(\kappa_2)\right|^q d\epsilon\right)^{\frac1q}\\
&+\left(\int_{\frac12}^1\left(\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)^{\frac1p}\left(\int_{\frac12}^1\epsilon^s\left|F''(\kappa_2)\right|^q+(1-\epsilon)^s\left|F''(\kappa_1)\right|^q d\epsilon\right)^{\frac1q}\\
&\left.+\left(\int_{\frac12}^1\left(\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)^{\frac1p}\left(\int_{\frac12}^1\epsilon^s\left|F''(\kappa_1)\right|^q+(1-\epsilon)^s\left|F''(\kappa_2)\right|^q d\epsilon\right)^{\frac1q}\right]\\
&=\frac{(\kappa_2-\kappa_1)^2}{2(\zeta+1)}\left[\left(\int_0^{\frac12}\left(\frac{\zeta+4}{3}\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)^{\frac1p}+\left(\frac1\zeta\mathcal{B}\left(p+1,\frac{p+1}{\zeta},1-\left(\frac12\right)^\zeta\right)\right)^{\frac1p}\right]\\
&\times\left[\left(\frac{\left(2^{s+1}-1\right)\left|F''(\kappa_2)\right|^q+\left|F''(\kappa_1)\right|^q}{2^{s+1}(s+1)}\right)^{\frac1q}+\left(\frac{\left(2^{s+1}-1\right)\left|F''(\kappa_1)\right|^q+\left|F''(\kappa_2)\right|^q}{2^{s+1}(s+1)}\right)^{\frac1q}\right]
\end{aligned}
$$

□

Remark 2.5. *(See [2]) If $s=1$ in Theorem 2.4, the following inequality is obtained.*

$$
\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\kappa_2-\kappa_1)^\zeta}[\mathfrak{I}^\zeta_{\kappa_1^+}F(\kappa_2)+\mathfrak{I}^\zeta_{\kappa_2^-}F(\kappa_1)]-\frac{1}{3}[2F(\kappa_1)-F(\frac{\kappa_1+\kappa_2}{2})+2F(\kappa_2)]\right|\\
\le\ &\frac{(\kappa_2-\kappa_1)^2}{2(\zeta+1)}\left[\left(\int_0^{\frac12}\left(\frac{\zeta+4}{3}\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)^{\frac1p}+\left(\frac1\zeta\mathcal{B}\left(p+1,\frac{p+1}{\zeta},1-\left(\frac12\right)^\zeta\right)\right)^{\frac1p}\right]\\
&\times\left[\left(\frac{3\left|F''(\kappa_2)\right|^q+\left|F''(\kappa_1)\right|^q}{8}\right)^{\frac1q}+\left(\frac{3\left|F''(\kappa_1)\right|^q+\left|F''(\kappa_2)\right|^q}{8}\right)^{\frac1q}\right]
\end{aligned}
$$

Let $\theta_1=\left|F''(\kappa_1)\right|^q$,$\beta_1=3\left|F''(\kappa_2)\right|^q$, $\theta_2=3\left|F''(\kappa_1)\right|^q$, $\beta_2=3\left|F''(\kappa_2)\right|^q$, leveraging the given facts that

$$\sum_{k=1}^n(\theta_k+\beta_k)^h\le\sum_{k=1}^n\theta_k^h+\sum_{k=1}^n\beta_k^h, 0\le h<1,$$

and $1+3^{\frac1q}\le 4$, the desired result can be obtained as follows.

$$
\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\kappa_2-\kappa_1)^\zeta}[\mathfrak{I}^\zeta_{\kappa_1^+}F(\kappa_2)+\mathfrak{I}^\zeta_{\kappa_2^-}F(\kappa_1)]-\frac{1}{3}[2F(\kappa_1)-F\Big(\frac{\kappa_1+\kappa_2}{2}\Big)+2F(\kappa_2)]\right|\\
\le\ &\frac{(\kappa_2-\kappa_1)^2}{2^{\frac3q-1}(\zeta+1)}\left[\left(\int_0^{\frac12}\left(\frac{\zeta+4}{3}\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)^{\frac1p}+\left(\frac1\zeta\mathcal{B}\left(p+1,\frac{p+1}{\zeta},1-\left(\frac12\right)^\zeta\right)\right)^{\frac1p}\right]\\
&\times(|F''(\kappa_2)|+|F''(\kappa_1)|)
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $\mathcal{B}$ *represents the incomplete beta function, defined as:*

$$\mathcal{B}(k,y,r)=\int_0^r \epsilon^{k-1}(1-\epsilon)^{y-1}d\epsilon.$$

Remark 2.6. *If* $s=1$ *and* $\zeta=1$ *in Theorem 2.4, the following inequality is obtained:*

$$\begin{aligned}&\left|\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}F(\epsilon)d\epsilon-\frac{1}{3}[2F(\kappa_1)-F(\frac{\kappa_1+\kappa_2}{2})+2F(\kappa_2)]\right|\\ \leq\;&\frac{(\kappa_2-\kappa_1)^2}{2^{\frac{3}{q}}}\left[\left(\int_0^{\frac{1}{2}}\left(\frac{5}{3}\epsilon-\epsilon^2\right)^p d\epsilon\right)^{\frac{1}{p}}+\left(\mathcal{B}\left(p+1,p+1,\frac{1}{2}\right)\right)^{\frac{1}{p}}\right]\\ &\times(|F''(\kappa_2)|+|F''(\kappa_1)|)\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $\mathcal{B}$ *represents the incomplete beta function, defined as:*

$$\mathcal{B}(k,y,r)=\int_0^r \epsilon^{k-1}(1-\epsilon)^{y-1}d\epsilon.$$

Theorem 2.7. *Assume the conditions of Lemma 1.7 are satisfied, if* $\left|F''\right|^q$, *where* $q>1$, *is* $s-$*convex in the second sense on the interval* $[\kappa_1,\kappa_2]$, *then:*

$$\begin{aligned}&\left|\frac{\Gamma(\zeta+1)}{2(\kappa_2-\kappa_1)^\zeta}[\mathfrak{I}^\zeta_{\kappa_1^+}F(\kappa_2)+\mathfrak{I}^\zeta_{\kappa_2^-}F(\kappa_1)]-\frac{1}{3}[2F(\kappa_1)-F(\frac{\kappa_1+\kappa_2}{2})+2F(\kappa_2)]\right|\\ \leq\;&=\frac{(\kappa_2-\kappa_1)^2}{(\zeta+1)}\times\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)\right.\\ &\left.+\frac{1}{q}\left(\frac{1}{\zeta}\mathcal{B}\left(p+1,\frac{p+1}{\zeta},1-\left(\frac{1}{2}\right)^\zeta\right)\right)+\frac{\left|F''(\kappa_2)\right|^q+\left|F''(\kappa_1)\right|^q}{q(s+1)}\right]\end{aligned}$$

for $s\in(0,1],\zeta>0$ *where* $\frac{1}{p}+\frac{1}{q}=1$ *and* $\mathcal{B}$ *represents the incomplete beta function, defined as:*

$$\mathcal{B}(k,y,r)=\int_0^r \epsilon^{k-1}(1-\epsilon)^{y-1}d\epsilon.$$

Proof. On applying the modulus operation to Lemma 1.7, we obtain:

$$\begin{aligned}&\left|\frac{\Gamma(\zeta+1)}{2(\kappa_2-\kappa_1)^\zeta}[\mathfrak{I}^\zeta_{\kappa_1^+}F(\kappa_2)+\mathfrak{I}^\zeta_{\kappa_2^-}F(\kappa_1)]-\frac{1}{3}[2F(\kappa_1)-F\Big(\frac{\kappa_1+\kappa_2}{2}\Big)+2F(\kappa_2)]\right|\\ \leq\;&\frac{(\kappa_2-\kappa_1)^2}{2(\zeta+1)}\left[\int_0^{\frac{1}{2}}\left|\epsilon^{\zeta+1}-\frac{\zeta+4}{3}\epsilon\right|\left|F''(\epsilon\kappa_2+(1-\epsilon)\kappa_1)\right|d\epsilon\right.\\ &+\int_0^{\frac{1}{2}}\left|\epsilon^{\zeta+1}-\frac{\zeta+4}{3}\epsilon\right|\left|F''(\epsilon\kappa_1+(1-\epsilon)\kappa_2)\right|d\epsilon\\ &+\int_{\frac{1}{2}}^1\left|\epsilon^{\zeta+1}-\epsilon\right|\left|F''(\epsilon\kappa_2+(1-\epsilon)\kappa_1)\right|d\epsilon\\ &\left.+\int_{\frac{1}{2}}^1\left|\epsilon^{\zeta+1}-\epsilon\right|\left|F''(\epsilon\kappa_1+(1-\epsilon)\kappa_2)\right|d\epsilon\right].\end{aligned}$$

When we apply Young's inequality to the above inequality, we obtain the following result:

$$
\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\kappa_2-\kappa_1)^{\zeta}}[\Im^{\zeta}_{\kappa_1^+}F(\kappa_2)+\Im^{\zeta}_{\kappa_2^-}F(\kappa_1)]-\frac{1}{3}[2F(\kappa_1)-F\Big(\frac{\kappa_1+\kappa_2}{2}\Big)+2F(\kappa_2)]\right| \\
\leq\;& \frac{(\kappa_2-\kappa_1)^2}{2(\zeta+1)}\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left|\epsilon^{\zeta+1}-\frac{\zeta+4}{3}\epsilon\right|^p d\epsilon\right)+\frac{1}{q}\left(\int_0^{\frac{1}{2}}\left|F''(\epsilon\kappa_2+(1-\epsilon)\kappa_1)\right|^q d\epsilon\right)\right. \\
&+\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left|\epsilon^{\zeta+1}-\frac{\zeta+4}{3}\epsilon\right|^p d\epsilon\right)+\frac{1}{q}\left(\int_0^{\frac{1}{2}}\left|F''(\epsilon\kappa_1+(1-\epsilon)\kappa_2)\right|^q d\epsilon\right) \\
&+\frac{1}{p}\left(\int_{\frac{1}{2}}^1\left|\epsilon^{\zeta+1}-\epsilon\right|^p d\epsilon\right)+\frac{1}{q}\left(\int_{\frac{1}{2}}^1\left|F''(\epsilon\kappa_2+(1-\epsilon)\kappa_1)\right|^q d\epsilon\right) \\
&\left.+\frac{1}{p}\left(\int_{\frac{1}{2}}^1\left|\epsilon^{\zeta+1}-\epsilon\right|^p d\epsilon\right)+\frac{1}{q}\left(\int_{\frac{1}{2}}^1\left|F''(\epsilon\kappa_1+(1-\epsilon)\kappa_2)\right|^q d\epsilon\right)\right].
\end{aligned}
$$

Utilizing the $s-$convex in the second sense of $\left|F''\right|^q$, we obtain

$$
\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\kappa_2-\kappa_1)^{\zeta}}[\Im^{\zeta}_{\kappa_1^+}F(\kappa_2)+\Im^{\zeta}_{\kappa_2^-}F(\kappa_1)]-\frac{1}{3}[2F(\kappa_1)-F\Big(\frac{\kappa_1+\kappa_2}{2}\Big)+2F(\kappa_2)]\right| \\
\leq\;& \frac{(\kappa_2-\kappa_1)^2}{2(\zeta+1)}\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)+\frac{1}{q}\left(\int_0^{\frac{1}{2}}\epsilon^s\left|F''(\kappa_2)\right|^q+(1-\epsilon)^s\left|F''(\kappa_1)\right|^q d\epsilon\right)\right. \\
&+\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)+\frac{1}{q}\left(\int_0^{\frac{1}{2}}\epsilon^s\left|F''(\kappa_1)\right|^q+(1-\epsilon)^s\left|F''(\kappa_2)\right|^q d\epsilon\right) \\
&+\frac{1}{p}\left(\int_{\frac{1}{2}}^1\left(\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)+\frac{1}{q}\left(\int_{\frac{1}{2}}^1\epsilon^s\left|F''(\kappa_2)\right|^q+(1-\epsilon)^s\left|F''(\kappa_1)\right|^q d\epsilon\right) \\
&\left.+\frac{1}{p}\left(\int_{\frac{1}{2}}^1\left(\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)+\frac{1}{q}\left(\int_{\frac{1}{2}}^1\epsilon^s\left|F''(\kappa_1)\right|^q+(1-\epsilon)^s\left|F''(\kappa_2)\right|^q d\epsilon\right)\right] \\
=\;& \frac{(\kappa_2-\kappa_1)^2}{(\zeta+1)}\times\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)\right. \\
&\left.+\frac{1}{q}\left(\frac{1}{\zeta}\mathcal{B}\left(p+1,\frac{p+1}{\zeta},1-\left(\frac{1}{2}\right)^{\zeta}\right)\right)+\frac{\left|F''(\kappa_2)\right|^q+\left|F''(\kappa_1)\right|^q}{q(s+1)}\right]
\end{aligned}
$$

□

Corollary 2.8. *If $s=1$ in Theorem 2.7, the following inequality is obtained:*

$$
\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\kappa_2-\kappa_1)^{\zeta}}[\Im^{\zeta}_{\kappa_1^+}F(\kappa_2)+\Im^{\zeta}_{\kappa_2^-}F(\kappa_1)]-\frac{1}{3}[2F(\kappa_1)-F\Big(\frac{\kappa_1+\kappa_2}{2}\Big)+2F(\kappa_2)]\right| \\
\leq\;& \frac{(\kappa_2-\kappa_1)^2}{(\zeta+1)}\times\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\epsilon-\epsilon^{\zeta+1}\right)^p d\epsilon\right)\right. \\
&\left.+\frac{1}{q}\left(\frac{1}{\zeta}\mathcal{B}\left(p+1,\frac{p+1}{\zeta},1-\left(\frac{1}{2}\right)^{\zeta}\right)\right)+\frac{\left|F''(\kappa_2)\right|^q+\left|F''(\kappa_1)\right|^q}{2q}\right]
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $\mathcal{B}$ *represents the incomplete beta function, defined as:*

$$\mathcal{B}(k,y,r)=\int_0^r \epsilon^{k-1}(1-\epsilon)^{y-1}d\epsilon.$$

Corollary 2.9. *If* $s=1$ *and* $\zeta=1$ *in Theorem 2.7, the following inequality is obtained.*

$$\begin{aligned}
&\left|\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}F(\epsilon)d\epsilon-\frac{1}{3}\left[2F(\kappa_1)-F\Big(\frac{\kappa_1+\kappa_2}{2}\Big)+2F(\kappa_2)\right]\right|\\
\leq\quad&\frac{(\kappa_2-\kappa_1)^2}{2}\times\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{5}{3}\epsilon-\epsilon^2\right)^p d\epsilon\right)\right.\\
&\left.+\frac{1}{q}\left(\mathcal{B}\left(p+1,p+1,\frac{1}{2}\right)\right)+\frac{\left|F''(\kappa_2)\right|^q+\left|F''(\kappa_1)\right|^q}{2q}\right]
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $\mathcal{B}$ *represents the incomplete beta function, defined as:*

$$\mathcal{B}(k,y,r)=\int_0^r \epsilon^{k-1}(1-\epsilon)^{y-1}d\epsilon.$$

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