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Analytical approach and stability results for Caputo generalized proportional fractional differential equation involving two different fractional orders

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Abstract. The results presented in this research paper investigate the existence, uniqueness and stability for a new class of Caputo generalized proportional fractional differential equation involving two different fractional orders. We expose and highlight some of the characteristics of the generalized proportional fractional derivative. We established the existence and uniqueness results by employing Schaefer fixed point theorem and Banach contraction principle, and also we investigate different kinds of stability such as Ulam-Hyers and generalized Ulam-Hyers stability. As application, we provide an example to demonstrate our theoretical results.

1. Introduction

Fractional differential equations (FDEs) have recently captured the attention of many mathematicians, because it can effectively represent a variety of scientific phenomena, and has been proven to be effective in physics, mechanics, biology, chemistry, and control theory, and other domains for example, see [1, 5, 7, 11, 12, 14, 15, 20–28].

There are numerous approaches to define fractional integrals and derivatives, however the most well-known ones are the Riemann-Liouville and the Caputo fractional integrals and derivatives, this derivatives had been effectively employed to develop models of long-term memory behaves and the challenges that emerged in numerous scientific and technological fields [4, 8–10], for more details for Caputo fractional derivative, we direct readers to the papers [3, 18, 31]. In [16], Jarad et al. as the modification of the conformable derivatives [2, 17], the authors introduced a novel kind of fractional derivative, that called generalized proportional fractional (GPF) derivative. Anderson et al. [6] were able to handle with the fact that the fractional conformable derivative does not tends to the original function where the order ρ tends to 0, by defining the proportional derivative of order ρ by

$$D_\tau^\rho \mathfrak{h}(\tau) = \xi_1(\rho,\tau)\mathfrak{h}(\tau) + \xi_2(\rho,\tau)\mathfrak{h}'(\tau),$$

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where $\xi_1, \xi_2 : [0,1] \times \mathbb{R} \longrightarrow [0,\infty)$ are continuous functions such that, for all $\tau \in \mathbb{R}$,

$$\lim_{\rho\to 0^+} \xi_1(\rho,\tau) = 1, \lim_{\rho\to 0^+} \xi_2(\rho,\tau) = 0, \lim_{\rho\to 1^-} \xi_1(\rho,\tau) = 0, \lim_{\rho\to 1^-} \xi_2(\rho,\tau) = 1,$$

and $\xi_1(\rho,\tau) \neq 0, \xi_2(\rho,\tau) \neq 0, \rho \in [0,1]$, by this modifications, the new proportional derivative tends to the initial function when ρ tends to 0.
In the previous few decades, authors are interested to study this new fractional derivative for that and motivated by the mentioned works, in this paper, we combine their ideas to investigate the existence, uniqueness and stability results for the problem of the form

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}^{\sigma,\chi}_{\gamma^+}\Big({}^{\mathfrak{C}}\mathfrak{D}^{\varpi,\chi}_{\gamma^+} w(\tau)\Big) = \mathfrak{h}(\tau, w(\tau)),\ \tau \in \Lambda := [\gamma,\delta], \\ w(\gamma) = 0\ ,\ \ w(\delta) = \displaystyle\sum_{j=1}^{m} \nu_j \mathfrak{I}^{\alpha_j,\chi}_{\gamma^+} w(\varrho_j) + \sum_{i=1}^{n} \iota_i \mathfrak{I}^{\beta_i,\chi}_{\gamma^+} w(\kappa_i). \end{cases} \tag{1}$$

Where ${}^{\mathfrak{C}}\mathfrak{D}^{\sigma,\chi}_{\gamma^+}$ and ${}^{\mathfrak{C}}\mathfrak{D}^{\varpi,\chi}_{\gamma^+}$ are the Caputo generalized proportional fractional derivative of order σ, $0 < \sigma \leq 1$ and ϖ, $0 < \varpi \leq 1$, respectively. $\mathfrak{I}^{\alpha_j,\chi}_{\gamma^+}$ and $\mathfrak{I}^{\beta_i,\chi}_{\gamma^+}$ are the generalized proportional fractional integral of order α_j, $\beta_i > 0$, $\chi \in (0,1]$, $\gamma \geq 0$, $\nu_j, \iota_i \in \mathbb{R}$, $j = 1,...,m$, $i = 1,...,n$, $\gamma < \varrho_1 < ... < \varrho_n < \delta$, $\gamma < \kappa_1 < ... < \kappa_n < \delta$ and $\mathfrak{h} \in C\big(\Lambda \times \mathbb{R}, \mathbb{R}\big)$.

The rest of this paper is organized as follows : In section 2, we recall some notations, definitions, and lemmas from fractional calculus that will be used in our study. In section 3, we discuss the existence results for the problem (1) by making use of Schaefer's fixed-point theorem and to deal with the uniqueness result we use Banach's contraction principle. In section 4, we discuss the Ulam–Hyers stability and the generalized Ulam-Hyers stability of solutions for the problem (1). Finally, an example is provided to illustrate the main results.

2. Preliminaries

Definition 2.1. *[6] For $\chi \in (0,1]$. Let $\xi_1, \xi_2 : [0,1] \times \mathbb{R} \longrightarrow [0,\infty)$ be continuous functions such that, for all $\tau \in \mathbb{R}$,*

$$\lim_{\chi\to 0^+} \xi_1(\chi,\tau) = 1, \lim_{\chi\to 0^+} \xi_2(\chi,\tau) = 0, \lim_{\chi\to 1^-} \xi_1(\chi,\tau) = 0, \lim_{\chi\to 1^-} \xi_2(\chi,\tau) = 1,$$

and $\xi_1(\chi,\tau) \neq 0, \xi_2(\chi,\tau) \neq 0, \chi \in [0,1]$, then the proportional derivative of order χ of $\mathfrak{h}$ is defined by

$$D^{\chi}\mathfrak{h}(\tau) = \xi_1(\chi,\tau)\mathfrak{h}(\tau) + \xi_2(\chi,\tau)\mathfrak{h}'(\tau), \tag{2}$$

By setting $\xi_1(\chi,\tau) = \chi - 1$ and $\xi_2(\chi,\tau) = \chi$, (2) becomes

$$D^{\chi}\mathfrak{h}(\tau) = (1-\chi)\mathfrak{h}(\tau) + \chi\mathfrak{h}'(\tau) \tag{3}$$

Definition 2.2. *[16] For $\chi \in (0,1]$, $\sigma \in \mathbb{C}$ with $\mathcal{R}e(\sigma) > 0$. The* GPF *integral of order σ of a function $\mathfrak{h} \in L^1([\gamma,\delta],\mathbb{R})$ is defined by*

$$\mathfrak{I}^{\sigma,\chi}_{\gamma^+}\mathfrak{h}(\tau) = \frac{1}{\chi^{\sigma}\Gamma(\sigma)} \int_{\gamma}^{\tau} e^{\frac{\chi-1}{\chi}(\tau-s)} \Big(\tau - s\Big)^{\sigma-1} \mathfrak{h}(s)ds \tag{4}$$

where $\Gamma(.)$ is the gamma function.

Definition 2.3. *[16] For $\chi \in (0,1]$, $\sigma \in \mathbb{C}$ with $\mathcal{R}e(\sigma) > 0$. The Caputo* GPF *derivative of order σ of a function $\mathfrak{h} \in C^n([\gamma,\delta],\mathbb{R})$ is defined by*

$${}^{\mathfrak{C}}\mathfrak{D}^{\sigma,\chi}_{\gamma^+}\mathfrak{h}(\tau) = \frac{1}{\chi^{n-\sigma}\Gamma(n-\sigma)} \int_{\gamma}^{\tau} e^{\frac{\chi-1}{\chi}(\tau-s)} \Big(\tau - s\Big)^{n-\sigma-1} D^{n,\chi}\mathfrak{h}(s)ds, \tag{5}$$

where $n-1 < \sigma < n$, $n = [\mathcal{R}e(\sigma)] + 1$ where $[\mathcal{R}e(\sigma)]$ is the integer part of $\mathcal{R}e(\sigma)$, and $(D^{n,\chi}\mathfrak{h})(\tau) = (D^{\chi}\mathfrak{h}(\tau))^n$ with $D^{\chi}\mathfrak{h}(\tau) = (1-\chi)\mathfrak{h}(\tau) + \chi\mathfrak{h}'(\tau)$.

Lemma 2.4. *[16] For $\chi \in (0,1]$, $\sigma, \varpi \in \mathbb{C}$ with $\mathcal{R}e(\sigma) > 0$ and $\mathcal{R}e(\varpi) > 0$. If $\mathfrak{h} \in C([\gamma,\delta],\mathbb{R})$ then we have*

$$\mathfrak{I}_{\gamma^+}^{\sigma,\chi}\mathfrak{I}_{\gamma^+}^{\varpi,\chi}\mathfrak{h}(\tau) = \mathfrak{I}_{\gamma^+}^{\sigma+\varpi,\chi}\mathfrak{h}(\tau), \tau > \gamma. \tag{6}$$

Lemma 2.5. *[16] Let $\chi \in (0,1]$, $n \in \mathbb{N}^+$, $\mathfrak{h} \in L^1([\gamma,\delta],\mathbb{R})$ and $\mathfrak{I}_{\gamma^+}^{\sigma,\chi}\mathfrak{h} \in AC^n([\gamma,\delta],\mathbb{R})$. Then*

$$\mathfrak{I}_{\gamma^+}^{\sigma,\chi}({}^{\mathfrak{C}}\mathfrak{D}_{\gamma^+}^{\sigma,\chi}\mathfrak{h})(\tau) = \mathfrak{h}(\tau) - e^{\frac{\chi-1}{\chi}(\tau-\gamma)}\sum_{k=1}^{n}\left(\mathfrak{I}_{\gamma^+}^{k-\sigma,\chi}\mathfrak{h}\right)(\gamma)\frac{(\tau-\gamma)^{\sigma-k}}{\chi^{\sigma-k}\Gamma(\sigma-k+1)}. \tag{7}$$

Lemma 2.6. *[16] Let $\sigma, \varpi \in \mathbb{C}$ with $\mathcal{R}e(\sigma) > 0$ and $\mathcal{R}e(\upsilon) > 0$. Then for each $\chi \in (0,1]$ and $n = [\mathcal{R}e(\sigma)] + 1$, we have*

$$(i)\left(\mathfrak{I}_{\gamma^+}^{\sigma,\chi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}(\tau-\gamma)^{\upsilon-1}\right)(\tau) = \frac{\Gamma(\upsilon)}{\chi^{\sigma}\Gamma(\upsilon+\sigma)}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}(\tau-\gamma)^{\upsilon+\sigma-1}, \quad \mathcal{R}e(\sigma) > 0.$$

$$(ii)\left({}^{\mathfrak{C}}\mathfrak{D}_{\gamma^+}^{\sigma,\chi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}(\tau-\gamma)^{\upsilon-1}\right)(\tau) = \frac{\chi^{\sigma}\Gamma(\upsilon)}{\Gamma(\upsilon-\sigma)}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}(\tau-\gamma)^{\upsilon-\sigma-1}, \quad \mathcal{R}e(\sigma) > n.$$

$$(iii)\left({}^{\mathfrak{C}}\mathfrak{D}_{\gamma^+}^{\sigma,\chi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}(\tau-\gamma)^{k}\right)(\tau) = 0, \quad \mathcal{R}e(\sigma) > n, \ k = 0,1\ldots,n-1.$$

Theorem 2.7. *(Schaefers's fixed point theorem)[30, 32]*
Let $\mathcal{X}$ be a Banach space and $\mathcal{K} : \mathcal{X} \longrightarrow \mathcal{X}$, be a completely continuous operator. If the set $\Upsilon_{\varepsilon} = \{w \in \mathcal{X} \mid w = \varepsilon\mathcal{K}w \,;\, 0 \le \varepsilon \le 1\}$ is bounded, then $\mathcal{K}$ has at least a fixed point in $\mathcal{X}$.

Theorem 2.8. *(Banach's fixed point theorem)[13]*
Let $\mathcal{X}$ be a Banach space, C a closed subset of $\mathcal{X}$. Then any contraction mapping $\mathcal{K}$ from C into itself has a unique fixed point.

3. Main Result

Lemma 3.1. *Let $\gamma \ge 0$, $0 < \sigma \le 1$, $0 < \varpi \le 1$, and $\mathfrak{f} \in C(\Lambda,\mathbb{R})$. Then the function w is a solution of the following boundary value problem:*

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}_{\gamma^+}^{\sigma,\chi}\left({}^{\mathfrak{C}}\mathfrak{D}_{\gamma^+}^{\varpi,\chi}w(\tau)\right) = \mathfrak{f}(\tau), \quad \tau \in \Lambda := [\gamma,\delta], \\ w(\gamma) = 0\,, \ w(\delta) = \displaystyle\sum_{j=1}^{m}\nu_j\mathfrak{I}_{\gamma^+}^{\alpha_j,\chi}w(\varrho_j) + \sum_{i=1}^{n}\iota_i\mathfrak{I}_{\gamma^+}^{\beta_i,\chi}w(\kappa_i), \ \gamma < \varrho_j,\kappa_i < \delta, \end{cases} \tag{8}$$

if and only if

$$\begin{aligned} w(\tau) &= \mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi}\mathfrak{f}(\tau) + \frac{(\tau-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)} \\ &\times\left[\mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi}\mathfrak{f}(\delta) - \sum_{i=1}^{n}\iota_i\mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\beta_i,\chi}\mathfrak{f}(\kappa_i) - \sum_{j=1}^{m}\nu_j\mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\alpha_j,\chi}\mathfrak{f}(\varrho_j)\right] \\ &= \frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\tau}e^{\frac{\chi-1}{\chi}(\tau-s)}(\tau-s)^{\varpi+\sigma-1}\mathfrak{f}(s)ds + \frac{(\tau-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)} \\ &\times\left[\frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\delta}e^{\frac{\chi-1}{\chi}(\delta-s)}(\delta-s)^{\varpi+\sigma-1}\mathfrak{f}(s)ds\right. \end{aligned} \tag{9}$$

$$- \sum_{i=1}^{n} \iota_i \frac{1}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i)} \int_{\gamma}^{\kappa_i} e^{\frac{\chi-1}{\chi}(\kappa_i-s)}(\kappa_i-s)^{\varpi+\sigma+\beta_i-1}\mathfrak{f}(s)ds$$
$$- \sum_{j=1}^{n} \nu_j \frac{1}{\chi^{\varpi+\sigma+\alpha_j}\Gamma(\varpi+\sigma+\alpha_j)} \int_{\gamma}^{\varrho_j} e^{\frac{\chi-1}{\chi}(\varrho_j-s)}(\varrho_j-s)^{\varpi+\sigma+\alpha_j-1}\mathfrak{f}(s)ds\Bigg],$$

where

$$\Theta = \sum_{i=1}^{n} \iota_i \frac{(\kappa_i-\gamma)^{\varpi+\beta_i} e^{\frac{\chi-1}{\chi}(\kappa_i-\gamma)}}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i+1)} + \sum_{j=1}^{n} \nu_j \frac{(\varrho_j-\gamma)^{\varpi+\alpha_j} e^{\frac{\chi-1}{\chi}(\varrho_j-\gamma)}}{\chi^{\varpi+\alpha_j}\Gamma(\varpi+\alpha_j+1)}$$
$$- \frac{(\delta-\gamma)^{\varpi} e^{\frac{\chi-1}{\chi}(\delta-\gamma)}}{\chi^{\varpi}\Gamma(\varpi+1)} \neq 0. \tag{10}$$

Proof. Let w be the solution of problem (8). By applying the GPF integral of order σ, ϖ and Lemma 2.5 with Lemma 2.6, the first equation of problem (8) can be expressed as

$$w(\tau) = \mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi}\mathfrak{f}(\tau) + d_0 \frac{(\tau-\gamma)^{\varpi} e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{\chi^{\varpi}\Gamma(\varpi+1)} + d_1 e^{\frac{\chi-1}{\chi}(\tau-\gamma)}, \tag{11}$$

where d_0 and d_1 are constants. Next, by using the boundary condition $w(\gamma) = 0$ in (11) we obtain $d_1 = 0$ then

$$w(\tau) = \mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi}\mathfrak{f}(\tau) + d_0 \frac{(\tau-\gamma)^{\varpi} e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{\chi^{\varpi}\Gamma(\varpi+1)}, \tag{12}$$

next, by using the boundary condition $w(\delta) = \sum_{j=1}^{m} \nu_j \mathfrak{I}_{\gamma^+}^{\alpha_j,\chi} w(\varrho_j) + \sum_{i=1}^{n} \iota_i \mathfrak{I}_{\gamma^+}^{\beta_i,\chi} w(\kappa_i)$ in (12) we obtain

$$d_0 = \frac{1}{\Theta}\Bigg[\mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi}\mathfrak{f}(\delta) - \sum_{i=1}^{n} \iota_i \mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\beta_i,\chi}\mathfrak{f}(\kappa_i) - \sum_{j=1}^{m} \nu_j \mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\alpha_j,\chi}\mathfrak{f}(\varrho_j)\Bigg], \tag{13}$$

where Θ is given by (10). Substituting the value of d_0 in (12) we obtain

$$w(\tau) = \mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi}\mathfrak{f}(\tau) + \frac{(\tau-\gamma)^{\varpi} e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)}$$
$$\times \Bigg[\mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi}\mathfrak{f}(\delta) - \sum_{i=1}^{n} \iota_i \mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\beta_i,\chi}\mathfrak{f}(\kappa_i) - \sum_{j=1}^{m} \nu_j \mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\alpha_j,\chi}\mathfrak{f}(\varrho_j)\Bigg]. \tag{14}$$

The converse follows by direct computation that the solution $w(\tau)$ given by (9) satisfies problem (8) under the given boundary conditions. □

4. Existence and Uniqueness Results for Problem (1)

In this section, we present the existence and uniqueness results for the problem (1).

In view of Lemma 3.1 we define the operator $\mathcal{K} : C \longrightarrow C$ by

$$(\mathcal{K}w)(\tau) = \mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi}\mathfrak{h}(\tau, w(\tau)) + \frac{(\tau-\gamma)^{\varpi} e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)}$$

$$\times\left[\Im_{\gamma^+}^{\varpi+\sigma,\chi}\mathfrak{h}(\delta,w(\delta))-\sum_{i=1}^{n}\iota_i\Im_{\gamma^+}^{\varpi+\sigma+\beta_i,\chi}\mathfrak{h}(\kappa_i,w(\kappa_i))-\sum_{j=1}^{m}\nu_j\Im_{\gamma^+}^{\varpi+\sigma+\alpha_j,\chi}\mathfrak{h}(\varrho_j,w(\varrho_j))\right]$$

$$\begin{aligned}&=\frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\tau}e^{\frac{\chi-1}{\chi}(\tau-s)}(\tau-s)^{\varpi+\sigma-1}\mathfrak{h}(s,w(s))ds\\&+\frac{(\tau-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)}\Bigg[\frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\delta}e^{\frac{\chi-1}{\chi}(\delta-s)}(\delta-s)^{\varpi+\sigma-1}\mathfrak{h}(s,w(s))ds\\&-\sum_{i=1}^{n}\iota_i\frac{1}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i)}\int_{\gamma}^{\kappa_i}e^{\frac{\chi-1}{\chi}(\kappa_i-s)}(\kappa_i-s)^{\varpi+\sigma+\beta_i-1}\mathfrak{h}(s,w(s))ds\\&-\sum_{j=1}^{n}\nu_j\frac{1}{\chi^{\varpi+\sigma+\alpha_j}\Gamma(\varpi+\sigma+\alpha_j)}\int_{\gamma}^{\varrho_j}e^{\frac{\chi-1}{\chi}(\varrho_j-s)}(\varrho_j-s)^{\varpi+\sigma+\alpha_j-1}\mathfrak{h}(s,w(s))ds\Bigg],\end{aligned}\tag{15}$$

where $C=C([\gamma,\delta],\mathbb{R})$ denotes the Banach space of all continuous functions from $[\gamma,\delta]$ into $\mathbb{R}$ with the norm $\|w\|:=sup\{|w(\tau)|;\tau\in[\gamma,\delta]\}$.

Now, to deal with the existence and uniqueness results for the problem (1), we use the following notations to simplify the computations

$$\mathfrak{A}=\frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}.\tag{16}$$

$$\begin{aligned}\mathfrak{B}=\frac{(\delta-\gamma)^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\Bigg[&\frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}+\sum_{i=1}^{n}|\iota_i|\frac{(\kappa_i-\gamma)^{\varpi+\sigma+\beta_i}}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i+1)}\\&+\sum_{j=1}^{n}|\nu_j|\frac{(\varrho_j-\gamma)^{\varpi+\sigma+\alpha_j}}{\chi^{\varpi+\sigma+\alpha_j}\Gamma(\varpi+\sigma+\alpha_j+1)}\Bigg].\end{aligned}\tag{17}$$

We reveal the principal results under the following hypotheses.

(H_1): $|\mathfrak{h}(\tau,v)-\mathfrak{h}(\tau,w)|\leq\mathcal{L}|v-w|$; $\mathcal{L}>0$, for each $\tau\in[\gamma,\delta]$ and $v,w\in\mathbb{R}$.

(H_2): there exist non-negatives continuous functions ψ_1 and ψ_2, such that $|\mathfrak{h}(\tau,w)|\leq\psi_1(\tau)+\psi_2(\tau)|w|$, $(\tau,w)\in[\gamma,\delta]\times\mathbb{R}$, with $\|\psi_1\|=sup_{\tau\in[\gamma,\delta]}|\psi_1(\tau)|$, $\|\psi_2\|=sup_{\tau\in[\gamma,\delta]}|\psi_2(\tau)|$.

(H_3): $\|\psi_2\|\left(\mathfrak{A}+\mathfrak{B}\right)<1$, where $\mathfrak{A}$ and $\mathfrak{B}$ are given by (16) and (17).

4.1. Existence result based on Schaefers fixed point theorem

Theorem 4.1. *Assume that* (H_2) *and* (H_3) *are satisfied. Then, there exists at least one solution for the problem* (1) *on* $[\gamma,\delta]$.

As a means of demonstrating Theorem 4.1, we will prove that the operator $\mathcal{K}$ satisfies the conditions of Theorem 2.7 (Schaefer's fixed point theorem).

Proof. Consider the operator $\mathcal{K}$ defined in (15), we will show that $\mathcal{K}$ is a completely continuous operator.
Step 1: $\mathcal{K}$ is continuous.
By using the continuity of function $\mathfrak{h}$, it follows that $\mathcal{K}$ is continuous.
Step 2: $\mathcal{K}$ is bounded.
Let $\mathcal{N}$ a bounded set, such that $\mathcal{N}\subset\mathcal{B}_\rho$, we will show that $\mathcal{K}(\mathcal{N})$ is bounded for all $w\in\mathcal{N}$. For each $\tau\in\Lambda$ and $w\in\mathcal{N}$, we have

$$
\begin{aligned}
\left|(\mathcal{K}w)(\tau)\right| &\leq \mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi}|\mathfrak{h}(\tau,w(\tau))| + \frac{(\tau-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\
&\times\left[\mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi}|\mathfrak{h}(\delta,w(\delta))| + \sum_{i=1}^{n}\iota_i\mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\beta_i,\chi}|\mathfrak{h}(\kappa_i,w(\kappa_i))| + \sum_{j=1}^{m}\nu_j\mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\alpha_j,\chi}|\mathfrak{h}(\varrho_j,w(\varrho_j))|\right], \\
&\leq \frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\tau} e^{\frac{\chi-1}{\chi}(\tau-s)}\left(\tau-s\right)^{\varpi+\sigma-1}|\mathfrak{h}(s,w(s))|ds \\
&+ \frac{(\tau-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\left[\frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\delta} e^{\frac{\chi-1}{\chi}(\delta-s)}\left(\delta-s\right)^{\varpi+\sigma-1}|\mathfrak{h}(s,w(s))|ds\right. \\
&+ \sum_{i=1}^{n}\iota_i\frac{1}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i)}\int_{\gamma}^{\kappa_i} e^{\frac{\chi-1}{\chi}(\kappa_i-s)}\left(\kappa_i-s\right)^{\varpi+\sigma+\beta_i-1}|\mathfrak{h}(s,w(s))|ds \\
&\left.+ \sum_{j=1}^{m}\nu_j\frac{1}{\chi^{\varpi+\sigma+\alpha_j}\Gamma(\varpi+\sigma+\alpha_j)}\int_{\gamma}^{\varrho_j} e^{\frac{\chi-1}{\chi}(\varrho_j-s)}\left(\varrho_j-s\right)^{\varpi+\sigma+\alpha_j-1}|\mathfrak{h}(s,w(s))|ds\right],
\end{aligned}
$$

using (H_2) and the property $e^{\frac{\chi-1}{\chi}(t-s)} \leq 1$ for $\gamma \leq s < t < \tau \leq \delta$ it leads to

$$
\begin{aligned}
\left|(\mathcal{K}w)(\tau)\right| &\leq \frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\left(\|\psi_1\| + |w|\|\psi_1\|\right) + \frac{(\delta-\gamma)^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\
&\times\left[\frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\left(\|\psi_1\| + |w|\|\psi_1\|\right)\right. \\
&+ \sum_{i=1}^{n}|\iota_i|\frac{(\kappa_i-\gamma)^{\varpi+\sigma+\beta_i}}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i+1)}\left(\|\psi_1\| + |w|\|\psi_1\|\right) \\
&\left.+ \sum_{j=1}^{m}|\nu_j|\frac{(\varrho_j-\gamma)^{\varpi+\sigma+\alpha_j}}{\chi^{\varpi+\sigma+\alpha_j}\Gamma(\varpi+\sigma+\alpha_j+1)}\left(\|\psi_1\| + |w|\|\psi_1\|\right)\right] \\
&\leq \left(\|\psi_1\| + \rho\|\psi_1\|\right)\frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)} + \frac{(\delta-\gamma)^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\
&\times\left[\frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)} + \sum_{i=1}^{n}|\iota_i|\frac{(\kappa_i-\gamma)^{\varpi+\sigma+\beta_i}}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i+1)}\right. \\
&\left.+ \sum_{j=1}^{m}|\nu_j|\frac{(\varrho_j-\gamma)^{\varpi+\sigma+\alpha_j}}{\chi^{\varpi+\sigma+\alpha_j}\Gamma(\varpi+\sigma+\alpha_j+1)}\right]\left(\|\psi_1\| + \rho\|\psi_1\|\right) \\
&\leq \left(\|\psi_1\| + \rho\|\psi_1\|\right)\left(\mathfrak{A}+\mathfrak{B}\right),
\end{aligned}
$$

then $\|\mathcal{K}w\| \leq \left(\|\psi_1\| + \rho\|\psi_1\|\right)\left(\mathfrak{A}+\mathfrak{B}\right)$, where $\mathfrak{A}$ and $\mathfrak{B}$ are given by (16) and (17).

Step 3: $\mathcal{K}$ is equicontinuous.

Let $\tau_1, \tau_2 \in [\gamma, \delta]$ such that $\tau_1 < \tau_2$, and for each $w \in \mathcal{N}$ we obtain

$$
\begin{aligned}
&|(\mathcal{K}w)(\tau_2) - (\mathcal{K}w)(\tau_1)| \\
&\quad\leq \frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\tau_1}\left[(\tau_2-s)^{\varpi+\sigma-1} - (\tau_1-s)^{\varpi+\sigma-1}\right]|\mathfrak{h}(s,w(s))|ds \\
&\quad+ \frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\varpi+\sigma-1}|\mathfrak{h}(s,w(s))|ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\left|(\tau_2-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau_2-\gamma)}-(\tau_1-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau_1-\gamma)}\right|}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\\
&\times\Bigg[\frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\delta}\left(\delta-s\right)^{\varpi+\sigma-1}|\mathfrak{h}(s,w(s))|ds\\
&+\sum_{i=1}^{n}\iota_i\frac{1}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i)}\int_{\gamma}^{\kappa_i}\left(\kappa_i-s\right)^{\varpi+\sigma+\beta_i-1}|\mathfrak{h}(s,w(s))|ds\\
&+\sum_{j=1}^{m}\nu_j\frac{1}{\chi^{\varpi+\sigma+\alpha_j}\Gamma(\varpi+\sigma+\alpha_j)}\int_{\gamma}^{\varrho_j}\left(\varrho_j-s\right)^{\varpi+\sigma+\alpha_j-1}|\mathfrak{h}(s,w(s))|ds\Bigg],\\
&\leq\frac{\left(\|\psi_1\|+\rho\|\psi_2\|\right)}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\tau_1}\left[(\tau_2-s)^{\varpi+\sigma-1}-(\tau_1-s)^{\varpi+\sigma-1}\right]ds\\
&+\frac{\left(\|\psi_1\|+\rho\|\psi_2\|\right)}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\varpi+\sigma-1}ds\\
&+\frac{\left|(\tau_2-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\Phi(\tau_2)-\Phi(\gamma))}-(\tau_1-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau_1-\gamma)}\right|}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\\
&\times\Bigg[\frac{\left(\|\psi_1\|+\rho\|\psi_2\|\right)}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\delta}\left(\delta-s\right)^{\varpi+\sigma-1}ds\\
&+\sum_{i=1}^{n}\iota_i\frac{\left(\|\psi_1\|+\rho\|\psi_2\|\right)}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i)}\int_{\gamma}^{\kappa_i}\left(\kappa_i-s\right)^{\varpi+\sigma+\beta_i-1}ds\\
&+\sum_{j=1}^{m}\nu_j\frac{\left(\|\psi_1\|+\rho\|\psi_2\|\right)}{\chi^{\varpi+\sigma+\alpha_j}\Gamma(\varpi+\sigma+\alpha_j)}\int_{\gamma}^{\varrho_j}\left(\varrho_j-s\right)^{\varpi+\sigma+\alpha_j-1}ds\Bigg],\\
&\leq\frac{\left(\|\psi_1\|+\rho\|\psi_2\|\right)}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\Big(2(\tau_2-\tau_1)^{\varpi+\sigma}+\left|(\tau_2-\gamma)^{\varpi+\sigma}-(\tau_1-\gamma)^{\varpi+\sigma}\right|\Big)\\
&+\frac{\left|(\tau_2-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau_2-\gamma)}-(\tau_1-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau_1-\gamma)}\right|}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\\
&\times\Bigg[\frac{\left(\|\psi_1\|+\rho\|\psi_2\|\right)}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\left(\delta-\gamma\right)^{\varpi+\sigma}\\
&+\sum_{i=1}^{n}\iota_i\frac{\left(\|\psi_1\|+\rho\|\psi_2\|\right)}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i+1)}\left(\kappa_i-\gamma\right)^{\varpi+\sigma+\beta_i}\\
&+\sum_{j=1}^{m}\nu_j\frac{\left(\|\psi_1\|+\rho\|\psi_2\|\right)}{\chi^{\varpi+\sigma+\alpha_j}\Gamma(\varpi+\sigma+\alpha_j+1)}\left(\varrho_j-\gamma\right)^{\varpi+\sigma+\alpha_j}\Bigg],
\end{aligned}
$$

the right hand side tends to zero as $\tau_2 \longrightarrow \tau_1$, indepently of $w \in \mathcal{N}$ which leads to $|(\mathcal{K}w)(\tau_2) - (\mathcal{K}w)(\tau_1)| \longrightarrow 0$ as $\tau_2 \longrightarrow \tau_1$ this implies that $\mathcal{K}(\mathcal{N})$ is equicontinuous. From step 1, step 2 and step 3 it follows by using the Arzelà-Ascoli theorem that the operator $\mathcal{K}$ is relatively compact, as consequence the operator $\mathcal{K}$ is completely continuous.

Step 4: The set $\Upsilon_\varepsilon = \{w \in C(\Lambda, \mathbb{R}) \mid w = \varepsilon\mathcal{K}w\ ;\ 0 \leq \varepsilon \leq 1\}$ is bounded.

We are going to show that the set Υ_ε is bounded. For all $w \in \Upsilon_\varepsilon$, by using (H_2), we have

$$\begin{aligned}
\left|w(\tau)\right| &\le \mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi}|\mathfrak{h}(\tau,w(\tau))| + \frac{(\tau-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\
&\times\Bigg[\mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi}|\mathfrak{h}(\delta,w(\delta))| + \sum_{i=1}^{n}\iota_i\mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\beta_i,\chi}|\mathfrak{h}(\kappa_i,w(\kappa_i))| + \sum_{j=1}^{m}\nu_j\mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\alpha_j,\chi}|\mathfrak{h}(\varrho_j,w(\varrho_j))|\Bigg],\\
&\le \frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\tau}e^{\frac{\chi-1}{\chi}(\tau-s)}\Big(\tau-s\Big)^{\varpi+\sigma-1}|\mathfrak{h}(s,w(s))|ds\\
&+\frac{(\tau-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\Bigg[\frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\delta}e^{\frac{\chi-1}{\chi}(\delta-s)}\Big(\delta-s\Big)^{\varpi+\sigma-1}|\mathfrak{h}(s,w(s))|ds\\
&+\sum_{i=1}^{n}\iota_i\frac{1}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i)}\int_{\gamma}^{\kappa_i}e^{\frac{\chi-1}{\chi}(\kappa_i-s)}\Big(\kappa_i-s\Big)^{\varpi+\sigma+\beta_i-1}|\mathfrak{h}(s,w(s))|ds\\
&+\sum_{j=1}^{n}\nu_j\frac{1}{\chi^{\varpi+\sigma+\alpha_j}\Gamma(\varpi+\sigma+\alpha_j)}\int_{\gamma}^{\varrho_j}e^{\frac{\chi-1}{\chi}(\varrho_j-s)}\Big(\varrho_j-s\Big)^{\varpi+\sigma+\alpha_j-1}|\mathfrak{h}(s,w(s))|ds\Bigg],
\end{aligned}$$

using (H_2) and the property $e^{\frac{\chi-1}{\chi}(t-s)} \le 1$ for $\gamma \le s < t < \tau \le \delta$ it leads to

$$\begin{aligned}
\left|w(\tau)\right| &\le \frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\Big(\|\psi_1\|+|w|\|\psi_2\|\Big)+\frac{(\delta-\gamma)^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\\
&\times\Bigg[\frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\Big(\|\psi_1\|+|w|\|\psi_2\|\Big)\\
&+\sum_{i=1}^{n}|\iota_i|\frac{(\kappa_i-\gamma)^{\varpi+\sigma+\beta_i}}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i+1)}\Big(\|\psi_1\|+|w|\|\psi_2\|\Big)\\
&+\sum_{j=1}^{n}|\nu_j|\frac{(\varrho_j-\gamma)^{\varpi+\sigma+\alpha_j}}{\chi^{\varpi+\sigma+\alpha_j}\Gamma(\varpi+\sigma+\alpha_j+1)}\Big(\|\psi_1\|+|w|\|\psi_2\|\Big)\Bigg]\\
&\le\Big(\|\psi_1\|+|w|\|\psi_2\|\Big)\frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}+\frac{(\delta-\gamma)^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\\
&\times\Bigg[\frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}+\sum_{i=1}^{n}|\iota_i|\frac{(\kappa_i-\gamma)^{\varpi+\sigma+\beta_i}}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i+1)}\\
&+\sum_{j=1}^{n}|\nu_j|\frac{(\varrho_j-\gamma)^{\varpi+\sigma+\alpha_j}}{\chi^{\varpi+\sigma+\alpha_j}\Gamma(\varpi+\sigma+\alpha_j+1)}\Bigg]\Big(\|\psi_1\|+|w|\|\psi_2\|\Big)\\
&\le\Big(\|\psi_1\|+|w|\|\psi_2\|\Big)\Big(\mathfrak{A}+\mathfrak{B}\Big),\\
&\le\Big(\mathfrak{A}+\mathfrak{B}\Big)\|\psi_1\|+|w|\|\psi_2\|\Big(\mathfrak{A}+\mathfrak{B}\Big),
\end{aligned}$$

Where $\mathfrak{A}$ and $\mathfrak{B}$ are given by (16) and (17). Thus, we have

$$\|w\| \le \frac{\Big(\mathfrak{A}+\mathfrak{B}\Big)\|\psi_1\|}{1-\|\psi_2\|\Big(\mathfrak{A}+\mathfrak{B}\Big)}.$$

This proves that the set Υ_ε is bounded in $C(\Lambda,\mathbb{R})$, by using Theorem 2.7, $\mathcal{K}$ has at least one fixed point which is the solution of the problem (1). □

4.2. Uniqueness result based on Banach fixed point theorem

The second existence and uniqueness result will be established using Banach fixed point theorem.

Theorem 4.2. *Assume that* (H_1) *is verified. If* $\mathcal{L}\Big(\mathfrak{A}+\mathfrak{B}\Big)<1$, *where* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are respectively given by* (16) *and* (17), *then the problem* (1) *has a unique solution on* $[\gamma,\delta]$.

Proof. Consider the operator $\mathcal{K}$ defined in (15). The problem (1) is then can be transformed into a fixed point problem $w=\mathcal{K}w$. By using Banach contraction principle we will show that $\mathcal{K}$ has a unique fixed point.
We set $sup_{\tau\in[\gamma,\delta]}=|\mathfrak{h}(\tau,0)|=\mathcal{M}<\infty$, and choose $\rho>0$ such that

$$\rho\geq\frac{\mathcal{M}\Big(\mathfrak{A}+\mathfrak{B}\Big)}{1-\mathcal{L}\Big(\mathfrak{A}+\mathfrak{B}\Big)},\tag{18}$$

$\mathcal{B}_\rho=\{w\in C([\gamma,\delta],\mathbb{R});\|w\|\leq\rho)\}$, where $\mathfrak{A},\mathfrak{B}$ are respectively given by (16) and (17).
Step 1: We show that $\mathcal{K}\mathcal{B}_\rho\subset\mathcal{B}_\rho$.
For any $w\in\mathcal{B}_\rho$ we have

$$\begin{aligned}|\mathfrak{h}(\tau,w(\tau))|&\leq|\mathfrak{h}(\tau,w(\tau))-\mathfrak{h}(\tau,0)|+|\mathfrak{h}(\tau,0)|\\&\leq\mathcal{L}|w(\tau)|+\mathcal{M}\\&\leq\mathcal{L}|w|+\mathcal{M},\end{aligned}$$

then we have

$$\begin{aligned}\Big|(\mathcal{K}w)(\tau)\Big|&\leq\mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi}|\mathfrak{h}(\tau,w(\tau))|+\frac{(\tau-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\\&\times\Bigg[\mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi}|\mathfrak{h}(\delta,w(\delta))|+\sum_{i=1}^{n}\iota_i\mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\beta_i,\chi}|\mathfrak{h}(\kappa_i,w(\kappa_i))|+\sum_{j=1}^{m}\nu_j\mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\alpha_j,\chi}|\mathfrak{h}(\varrho_j,w(\varrho_j))|\Bigg],\\&\leq\frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_\gamma^\tau e^{\frac{\chi-1}{\chi}(\tau-s)}\Big(\tau-s\Big)^{\varpi+\sigma-1}|\mathfrak{h}(s,w(s))|ds\\&+\frac{(\tau-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\Bigg[\frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_\gamma^\delta e^{\frac{\chi-1}{\chi}(\delta-s)}\Big(\delta-s\Big)^{\varpi+\sigma-1}|\mathfrak{h}(s,w(s))|ds\\&+\sum_{i=1}^{n}\iota_i\frac{1}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i)}\int_\gamma^{\kappa_i}e^{\frac{\chi-1}{\chi}(\kappa_i-s)}\Big(\kappa_i-s\Big)^{\varpi+\sigma+\beta_i-1}|\mathfrak{h}(s,w(s))|ds\\&+\sum_{j=1}^{n}\nu_j\frac{1}{\chi^{\varpi+\sigma+\alpha_j}\Gamma(\varpi+\sigma+\alpha_j)}\int_\gamma^{\varrho_j}e^{\frac{\chi-1}{\chi}(\varrho_j-s)}\Big(\varrho_j-s\Big)^{\varpi+\sigma+\alpha_j-1}|\mathfrak{h}(s,w(s))|ds\Bigg],\end{aligned}$$

using (H_1) and the property $e^{\frac{\chi-1}{\chi}(t-s)}\leq1$ for $\gamma\leq s<t<\tau\leq\delta$ it leads to

$$\begin{aligned}\Big|(\mathcal{K}w)(\tau)\Big|\leq&\frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\Big(\mathcal{L}|w|+\mathcal{M}\Big)+\frac{(\delta-\gamma)^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\\&\times\Bigg[\frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\Big(\mathcal{L}|w|+\mathcal{M}\Big)\\&+\sum_{i=1}^{n}|\iota_i|\frac{(\kappa_i-\gamma)^{\varpi+\sigma+\beta_i}}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i+1)}\Big(\mathcal{L}|w|+\mathcal{M}\Big)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}|\nu_j|\frac{(\varrho_j-\gamma)^{\varpi+\sigma+\alpha_j}}{\chi^{\varpi+\sigma+\alpha_j}\Gamma(\varpi+\sigma+\alpha_j+1)}\Big(\mathcal{L}|w|+\mathcal{M}\Big)\Bigg]\\
&\leq\Big(\mathcal{L}|w|+\mathcal{M}\Big)\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}+\frac{(\delta-\gamma)^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\\
&\times\Bigg[\frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}+\sum_{i=1}^{n}|\iota_i|\frac{(\kappa_i-\gamma)^{\varpi+\sigma+\beta_i}}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i+1)}\\
&+\sum_{j=1}^{n}|\nu_j|\frac{(\varrho_j-\gamma)^{\varpi+\sigma+\alpha_j}}{\chi^{\varpi+\sigma+\alpha_j}\Gamma(\varpi+\sigma+\alpha_j+1)}\Bigg]\Big(\mathcal{L}|w|+\mathcal{M}\Big)\\
&\leq\Big(\mathcal{L}|w|+\mathcal{M}\Big)\Big(\mathfrak{A}+\mathfrak{B}\Big),\\
&\leq\Big(\mathcal{L}\rho+\mathcal{M}\Big)\Big(\mathfrak{A}+\mathfrak{B}\Big),\\
&\leq\rho,
\end{aligned}
$$

which implies that $\mathcal{K}\mathcal{B}_\rho\subset\mathcal{B}_\rho$. Where $\mathfrak{A},\mathfrak{B}$ are respectively given by (16) and (17).

Step 2: We show that the operator $\mathcal{K}$ is a contraction.
For any $v,w\in C$, and for $\tau\in[\gamma,\delta]$, we have

$$
\begin{aligned}
&\Big|(\mathcal{K}v)(\tau)-(\mathcal{K}w)(\tau)\Big|\leq\mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi}|\mathfrak{h}(\tau,v(\tau))-\mathfrak{h}(\tau,w(\tau))|+\frac{(\tau-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)}\\
&\times\Bigg[\mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi}|\mathfrak{h}(\delta,v(\delta))-\mathfrak{h}(\delta,w(\delta))|+\sum_{i=1}^{n}\iota_i\mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\beta_i,\chi}|\mathfrak{h}(\kappa_i,v(\kappa_i))-\mathfrak{h}(\kappa_i,w(\kappa_i))|\\
&+\sum_{j=1}^{m}\nu_j\mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\alpha_j,\chi}|\mathfrak{h}(\varrho_j,v(\varrho_j))-\mathfrak{h}(\varrho_j,w(\varrho_j))|\Bigg],\\
&\leq\frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\tau}e^{\frac{\chi-1}{\chi}(\tau-s)}\Big(\tau-s\Big)^{\varpi+\sigma-1}|\mathfrak{h}(s,v(s))-\mathfrak{h}(s,w(s))|ds\\
&+\frac{(\tau-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\Bigg[\frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\delta}e^{\frac{\chi-1}{\chi}(\delta-s)}\Big(\delta-s\Big)^{\varpi+\sigma-1}|\mathfrak{h}(s,v(s))-\mathfrak{h}(s,w(s))|ds\\
&+\sum_{i=1}^{n}\iota_i\frac{1}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i)}\int_{\gamma}^{\kappa_i}e^{\frac{\chi-1}{\chi}(\kappa_i-s)}\Big(\kappa_i-s\Big)^{\varpi+\sigma+\beta_i-1}|\mathfrak{h}(s,v(s))-\mathfrak{h}(s,w(s))|ds\\
&+\sum_{j=1}^{n}\nu_j\frac{1}{\chi^{\varpi+\sigma+\alpha_j}\Gamma(\varpi+\sigma+\alpha_j)}\int_{\gamma}^{\varrho_j}e^{\frac{\chi-1}{\chi}(\varrho_j-s)}\Big(\varrho_j-s\Big)^{\varpi+\sigma+\alpha_j-1}|\mathfrak{h}(s,v(s))-\mathfrak{h}(s,w(s))|ds\Bigg],\\
&\leq\Big(\mathcal{L}|v-w|\Big)\frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}+\frac{(\delta-\gamma)^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\\
&\times\Bigg[\frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}+\sum_{i=1}^{n}|\iota_i|\frac{(\kappa_i-\gamma)^{\varpi+\sigma+\beta_i}}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i+1)}\\
&+\sum_{j=1}^{n}|\nu_j|\frac{(\varrho_j-\gamma)^{\varpi+\sigma+\alpha_j}}{\chi^{\varpi+\sigma+\alpha_j}\Gamma(\varpi+\sigma+\alpha_j+1)}\Bigg]\Big(\mathcal{L}|v-w|\Big),\\
&\leq\mathcal{L}\Big(\mathfrak{A}+\mathfrak{B}\Big)|v-w|,
\end{aligned}
$$

which implies, $\|\mathcal{K}v-\mathcal{K}w\|\leq\mathcal{L}\Big(\mathfrak{A}+\mathfrak{B}\Big)\|v-w\|$. As $\mathcal{L}\Big(\mathfrak{A}+\mathfrak{B}\Big)<1$, then $\mathcal{K}$ is a contraction. Therefore, by

Banach fixed-point theorem, the operator $\mathcal{K}$ has a unique fixed point which is indeed the unique solution of problem (1). □

5. Ulam–Hyers and generalized Ulam–Hyers stability analysis

In this section, we are interested to study the Ulam–Hyers (U-H) and the generalized Ulam–Hyers (G-U-H) stability of problem (1).

Let $\varepsilon > 0$, we consider the following inequality

$$\left| {}^{\mathfrak{C}}\mathfrak{D}^{\sigma,\chi}_{\gamma^+}\left({}^{\mathfrak{C}}\mathfrak{D}^{\varpi,\chi}_{\gamma^+}\tilde{w}(\tau)\right) - \mathfrak{h}(\tau, \tilde{w}(\tau))\right| \leq \varepsilon, \quad \tau \in \Lambda := [\gamma, \delta], \tag{19}$$

Definition 5.1. *[29] The problem* (1) *is U-H stable if there exists $\lambda > 0$, such that for each $\varepsilon > 0$ and for each solution $\tilde{w} \in C$ of inequality* (19)*, there exists $w, \in C$ solution of the problem* (1) *complying with*

$$\|\tilde{w} - w\| \leq \lambda\varepsilon. \tag{20}$$

Definition 5.2. *[29] The problem* (1) *is G-U-H stable if there exists $\varphi \in C$ with $\varphi(0) = 0$, such that for each $\varepsilon > 0$ and for each solution $\tilde{w} \in C$ of inequality* (19)*, there exists $w \in C$ solution of the problem* (1) *complying with.*

$$\|\tilde{w} - w\| \leq \varphi(\varepsilon). \tag{21}$$

Remark 5.3. *A function $\tilde{w} \in C$ is a solution of inequalities* (19) *if and only if there exists a function $\mathfrak{g} \in C$ such that*

i- $|\mathfrak{g}(\tau)| \leq \varepsilon$,

ii- for $\tau \in [\gamma, \delta]$:

$$ {}^{\mathfrak{C}}\mathfrak{D}^{\sigma,\chi}_{\gamma^+}\left({}^{\mathfrak{C}}\mathfrak{D}^{\varpi,\chi}_{\gamma^+}\tilde{w}(\tau)\right) = \mathfrak{h}(\tau, \tilde{w}(\tau)) + \mathfrak{g}(\tau). \tag{22}$$

To simplify the computations, we use the following notations:

$$\begin{aligned}\Omega_1 = \mathcal{L}\Bigg\{ &\frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)} + \frac{(\delta-\gamma)^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\Bigg[\sum_{i=1}^{n}|\iota_i|\frac{(\kappa_i-\gamma)^{\varpi+\sigma+\beta_i}}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i+1)} \\ &+ \frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)} + \sum_{j=1}^{n}|\nu_j|\frac{(\varrho_j-\gamma)^{\varpi+\sigma+\alpha_j}}{\chi^{\varpi+\sigma+\alpha_j}\Gamma(\varpi+\sigma+\alpha_j+1)}\Bigg]\Bigg\},\end{aligned} \tag{23}$$

$$\Omega_2 = \frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}, \tag{24}$$

Theorem 5.4. *Assume that (H_1) hold, if $\Omega_1 < 1$ then the problem* (1) *is Ulam–Hyers stable on $[\gamma, \delta]$ and consequently is generalized Ulam–Hyers stable, where Ω_1 is given by* (23).

Proof. Let $\varepsilon > 0$, and $\tilde{w} \in C$ satisfies inequality (19), and $w \in C$ be the unique solution of the problem (1) with the conditions $\tilde{w}(\gamma) = w(\gamma)$, $\tilde{w}(\delta) = w(\delta)$, then by Lemma 2.5, we obtain

$$\begin{aligned}w(\tau) = {}&\mathfrak{I}^{\varpi+\sigma,\chi}_{\gamma^+}\mathfrak{h}(\tau, w(\tau)) + \frac{(\tau-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)} \\ &\times\left[\mathfrak{I}^{\varpi+\sigma,\chi}_{\gamma^+}\mathfrak{h}(\delta, w(\delta)) - \sum_{i=1}^{n}\iota_i\mathfrak{I}^{\varpi+\sigma+\beta_i,\chi}_{\gamma^+}\mathfrak{h}(\kappa_i, w(\kappa_i)) - \sum_{j=1}^{m}\nu_j\mathfrak{I}^{\varpi+\sigma+\alpha_j,\chi}_{\gamma^+}\mathfrak{h}(\varrho_j, w(\varrho_j))\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)} \int_{\gamma}^{\tau} e^{\frac{\chi-1}{\chi}(\tau-s)} (\tau-s)^{\varpi+\sigma-1} \mathfrak{h}(s, w(s)) ds \\
&+ \frac{(\tau-\gamma)^{\varpi} e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{\Theta \chi^{\varpi}\Gamma(\varpi+1)} \\
&\times \Bigg[\frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)} \int_{\gamma}^{\delta} e^{\frac{\chi-1}{\chi}(\delta-s)} (\delta-s)^{\varpi+\sigma-1} \mathfrak{h}(s, w(s)) ds \\
&- \sum_{i=1}^{n} \iota_i \frac{1}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i)} \int_{\gamma}^{\kappa_i} e^{\frac{\chi-1}{\chi}(\kappa_i-s)} (\kappa_i-s)^{\varpi+\sigma+\beta_i-1} \mathfrak{h}(s, w(s)) ds \\
&- \sum_{j=1}^{n} \nu_j \frac{1}{\chi^{\varpi+\sigma+\alpha_j}\Gamma(\varpi+\sigma+\alpha_j)} \int_{\gamma}^{\varrho_j} e^{\frac{\chi-1}{\chi}(\varrho_j-s)} (\varrho_j-s)^{\varpi+\sigma+\alpha_j-1} \mathfrak{h}(s, w(s)) ds \Bigg],
\end{aligned} \tag{25}
$$

Since, $\tilde{w} \in C$ satisfies inequality (19) by using Remark 5.3 we have

$$
\begin{cases}
{}^{\mathfrak{C}}\mathfrak{D}_{\gamma^+}^{\sigma,\chi}\left({}^{\mathfrak{C}}\mathfrak{D}_{\gamma^+}^{\varpi,\chi} \tilde{w}(\tau)\right) = \mathfrak{h}(\tau, \tilde{w}(\tau)) + \mathfrak{g}(\tau), \ \tau \in \Lambda := [\gamma, \delta], \\
\tilde{w}(\gamma) = w(\gamma) \quad , \quad \tilde{w}(\delta) = w(\delta),
\end{cases} \tag{26}
$$

then by Lemma 2.5, we obtain

$$
\begin{aligned}
\tilde{w}(\tau) &= \mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi} \mathfrak{h}(\tau, \tilde{w}(\tau)) + \frac{(\tau-\gamma)^{\varpi} e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{\Theta \chi^{\varpi}\Gamma(\varpi+1)} \\
&\times \left[\mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi} \mathfrak{h}(\delta, \tilde{w}(\delta)) - \sum_{i=1}^{n} \iota_i \mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\beta_i,\chi} \mathfrak{h}(\kappa_i, \tilde{w}(\kappa_i)) - \sum_{j=1}^{m} \nu_j \mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\alpha_j,\chi} \mathfrak{h}(\varrho_j, \tilde{w}(\varrho_j)) \right] \\
&= \frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)} \int_{\gamma}^{\tau} e^{\frac{\chi-1}{\chi}(\tau-s)} (\tau-s)^{\varpi+\sigma-1} \mathfrak{h}(s, \tilde{w}(s)) ds \\
&+ \frac{(\tau-\gamma)^{\varpi} e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{\Theta \chi^{\varpi}\Gamma(\varpi+1)} \Bigg[\frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)} \int_{\gamma}^{\delta} e^{\frac{\chi-1}{\chi}(\delta-s)} (\delta-s)^{\varpi+\sigma-1} \mathfrak{h}(s, \tilde{w}(s)) ds \\
&- \sum_{i=1}^{n} \iota_i \frac{1}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i)} \int_{\gamma}^{\kappa_i} e^{\frac{\chi-1}{\chi}(\kappa_i-s)} (\kappa_i-s)^{\varpi+\sigma+\beta_i-1} \mathfrak{h}(s, \tilde{w}(s)) ds \\
&- \sum_{j=1}^{n} \nu_j \frac{1}{\chi^{\varpi+\sigma+\alpha_j}\Gamma(\varpi+\sigma+\alpha_j)} \int_{\gamma}^{\varrho_j} e^{\frac{\chi-1}{\chi}(\varrho_j-s)} (\varrho_j-s)^{\varpi+\sigma+\alpha_j-1} \mathfrak{h}(s, \tilde{w}(s)) ds \Bigg] \\
&+ \frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)} \int_{\gamma}^{\tau} e^{\frac{\chi-1}{\chi}(\tau-s)} (\tau-s)^{\varpi+\sigma-1} \mathfrak{g}(s) ds,
\end{aligned} \tag{27}
$$

for each $\tau \in [\gamma, \delta]$, we have

$$
\begin{aligned}
&\left| \tilde{w}(\tau) - w(\tau) \right| \\
&\leq \mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi} \left| \mathfrak{h}(\tau, \tilde{w}(\tau)) - \mathfrak{h}(\tau, w(\tau)) \right| + \frac{(\tau-\gamma)^{\varpi} e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{|\Theta| \chi^{\varpi}\Gamma(\varpi+1)} \\
&\times \Bigg[\mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi} \left| \mathfrak{h}(\delta, \tilde{w}(\delta)) - \mathfrak{h}(\delta, w(\delta)) \right| + \sum_{i=1}^{n} |\iota_i| \mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\beta_i,\chi} \left| \mathfrak{h}(\kappa_i, \tilde{w}(\kappa_i)) - \mathfrak{h}(\kappa_i, w(\kappa_i)) \right| \\
&+ \sum_{j=1}^{m} \nu_j \mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\alpha_j,\chi} |\mathfrak{h}(\varrho_j, \tilde{w}(\varrho_j) - \mathfrak{h}(\varrho_j, \tilde{w}(\varrho_j)| \Bigg] + \mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi} |\mathfrak{g}(\tau)|,
\end{aligned}
$$

using (H_1), the property $e^{\frac{\chi-1}{\chi}(t-s)} \leq 1$ for $\gamma \leq s < t < \tau \leq \delta$ and Remark 5.3 leads to

$$\begin{aligned}
\|\tilde{w} - w\| &\leq \frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\mathcal{L}\|\tilde{w} - w\| \\
&+ \frac{(\delta-\gamma)^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\Bigg[\sum_{i=1}^{n}|\iota_i|\frac{(\kappa_i-\gamma)^{\varpi+\sigma+\beta_i}}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i+1)}\mathcal{L}\|\tilde{w} - w\| \\
&+ \frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\mathcal{L}\|\tilde{w} - w\| + \sum_{j=1}^{n}|\nu_j|\frac{(\varrho_j-\gamma)^{\varpi+\sigma+\alpha_j}}{\chi^{\varpi+\sigma+\alpha_j}\Gamma(\varpi+\sigma+\alpha_j+1)}\mathcal{L}\|\tilde{w} - w\|\Bigg] \\
&+ \frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\varepsilon, \\
&\leq \|\tilde{w} - w\|\mathcal{L}\Bigg\{\frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)} + \frac{(\delta-\gamma)^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\
&\times\Bigg[\sum_{i=1}^{n}|\iota_i|\frac{(\kappa_i-\gamma)^{\varpi+\sigma+\beta_i}}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i+1)} + \frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)} \\
&+ \sum_{j=1}^{n}|\nu_j|\frac{(\varrho_j-\gamma)^{\varpi+\sigma+\alpha_j}}{\chi^{\varpi+\sigma+\alpha_j}\Gamma(\varpi+\sigma+\alpha_j+1)}\Bigg]\Bigg\} + \frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\varepsilon, \\
&\leq \|\tilde{w} - w\|\Omega_1 + \Omega_2\varepsilon, \\
&\leq \Omega_1\|\tilde{w} - w\| + \Omega_2\varepsilon, \\
&\leq \frac{\Omega_2}{1-\Omega_1}\varepsilon,
\end{aligned}$$

which implies,

$$\|\tilde{w} - w\| \leq \frac{\Omega_2}{1-\Omega_1}\varepsilon. \tag{28}$$

By setting $\lambda = \dfrac{\Omega_2}{1-\Omega_1}$, where Ω_1 and Ω_2 are given by (23) and (24), we obtain

$$\|\tilde{w} - w\| \leq \lambda\varepsilon. \tag{29}$$

This proves that the problem (1), is U-H stable.
consequently, by setting $\varphi(\varepsilon) = \lambda\varepsilon$ with $\varphi(0) = 0$ we get

$$\|\tilde{w} - w\| \leq \varphi(\varepsilon). \tag{30}$$

This shows that the problem (1) is G-U-H stable. □

6. Example

Consider the following problem

$$\begin{cases}
{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\frac{3}{7},\frac{1}{2}}\left({}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\frac{5}{7},\frac{1}{2}}w(\tau)\right) = \dfrac{e^{-\tau}}{4+e^{\tau}}\Big(\dfrac{|w(\tau)|}{1+|w(\tau)|}\Big),\ \tau \in \Lambda := [0,1], \\
w(0) = 0\,,\ \ w(1) = \dfrac{1}{5}\mathfrak{I}^{\frac{2}{7},\frac{1}{2}}w(\dfrac{1}{3}) + \dfrac{2}{5}\mathfrak{I}^{\frac{3}{7},\frac{1}{2}}w(\dfrac{2}{3}) + \dfrac{3}{5}\mathfrak{I}^{\frac{4}{7},\frac{1}{2}}w(\dfrac{4}{3}) \\
\qquad + \dfrac{3}{7}\mathfrak{I}^{\frac{2}{3},\frac{1}{2}}w(\dfrac{1}{4}) + \dfrac{5}{7}\mathfrak{I}^{\frac{4}{3},\frac{1}{2}}w(\dfrac{3}{4}).
\end{cases} \tag{31}$$

Where $\sigma = \frac{3}{7}$, $\varpi = \frac{5}{7}$, $\chi = \frac{1}{2}$, $\gamma = 0$, $\delta = 1$, $\Lambda = [0,1]$, $m = 3$, $\nu_1 = \frac{1}{5}$, $\nu_2 = \frac{2}{5}$, $\nu_3 = \frac{3}{5}$, $\alpha_1 = \frac{2}{7}$, $\alpha_2 = \frac{3}{7}$, $\alpha_3 = \frac{4}{7}$, $\varrho_1 = \frac{1}{3}$, $\varrho_2 = \frac{2}{3}$, $\varrho_3 = \frac{4}{3}$, $n = 2$, $\iota_1 = \frac{3}{7}$, $\iota_2 = \frac{5}{7}$, $\beta_1 = \frac{4}{3}$, $\beta_2 = \frac{4}{3}$, $\kappa_1 = \frac{1}{4}$ and $\kappa_2 = \frac{3}{4}$.

For $(\tau, w) \in [0,1] \times \mathbb{R}_+$, we define $\mathfrak{h}(\tau, w) = \frac{e^{-\tau}}{6+e^{\tau}}\Big(\frac{w}{1+w}\Big)$. $\mathfrak{h}$ is a continuous function, furthermore for every $\tau \in [0,1]$ and $v, w \in \mathbb{R}_+$we have

$$
\begin{aligned}
|\mathfrak{h}(\tau, v) - \mathfrak{h}(\tau, w)| &\leq \Big|\frac{1}{6+e^{\tau}}\Big|\Big|\frac{v-w}{(1+v)(1+w)}\Big| \\
&\leq \frac{1}{7}|v-w|.
\end{aligned}
$$

By setting $\mathcal{L} = \frac{1}{7} > 0$ the hypotheses (H_1) holds. Next by using the data given above, we get : $|\Theta| = 0.674256$, $\mathfrak{A} = 0.013065$, $\mathfrak{B} = 0.756731$.
Then

$$
\mathcal{L}\Big(\mathfrak{A} + \mathfrak{B}\Big) = 0,142857 \times \Big(0.013065 + 0.756731\Big) \simeq 0.109971 < 1.
$$

The problem (31) satisfies all the hypothesis of Theorem 4.2, thus, the problem (31) has a unique solution on $[0,1]$. Additionally, $\Omega_1 = 0.643688 < 1$. Hence, using Theorem 5.4, the problem (31) is both Ulam–Hyers and also generalized Ulam–Hyers stable on $[0,1]$.

7. Conclusion

In this paper, we have studied and investigated the existence, uniqueness and stability results for a new class of Caputo generalized proportional fractional differential equation involving two different fractional orders. The novelty of the considered problem is that it has been studied under Caputo generalized proportional fractional derivative, which is more general than the works based on the well-known Caputo fractional derivative. In this work we established the existence and uniqueness results for our problem, by using a standard fixed point theorems (Schaefer fixed point theorem and Banach contraction principle) and also we examined the stability of our problem by using Ulam-Hyers and generalized Ulam-Hyers stability. Finally a numerical example is presented to clarify the obtained results.

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Data Availability

The data used to support the finding of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflict of interest.

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