Filomat 39:10 (2025), 3453–3461
https://doi.org/10.2298/FIL2510453L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Spectral conditions for graphs to be *IM*-extendable or *k*-critical-bipartite

Xiaoyun Lv[a], Jianxi Li[b], Shou-Jun Xu[a,*]

[a] *School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu, P.R. China*
[b] *School of Mathematics and Statistics, Minnan Normal University, Zhangzhou, Fujian, P.R. China*

Abstract. A graph G is induced matching extendable or *IM*-extendable if every induced matching of G is contained in a perfect matching of G. For a bipartite graph $G[U, V]$ with $|U| = n_1$, $|V| = n_2$, and $n_1 > n_2 > 1$, we say G is k-critical-bipartite if deleting at most $k = n_1 - n_2$ vertices from U yields G' that has a perfect matching. Previously, Yuan (1998) and Laroche (2014) provided the structural characterizations for *IM*-extendable graphs and k-critical-bipartite graphs, respectively. Since the eigenvalues of a graph are closely related to its structural properties, we now characterize them from the perspective of the spectrum of graphs. In this paper, we first provide a tight spectral radius condition that guarantees a graph with minimum degree is *IM*-extendable. Furthermore, we present a tight spectral radius condition that ensures a bipartite graph with minimum degree is k-critical-bipartite.

1. Introduction

In this paper, we only consider finite undirected simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, we use n and m to denote its order and size, respectively. For $v \in V(G)$, let $d_G(v)$ be the degree of v. We use $\delta(G)$ (or δ for short) to denote the minimum degree of a graph G. For a subset $S \subseteq V(G)$, let $o(G - S)$ denote the number of odd components of $G - S$. A graph $G[X, Y]$ is *bipartite* if its vertex set can be partitioned into two disjoint subsets X and Y such that every edge has one end in X and the other end in Y. For two disjoint graphs G_1 and G_2, we use $G_1 \cup G_2$ and $G_1 \nabla G_2$ to denote respectively the *union* and *join* of G_1 and G_2. Given two bipartite graphs $G_1[X_1, Y_1]$ and $G_2[X_2, Y_2]$, let $G_1 \nabla_1 G_2$ denote the graph obtained from $G_1 \cup G_2$ by adding all possible edges between X_1 and Y_2. The *adjacency matrix* of a graph G is defined as $A(G) = (a_{i,j})$, where $a_{i,j} = 1$ if i and j are adjacent in G, and $a_{i,j} = 0$ otherwise. The largest eigenvalue of $A(G)$ is called the *spectral radius* of G, denoted by $\rho(G)$.

An edge set $M \subset E(G)$ is called a *matching* of G if any pair of edges in M are not adjacent. Moreover, a matching M is *perfect* if $V(M) = V(G)$. A matching in G is called *induced* if no two edges in the matching are joined by an edge in G. A graph G is called *induced matching extendable* if every induced matching of G can be extended to a perfect matching, which was introduced by Yuan [10]. For convenience, we refer

2020 *Mathematics Subject Classification*. Primary 05C50.
Keywords. Spectral radius, *IM*-extendable, *k*-critical-bipartite, minimum degree.
Received: 02 November 2024; Accepted: 29 January 2025
Communicated by Paola Bonacini
The research is partially supported by NSFC (Nos. 12071194, 12171089, 12271235), NSF of Fujian (No. 2021J02048).
* Corresponding author: Shou-Jun Xu
Email addresses: 120220907940@lzu.edu.cn (Xiaoyun Lv), ptjxli@hotmail.com (Jianxi Li), shjxu@lzu.edu.cn (Shou-Jun Xu)
ORCID iDs: https://orcid.org/0009-0007-7235-2848 (Xiaoyun Lv), https://orcid.org/0000-0001-7034-9062 (Jianxi Li), https://orcid.org/0000-0002-2046-3040 (Shou-Jun Xu)

to G as *IM-extendable*, which means that G is induced matching extendable. It is easy to see that every *IM*-extendable graph must have an even number of vertices.

There are several sufficient conditions for a graph to be *IM*-extendable. For example, Liu, Yuan and Wang [8] studied *IM*-extendable graphs in terms of degree. Wang and Yuan [9] proved a degree sum condition that ensures a graph is *IM*-extendable. In particular, Yuan [10] gave the following structural characterization of a graph that is *IM*-extendable.

Theorem 1.1 ([10]). *A graph G is IM-extendable if and only if for every induced matching M of G and every $S \subseteq V(G)\backslash V(M)$, $o(G - V(M) - S) \leq |S|$.*

On the other hand, note that the eigenvalues of a graph are closely related to its structural properties, so it is natural and interesting to ask "*Are there some conditions from spectral viewpoints to ensure that a graph G is IM-extendable?* "

Using Theorem 1.1, we first provide a tight spectral radius condition that guarantees a graph with minimum degree is *IM*-extendable.

Theorem 1.2. *For any even n, let G be a connected graph of order n with minimum degree $\delta \geq 5r + 1$, where $r \geq 1$ is an integer. If $n \geq \max\{2r + 32\delta + 12, \delta^3 - \frac{2r-3}{2}\delta^2 - 2(r^2 - r - 1)\delta - 2r^2 + 3r + \frac{3}{2}\}$ and*

$$\rho(G) \geq \rho(K_\delta \nabla (rK_2 \nabla (K_{n-2\delta-2r-1} \cup (\delta + 1)K_1))),$$

then G is IM-extendable unless $G = K_\delta \nabla (rK_2 \nabla (K_{n-2\delta-2r-1} \cup (\delta + 1)K_1))$.

For a bipartite graph $G[U, V]$ with $|U| = n_1$, $|V| = n_2$, and $n_1 > n_2 > 1$, we say G is k-critical-bipartite if deleting at most $k = n_1 - n_2$ vertices from U yields G' that has a perfect matching. Up to now, much attention has been paid to k-critical-bipartite graphs. For example, Cichacz et al. [2, 3] studied the problem of finding a minimum k-critical-bipartite graph of order (n_1, n_2). Li and Nie [7] gave the characterization of k-critical-bipartite graphs and they also described the connectivity of k-critical-bipartite graphs. On the other hand, Laroche et al. [6] gave a Hall-style characterization of k-critical-bipartite graphs.

Theorem 1.3. *A bipartite graph $G[A, B]$ is k-critical-bipartite if and only if $|N(X)| \geq |X| + k$ for all nonempty subset $X \subseteq B$ with $|X| \leq |A| - k$.*

In fact, the definition of k-factor-critical graphs (for a nonnegative integer k, a graph G is said to be *k-factor-critical* if $G - T$ has a perfect matching for any $T \subseteq V(G)$ with $|T| = k$) provides us with a framework. Within this context, a k-critical bipartite graph can be viewed as a special case of a k-factor-critical graph when considering only bipartite graphs. In [11], Zheng, Li, Luo and Wang provided a spectral radius condition for a general graph with minimum degree to be k-factor-critical. In particular, we consider the case of a bipartite graph in the following. Inspired by the result in [11] and utilizing Theorem 1.3, we provide a tight spectral radius condition that ensures a bipartite graph with minimum degree is a k-critical-bipartite graph.

Theorem 1.4. *Let k, s, n_1, n_2, n be integers, where $1 \leq k \leq \frac{n}{2} - 1$, $n_1 \geq n_2 \geq 2s + k + 1$ and $n_1 + n_2 = n$. If $G[A, B]$ ($|A| = n_1$, $|B| = n_2$) be a bipartite graph of order n with minimum degree δ and*

$$\rho(G) \geq \rho(K_{\delta+k+1,\delta} \nabla_1 K_{n_1-\delta-k-1,n_2-\delta}),$$

then G is k-critical-bipartite unless $G = K_{\delta+k+1,\delta} \nabla_1 K_{n_1-\delta-k-1,n_2-\delta}$.

The remainder of the paper is organized as follows. In Section 2, we present some preliminary results, which will be used in the subsequent sections. In Section 3, we will give the proofs of Theorems 1.2 and 1.4.

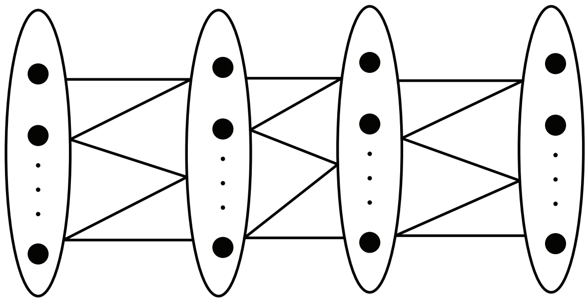

Figure 1: Graph $K_{\delta+k+1,\delta}\nabla_1 K_{n_1-\delta-k-1,n_2-\delta}$.

2. Preliminary

In this section, we present some preliminary results and lemmas which will be used in the subsequent sections.

The eigenvalues of an $n \times n$ real symmetric matrix M are denoted by $\lambda_1(M) \geq \lambda_2(M) \geq \ldots \geq \lambda_n(M)$, where we always assume the eigenvalues to be arranged in nonincreasing order. Given a partition $\pi = (X_1, X_2, \ldots, X_t)$ of the set $\{1, 2, \ldots, n\}$ and a matrix B whose rows and columns are labelled with elements in $\{1, 2, \ldots, n\}$, B can be expressed as the following partitioned matrix

$$B = \begin{pmatrix} B_{11} & \ldots & B_{1t} \\ \vdots & \vdots & \vdots \\ B_{t1} & \ldots & B_{tt} \end{pmatrix}$$

with respect to π. The *quotient matrix* B_π of B with respect to π is the t by t matrix (b_{ij}) such that each entry b_{ij} is the average row sum of B_{ij}. If the row sum of each block B_{ij} is a constant, then the partition is *equitable*.

Lemma 2.1 (Lemma 2.3.1 in[1], Lemma 9.3.1 in[5]). *Let M be a real symmetric matrix, and let $\lambda_1(M)$ be the largest eigenvalue of M. If B_π is an equitable quotient matrix of M, then the eigenvalues of B_π are also eigenvalues of M. Furthermore, if M is nonnegative and irreducible, then $\lambda_1(M) = \lambda_1(B_\pi)$.*

Lemma 2.2 (Theorem 2.5.1 in[1]). *Let G be a connected graph and let H be a spanning (or proper) subgraph of G. Then $\rho(H) \leq \rho(G)$ (or $\rho(H) < \rho(G)$), with equality holds if and only if $G = H$.*

Lemma 2.3 (Lemma 3.1 in[4]). *Let $n = \sum_{i=1}^{t} n_i + s$. If $n_1 \geq n_2 \geq \ldots \geq n_t \geq 1$ and $n_1 < n - s - t + 1$, then*

$$\rho(K_s \nabla (K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_t})) < \rho(K_s \nabla (K_{n-s-t+1} \cup (t-1)K_1)).$$

Lemma 2.4. *Let n be an even positive integer. The graph $K_\delta \nabla (rK_2 \nabla (K_{n-2\delta-2r-1} \cup (\delta+1)K_1))$ is not IM-extendable.*

Proof. Let $G = K_\delta \nabla (rK_2 \nabla (K_{n-2\delta-2r-1} \cup (\delta+1)K_1))$. Without loss of generality, let $V(M) = V(rK_2)$ and $S = V(K_\delta)$. Then

$$o(G - V(M) - S) = o(G - V(rK_2) - V(K_\delta)) = \delta + 2 > \delta = |S|.$$

By Theorem 1.1, G is not *IM*-extendable. $\square$

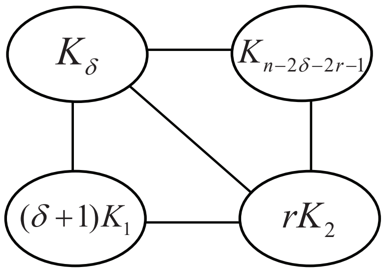

Figure 2: Graph $K_\delta\nabla(rK_2\nabla(K_{n-2\delta-2r-1}\cup(\delta+1)K_1))$.

Lemma 2.5. *Let n_1, n_2, s, k be integers, where $n_1 \geq n_2 \geq 2s+k+1$ and $k \geq 1$. Then we have*

$$\rho(K_{s+k+1,s}\nabla_1 K_{n_1-s-k-1,n_2-s}) < \rho(K_{s+k,s-1}\nabla_1 K_{n_1-s-k,n_2-s+1}).$$

Proof. Note that $A(K_{s+k+1,s}\nabla_1 K_{n_1-s-k-1,n_2-s})$ has the equitable quotient matrix

$$B_\pi^s = \begin{pmatrix} 0 & 0 & s & n_2-s \\ 0 & 0 & 0 & n_2-s \\ s+k+1 & 0 & 0 & 0 \\ s+k+1 & n_1-s-k-1 & 0 & 0 \end{pmatrix}.$$

By a simple computation, the characteristic polynomial of B_π^s is

$$\varphi(B_\pi^s, x) = x^4 - [n_1(n_2-s) + s(s+k+1)]x^2 + s(n_2-s)(s+k+1)(n_1-s-k-1).$$

Observe that $A(K_{s+k,s-1}\nabla_1 K_{n_1-s-k,n_2-s+1})$ has the equitable quotient matrix B_π^{s-1}, which is obtained by replacing s with $s-1$ in B_π^s. Then by $n_1 \geq n_2 \geq 2s+k+1$, we obtain

$$\begin{aligned}\varphi(B_\pi^s, x) - \varphi(B_\pi^{s-1}, x) &= (n_1-2s-k)(2n_2s - 3s^2 - 2sk + n_2k + s + k + x^2) \\ &\quad + n_2s(s-1) - 2s^3 - s^2k + 2s^2 + sk \\ &\geq k^2 + 2s^2 + 2s + 2k(s+1) + x^2 \\ &> 0,\end{aligned}$$

which leads to $\lambda_1(B_\pi^s) < \lambda_1(B_\pi^{s-1})$. It follows that

$$\rho(K_{s+k+1,s}\nabla_1 K_{n_1-s-k-1,n_2-s}) < \rho(K_{s+k,s-1}\nabla_1 K_{n_1-s-k,n_2-s+1})$$

by Lemma 2.1. This completes the proof. □

3. Proofs of Theorems 1.2 and 1.4

By the Perron-Frobenius theorem, $\rho(G)$ is always a positive number (unless G is an empty graph), and there exists an unique positive unit eigenvector corresponding to $\rho(G)$, which is called the *Perron vector* of G.

Firstly, we give the proof of Theorem 1.2.

Proof of Theorem 1.2. Suppose, to the contrary, that G is not IM-extendable. By Theorem 1.1, there exists a vertex subset $S \subseteq V(G)\backslash V(M)$ such that $o(G-V(M)-S) > |S|$. Since n is even, we have $o(G-V(M)-S) \equiv |S|$

(mod 2). Thus $o(G - V(M) - S) \geq |S| + 2$. For convenience, let $|V(M)| = 2r$ and $|S| = s$. It is clear that G is a spanning subgraph of $G' = rK_2\nabla(K_s\nabla(K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_{s+2}}))$ for some odd integers $n_1 \geq n_2 \geq \ldots \geq n_{s+2} > 0$ with $\sum_{i=1}^{s+2} n_i = n - 2r - s$. Then by Lemma 2.2, we have

$$\rho(G) \leq \rho(G'), \tag{1}$$

where equality holds if and only if $G = G'$. Let $G'' = K_s\nabla(rK_2\nabla(K_{n-2s-2r-1} \cup (s+1)K_1))$. By Lemma 2.3, we obtain that

$$\rho(G') \leq \rho(G''), \tag{2}$$

where equality holds if and only if $(n_1, n_2, \ldots, n_{s+2}) = (n - 2s - 2r - 1, 1, \ldots, 1)$.

Case 1. $s = \delta$.

Combining (1) and (2), we have

$$\rho(G) \leq \rho(G') \leq \rho(G'') = \rho(K_\delta\nabla(rK_2\nabla(K_{n-2\delta-2r-1} \cup (\delta+1)K_1))).$$

By the assumption $\rho(G) \geq \rho(K_\delta\nabla(rK_2\nabla(K_{n-2\delta-2r-1} \cup (\delta+1)K_1)))$, we have $G = K_\delta\nabla(rK_2\nabla(K_{n-2\delta-2r-1} \cup (\delta+1)K_1))$. By Lemma 2.4, $K_\delta\nabla(rK_2\nabla(K_{n-2\delta-2r-1} \cup (\delta+1)K_1))$ is not IM-extendable. Therefore, $G = K_\delta\nabla(rK_2\nabla(K_{n-2\delta-2r-1} \cup (\delta+1)K_1))$.

Case 2. $s \geq \delta + 1$.

Recall that $G'' = K_s\nabla(rK_2\nabla(K_{n-2s-2r-1} \cup (s+1)K_1))$. The vertex set of G'' can be devided into $V(G'') = V(K_s) \cup V((s+1)K_1) \cup V(rK_2) \cup V(K_{n-2s-2r-1})$, where $V(K_s) = \{u_1, u_2, \ldots, u_s\}$, $V((s+1)K_1) = \{v_1, v_2, \ldots, v_{s+1}\}$, $V(rK_2) = \{w_1, w_2, \ldots, w_{2r}\}$ and $V(K_{n-2s-2r-1}) = \{z_1, z_2, \ldots, z_{n-2s-2r-1}\}$.

Let $G^* = G''+E_1-E_2$, where $E_1 = \{v_iz_j | \delta+2 \leq i \leq s+1, 1 \leq j \leq n-2s-2r-1\} \cup \{v_iv_j | \delta+2 \leq i \leq s, i+1 \leq j \leq s+1\}$ and $E_2 = \{u_iv_j | \delta+1 \leq i \leq s, 1 \leq j \leq \delta+1\}$. Obviously, $G^* = K_\delta\nabla(rK_2\nabla(K_{n-2\delta-2r-1} \cup (\delta+1)K_1))$. Let x be the Perron vector of $A(G'')$ with respect to $\rho'' = \rho(G'')$. By symmetry, x takes the same value (denoted as x_1, x_2, x_3 and x_4) on the vertices of $V(K_s)$, $V((s+1)K_1)$, $V(rK_2)$ and $V(K_{n-2s-2r-1})$, respectively. By $A(G'')x = \rho''x$, we have

$$\rho''x_2 = sx_1 + 2rx_3, \tag{3}$$

$$\rho''x_3 = sx_1 + (s+1)x_2 + x_3 + (n - 2s - 2r - 1)x_4, \tag{4}$$

$$\rho''x_4 = sx_1 + 2rx_3 + (n - 2s - 2r - 2)x_4. \tag{5}$$

Observe that $n \geq 2s + 2r + 2$. According to (3) and (5), we obtain that $x_4 \geq x_2$. By (4) and (5), we have $\rho''x_3 - \rho''x_4 = (s+1)x_2 - (2r-1)x_3 + x_4$. It follows that $x_4 = \frac{(\rho''+2r-1)x_3-(s+1)x_2}{\rho''+1} \geq x_2$. Then we have $x_3 \geq \frac{\rho''+s+2}{\rho''+2r-1}x_2$. Note that $s \geq \delta + 1$ and $\delta \geq 5r + 1$. Then $\rho'' + s + 2 \geq \rho'' + \delta + 3 > \rho'' + 2r - 1$, and hence $x_3 > x_2$. Combining (3), we have

$$x_2 > \frac{sx_1}{\rho'' - 2r}. \tag{6}$$

Recall that $G^* = K_\delta\nabla(rK_2\nabla(K_{n-2\delta-2r-1} \cup (\delta+1)K_1))$. Note that G^* contains $K_{n-2\delta-2r-1}$ as a proper subgraph. Then $\rho(G^*) > n - 2\delta - 2r - 2$. Similarly, let y be the Perron vector of $A(G^*)$, and let $\rho^* = \rho(G^*)$. By symmetry, y takes the same value (denoted as y_1, y_2, y_3 and y_4) on the vertices of $V(K_\delta)$, $V((\delta+1)K_1)$, $V(rK_2)$ and $V(K_{n-2\delta-2r-1})$. By $A(G^*)y = \rho^*y$, we have

$$\rho^*y_2 = \delta y_1 + 2ry_3, \tag{7}$$

$$\rho^*y_4 = \delta y_1 + 2ry_3 + (n - 2\delta - 2r - 2)y_4. \tag{8}$$

Combining (7) and (8), we have

$$y_4 = \frac{\rho^* y_2}{\rho^* - (n - 2\delta - 2r - 2)}. \tag{9}$$

Note that $n \geq 2s + 2r + 2$. Then $\delta + 1 \leq s \leq \frac{n-2r-2}{2}$. Since G'' is not a complete graph, $\rho'' < n - 1$.

We claim that $\rho'' < \rho^*$. Suppose to the contrary that $\rho'' \geq \rho^*$. By $x_4 \geq x_2$, (6) and (9), we derive

$$\begin{aligned}
y^T(\rho^* - \rho'')x &= y^T(A(G^*) - A(G''))x \\
&= \sum_{i=\delta+2}^{s+1} \sum_{j=1}^{n-2s-2r-1} (x_{v_i} y_{z_j} + x_{z_j} y_{v_i}) + \sum_{i=\delta+2}^{s} \sum_{j=i+1}^{s+1} (x_{v_i} y_{v_j} + x_{v_j} y_{v_i}) - \sum_{i=\delta+1}^{s} \sum_{j=1}^{\delta+1} (x_{u_i} y_{v_j} \\
&\quad + x_{v_j} y_{u_i}) \\
&= (n - 2s - 2r - 1)(s - \delta)(x_2 y_4 + x_4 y_4) + (s - \delta - 1)(s - \delta)x_2 y_4 - (s - \delta)(\delta \\
&\quad + 1)(x_1 y_2 + x_2 y_4) \\
&\geq (s - \delta)[2(n - 2s - 2r - 1)x_2 y_4 + (s - \delta - 1)x_2 y_4 - (\delta + 1)x_2 y_4 - (\delta + 1)x_1 y_2] \\
&= (s - \delta)[(2n - 3s - 4r - 2\delta - 4)x_2 y_4 - (\delta + 1)x_1 y_2] \\
&> (s - \delta)[(2n - 3s - 4r - 2\delta - 4) \cdot \frac{s x_1}{\rho'' - 2r} \cdot \frac{\rho^* y_2}{\rho^* - (n - 2\delta - 2r - 2)} - (\delta + 1)x_1 y_2] \\
&= \frac{(s - \delta)x_1 y_2}{(\rho'' - 2r)(\rho^* - (n - 2\delta - 2r - 2))}[(2n - 3s - 4r - 2\delta - 4)s\rho^* - (\delta + 1)(\rho'' \\
&\quad - 2r)(\rho^* - (n - 2\delta - 2r - 2))] \\
&= \frac{(\delta + 1)(s - \delta)x_1 y_2}{(\rho'' - 2r)(\rho^* - (n - 2\delta - 2r - 2))}[\rho^*(2n - 3s - 4r - 2\delta - 4)\frac{s}{\delta + 1} - (\rho'' \\
&\quad - 2r)(\rho^* - (n - 2\delta - 2r - 2))].
\end{aligned}$$

Note that $s \geq \delta + 1$, $\rho'' \geq \rho^*$ and $\rho^* > \rho^*(K_\delta) = \delta - 1 \geq 5r$. Then

$$\begin{aligned}
y^T(\rho^* - \rho'')x &> \frac{(\delta + 1)(s - \delta)x_1 y_2}{(\rho'' - 2r)(\rho^* - (n - 2\delta - 2r - 2))}[\rho^*(2n - 3s - 4r - 2\delta - 4) - \rho''\rho^* + \rho''(n - 2\delta \\
&\quad - 2r - 2) + 2r\rho^* - 2r(n - 2\delta - 2r - 2)] \\
&= \frac{\rho^*(\delta + 1)(s - \delta)x_1 y_2}{(\rho'' - 2r)(\rho^* - (n - 2\delta - 2r - 2))}[(2n - 3s - 4r - 2\delta - 4) - \rho'' + \frac{\rho''}{\rho^*}(n - 2\delta \\
&\quad - 2r - 2) + 2r - \frac{2r}{\rho^*}(n - 2\delta - 2r - 2)] \\
&> \frac{\rho^*(\delta + 1)(s - \delta)x_1 y_2}{(\rho'' - 2r)(\rho^* - (n - 2\delta - 2r - 2))}[(2n - 3s - 4r - 2\delta - 4) - \rho'' + (n - 2\delta \\
&\quad - 2r - 2) + 2r - \frac{2}{5}(n - 2\delta - 2r - 2)] \\
&= \frac{\rho^*(\delta + 1)(s - \delta)x_1 y_2}{(\rho'' - 2r)(\rho^* - (n - 2\delta - 2r - 2))}(\frac{13}{5}n - 3s - \frac{16}{5}r - \frac{16}{5}\delta - \frac{26}{5} - \rho'').
\end{aligned}$$

Since K_s is a proper subgraph of G'' and $\delta \geq 5r + 1$, $\rho'' > \rho(K_s) = s - 1 \geq \delta > 2r$. Note that $s \leq \frac{n-2r-2}{2}$,

$\rho'' < n-1$, $\rho^* > n-2\delta-2r-2$ and $n \geq 2r+32\delta+12$. Then

$$\begin{aligned} y^T(\rho^*-\rho'')x &> \frac{\rho^*(\delta+1)(s-\delta)x_1y_2}{(\rho''-2r)(\rho^*-(n-2\delta-2r-2))}\Big(\frac{13}{5}n - 3\cdot\frac{n-2r-2}{2} - \frac{16}{5}r - \frac{16}{5}\delta \\ &\quad - \frac{26}{5} - \rho''\Big) \\ &= \frac{\rho^*(\delta+1)(s-\delta)x_1y_2}{(\rho''-2r)(\rho^*-(n-2\delta-2r-2))}\Big(\frac{1}{10}n - \frac{1}{5}r - \frac{16}{5}\delta - \frac{6}{5} + (n-1) - \rho''\Big) \\ &> \frac{\rho^*(\delta+1)(s-\delta)x_1y_2}{(\rho''-2r)(\rho^*-(n-2\delta-2r-2))}\cdot\frac{n-2r-32\delta-12}{10} \\ &\geq 0. \end{aligned}$$

This implies that $\rho^* > \rho''$, which contradicts the assumption $\rho'' \geq \rho^*$.

By $\rho^* > \rho''$, (1) and (2), we conclude

$$\rho(G) \leq \rho(G') \leq \rho(G'') < \rho(G^*) = \rho(K_\delta\nabla(rK_2\nabla(K_{n-2\delta-2r-1}\cup(\delta+1)K_1))),$$

which contradicts $\rho(G) \geq \rho(K_\delta\nabla(rK_2\nabla(K_{n-2\delta-2r-1}\cup(\delta+1)K_1)))$.

Case 3. $s < \delta$.

Recall that $G' = rK_2\nabla(K_s\nabla(K_{n_1}\cup K_{n_2}\cup\ldots\cup K_{n_{s+2}}))$. Then $d_{G'}(v) = n_{s+2}-1+s+2r \geq \delta$ for $v \in V(K_{n_{s+2}})$, and hence $n_{s+2} \geq \delta-s-2r+1$. Let $G''' = rK_2\nabla(K_s\nabla(K_{n-s-2r-(s+1)(\delta-s-2r+1)}\cup(s+1)K_{\delta-s-2r+1}))$. By Lemma 2.3, we have

$$\rho(G') \leq \rho(G'''), \tag{10}$$

where equality holds if and only if $(n_1, n_2, \ldots, n_{s+2}) = (n-s-2r-(s+1)(\delta-s-2r+1), \delta-s-2r+1, \delta-s-2r+1, \ldots, \delta-s-2r+1)$.

Let $\rho''' = \rho(G''')$. We claim that $\rho''' < n-2r-1-(s+1)(\delta-s+1)$. Suppose to the contrary that $\rho''' \geq n-2r-1-(s+1)(\delta-s+1)$. Let z be the Perron vector of $A(G''')$. By symmetry, z takes the same values z_1, z_2, z_3 and z_4 on the vertices of $V(K_s)$, $V((s+1)K_{\delta-s-2r+1})$, $V(rK_2)$ and $V(K_{n-s-2r-(s+1)(\delta-s-2r+1)})$, respectively. By $A(G''')z = \rho'''z$, we obtain

$$\begin{aligned} \rho'''z_1 &= (s-1)z_1 + (s+1)(\delta-s-2r+1)z_2 + 2rz_3 \\ &\quad + (n-s-2r-(s+1)(\delta-s-2r+1))z_4, \end{aligned} \tag{11}$$

$$\rho'''z_2 = sz_1 + (\delta-s-2r)z_2 + 2rz_3, \tag{12}$$

$$\begin{aligned} \rho'''z_3 &= sz_1 + (s+1)(\delta-s-2r+1)z_2 + z_3 \\ &\quad + (n-s-2r-(s+1)(\delta-s-2r+1))z_4, \end{aligned} \tag{13}$$

$$\rho'''z_4 = sz_1 + 2rz_3 + (n-s-2r-1-(s+1)(\delta-s-2r+1))z_4. \tag{14}$$

By (11) and (13), we have

$$z_3 = \frac{(\rho'''+1)z_1}{\rho'''+2r-1}. \tag{15}$$

Substituting (15) into (12) and (14), we have

$$z_2 = \frac{sz_1 + \frac{2r(\rho'''+1)}{\rho'''+2r-1}z_1}{\rho'''-(\delta-s-2r)}, \tag{16}$$

$$z_4 = \frac{sz_1 + \frac{2r(\rho'''+1)}{\rho'''+2r-1}z_1}{\rho''' - [n - s - 2r - 1 - (s+1)(\delta - s - 2r + 1)]}. \tag{17}$$

Since $n \geq \delta^3 - \frac{2r-3}{2}\delta^2 - 2(r^2 - r - 1)\delta - 2r^2 + 3r + \frac{3}{2}$, we have $\rho''' \geq n - 2r - 1 - (s+1)(\delta - s + 1) > \delta - 2r + 1$. Substituting (15), (16) and (17) into (11), we derive

$$\begin{aligned}
\rho''' + 1 &= s + \frac{[n - s - 2r - (s+1)(\delta - s - 2r + 1)](s + \frac{2r(\rho'''+1)}{\rho'''+2r-1})}{\rho''' - [n - s - 2r - 1 - (s+1)(\delta - s - 2r + 1)]} + \frac{2r(\rho'''+1)}{\rho'''+2r-1} \\
&\quad + \frac{(s+1)(\delta - s - 2r + 1)(s + \frac{2r(\rho'''+1)}{\rho'''+2r-1})}{\rho''' - (\delta - s - 2r)} \\
&\leq s + \frac{[n - s - 2r - (s+1)(\delta - s - 2r + 1)](s + 2r)}{\rho''' - [n - s - 2r - 1 - (s+1)(\delta - s - 2r + 1)]} + 2r + \frac{(s+1)(\delta - s - 2r + 1)(s + 2r)}{\rho''' - (\delta - s - 2r)} \\
&< n - 2r - 1 - (s+1)(\delta - s + 1) - \frac{1}{2sr - s + 2r}[2n(sr + 2r) + (4r-1)s^3 + (8r^2 - 4\delta r \\
&\quad + \delta + 1)s^2 + (8r^3 - 4\delta r^2 - 6\delta r - 4r^2 - 4r + 1)s + 8r^3 - 4\delta r^2 - 12r^2 - 4\delta r - 6r].
\end{aligned}$$

Let

$$\begin{aligned}
f(n) &= 2n(sr + 2r) + (4r-1)s^3 + (8r^2 - 4\delta r + \delta + 1)s^2 + (8r^3 - 4\delta r^2 - 6\delta r - 4r^2 - 4r + 1)s \\
&\quad + 8r^3 - 4\delta r^2 - 12r^2 - 4\delta r - 6r.
\end{aligned}$$

We assert that $f(n) \geq 0$. In fact, suppose that $f(n) < 0$. Note that $0 \leq s < \delta$, $-4r + 1 < 0$, $-8r^2 + 4\delta r - \delta - 1 > 0$ and $-8r^3 + 4\delta r^2 + 6\delta r + 4r^2 + 4r - 1 > 0$. Then

$$\begin{aligned}
n &< \frac{1}{2(sr + 2r)}[(-4r+1)s^3 + (-8r^2 + 4\delta r - \delta - 1)s^2 + (-8r^3 + 4\delta r^2 + 6\delta r + 4r^2 + 4r - 1)s \\
&\quad - 8r^3 + 4\delta r^2 + 12r^2 + 4\delta r + 6r] \\
&< \frac{1}{4r}[(-8r^2 + 4\delta r - \delta - 1)\delta^2 + (-8r^3 + 4\delta r^2 + 6\delta r + 4r^2 + 4r - 1)\delta - 8r^3 + 4\delta r^2 + 12r^2 \\
&\quad + 4\delta r + 6r] \\
&= \frac{1}{4r}[(4r-1)\delta^3 + (-4r^2 + 6r - 1)\delta^2 + (-8r^3 + 8r^2 + 8r - 1)\delta - 8r^3 + 12r^2 + 6r] \\
&< \frac{1}{4r}[4r\delta^3 + (-4r^2 + 6r)\delta^2 + (-8r^3 + 8r^2 + 8r)\delta - 8r^3 + 12r^2 + 6r] \\
&= \delta^3 - \frac{2r-3}{2}\delta^2 - 2(r^2 - r - 1)\delta - 2r^2 + 3r + \frac{3}{2},
\end{aligned}$$

which contradicts $n \geq \delta^3 - \frac{2r-3}{2}\delta^2 - 2(r^2 - r - 1)\delta - 2r^2 + 3r + \frac{3}{2}$. Hence, $f(n) \geq 0$. Then

$$\begin{aligned}
\rho''' + 1 &< n - 2r - 1 - (s+1)(\delta - s + 1) - \frac{1}{2sr - s + 2r} \cdot f(n) \\
&< n - 2r - 1 - (s+1)(\delta - s + 1) \\
&\leq \rho''',
\end{aligned}$$

a contradiction. Therefore, we have $\rho''' < n - 2r - 1 - (s+1)(\delta - s + 1) = n - \delta - 2r - 1 - [(\delta - s)s + 1] < n - \delta - 2r - 1$. Note that $K_{n-\delta-2r}$ is a proper subgraph of $K_\delta \nabla (rK_2 \nabla (K_{n-2\delta-2r-1} \cup (\delta+1)K_1))$. Then $n - \delta - 2r - 1 = \rho(K_{n-\delta-2r}) < \rho(K_\delta \nabla (rK_2 \nabla (K_{n-2\delta-2r-1} \cup (\delta+1)K_1)))$. Combining (1) and (10), we obtain

$$\rho(G) \leq \rho(G') \leq \rho(G''') < n - \delta - 2r - 1 < \rho(K_\delta \nabla (rK_2 \nabla (K_{n-2\delta-2r-1} \cup (\delta+1)K_1))),$$

which contradicts $\rho(G) \geq \rho(K_\delta \nabla (rK_2 \nabla (K_{n-2\delta-2r-1} \cup (\delta+1)K_1)))$. This completes the proof. □

Now, we give the proof of Theorem 1.4.

Proof of Theorem 1.4. Let $G[A,B]$ be a bipartite graph of order n with minimum degree δ. Suppose that G is not k-critical-bipartite, by Theorem 1.3, there exists some nonempty subset $S \subseteq B$ with $s = |S| \leq |A| - k$ such that $|N(S)| < |S| + k$. Then G is a spanning subgraph of $K_{s+k+1,s}\nabla_1 K_{n_1-s-k-1,n_2-s}$ for some s with $\delta \leq s \leq \frac{n_1-k-1}{2}$. By Lemmas 2.2 and 2.5, we have

$$\rho(G) \leq \rho(K_{s+k+1,s}\nabla_1 K_{n_1-s-k-1,n_2-s}) \leq \rho(K_{\delta+k+1,\delta}\nabla_1 K_{n_1-\delta-k-1,n_2-\delta}),$$

where the first equality holds if and only if $G = K_{s+k+1,s}\nabla_1 K_{n_1-s-k-1,n_2-s}$, and the second equality holds if and only if $s = \delta$. Note that $K_{s+k+1,s}\nabla_1 K_{n_1-s-k-1,n_2-s}$ is not k-critical-bipartite. Thus the result follows. □

Acknowledgements This work was funded in part by National Natural Science Foundation of China (Grants Nos. 12071194, 12171089, 12271235) and National Science Foundation of Fujian (No. 2021J02048). The authors would like to express their sincere gratitude to all the referees for their careful reading and insightful suggestions.

Data Availibility Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

References

[1] A.E. Brouwer, W.H. Haemers, *Spectra of Graphs*, Springer, Berlin, 2011.
[2] S. Cichacz, K. Suchan, *Minimum k-critical-bipartite graphs*, Discrete Math. **302** (2021), 54–66.
[3] S. Cichacz, A. Görlich, K. Suchan, *Minimum k-critical-bipartite graphs: the irregular Case*, https://doi.org/10.48550/arXiv.2307.07315
[4] D. Fan, S. Goryainov, X. Huang, H. Lin, *The spanning k-trees, perfect matchings and spectral radius of graphs*, Linear Multilinear Algebra **70** (2022), 7264–7275.
[5] C. Godsil, G. Royle, *Algebraic Graph Theory*, Graduate Texts in Mathematices, 207, Springer-Verlag, New York, 2001.
[6] P. Laroche, F. Marchetti, S. Martin, and Z. Róka, *Bipartite complete matching vertex interdiction problem: Application to robust nurse assignment*. In Proceedings of the 2014 International Conference on Control, Decision and Information Technologies, CoDIT, pages 182–187. Institute of Electrical and Electronics Engineers, 2014.
[7] Y. Li, Z. Nie. *A note on n-critical bipartite graphs and its application*. In Proceedings of the 3rd Conference on Combinatorial Optimization and Applications, COCOA, pages 279–286. Springer, 2009.
[8] Y. Liu, J. Yuan, S. Wang, *Degree conditions of IM-extendable graphs*, Appl. Math. JCU. **15** (2000), 1–6.
[9] Q. Wang, J. Yuan, *Degree sum conditions of induced matching extendable graphs*, J. Zhengzhou Univ. **32** (2000), 19–21.
[10] J. Yuan, *Induced matching extendable graphs*, J. Graph Theory **28** (1998)(4), 203–213.
[11] L. Zheng, S. Li, X. Luo, G. Wang, *Some sufficient conditions for a graph with minimum degree to be k-factor-critical*, Discrete Appl. Math. **348** (2024), 279–291.