Filomat 39:10 (2025), 3227–3237
https://doi.org/10.2298/FIL2510227R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

η-*Einstein Hopf real hypersurfaces in the complex space forms

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Abstract. Einstein's metrics and their generalizations have attracted the attention of Mathematicians due to their applications in physics and other natural sciences. The generalization of Einstein metrics is Ricci solitons, η-Einstein metrics, pseudo-Einstein metrics, and Miao-Tam critical metrics. Given the established non-existence of Einstein real hypersurfaces in a non-flat complex space form $\hat{M}_n(c)$ [2, 13], motivated our investigation into the properties of η-*Einstein real hypersurface in $\hat{M}_n(c)$.

In this paper, we examine the η-*Einstein Hopf real hypersurface in the complex space form. We prove that there exist η-*Einstein Hopf real hypersurfaces.

1. Introduction

In 1982, Cecil and Ryan established the non-existence of Einstein real hypersurfaces in $\mathbb{C}P^n$ for $n \geq 3$ [2], and in 1985, Montiel gave an analogous result in $\mathbb{C}H^n$ for $n \geq 3$ [13]. In 2002, Hamada gave a classification of Hopf *Einstein real hypersurfaces of $\hat{M}_n(c)$ [7]. In 2010, Ivey and Ryan [9] provided an updated classification of the work of Hamada [7] in $\mathbb{C}P^n$ and $\mathbb{C}H^n$.

The generalization of Einstein metrics is Ricci solitons, η-Einstein metrics, pseudo-Einstein metrics, and Miao-Tam critical metrics. Pérez and Suh [17] proved that no Hopf real hypersurface in $\mathbb{C}P^n$ $(n \geq 3)$, possess Lie $\mathbb{D}$-parallel structure Jacobi operators. Given the non-existence of Einstein real hypersurfaces in $\hat{M}_n(c)$ [2, 13], Chen [3] investigated real hypersurfaces endowed with Miao-Tam critical metrics of complex space forms in 2018 and obtained some existence/non-existence results.

Takagi [24] initially classified homogeneous real hypersurfaces in $\mathbb{C}P^n$ into types A_1, A_2, B, C, D, and E, and the classification of such types of hypersurfaces in $\mathbb{C}P^n$ was completed by Kimura [12]. However, Montiel [13] classified real hypersurfaces in $\mathbb{C}H^n$ into types A_0, A_1, A_2, and B, and for complete classification please see Cecil and Ryan [1]. Suh [21] introduced the idea of pseudo-Einstein real hypersurfaces in the complex quadric and provided a complete classification of these hypersurfaces. Moreover, Pérez and López [16] investigated real hypersurfaces in $\mathbb{C}P^n$ with some conditions on the shape operator.

2020 *Mathematics Subject Classification*. Primary 53C25; Secondary 53D15.
Keywords. Gradient Ricci soliton, η-*Einstein, complex space form, contact hypersurface, Hopf real hypersurface.
Received: 11 January 2025; Accepted: 12 January 2025
Communicated by Ljubica Velimirović
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On the other hand, Hamilton pioneered the study of manifolds with positive curvature by employing an efficient approach of Ricci flow [8]. Ricci solitons are a special class of solutions to this flow. They are often referred to as η-Einstein metrics in physics and have significant applications within this field. A Riemannian metric g is called a Ricci soliton, if

$$\frac{1}{2}\mathcal{L}_X g + Ric = \nu g, \tag{1}$$

where X denotes the potential vector field, Ric is the Ricci curvature tensor, and ν is a real constant. If $X = \nabla f$, $f \in C^{\infty}(M)$, then (1) is called a gradient Ricci soliton. Cho and Kimura [4, 5] proved the non-existence of gradient Ricci soliton in Hopf or a non-Hopf hypersurface of $\hat{M}_n(c)$.

The Ricci curvature tensor S and Ricci operator Q is defined as [10]

$$S(U, V) = g(QU, V) = \sum_{i=1}^{2n-1} g(R(e_i, U)V, e_i), \tag{2}$$

$\forall\ U, V \in TM$, where R is a Riemann curvature tensor and e_i are local orthonormal vector fields on M^{2n-1}.

Tachibana [23] introduced the concept of the $*$-Ricci tensor within the framework of almost Hermitian manifolds. Subsequently, Hamada [7] extended this notion to real hypersurfaces in $\hat{M}_n(c \neq 0)$ and defined it on an almost contact metric manifold M as follows :

$$S^*(U, V) = g(Q^*U, V) = \frac{1}{2}\text{trace}(Z \mapsto R(U, \phi V)\phi Z),\ \text{for any}\ U, V, Z \in TM, \tag{3}$$

where S^* is the $*$-Ricci tensor, Q^* is a $*$-Ricci operator, and ϕ is a $(1, 1)$-tensor field.

Kaimakamis and Panagiotidou [10] defined the $*$-Ricci soliton on a Riemannian manifold (M, g) as

$$\frac{1}{2}\mathcal{L}_X g + Ric^* = \nu g, \tag{4}$$

where Ric^* is the $*$-Ricci tensor, ν is a real constant, and X is a potential field. The $*$-Ricci soliton of real hypersurfaces in $\hat{M}_n(c \neq 0)$ with potential structure vector field ξ was explored by them in [10].

In addition to the usual Ricci tensor, Riemannian manifolds equipped with additional structures (almost Hermitian, almost contact, etc.) allow other possible contractions of the curvature tensor. The *-Ricci tensor (which is obtained by contracting the curvature tensor jointly with the complex structure) coincides with the usual Ricci tensor for Kahler manifolds. However, Ric^* is essentially different for more general, almost Hermitian manifolds. In fact, Ric^* is not necessarily symmetric in the generic situation, but it is symmetric for manifolds admitting the η-*Einstein metric (5) defined below as we get $Ric^*(U, V) = Ric^*(V, U)$.

We define, (M, g, f, m) as $(m-)$ η-*Einstein if

$$Ric^* + Hess f - \frac{1}{m} df \otimes df = \nu g, \tag{5}$$

where (M, g) is a Riemannian manifold, $m \in \mathbb{Z}^+$, and $f \in C^{\infty}(M)$. $Hess\, f$ represents the Hessian of f. If m approaches ∞ then (5) yields the gradient $*$-Ricci soliton. An η-*Einstein metric reduces to an *Einstein metric when f is constant. Furthermore, an η-*Einstein metric is classified as expanding ($\nu < 0$), steady ($\nu = 0$), or shrinking ($\nu > 0$). Wang [25] examined $\mathbb{D}$-recurrent $*$-Ricci tensor on real hypersurface in $\mathbb{C}H^2(c \neq 0)$. Recently, the authors [6, 18–20] examined Ricci solitons, $*$-Ricci solitons, and generalization of *Einstein metrics on almost contact metric manifolds. Also, Suh [22] investigated existence/non-existence conditions for Ricci solitons and pseudo-Einstein real hypersurfaces in the complex hyperbolic quadric.

In light of the fact that Einstein Hopf real hypersurfaces do not exist in $\hat{M}_n(c \neq 0)$, this paper investigates the existence/non-existence of η-*Einstein Hopf real hypersurfaces in the complex space forms. The key findings of this work are as follows:

Theorem 1.1. *Let M be a Hopf real hypersurface in $\hat{M}_n(c \neq 0)$, with $A\xi = 0$. Then, M does not admit an η-*Einstein metric.*

Theorem 1.2. *Let M be a Hopf real hypersurface in $\hat{M}_n(c \neq 0)$, with $A\xi = \alpha\xi, \alpha \neq 0$. Then, M admits an η-*Einstein metric in $\mathbb{C}H^n(n \geq 2)$ only. Moreover, $2n = \coth^2 r$ and M is locally congruent to a geodesic hypersphere either with steady *Ricci flat metric, or with shrinking η-*Einstein metric.*

Theorem 1.1 establishes the non-existence of η-*Einstein metrics for Hopf real hypersurfaces in the complex space forms under the condition that $A\xi = 0$. In contrast, Theorem 1.2 demonstrates that for Hopf real hypersurfaces with $A\xi = \alpha\xi$ and $\alpha \neq 0$, an η-*Einstein metric exists only in the context of the complex hyperbolic space $\mathbb{C}H^n$. Furthermore, it is shown that such a hypersurface is congruent to a geodesic hypersphere, either with a steady *Ricci flat metric or a shrinking η-*Einstein metric. These theorems together highlight a distinct difference in behavior depending on the structure of the shape operator A.

As a direct consequence of these results, we deduce the non-existence of η-*Einstein Hopf real hypersurfaces in the complex projective space $\mathbb{C}P^n$. This conclusion is formalized in the following corollary:

Corollary 1.3. *There do not exist η-*Einstein Hopf real hypersurfaces in $\mathbb{C}P^n$, $n \geq 2$.*

For the real Hopf hypersurfaces in $\mathbb{C}^n$ with an η-*Einstein metric we have:

Theorem 1.4. *Let M be a complete contact hypersurface in $\mathbb{C}^n$. Then M admits an η-*Einstein metric. Moreover, M is locally congruent*

(*i*) *either to a generalized cylinder $S^{n-1} \times \mathbb{R}^n$ such that either M admits steady *Ricci flat metric or admits steady η-*Einstein metric and $\xi\xi f = \frac{(\xi f)^2}{m}$,*

(*ii*) *or to $\mathbb{R}^{2n-2} \times S^1$ such that either M admits steady *Ricci flat metric or admits steady η-*Einstein metric and $\xi\xi f = \frac{(\xi f)^2}{m}$.*

Theorem 1.4 considers real Hopf hypersurfaces in Euclidean complex space $\mathbb{C}^n$ and establishes the existence of η-*Einstein metrics for complete contact hypersurfaces. The result characterizes such hypersurfaces as locally congruent to generalized cylinders or products of spheres and Euclidean spaces, with either steady *Ricci flat metrics or steady η-*Einstein metrics. The corollaries below extend this result by classifying specific cases of real hypersurfaces, depending on whether $A\xi = \alpha\xi$ or $A\xi = 0$, and further describe the geometric structure of these hypersurfaces:

Corollary 1.5. *Let M be a complete real hypersurface with $A\xi = \alpha\xi$, $\alpha \neq 0$ in $\mathbb{C}^n$ complying with (5). Then M is locally congruent to $\mathbb{R}^{2n-2} \times S^1$ such that either M is steady η-*Einstein metric or M is steady *Ricci flat metric and $\xi\xi f = \frac{(\xi f)^2}{m}$.*

Corollary 1.6. *Let M be a complete real hypersurface with $A\xi = 0$ of $\mathbb{C}^n$ complying with (5). Then, M is locally congruent either to $S^{n-1} \times \mathbb{R}^n$ such that either M admits steady *Ricci flat metric or admits steady η-*Einstein metric or locally congruent to $\mathbb{R}^{2n-1}$ and $\xi\xi f = \frac{(\xi f)^2}{m}$.*

Our approach to obtain the results of this paper is as follows: Using η-*Einstein condition, Gauss equation, and Codazzi equation, we express relationships in terms of $(1,1)$ tensor field ϕ, shape operator A, and principal curvature α of the structure vector field ξ for Hopf real hypersurfaces in $\hat{M}_n(c)$ in a simple form (see Lemma 3.3). Next, using Lemma 3.3 and eigenvalues of homogeneous real hypersurfaces in $\mathbb{C}P^n$ [12, 15, 24] and $\mathbb{C}H^n$ [1, 13, 15] after suitable changes according to constant holomorphic sectional curvatures, we analyze all the possible cases of Hopf real hypersurfaces in $\hat{M}_n(c \neq 0)$ to obtain the nature of η-*Einstein metric (see Theorem 1.2). Similarly, we obtain the results for real hypersurfaces in $\mathbb{C}^n$ admitting η-*Einstein metric. In the results, γ, δ, λ, μ denote the principal curvatures of holomorphic distribution and α denotes the principal curvature of ξ distribution.

The paper is organised as follows: In section 2, we present essential definitions and fundamental results useful in subsequent sections. Section 3 is devoted to the study of the existence of Hopf real hypersurfaces admitting η-*Einstein metric in $\mathbb{C}P^n$ and $\mathbb{C}H^n$. In section 4, we study real hypersurfaces in $\mathbb{C}^n$ which satisfy (5).

2. Preliminaries and some basic results

A complex space form is defined as a Kahler manifold with constant holomorphic sectional curvature c. A complete, simply connected complex space form is analytically isometric to a complex Euclidean space ($\mathbb{C}^n$), a complex projective space ($\mathbb{C}P^n$) and a complex hyperbolic space ($\mathbb{C}H^n$) if $c = 0$, $c > 0$, and $c < 0$ respectively.

Let $\hat{M}_n(c)$ be a non-flat complex space form with complex structure J and M denote a real hypersurface without boundary immersed in $\hat{M}_n(c)$. For any vector field U tangent to M, we define

$$JU = \phi U + \eta(U)N, \quad \xi = -JN, \tag{6}$$

where ϕU is the tangential part of JU, ϕ is a $(1,1)$ tensor field, N is a locally defined unit normal vector, ξ is the unit structure vector field, η is a 1-form on M.

Further, we have

$$\phi^2 U = -U + \eta(U)\xi, \;\; \eta(\xi) = 1, \;\; \phi \circ \xi = 0, \;\; \eta \circ \phi = 0, \tag{7}$$

$$g(U,\xi) = \eta(U), \qquad g(\phi U, \phi V) = g(U,V) - \eta(U)\eta(V), \tag{8}$$

where $U, V \in TM$ and g is the Riemannian metric induced on M from $\hat{g}$ of ambient space. From (7) and (8), (ϕ, η, ξ, g) defines an almost contact metric structure on M.

If $\hat{\nabla}$ and ∇ denote the linear connections on $\hat{M}$ and M, respectively, we have the Gauss and Weingarten formulae

$$\hat{\nabla}_U V = \nabla_U V + g(AU, V)N, \qquad \hat{\nabla}_U N = -AU, \tag{9}$$

respectively, where A is the shape operator of M. Also, for the almost contact metric structure on M, we have

$$\nabla_U \xi = \phi A U, \qquad (\nabla_U \phi)V = \eta(V)AU - g(AU, V)\xi. \tag{10}$$

Let R denotes the Riemann curvature tensor field of M. Then, we have the Gauss and Codazzi equations

$$\begin{aligned} R(U,V)Z \;\; = \;\; & \frac{c}{4}\Big(g(V,Z)U - g(U,Z)V + g(\phi V, Z)\phi U - g(\phi U, Z)\phi V \\ & +2g(U, \phi V)\phi Z\Big) + g(AV, Z)AU - g(AU, Z)AV, \end{aligned} \tag{11}$$

$$(\nabla_U A)V - (\nabla_V A)U = \frac{c}{4}\Big(\eta(U)\phi V - \eta(V)\phi U - 2g(\phi U, V)\xi\Big), \tag{12}$$

respectively, for any $U, V, Z \in TM$.

From (2) and (11), the Ricci operator Q on M is given by:

$$QU = \frac{c}{4}\Big((2n+1)U - 3\eta(U)\xi\Big) + hAU - A^2 U, \tag{13}$$

where h denotes the trace of A.

From (3), (7), and (11), the $*$-Ricci operator Q^* on M is given by:

$$Q^* U = \frac{nc}{2}\Big(U - \eta(U)\xi\Big) - (\phi A)^2 U. \tag{14}$$

Using (14) in (13), we find that

$$QU = Q^* U + \frac{(2n-3)c}{4}\eta(U)\xi + \frac{c}{4}U + hAU - A^2 U + (\phi A)^2 U. \tag{15}$$

3. η-*Einstein Hopf real hypersurfaces

Let M be a Hopf hypersurface in $\hat{M}_n(c)$. Then the shape operator A of M complies

$$A\xi = \alpha\xi, \tag{16}$$

where α is a constant (cf. [15], Theorem 2.1).

Differentiating (16) with respect to U, we find

$$(\nabla_U A)\xi = \alpha\phi AU - A\phi AU. \tag{17}$$

Using (17) in (12), we get

$$(\nabla_\xi A)U = \alpha\phi AU - A\phi AU + \frac{c}{4}\phi U, \tag{18}$$

for any $U \in TM$. Because of self-adjointness of $\nabla_\xi A$, the antisymmetry part of (18), gives

$$2A\phi AU - \frac{c}{2}\phi U = \alpha(A\phi + \phi A)U. \tag{19}$$

The decomposition of tangent bundle TM is

$$TM = \langle\xi\rangle \oplus \mathfrak{D}, \tag{20}$$

where $\mathfrak{D} = \{U \in TM\colon U \perp \xi\}$. Since $A\xi = \alpha\xi$, hence $A\mathfrak{D} \subset \mathfrak{D}$; so, we can choose $U \in \mathfrak{D}$ such that

$$AU = \mu_i U, \quad i = 1, 2, \ldots, n-1 \tag{21}$$

for some function $\mu_i \in C^\infty(M)$. Then from (19), we get

$$(\alpha - 2\mu_i)A\phi U = -(\mu_i\alpha + \frac{c}{2})\phi U. \tag{22}$$

Now, suppose that

$$A\phi U = \lambda_i\phi U, \quad i = 1, 2, \ldots, n-1 \tag{23}$$

for $U \in \mathfrak{D}$, where λ_i are eigenvalue of A corresponding to ϕU. Then from (22), we have

$$(2\mu_i - \alpha)(2\lambda_i - \alpha) = \alpha^2 + c. \tag{24}$$

The following Lemma is crucial for proving our results.

Lemma 3.1. *[11] Let M be a real hypersurface of a complex space form $\hat{M}_n(c)$. If $\phi A + A\phi = 0$, then $c = 0$.*

Lemma 3.2. *Let M be a real hypersurface with η-*Einstein metric in $\hat{M}_n(c)$. Then, Riemann curvature tensor R of M satisfies*

$$R(U,V)\nabla f = (\nabla_V Q^*)U - (\nabla_U Q^*)V + \frac{1}{m}\Big(U(f)Q^*V - V(f)Q^*U\Big) - \frac{\nu}{m}\Big(U(f)V - V(f)U\Big), \tag{25}$$

for any $U, V \in TM$.

Proof. Equation (5) yields

$$Q^*V + \nabla_V\nabla f = \nu V + \frac{1}{m}(Vf)\nabla f. \tag{26}$$

Using $R(U,V) + \nabla_{[U,V]} = \nabla_U\nabla_V - \nabla_V\nabla_U$, and (26) repeatedly, we obtain (25). □

Lemma 3.3. *Let M be an η-*Einstein Hopf real hypersurface in $\hat{M}_n(c)$. Then,*

$$\Big(\frac{nc}{2}+\frac{c}{4}\Big)g((\phi A+A\phi)U,V)+\frac{\alpha}{2}\Big(g(AV,(\phi A+A\phi)U)-g(AU,(\phi A+A\phi)V)\Big) = \Big(\frac{c}{4}-\frac{\nu}{m}\Big)\Big(V(f)\eta(U)-U(f)\eta(V)\Big)+\alpha\Big(AV(f)\eta(U)-AU(f)\eta(V)\Big), \quad (27)$$

$$\Big(\frac{\alpha^2}{2}+\frac{nc}{2}+\frac{c}{4}\Big)(A\phi+\phi A)+\frac{\alpha}{2}(\phi A^2+A^2\phi)+\frac{\alpha c\phi}{4}=0. \quad (28)$$

Proof. Replacing Z by ∇f in (11), we obtain

$$R(U,V)\nabla f=\frac{c}{4}\Big(\phi V(f)\phi U-\phi U(f)\phi V+V(f)U-U(f)V+2g(U,\phi V)\phi\nabla f\Big)+AV(f)AU-AU(f)AV. \quad (29)$$

Utilising (29) in (25), we find

$$(\nabla_V Q^*)U-(\nabla_U Q^*)V+\frac{1}{m}\Big(U(f)Q^*V-V(f)Q^*U\Big)=\Big(\frac{c}{4}-\frac{\nu}{m}\Big)\Big(V(f)U-U(f)V\Big) + \frac{c}{4}\Big(\phi V(f)\phi U-\phi U(f)\phi V+2g(U,\phi V)\phi\nabla f\Big)+AV(f)AU-AU(f)AV. \quad (30)$$

Differentiating (14) along V on TM, we get

$$\begin{aligned}(\nabla_V Q^*)U &= \frac{nc}{2}\Big(-g(\nabla_V\xi,U)\xi-\eta(U)\nabla_V\xi\Big)-(\nabla_V\phi)A\phi AU \\ &- \phi(\nabla_V A)\phi AU-\phi A(\nabla_V\phi)AU-\phi A\phi(\nabla_V A)U.\end{aligned} \quad (31)$$

This gives

$$\begin{aligned}(\nabla_V Q^*)U-(\nabla_U Q^*)V &= \frac{nc}{2}\Big(g(\nabla_U\xi,V)\xi+\eta(V)\nabla_U\xi\Big)+\frac{nc}{2}\Big(-g(\nabla_V\xi,U)\xi \\ &- \eta(U)\nabla_V\xi\Big)-(\nabla_V\phi)A\phi AU+(\nabla_U\phi)A\phi AV \\ &+ \phi\Big((\nabla_U A)\phi AV-(\nabla_V A)\phi AU\Big)+\phi A\Big((\nabla_U\phi)AV \\ &- (\nabla_V\phi)AU\Big)+\phi A\phi\Big((\nabla_U A)V-(\nabla_V A)U\Big).\end{aligned} \quad (32)$$

Using (10), (12) and (16) in (32), we find

$$\begin{aligned}(\nabla_V Q^*)U-(\nabla_U Q^*)V &= \frac{nc}{2}\Big(g(\phi AU,V)\xi+\eta(V)\phi AU-g(\phi AV,U)\xi \\ &- \eta(U)\phi AV\Big)+g(AV,A\phi AU)\xi-g(AU,A\phi AV)\xi \\ &+ \phi(\nabla_U A)\phi AV-\phi(\nabla_V A)\phi AU+\alpha\phi A\Big(\eta(V)AU \\ &- \eta(U)AV\Big)+\frac{c}{4}\Big(\eta(V)\phi AU-\eta(U)\phi AV\Big).\end{aligned} \quad (33)$$

Taking the inner product of (30) with ξ and further utilising (14) and (33), we obtain

$$\begin{aligned}&\frac{nc}{2}\Big(g(\phi AU,V)-g(\phi AV,U)\Big)+g(AV,A\phi AU)-g(AU,A\phi AV)= \\ &\Big(\frac{c}{4}-\frac{\nu}{m}\Big)\Big(V(f)\eta(U)-U(f)\eta(V)\Big)+\alpha\Big(AV(f)\eta(U)-AU(f)\eta(V)\Big).\end{aligned} \quad (34)$$

Using (19) in (34), we obtain (27).
Putting $U=\phi X$ and $V=\phi Y$ in (27) and using (19) again, we obtain (28). Thus proof is complete. □

4. Proof of Theorems 1.1 and 1.2

Let M be a hypersurface in $\hat{M}_n(c)$ with an η-*Einstein metric such that $A\xi = 0$.
Then (28) implies

$$c\Big(\frac{2n+1}{4}\Big)(A\phi + \phi A) = 0.$$

Since $c \neq 0$, therefore $A\phi+\phi A = 0$. From Lemma 3.1, we see that $A\phi+\phi A = 0$ gives $c = 0$, a contradiction. This concludes the proof of Theorem 1.1.

Taking $V = \xi$ and $U \in \mathfrak{D}$ in (27) and using (16) and (21), we get

$$\Big(\frac{c}{4} - \frac{\nu}{m} + \alpha\mu_i\Big)U(f) = 0. \tag{35}$$

Putting $U = V = \xi$ in (5), we get

$$\xi\xi f - \frac{(\xi f)^2}{m} = \nu. \tag{36}$$

Putting $U = V \in \mathfrak{D}$ in (5), we find

$$Ric^*(U,U) + Hess f(U,U) - \frac{(Uf)^2}{m} = \nu g(U,U). \tag{37}$$

Using (14), (21) and (23) in (37), we get

$$\Big(\frac{nc}{2} + \mu_i\lambda_i - \nu\Big)g(U,U) + g(\nabla_U\nabla f, U) - \frac{(Uf)^2}{m} = 0. \tag{38}$$

Taking $V = \phi U$ in (27) and using (21) and (23), we find

$$(\mu_i + \lambda_i)\Big(nc + \frac{c}{2} + \alpha\lambda_i + \alpha\mu_i\Big) = 0. \tag{39}$$

We claim that $\mu_i + \lambda_i \neq 0$. In fact, if

$$\mu_i + \lambda_i = 0, \tag{40}$$

then using this in (24), we obtain $\frac{c}{4} = -\mu_i^2$. Hence, there exists real hypersurfaces in $\mathbb{C}H^n$ only. As Hopf hypersurfaces in $\mathbb{C}H^n$ have atmost three distinct eigenvalues, so we consider three distinct eigenvalues α, μ, $-\mu$ such that $\mu = \mu_1 = \mu_2 = \cdots = \mu_{n-1}$ and $-\mu = \lambda_1 = \lambda_2 = \cdots = \lambda_{n-1}$. Therefore, we discuss hypersurfaces of type A_2 and B.

Let M is of type A_2. Substituting eigenvalues (see Cecil and Ryan [1], Montiel [13], or Niebergall and Ryan [15]), $\alpha = 2\coth 2r$, $\lambda = \tanh r$, $\mu = \coth r$ in (40), we find $\tanh^2 r = -1$, a contradiction since $0 < \tanh^2 r < 1$.

Next, if M is of type B. Utilising eigenvalues (see Cecil and Ryan [1], Montiel [13], or Niebergall and Ryan [15]), $\alpha = 2\tanh 2r$, $\lambda = \coth r$, $\mu = \tanh r$ in (40), we get $\tanh^2 r = -1$, a contradiction. Therefore, from (39), we get

$$\Big(n + \frac{1}{2}\Big)c + (\mu_i + \lambda_i)\alpha = 0. \tag{41}$$

Now, in view of (24), we consider the following cases:

Case I: Let $\alpha^2 + c \neq 0$. So from (24), we get that $\mu_i \neq \frac{\alpha}{2}$.
Using (41) in (24), we obtain

$$\frac{2n+1}{n} = \frac{(\lambda_i + \mu_i)\alpha}{\mu_i\lambda_i}. \tag{42}$$

Now, if $\mu_i \neq \lambda_i$, then we can discuss the case of three and five distinct eigenvalues in $\mathbb{C}P^n$. Suppose there are five distinct eigenvalues α, $\lambda = \lambda_1 = \lambda_2 = \cdots = \lambda_p$, $\delta = \lambda_{p+1} = \cdots = \lambda_{n-1}$, $\mu = \mu_1 = \mu_2 = \cdots = \mu_p$, and $\gamma = \mu_{p+1} = \cdots = \mu_{n-1}$, therefore we discuss M as η-*Einstein hypersurfaces of three types C, D, and E in $\mathbb{C}P^n$ (see [12], [15], or [24]). As these hypersurfaces have same eigenvalues so we will discuss for any one of the three hypersurfaces say C. As the principal spaces of λ and μ are ϕ-invariant and the principal spaces of γ and δ are interchanged by ϕ. So there will be 16 combinations for choices of eigenvalues λ, μ, γ, and δ. We will consider one of the cases and other can be seen easily. Using eigenvalues (see Kimura [12], Niebergall and Ryan [15], or Takagi [24]), $\alpha = -2\cot 2r$, $\lambda = \tan r$, $\delta = \cot(\frac{3\pi}{4} - r)$, $\mu = -\cot r$, and $\gamma = \cot(\frac{\pi}{4} - r)$ in (42), we obtain

$$\frac{2n+1}{n} = \frac{(\tan r - \cot r)(-2\cot 2r)}{(\tan r)(-\cot r)}, \tag{43}$$

and

$$\frac{2n+1}{n} = \frac{(\cot(\frac{3\pi}{4} - r) + \cot(\frac{\pi}{4} - r))(-2\cot 2r)}{(\cot(\frac{3\pi}{4} - r))(\cot(\frac{\pi}{4} - r))}. \tag{44}$$

From (43) and (44), we obtain

$$\tan^2 r = -1,$$

which is not possible.

Further, if we have three distinct eigenvalues α, λ and μ, therefore we discuss M as η-*Einstein hypersurfaces of type A_2 and B in $\mathbb{C}P^n$ [12, 15, 24] and $\mathbb{C}H^n$ [1, 13, 15].

Let M is of type A_2 in $\mathbb{C}P^n$. Using eigenvalues (see Kimura [12], Niebergall and Ryan [15], or Takagi [24]), $\lambda = -\tan r$, $\mu = \cot r, \alpha = 2\cot 2r$ in (42), we obtain

$$n = -\frac{\sin^2 r \cos^2 r}{\cos^4 r + \sin^4 r},$$

which gives $n < 0$, a contradiction.

Next, if M is of type B in $\mathbb{C}P^n$. Substituting eigenvalues (see Kimura [12], Niebergall and Ryan [15], or Takagi [24]), $\lambda = -\cot r$, $\mu = \tan r$, $\alpha = 2\tan 2r$ in (42), we get $n = \frac{1}{2}$, a contradiction.

Now, let M is of type A_2 in $\mathbb{C}H^n$. Substituting eigenvalues (see Cecil and Ryan [1], Montiel [13], or Niebergall and Ryan [15]), $\lambda = \tanh r$, $\mu = \coth r$, $\alpha = 2\coth 2r$ in (42), we find $n = \frac{\tanh^2 r}{1+\tanh^4 r}$, which gives $\tanh^2 r = \frac{1 \pm \sqrt{1-4n^2}}{2n}$, a contradiction.

Next, if M is of type B in $\mathbb{C}H^n$. Substituting eigenvalues (see Cecil and Ryan [1], Montiel [13], or Niebergall and Ryan [15]), $\lambda = \coth r$, $\mu = \tanh r$, $\alpha = 2\tanh 2r$ in (42), we get $n = \frac{1}{2}$, a contradiction.

Hence there is no hypersurface with η-*Einstein metric in $\mathbb{C}P^n$ and $\mathbb{C}H^n$ with three and five distinct eigenvalues.

Now, we discuss the case if $\mu_i = \lambda_i$, then from (24) and (41), we get

$$\frac{nc}{2} = -\mu_i^2. \tag{45}$$

Putting $\lambda_i = \mu_i$ and $c = -\frac{2\mu_i^2}{n}$ in (24), we obtain

$$\mu_i[(2n+1)\mu_i - 2\alpha n] = 0, \tag{46}$$

which gives

$$\mu_i = \frac{2\alpha n}{2n+1}, \tag{47}$$

as using $\mu_i = 0$ in (45) gives a contradiction $c = 0$.

From (45), we see that there exist η-*Einstein hypersurfaces in $\mathbb{C}H^n$ only. Moreover, we find that $A\phi = \phi A$ as $\mu_i = \lambda_i$. Hence from Theorem 5.1 (cf. [14]), M is locally congruent to type A_1 in $\mathbb{C}H^n$.

Let M is of type $A_{1,2}$ in $\mathbb{C}H^n$. Putting the eigenvalues of M (see Cecil and Ryan [1], Montiel [13], or Niebergall and Ryan [15]), $\mu = \tanh r$ and $\alpha = 2\coth 2r$ in (47), we get

$$(2n+1)\tanh r = 4n\coth 2r, \tag{48}$$

which gives $2n = \tanh^2 r$, a contradiction.

Now, if M is of type $A_{1,1}$ in $\mathbb{C}H^n$. Then, substituting the eigenvalues (see Cecil and Ryan [1], Montiel [13], or Niebergall and Ryan [15]), $\mu = \coth r$ and $\alpha = 2\coth 2r$ in (47), we get

$$(2n+1)\coth r = 4n\coth 2r, \tag{49}$$

which gives $2n = \coth^2 r$. Hence M is a geodesic hypersphere having an η-*Einstein metric with $2n = \coth^2 r$.

Hence there exists a geodesic hypersphere with an η-* Einstein metric in $\mathbb{C}H^n$ having two distinct eigenvalues with $2n = \coth^2 r$.

Now, in view of (35), we discuss the following subcases :

Subcase A: From (35), if $U(f) = 0$ then, we have

$$\nabla f = \langle \nabla f, \xi \rangle \xi. \tag{50}$$

Differentiating (50) along $Z \in TM$ and utilising (10), yields

$$\nabla_Z \nabla f = \langle \nabla f, \xi \rangle \phi AZ + Z(\langle \nabla f, \xi \rangle)\xi. \tag{51}$$

We know that $g(\nabla_W \nabla f, Z) = g(\nabla_Z \nabla f, W)$ for $W, Z \in TM$. Hence using (51) in it, we get

$$g(\xi(f)\phi AW + W(\xi(f))\xi, Z) = g(Z(\xi(f))\xi + \xi(f)\phi AZ, W). \tag{52}$$

Replacing Z by ϕX and W by ϕY in (52) and using (7), (8), and (16), we obtain

$$\xi(f)(A\phi X + \phi AX) = 0, \tag{53}$$

which yields $\xi(f) = 0$, as $A\phi + \phi A = 0$ gives $c = 0$ (cf. [11], Lemma 2.1), a contradiction. Hence f is constant, and using this in (5) and (36), we get M is *Ricci flat with steady soliton.

Thus, there exists a geodesic Hopf hypersphere M in $\mathbb{C}H^n$ having steady *Ricci flat metric with $2n = \coth^2 r$.

Subcase B: Now, we discuss if $U(f) \neq 0$, then, from (35), we get

$$\frac{c}{4} - \frac{\nu}{m} + \alpha\mu_i = 0. \tag{54}$$

Since there exists a geodesic hypersphere with η-* Einstein metric in $\mathbb{C}H^n$ with two distinct eigenvalues. Hence taking $\lambda_i = \mu_i$ in (24) and (54), we find

$$\nu = m\mu_i^2, \tag{55}$$

which gives a geodesic hypersphere M in $\mathbb{C}H^n$ with a shrinking η-*Einstein metric.

Subcase C: If both $U(f) = 0$ and $\frac{c}{4} - \frac{\nu}{m} + \alpha\mu_i = 0$ in (35). Then, we get $\nu = 0$ and $\nu = m\mu_i^2$, which gives $\mu_i = 0$. Then, from (45), we get $c = 0$, a contradiction.

Case II: Let $\alpha^2 + c = 0$. Then, ambient space is $\mathbb{C}H^n$ only, as $c = -\alpha^2$. Now, if $\mu_i \neq \frac{\alpha}{2}$ in (24), then $\lambda_i = \frac{\alpha}{2}$. Using value of c and λ_i in (41), we obtain

$$n = \frac{\mu_i}{\alpha}. \tag{56}$$

Using $\alpha = 2\lambda_i$ in (56), we get

$$n = \frac{\mu_i}{2\lambda_i}. \tag{57}$$

Now, M has three distinct eigenvalues α, $\mu = \mu_i$ for $i = 1, 2, \dots, n-1$ and $\frac{\alpha}{2}$, therefore M can be a hypersurface of two types A_2 and B in $\mathbb{C}H^n$.

Let M is of type A_2 in $\mathbb{C}H^n$. Using eigenvalues $\lambda = \lambda_i = \tanh r$, for $i = 1, 2, \dots, n-1$; $\mu = \mu_i = \coth r$, for $i = 1, 2, \dots, n-1$; $\alpha = 2\coth 2r$ in (56), we find $\tanh^2 r = \frac{1-n}{n}$, a contradiction.

Next, if M is of type B in $\mathbb{C}H^n$. Putting eigenvalues $\lambda = \lambda_i = \coth r$, for $i = 1, 2, \dots, n-1$; $\mu = \mu_i = \tanh r$, for $i = 1, 2, \dots, n-1$; $\alpha = 2\tanh 2r$ in (56), we get $\tanh^2 r = 4n - 1$, a contradiction as $0 < \tanh^2 r < 1$.

So, now we discuss the case $\lambda_i = \mu_i = \frac{\alpha}{2}$. In this case M is a horosphere in $\mathbb{C}H^n$ [1]. Putting $c = -\alpha^2$ and the eigenvalues $\mu_i = 1$, $\lambda_i = 1$, $\alpha = 2$ in (56), we find $n = \frac{1}{2}$, a contradiction. This concludes the proof of Theorem 1.2.

5. Proof of Theorem 1.4

We know that a hypersurface in $\mathbb{C}^n$ is said to be contact if $\phi A + A\phi = 2\zeta\phi$, for a smooth function $\zeta > 0$.

As M is a contact hypersurface so it is Hopf and $\alpha = \eta(A\xi)$ is constant (cf. [4], Lemma 3.1). If A has only one principal curvature $\frac{\alpha}{2} \in \mathfrak{D}$, such that $\alpha \neq 0$.

Let $U \in \mathfrak{D}$ such that $AU = (\frac{\alpha}{2})U$ and taking $V = \phi U$ in (27), we get

$$c = -\frac{2\alpha^2}{2n+1}, \tag{58}$$

which gives $\alpha = 0$, a contradiction, as $c = 0$.

Next, let $U \in \mathfrak{D}$ such that $AU = \mu_i U$, $A\phi U = \lambda_i \phi U$, $i = 1, 2, \dots, n-1$ with $\mu_i \neq \frac{\alpha}{2}$, so from (28), we have

$$\Big(\frac{\alpha^2}{2} + \frac{nc}{2} + \frac{c}{4}\Big)(\mu_i + \lambda_i) + \frac{\alpha}{2}(\mu_i^2 + \lambda_i^2) + \frac{\alpha c}{4} = 0. \tag{59}$$

Eliminating λ_i, using (24) and (59), we get

$$2\alpha\mu_i^4 + (2nc + c)\mu_i^3 + \Big(-\alpha nc + \frac{c\alpha}{2}\Big)\mu_i^2 + \Big(\frac{nc^2}{2} + \frac{c^2}{4}\Big)\mu_i - \frac{n\alpha c^2}{4} = 0. \tag{60}$$

Putting $c = 0$ in (60), we find

$$\alpha\mu_i^4 = 0. \tag{61}$$

Putting $c = 0$ and $V = \xi$, in (27), we find

$$\frac{\nu}{m}(\nabla f - \xi(f)\xi) - \alpha(A\nabla f - \alpha\xi(f)\xi) = 0. \tag{62}$$

Computing the inner product of (62) with $U \in \mathfrak{D}$, we obtain

$$\Big(\frac{\nu}{m} - \alpha\mu_i\Big)U(f) = 0. \tag{63}$$

Using (61) in (63), we get

$$\nu U(f) = 0. \tag{64}$$

Now, from (64), if $U(f) = 0$, then from the proof of Theorem 1.2, we can see that f is constant as $A\phi + \phi A \neq 0$. Using this in (5) and (36), we get M is *Ricci flat with steady soliton.

Next, if $\nu = 0$ and $U(f) \neq 0$, then M is steady soliton with η-*Einstein metric.

From (61), if $\alpha = 0$, $\mu_i \neq 0$. Then using this and $c = 0$ in (24), we get $\lambda_i = 0$. Hence M is locally congruent to $S^{n-1} \times \mathbb{R}^n$.

From (61), if $\alpha \neq 0$ and $\mu_i = 0$. Then using this and $c = 0$ in (24), we find $\lambda_i = 0$. Hence M is locally congruent to $\mathbb{R}^{2n-2} \times S^1$. Thus proof is complete.

Acknowledgements

The first author acknowledges the support of research fellowship F.No. GGSIPU/DRC/2021/685 from Guru Gobind Singh Indraprastha University. The second author is partially supported by FRGS grant for the year 2024-25, F. No. GGSIPU/RDC/FRGS/2024/2437/7. The third author was supported by grants Proj. No. RS-2018-NR033208 and RS-2021-NR061954 from National Research Foundation of Korea.

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