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Uncertainty measures and concomitants of generalized order statistics in the Sarmanov family

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Abstract. The objective of this work is to examine, in a broad context, the statistical characteristics of the concomitants of generalized order statistics from the Sarmanov family of bivariate distributions. Additionally, a number of recent information measures for concomitants of generalized order statistics based on the Sarmanov and Farlie-Gumbel-Morgenstern families are investigated with comparisons. These measures include weighted entropy, weighted extropy, past extropy, residual extropy, cumulative residual extropy, and negative cumulative residual extropy. In addition, based on concomitants of generalized order statistics, the issue of estimating cumulative residual extropy is investigated using the empirical technique. Also, numerical examples are conducted to illustrate our theoretical findings. Finally, a bivariate real-world data set has been examined for illustration purposes, and the performance is commendable.

1. Introduction

The concept of entropy was introduced by Shannon [36] as a mathematical measure of information that measures the average reduction of uncertainty or variability associated with a random variable (RV), which is widely used in the fields of information theory, statistics, probability, and so forth. Suppose that Y is a non-negative continuous RV having a probability density function (PDF), then the entropy (uncertainty measure) of Y is defined as

$$N(Y) = \int_{-\infty}^{\infty} (-f_Y(y)) \log f_Y(y) dy.$$

A recent generalization of classical entropy called weighted entropy (WE), which is a measure of the amount of information produced by Y has been proposed in the literature as follows:

$$N^{(\omega)}(Y) = \int_{-\infty}^{\infty} (-y f_Y(y)) \log f_Y(y) dy. \tag{1}$$

For more details about this measure, see [15] and [21].

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Extropy is an alternative measure of uncertainty proposed by Lad et al. [28] as a complementary dual-function measure of entropy. In the last five years, this measure of uncertainty has received considerable attention, it is defined by (cf. [24])

$$\xi(Y) = \frac{-1}{2}\int_{-\infty}^{\infty} f_Y^2(y)dy = \frac{-1}{2}\int_0^1 f_Y\left(F_Y^{-1}(v)\right)dv \le 0.$$

In a study by Qiu and Jia [33], residual extropy (RE) was proposed as a measure of residual uncertainty of an RV Y as follows:

$$\xi_t(Y) = \frac{-1}{2\overline{F}_Y^2(t)}\int_t^{\infty} f_Y^2(y)dy. \tag{2}$$

The $\xi_t(Y)$ is appropriate for measuring information when the uncertainty is related to the future. An important dimension of the $\xi_t(Y)$ is the past extropy (PE), which refers to uncertainty about the past lifetime of a system and for RV Y is defined as:

$$\xi^{(t)}(Y) = \frac{-1}{2F_Y^2(t)}\int_0^t f_Y^2(y)dy. \tag{3}$$

The weighted extropy (WEX) measure was proposed by Abdul Sathar and Nair [34] as

$$\xi^{(\omega)}(Y) = \frac{-1}{2}\int_{-\infty}^{\infty} y f_Y^2(y)dy. \tag{4}$$

The cumulative residual extropy (CREX) is defined as (see [6], and [24])

$$\xi^*(Y) = \frac{-1}{2}\int_0^{\infty} \overline{F}^2(y)dy \le 0. \tag{5}$$

Hence, the negative CREX (NCREX) can be presented as (cf. [24])

$$\tilde{\xi}(Y) = \frac{1}{2}\int_0^{\infty} \overline{F}^2(y)dy. \tag{6}$$

There has been increasing interest in studying generalized order statistics (GOSs) as a unified model for ascendingly ordered RVs. Kamps [29] introduced the GOSs model, which consists of many relevant models of ordered RVs, including order statistics (OSs), record values, sequential OSs (SOSs), and progressive censored type-II OSs (POS-II). The RVs $Y(r,n,\underline{m},\kappa), r = 1,2,...,n$, are called GOSs based on a continuous distribution function (DF) F_Y with the PDF f_Y, if their joint PDF has the form

$$f_{1,\ldots,n:n}^{(\underline{m},\kappa)}(y_1,...,y_n) = \kappa F_Y^{\gamma_n - 1}(y_n) f_Y(y_n)\prod_{i=1}^{n-1}\gamma_i F_Y^{\gamma_i-\gamma_{i+1}-1}(y_i)f_Y(y_i),$$

where $F^{-1}(0) \le y_1 \le ... \le y_n \le F^{-1}(1)$, $\kappa > 0, \gamma_i = n + \kappa - i + \sum_{t=i}^{n-1} m_t > 0, i = 1,\ldots,n-1$, and $\underline{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{R}$. In this paper, we assume that the parameters $\gamma_1,\ldots,\gamma_{n-1}$, and $\gamma_n = \kappa$, are pairwise different, i.e., $\gamma_t \ne \gamma_s, t \ne s,\ t,s = 1,2,...,n$. We obtain a very wide subclass of GOSs that contains m-GOSs (where $m_1 = ... = m_{n-1} = m$), OSs, POS-II, and SOSs. The PDF of the rth GOS is given by (cf. [30])

$$f_{Y(r,n,\underline{m},\kappa)}(y) = C_r\sum_{i=1}^{r}\alpha_{i;r}\overline{F}_Y^{\gamma_i-1}(y)f_Y(y),\quad y\in\mathbb{R},\quad 1\le r\le n, \tag{7}$$

where $\overline{F} = 1 - F$ is the survival function (SF) of F, $C_r = \prod_{i=1}^{r}\gamma_i$, and $\alpha_{i;r} = \prod_{\substack{j=1\\ j\ne i}}^{r}\frac{1}{\gamma_j-\gamma_i}$, $1 \le i \le r \le n$. The bivariate (multivariate) DFs with specified marginals are ideal for modeling bivariate (multivariate) data when only marginal distributions are all available. In such situations, it is often advantageous to use a flexible family of bivariate DFs, such as the Farlie-Gumbel-Morgenstern (FGM) family. The DF and PDF of the FGM family are defined, respectively, by

$$F_{Y,Z}(y,z) = F_Y(y)F_Z(z)\left[1 + \lambda\overline{F}_Y(y)\overline{F}_Z(z)\right]$$

and

$$f_{Y,Z}(y,z) = f_Y(y)f_Z(z)\left[1 + \lambda\left(2F_Y(y) - 1\right)\left(2F_Z(z) - 1\right)\right],\ -1 \le \lambda \le 1,$$

where $f_Y(y)$, $f_Z(z)$, and $F_Y(y)$, $F_Z(z)$ are the marginal PDFs and DFs of the RVs Y and Z, respectively, while $\overline{F}_Y$ and $\overline{F}_Z$ are the corresponding SFs. Generally, the FGM family has low dependence between variables, with Spearman's Rho $\rho \in (-0.33, 0.33)$. Thus, FGM can be useful in applications if there is little correlation between them. To improve the correlation level between family FGM and the literature, several extensions have been made available. Various applications of these extensions have been studied in [2], [5], [8], [9], [10], [14], and [25].

The Sarmanov family, denoted by SAR(.), has been examined recently in [3], [11], [12], and [26], who discovered that it is a significant rival to all known FGM extensions. The DF and PDF of SAR(ν) are given, respectively, by

$$F_{Y,Z}(y,z) = F_Y(y)F_Z(z)\left[1 + 3\nu\bar{F}_Y(y)\bar{F}_Z(z) + 5\nu^2\left(2F_Y(y) - 1\right)\left(2F_Z(z) - 1\right)\bar{F}_Y(y)\bar{F}_Z(z)\right]$$

and

$$\begin{aligned} f_{Y,Z}(y,z) &= f_Y(y)f_Z(z)\left[1 + 3\nu\left(2F_Y(y) - 1\right)\left(2F_Z(z) - 1\right)\right. \\ &+ \left.\frac{5}{4}\nu^2\left(3\left(2F_Y(y) - 1\right)^2 - 1\right)\left(3\left(2F_Z(z) - 1\right)^2 - 1\right)\right],\ |\nu| \le \frac{\sqrt{7}}{5}. \end{aligned} \tag{8}$$

Moreover, when the marginals are uniform then, the correlation coefficient is ν. Thus, in this case, the minimal and maximal correlation coefficient ρ of this copula are -0.529 and 0.529, respectively (cf. [7]; page 74).

The concomitants are a vital tool when selection and prediction problems are involved. The idea of concomitants of OSs (COSs) was first proposed by David [17]. To understand the concomitants in-depth, see [18]. Many studies have been published on the concomitants of the $m-$GOSs model. Researchers such as [2], [4], [13], and [19] have studied this issue. The concomitants of generalized order statistics (CGOSs), however, have only been studied in a restricted number of studies when $\gamma_t \ne \gamma_s, t \ne s,\ t, s = 1, 2, ..., n$. These include [1], [2], and [31].

Let $(Y_i, Z_i), i = 1, 2, ..., n,$ be a random sample from a continuous bivariate DF $F_{Y,Z}(y,z)$. If we denote $Y(r, n, \underline{m}, \kappa)$ as the rth GOS of the Y sample values, then the Z values associated with $Y(r, n, \underline{m}, \kappa)$ is called the concomitant of the rth GOS and is denoted by $Z_{[r,n,\underline{m},\kappa]}, r = 1, 2, ..., n$. The PDF of the concomitant of rth GOS is given by

$$f_{[r,n,\underline{m},\kappa]}(z) = \int_{-\infty}^{\infty} f_{Z|Y}(z|y) f_{Y(r,n,\underline{m},\kappa)}(y)dy. \tag{9}$$

The remainder of this paper is structured as follows: Under a broad framework, we obtain various features of CGOSs in SAR(ν) in Section 2. Section 3 examines various uncertainty measures for $Z_{[r,n,\underline{m},\kappa]}$ in SAR(ν), including WE, WEX, PE, RE, CREX, and NCREX, along with versatile illustrated examples and a comparison to some of those measures for the FGM family. We discuss the problem of estimating the CREX in Section 4. In Section 5, a real data set is analyzed for illustration. Finally, Section 6 presents the conclusion and future work of the study.

2. Properties of Concomitants of Generalized Order Statistics in SAR(ν)

In this section, we obtain some statistical properties of CGOSs in SAR(ν) such as marginal distribution, SF, marginal hazard rate function (HRF), reversed HRF, moment generating function (MGF), and moments. The notation $X \sim F$ is used in this section and the sequel to indicate that the RV X has the DF F.

2.1. Marginal distributions

The following theorem gives the marginal PDF for $Z_{[r,n,\underline{m},\kappa]}$.

Theorem 2.1. *Let $V_i \sim F_Z^{i+1}$, $i = 1, 2$. Then*

$$f_{[r,n,\underline{m},\kappa]}(z) = \left(1 - 3\Psi^{(\underline{m},\kappa)}_{r,n:1} + \frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right) f_Z(z) + 3\left(\Psi^{(\underline{m},\kappa)}_{r,n:1} - \frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right) f_{V_1}(z) + 5\Psi^{(\underline{m},\kappa)}_{r,n:2} f_{V_2}(z), \tag{10}$$

where $\Psi^{(\underline{m},\kappa)}_{r,n:1} = \nu\left(1 - 2C_r \sum_{i=1}^{r} \frac{\alpha_{i;r}}{\gamma_{i+1}}\right)$ and $\Psi^{(\underline{m},\kappa)}_{r,n:2} = 2\nu^2\left(1 - 6C_r \sum_{i=1}^{r} \frac{\alpha_{i;r}}{(\gamma_i+2)(\gamma_i+1)}\right)$.

Proof. *By using* (7), (8), (9), *and simple algebra, we get*

$$\begin{aligned} f_{[r,n,\underline{m},\kappa]}(z) &= \int_{-\infty}^{\infty} f_Z(z)\Bigg(1 + 3\nu(2F_Y(y) - 1)(2F_Z(z) - 1) + \frac{5}{4}\nu^2\left[3\left(2F_Y(y) - 1\right)^2 - 1\right] \\ &\times \left[3\left(2F_Z(z) - 1\right)^2 - 1\right]\Bigg) C_r \sum_{i=1}^{r} \alpha_{i;r} \overline{F}_Y^{\gamma_i - 1}(y) f_Y(y)dy \\ &= f_Z(z) + 3\left(f_{V_1}(z) - f_Z(z)\right) I_1 + \frac{5}{2}\left[2f_{V_2}(z) - 3f_{V_1}(z) + f_Z(z)\right] I_2, \end{aligned}$$

where

$$I_1 = \nu \int_{-\infty}^{\infty} \left(2F_Y(y) - 1\right) C_r \sum_{i=1}^{r} \alpha_{i;r} \overline{F}_Y^{\gamma_i - 1}(y) f_Y(y)dy = \nu\left[1 - 2C_r \sum_{i=1}^{r} \frac{\alpha_{i;r}}{\gamma_{i+1}}\right] = \Psi^{(\underline{m},\kappa)}_{r,n:1}$$

and

$$\begin{aligned} I_2 &= \nu^2 \int_{-\infty}^{\infty} \left[3\left(2F_Y(y) - 1\right)^2 - 1\right] C_r \sum_{i=1}^{r} \alpha_{i;r} \overline{F}_Y^{\gamma_i - 1}(y) f_Y(y)dy \\ &= \nu^2\left[12C_r\left(\sum_{i=1}^{r} \frac{\alpha_{i;r}}{\gamma_{i+2}} - \sum_{i=1}^{r} \frac{\alpha_{i;r}}{\gamma_{i+1}}\right) + 2\right] = \Psi^{(\underline{m},\kappa)}_{r,n:2}. \end{aligned}$$

This completes the proof. □

By using Theorem 2.1 and relying on (10), the DF, SF, HRF, and reversed HRF of $Z_{[r,n,\underline{m},\kappa]}$ based on SAR(ν) are, respectively, given by

$$F_{[r,n,\underline{m},\kappa]}(z) = \left[\left(1 - 3\Psi^{(\underline{m},\kappa)}_{r,n:1} + \frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right) + \left(3\Psi^{(\underline{m},\kappa)}_{r,n:1} - \frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right) F_Z(z) + 5\Psi^{(\underline{m},\kappa)}_{r,n:2} F_Z^2(z)\right] F_Z(z),$$

$$\overline{F}_{[r,n,\underline{m},\kappa]}(z) = \left[1 + \left(3\Psi^{(\underline{m},\kappa)}_{r,n:1} - \frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right) F_Z(z) + 5\Psi^{(\underline{m},\kappa)}_{r,n:2} F_Z^2(z)\right] \overline{F}_Z(z), \tag{11}$$

$$R_{[r,n,\underline{m},\kappa]}(z) = \frac{f_{[r,n,\underline{m},\kappa]}(z)}{\overline{F}_{[r,n,\underline{m},\kappa]}(z)} = \frac{\left[1 + 3\Psi^{(\underline{m},\kappa)}_{r,n:1}\left(2F_Z(z) - 1\right) + \frac{5}{4}\Psi^{(\underline{m},\kappa)}_{r,n:2}\left(3\left(2F_Z(z) - 1\right)^2 - 1\right)\right] R_Z(z)}{\left[1 + \left(3\Psi^{(\underline{m},\kappa)}_{r,n:1} - \frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right) F_Z(z) + 5\Psi^{(\underline{m},\kappa)}_{r,n:2} F_Z^2(z)\right]},$$

and

$$Q_{[r,n,\underline{m},\kappa]}(z) = \frac{f_{[r,n,\underline{m},\kappa]}(z)}{F_{[r,n,\underline{m},\kappa]}(z)} = \frac{\left[1+3\Psi^{(\underline{m},\kappa)}_{r,n:1}\left(2F_Z(z)-1\right)+\frac{5}{4}\Psi^{(\underline{m},\kappa)}_{r,n:2}\left(3\left(2F_Z(z)-1\right)^2-1\right)\right]Q_Z(z)}{\left[\left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)+\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)F_Z(z)+5\Psi^{(\underline{m},\kappa)}_{r,n:2}F_Z^2(z)\right]},$$

where $R(Z)$ and $Q(Z)$ are the HRF and reversed HRF of Z, respectively.

Proposition 2.2. *If $\Omega(z)$ is a function of z, then (10) yields*

$$\begin{aligned} E\left[\Omega(Z_{[r,n,\underline{m},\kappa]})\right] &= \left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)E\left[\Omega(Z)\right]+3\left(\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)E\left[\Omega(V_1)\right] \\ &+ 5\Psi^{(\underline{m},\kappa)}_{r,n:2}E\left[\Omega(V_2)\right], \end{aligned}$$

provided the expectations exist. Thus, we get the following general recurrence relation:

$$\begin{aligned} &E\left[\Omega(Z_{[r,n,\underline{m},\kappa]})\right]-E\left[\Omega(Z_{[r-1,n,\underline{m},\kappa]})\right]=\left(\frac{5}{2}\left(\Psi^{(\underline{m},\kappa)}_{r,n:2}-\Psi^{(\underline{m},\kappa)}_{r-1,n:2}\right)-3\left(\Psi^{(\underline{m},\kappa)}_{r,n:1}-\Psi^{(\underline{m},\kappa)}_{r-1,n:1}\right)\right)E\left[\Omega(Z)\right] \\ &+3E\left[\Omega(V_1)\right]\left(\left(\Psi^{(\underline{m},\kappa)}_{r,n:1}-\Psi^{(\underline{m},\kappa)}_{r-1,n:1}\right)+\frac{5}{2}\left(\Psi^{(\underline{m},\kappa)}_{r,n:2}-\Psi^{(\underline{m},\kappa)}_{r-1,n:2}\right)\right)+5E\left[\Omega(V_2)\right]\left(\Psi^{(\underline{m},\kappa)}_{r,n:2}-\Psi^{(\underline{m},\kappa)}_{r-1,n:2}\right) \\ &=6\nu C_{r-1}\left(15\nu\left[\sum_{i=1}^{r}\frac{\gamma_i\alpha_{i;r}}{(\gamma_i+1)(\gamma_i+2)}\right]-\left[\sum_{i=1}^{r}\frac{\gamma_i\alpha_{i;r}}{\gamma_i+1}\right]\right)E\left[\Omega(V_1)\right]-6\nu C_{r-1}E\left[\Omega(Z)\right] \\ &\times\left(5\nu\left[\sum_{i=1}^{r}\frac{\gamma_i\alpha_{i;r}}{(\gamma_i+1)(\gamma_i+2)}\right]-\left[\sum_{i=1}^{r}\frac{\gamma_i\alpha_{i;r}}{\gamma_i+1}\right]\right)-60\nu^2C_{r-1}\left[\sum_{i=1}^{r}\frac{\gamma_i\alpha_{i;r}}{(\gamma_i+1)(\gamma_i+2)}\right]E\left[\Omega(V_2)\right]. \end{aligned}$$

2.2. Moment generating function, moments of concomitants of generalized order statistics, and some recurrence relations

Using Theorem 2.1 and relying on (10), the MGF, moments and some recurrence relations in SAR(ν) are given by

$$M_{[r,n,\underline{m},\kappa]}(t)=\left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)M_Z(t)+\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)M_{V_1}(t)+5\Psi^{(\underline{m},\kappa)}_{r,n:2}M_{V_2}(t),$$

where $M_Z(t), M_{V_1}(t)$, and $M_{V_2}(t)$ are the MGFs of the RVs Z, V_1, and V_2, respectively. Thus, by using (10), the ℓth moment of $Z_{[r,n,\underline{m},\kappa]}$ in SAR(ν) is given by

$$\mu^{(\ell)}_{[r,n,\underline{m},\kappa]} = \left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\mu^{(\ell)}_Z+\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\mu^{(\ell)}_{V_1}+5\Psi^{(\underline{m},\kappa)}_{r,n:2}\mu^{(\ell)}_{V_2},$$

where $\mu^{(\ell)}_Z = E[Z^\ell]$, $\mu^{(\ell)}_{V_1} = E[{V_1}^\ell]$, and $\mu^{(\ell)}_{V_2} = E[{V_2}^\ell]$. From (10), we get the following general recurrence relation:

$$\begin{aligned} f_{[r,n,\underline{m},\kappa]}(z)-f_{[r-t,n-p,\underline{m},\kappa]}(z) &= 6f_{V_1}(t)\left(\Psi^{(\underline{m},\kappa)}_{r,n;t,p:1}-15\Psi^{(\underline{m},\kappa)}_{r,n;t,p:2}\right)-6f_Z(t)\left(\Psi^{(\underline{m},\kappa)}_{r,n;t,p:1}-5\Psi^{(\underline{m},\kappa)}_{r,n;t,p:2}\right) \\ &+ 60f_{V_2}(t)\Psi^{(\underline{m},\kappa)}_{r,n;t,p:2}, \end{aligned} \tag{12}$$

where

$$\Psi^{(\underline{m},\kappa)}_{r,n;t,p:1}=\nu C_r\left[\sum_{i=1}^{r}\frac{\alpha_{i;r}}{\gamma_i+1}-\sum_{i=p+1}^{r+p-t}\frac{\alpha_{i;r}}{\gamma_i+1}\right],\ 0\le t\le r-1,\ 0\le p\le n-r+1,$$

and

$$\Psi^{(\underline{m},\kappa)}_{r,n;t,p:2}=\nu^2 C_r\left[\sum_{i=1}^{r}\frac{\alpha_{i;r}}{(\gamma_i+2)(\gamma_i+1)}-\sum_{i=p+1}^{r+p-t}\frac{\alpha_{i;r}}{(\gamma_i+2)(\gamma_i+1)}\right],\ 0\le t\le r-1,\ 0\le p\le n-r+1.$$

Using (12), we get the following recurrence relations between the MGFs and moments, for the CGOSs in SAR(ν), respectively, by

$$\begin{aligned} M_{[r,n,\underline{m},\kappa]}(t) - M_{[r-t,n-p,\underline{m},\kappa]}(t) &= 6M_{V_1}(t)\left(\Psi^{(\underline{m},\kappa)}_{r,n;t,p:1} - 15\Psi^{(\underline{m},\kappa)}_{r,n;t,p:2}\right) - 6M_Z(t)\left(\Psi^{(\underline{m},\kappa)}_{r,n;t,p:1} - 5\Psi^{(\underline{m},\kappa)}_{r,n;t,p:2}\right) \\ &+ 60M_{V_2}(t)\Psi^{(\underline{m},\kappa)}_{r,n;t,p:2} \end{aligned}$$

and

$$\begin{aligned} \mu^{(\ell)}_{[r,n,\underline{m},\kappa]} - \mu^{(\ell)}_{[r-t,n-p,\underline{m},\kappa]} &= 6\mu^{(\ell)}_{V_1}(t)\left(\Psi^{(\underline{m},\kappa)}_{r,n;t,p:1} - 15\Psi^{(\underline{m},\kappa)}_{r,n;t,p:2}\right) - 6\mu^{(\ell)}_Z(t)\left(\Psi^{(\underline{m},\kappa)}_{r,n;t,p:1} - 5\Psi^{(\underline{m},\kappa)}_{r,n;t,p:2}\right) \\ &+ 60\mu^{(\ell)}_{V_2}(t)\Psi^{(\underline{m},\kappa)}_{r,n;t,p:2}. \end{aligned} \tag{13}$$

The following two theorems give some useful recurrence relations satisfied by the ℓth moments of CGOSs in SAR(ν) for any arbitrary distribution.

Theorem 2.3. *For any $\ell \in \Re^+$ and $1 \le r \le n-2$, we have*

$$\frac{\mu^{(\ell)}_{[r+2,n,\underline{m},\kappa]} - \mu^{(\ell)}_{[r,n,\underline{m},\kappa]}}{\mu^{(\ell)}_{[r+1,n,\underline{m},\kappa]} - \mu^{(\ell)}_{[r,n,\underline{m},\kappa]}} = \frac{6\Psi^{(\underline{m},\kappa)}_{r+2,n;2,0:1}\left(\mu^{(\ell)}_{V_1} - \mu^{(\ell)}_Z\right) + 30\Psi^{(\underline{m},\kappa)}_{r+2,n;2,0:2}\left(\mu^{(\ell)}_Z - 3\mu^{(\ell)}_{V_1} + 2\mu^{(\ell)}_{V_2}\right)}{6\Psi^{(\underline{m},\kappa)}_{r+1,n;1,0:1}\left(\mu^{(\ell)}_{V_1} - \mu^{(\ell)}_Z\right) + 30\Psi^{(\underline{m},\kappa)}_{r+1,n;1,0:2}\left(\mu^{(\ell)}_Z - 3\mu^{(\ell)}_{V_1} + 2\mu^{(\ell)}_{V_2}\right)} \tag{14}$$

and

$$\begin{aligned} \mu^{(\ell)}_{[r+2,n,\underline{m},\kappa]} &+ \mu^{(\ell)}_{[r+1,n,\underline{m},\kappa]} - 2\mu^{(\ell)}_{[r,n,\underline{m},\kappa]} = 6\left(\Psi^{(\underline{m},\kappa)}_{r+1,n;1,0:1} + \Psi^{(\underline{m},\kappa)}_{r+2,n;2,0:1}\right)\left(\mu^{(\ell)}_{V_1} - \mu^{(\ell)}_Z\right) \\ &+ 30\left(\Psi^{(\underline{m},\kappa)}_{r+1,n;1,0:2} + \Psi^{(\underline{m},\kappa)}_{r+2,n;2,0:2}\right)\left(\mu^{(\ell)}_Z - 3\mu^{(\ell)}_{V_1} + 2\mu^{(\ell)}_{V_2}\right). \end{aligned} \tag{15}$$

Proof. *Put $t = 2, p = 0$, and replace r by $r+2$ in* (13), *we get*

$$\mu^{(\ell)}_{[r+2,n,\underline{m},\kappa]} - \mu^{(\ell)}_{[r,n,\underline{m},\kappa]} = 6\Psi^{(\underline{m},\kappa)}_{r+2,n;2,0:1}\left(\mu^{(\ell)}_{V_1} - \mu^{(\ell)}_Z\right) + 30\Psi^{(\underline{m},\kappa)}_{r+2,n;2,0:2}\left(\mu^{(\ell)}_Z - 3\mu^{(\ell)}_{V_1} + 2\mu^{(\ell)}_{V_2}\right). \tag{16}$$

On the other hand, put $t = 1, p = 0$, and replace r by $r+1$ in (13), *we get*

$$\mu^{(\ell)}_{[r+1,n,\underline{m},\kappa]} - \mu^{(\ell)}_{[r,n,\underline{m},\kappa]} = 6\Psi^{(\underline{m},\kappa)}_{r+1,n;1,0:1}\left(\mu^{(\ell)}_{V_1} - \mu^{(\ell)}_Z\right) + 30\Psi^{(\underline{m},\kappa)}_{r+1,n;1,0:2}\left(\mu^{(\ell)}_Z - 3\mu^{(\ell)}_{V_1} + 2\mu^{(\ell)}_{V_2}\right). \tag{17}$$

Now by dividing (16) *by* (17) *we obtain* (14). *Relation* (15) *follows by adding* (16) *to* (17). □

Theorem 2.4. *For any $\ell \in \Re^+$ and $1 \le r \le n-2$, we have*

$$\frac{\mu^{(\ell)}_{[r,n,\underline{m},\kappa]} - \mu^{(\ell)}_{[r,n-2,\underline{m},\kappa]}}{\mu^{(\ell)}_{[r,n,\underline{m},\kappa]} - \mu^{(\ell)}_{[r,n-1,\underline{m},\kappa]}} = \frac{6\Psi^{(\underline{m},\kappa)}_{r,n;0,2:1}\left(\mu^{(\ell)}_{V_1} - \mu^{(\ell)}_Z\right) + 30\Psi^{(\underline{m},\kappa)}_{r,n;0,2:2}\left(\mu^{(\ell)}_Z - 3\mu^{(\ell)}_{V_1} + 2\mu^{(\ell)}_{V_2}\right)}{6\Psi^{(\underline{m},\kappa)}_{r,n;0,1:1}\left(\mu^{(\ell)}_{V_1} - \mu^{(\ell)}_Z\right) + 30\Psi^{(\underline{m},\kappa)}_{r,n;0,1:2}\left(\mu^{(\ell)}_Z - 3\mu^{(\ell)}_{V_1} + 2\mu^{(\ell)}_{V_2}\right)} \tag{18}$$

and

$$\begin{aligned} 2\mu^{(\ell)}_{[r,n,\underline{m},\kappa]} &- \mu^{(\ell)}_{[r,n-2,\underline{m},\kappa]} - \mu^{(\ell)}_{[r,n-1,\underline{m},\kappa]} = 6\left(\Psi^{(\underline{m},\kappa)}_{r,n;0,1:1} + \Psi^{(\underline{m},\kappa)}_{r,n;0,2:1}\right)\left(\mu^{(\ell)}_{V_1} - \mu^{(\ell)}_Z\right) \\ &+ 30\left(\Psi^{(\underline{m},\kappa)}_{r,n;0,2:2} + \Psi^{(\underline{m},\kappa)}_{r,n;0,1:2}\right)\left(\mu^{(\ell)}_Z - 3\mu^{(\ell)}_{V_1} + 2\mu^{(\ell)}_{V_2}\right). \end{aligned} \tag{19}$$

Proof. First, the representation (13) with $t = 0$ and $p = 2$ yields

$$\mu^{(\ell)}_{[r,n,\underline{m},\kappa]} - \mu^{(\ell)}_{[r,n-2,\underline{m},\kappa]} = 6\Psi^{(\underline{m},\kappa)}_{r,n;0,2:1}\left(\mu^{(\ell)}_{V_1} - \mu^{(\ell)}_Z\right) + 30\Psi^{(\underline{m},\kappa)}_{r,n;0,2:2}\left(\mu^{(\ell)}_Z - 3\mu^{(\ell)}_{V_1} + 2\mu^{(\ell)}_{V_2}\right). \tag{20}$$

On the other hand, use the representation (13) with $t = 0$ and $p = 1$, we get

$$\mu^{(\ell)}_{[r,n,\underline{m},\kappa]} - \mu^{(\ell)}_{[r,n-1,\underline{m},\kappa]} = 6\Psi^{(\underline{m},\kappa)}_{r,n;0,1:1}\left(\mu^{(\ell)}_{V_1} - \mu^{(\ell)}_Z\right) + 30\Psi^{(\underline{m},\kappa)}_{r,n;0,1:2}\left(\mu^{(\ell)}_Z - 3\mu^{(\ell)}_{V_1} + 2\mu^{(\ell)}_{V_2}\right). \tag{21}$$

Now, by dividing (20) by (21) we obtain (18). Finally, adding (20) to (21), we get (19). □

3. Some Uncertainty Measures in Sarmanov Family Along with Versatile Examples

In this section, we discuss the measures of WE, WEX, RE, PE, CREX, and NCREX for the concomitant $Z_{[r,n,\underline{m},\kappa]}$ of GOSs in SAR(ν) under a general framework. We consider the extended Weibull (EW) family of distributions, which was developed by Gurvich et al. [23] as a case study. The DF of EW is given by

$$F_Y(y) = 1 - \exp(-\tau H(y;\varepsilon)),\ y > 0, \tau > 0,$$

where $H(y;\varepsilon)$ is differentiable, nonnegative, continuous, and monotone increasing function when y depends on the parameter vector ε. Also, $H(y;\varepsilon) \longrightarrow 0^+$ as $y \longrightarrow 0^+$ and $H(y;\varepsilon) \longrightarrow +\infty$ as $y \longrightarrow +\infty$. This DF is denoted by EW (τ, ε) and has the following PDF

$$f_Y(y) = (\tau h(y;\varepsilon)) \exp(-\tau H(y;\varepsilon)),\ y > 0,$$

where $h(y;\varepsilon)$ is the derivative of $H(y;\varepsilon)$ with respect to y. The EW has a number of significant models, such as the Rayleigh, Pareto, Weibull, uniform, and exponential distributions. Refer to [27] for additional information about this family.

3.1. The weighted entropy and weighted extropy for Sarmanov and FGM families

Although there is no value for the shape parameter ν in (8) that causes the family to move to the FGM family, Barakat et al. [11] demonstrated that the Sarmanov family is an extension of the FGM family. Consequently, we are unable to draw any conclusions about the FGM family from the Sarmanov family. However, in this subsection, the conclusion about WE and WEX is dependent upon the radially symmetric characteristic, which is only met by the FGM and Sarmanov families (see [11]). A radially symmetric copula is a special type of copula that exhibits symmetry with respect to the center, $(\frac{1}{2}, \frac{1}{2})$, of the unit square $[0,1]^{\frac{1}{2}}$. Specifically, a copula $C(u,v)$ is said to be radially symmetric if $C(u,v) = 1 - u - v + C(1-u, 1-v)$, for all $(u,v) \in [0,1]^{\frac{1}{2}}$. We compare the WE and WEX for the FGM and Sarmanov families because of these reasons.

We have the following general result concerning any radially symmetric copula and especially concerning the FGM and Sarmanov copulas.

Proposition 3.1. *For any radially symmetric copula about $(\frac{1}{2}, \frac{1}{2})$ with density $\mathcal{L}(u,v)$, the WE*

$$N^{(\omega)}(Z_{[r,n]}) = \int_0^1 (-v\, \mathcal{L}_{[r,n,0,1]}(v)) \log \mathcal{L}_{[r,n,0,1]}(v) dv, \tag{22}$$

where $\mathcal{L}_{[r,n,0,1]}(.)$ is the PDF of the rth COS $Z_{[r,n,0,1]} := Z_{[r,n]}$ in $\mathcal{L}(u,v)$, satisfies the relation

$$N^{(\omega)}(Z_{[r,n]}) = N^{(\omega)}(Z_{[n-r+1,n]}).$$

Proof. Taking the transformation $v = \frac{1}{2} - E$ in (22), we get

$$\begin{aligned} N^{(\omega)}(Z_{[r,n]}) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(-\left(\frac{1}{2} - E\right) \mathcal{L}_{[r,n,0,1]}\left(\frac{1}{2} - E\right)\right) \log \mathcal{L}_{[r,n,0,1]}\left(\frac{1}{2} - E\right) dE \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(-\left(\frac{1}{2} + E\right) \mathcal{L}_{[n-r+1,n,0,1]}\left(\frac{1}{2} + E\right)\right) \log \mathcal{L}_{[n-r+1,n,0,1]}\left(\frac{1}{2} + E\right) dE. \end{aligned}$$

Now, let $\frac{1}{2} + E = \vartheta$, we obtain

$$N^{(\omega)}(Z_{[r,n]}) = \int_0^1 (-\vartheta \mathcal{L}_{[n-r+1,n,0,1]}(\vartheta)) \log \mathcal{L}_{[n-r+1,n,0,1]}(\vartheta) d\vartheta = N^{(\omega)}(Z_{[n-r+1,n]}).$$

This proves the proposition. $\square$

Proposition 3.2. *For any radially symmetric copula about $(\frac{1}{2}, \frac{1}{2})$, we get*

$$\xi^{(\omega)}(Z_{[r,n]}) = \xi^{(\omega)}(Z_{[n-r+1,n]}).$$

Proof. Clearly, we have

$$\xi^{(\omega)}(Z_{[r,n]}) = \frac{-1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2}+\theta\right)\left[\mathcal{L}_{[n-r+1,n,0,1]}\left(\frac{1}{2}+\theta\right)\right]^2 d\theta.$$

Thus, upon using the transformation $\tau = \frac{1}{2}+\theta$, we get

$$\xi^{(\omega)}(Z_{[r,n]}) = \frac{-1}{2}\int_0^1 \tau\left[\ \mathcal{L}_{[n-r+1,n,0,1]}(\tau)\ \right]^2 d\tau = \xi^{(\omega)}(Z_{[n-r+1,n]}).$$

This completes the proof. □

Remark 3.3. *According to [35], the marginal PDF for the COS $Z^*_{[r,n,0,1]} := Z^*_{[r:n]}$ in the FGM family is given by*

$$f^*_{[r:n]}(z) = f_Z(z)\left(1+\frac{\lambda(n-2r+1)}{n+1}\left(1-2F_Z(z)\right)\right). \tag{23}$$

Weighted entropy for Sarmanov and FGM families

Theorem 3.4. *Let $Z_{[r,n,\underline{m},\kappa]}$ be the CGOS, then from* (1) *and* (10) *the WE in SAR(ν) is given by*

$$\begin{aligned} N^{(\omega)}(Z_{[r,n,\underline{m},\kappa]}) &= N^{(\omega)}(Z)\left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)-\phi^{(\omega)}(1)\left(6\Psi^{(\underline{m},\kappa)}_{r,n:1}-15\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\\ &- \phi^{(\omega)}(2)\left(15\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)+\delta^{(\omega)}_{[r,n,\underline{m},\kappa]}, \end{aligned}$$

where $N^{(\omega)}(Z)$ is defined in (1), $\phi^{(\omega)}(p) = \int_{-\infty}^{\infty} zF_Z^p(z)f_Z(z)\log f_Z(z)dz$, and

$\delta^{(\omega)}_{[r,n,\underline{m},\kappa]} = E\left(-Z\log\left(1+3\Psi^{(\underline{m},\kappa)}_{r,n:1}\left(2F_Z(Z)-1\right)+\frac{5}{4}\Psi^{(\underline{m},\kappa)}_{r,n:2}\left(3\left(2F_Z(Z)-1\right)^2-1\right)\right)\right)$

(note that $U = F_Z(Z)$ is uniformly distributed RV).

Proof. The WE of $Z_{[r,n,\underline{m},\kappa]}$ is given by

$$\begin{aligned} N^{(\omega)}(Z_{[r,n,\underline{m},\kappa]}) &= \int_{-\infty}^{\infty}\left(-zf_{[r,n,\underline{m},\kappa]}(z)\right)\log f_{[r,n,\underline{m},\kappa]}(z)dz\\ &= \int_{-\infty}^{\infty}(-zf_Z(z))\left[1+3\Psi^{(\underline{m},\kappa)}_{r,n:1}\left(2F_Z(z)-1\right)+\frac{5}{4}\Psi^{(\underline{m},\kappa)}_{r,n:2}\left(3\left(2F_Z(z)-1\right)^2-1\right)\right]\\ &\times \log\left[f_Z(z)\left(1+3\Psi^{(\underline{m},\kappa)}_{r,n:1}\left(2F_Z(z)-1\right)+\frac{5}{4}\Psi^{(\underline{m},\kappa)}_{r,n:2}\left(3\left(2F_Z(z)-1\right)^2-1\right)\right)\right]dz\\ &= N^{(\omega)}(z)\left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)-\phi^{(\omega)}(1)\left(6\Psi^{(\underline{m},\kappa)}_{r,n:1}-15\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\\ &- \phi^{(\omega)}(2)\left(15\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)+\delta^{(\omega)}_{[r,n,\underline{m},\kappa]}. \end{aligned}$$

This completes the proof of the theorem. □

Remark 3.5. *Assume that Y and Z are EW based on SAR(ν) (SAR-EW). Then, the WE of CGOS is given by*

$$\begin{aligned} N^{(\omega)}_{EW}(Z_{[r,n,\underline{m},\kappa]}) &= N^{(\omega)}_{EW}(Z)\left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)-\phi^{(\omega)}_{EW}(1)\left(6\Psi^{(\underline{m},\kappa)}_{r,n:1}-15\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\\ &- \phi^{(\omega)}_{EW}(2)\left(15\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)+\delta^{(\omega)}_{EW[r,n,\underline{m},\kappa]}, \end{aligned}$$

where

$$N^{(\omega)}_{EW}(Z) = \int_0^{\infty}\left(-z\left[\ (\tau_2h(z;\varepsilon_2))\exp(-\tau_2H(z;\varepsilon_2))\ \right]\right)\log\left[\ (\tau_2h(z;\varepsilon_2))\exp(-\tau_2H(z;\varepsilon_2))\ \right]dz,$$

$$\phi^{(\omega)}_{EW}(p) = \int_0^\infty z\Big[1-\exp(-\tau_2 H(z;\varepsilon_2))\Big]^p\Big[(\tau_2 h(z;\varepsilon_2))\exp(-\tau_2 H(z;\varepsilon_2))\Big] \times \log\Big[(\tau_2 h(z;\varepsilon_2))\exp(-\tau_2 H(z;\varepsilon_2))\Big]dz,$$

and

$$\delta^{(\omega)}_{EW[r,n,\underline{m},\kappa]} = E\Big(-Z\log\Big(1+3\Psi^{(\underline{m},\kappa)}_{r,n:1}\Big(1-2\exp(-\tau_2 H(z;\varepsilon_2))\Big) + \frac{5}{4}\Psi^{(\underline{m},\kappa)}_{r,n:2}\Big(3\Big(1-2\exp(-\tau_2 H(z;\varepsilon_2))\Big)^2-1\Big)\Big)\Big).$$

Example 3.6. *Based on Remark 3.5, by choosing $H(z;\varepsilon_2) = -\log(1-z)$, and $\tau_2 = 1$ we get the Sarmanov copula, then we have $N^{(\omega)}(Z) = \phi^{(\omega)}(1) = \phi^{(\omega)}(2) = 0$. Thus, the WE of $Z_{[r,n,\underline{m},\kappa]}$ is given by $N^{(\omega)}(Z_{[r,n,\underline{m},\kappa]}) = E\Big(-Z\log\Big(1+3\Psi^{(\underline{m},\kappa)}_{r,n:1}\Big(2Z-1\Big)+\frac{5}{4}\Psi^{(\underline{m},\kappa)}_{r,n:2}\Big(3\Big(2Z-1\Big)^2-1\Big)\Big)\Big)$.*

Theorem 3.7. *Suppose that $Z^*_{[r:n]}$ be the COS based on FGM family, then from* (1) *and* (23)*, the WE is given by*

$$N^{(\omega)}(Z^*_{[r:n]}) = N^{(\omega)}(Z)\left(1+\frac{\lambda(n-2r+1)}{n+1}\right)+\left(\frac{2\lambda(n-2r+1)}{n+1}\right)\phi^{(\omega)}(1)+\psi^{(\omega)}_{[r:n]},$$

where $\psi^{(\omega)}_{[r:n]} = E\Big(-Z\log\Big(1+\frac{\lambda(n-2r+1)}{n+1}\Big(1-2F_Z(Z)\Big)\Big)\Big)$.

Proof. *The WE of $Z^*_{[r:n]}$ is given by*

$$\begin{aligned} N^{(\omega)}(Z^*_{[r:n]}) &= \int_0^\infty \left(-zf^*_{[r:n]}(z)\right)\log f^*_{[r:n]}(z)dz \\ &= \int_0^\infty (-zf_Z(z))\left(1+\frac{\lambda(n-2r+1)}{n+1}(1-2F_Z(z))\right)\log\left[f_Z(z)\left(1+\frac{\lambda(n-2r+1)}{n+1}(1-2F_Z(z))\right)\right]dz \\ &= N^{(\omega)}(Z)\left(1+\frac{\lambda(n-2r+1)}{n+1}\right)+\left(\frac{2\lambda(n-2r+1)}{n+1}\right)\phi^{(\omega)}(1)+\psi^{(\omega)}_{[r:n]}. \end{aligned}$$

This completes the proof of the theorem. □

Example 3.8. *For the FGM copula, we have $N^{(\omega)}(Z) = \phi^{(\omega)}(1) = 0$. Thus, the WE of $Z^*_{[r:n]}$ is given by $N^{(\omega)}(Z^*_{[r:n]}) = E\Big(-Z\log\Big(1+\frac{\lambda(n-2r+1)}{n+1}(1-2Z)\Big)\Big)$.*

Table 1 displays a comparison between the WE of the rth concomitant $Z_{[r,n,\underline{m},\kappa]}$ in the Sarmanov copula and the rth concomitant $Z^*_{[r:n]}$ in the FGM copula. The choices of ν and λ in Table 1 (as well as in the next table 4) are set so that the correlations of the Sarmanov and FGM copulas, $\rho = \nu$ and $\rho^* = \frac{\lambda}{3}$, are equal. With the same degrees of correlation, this action enables us to compare the various uncertainty measures between the two families. Table 2 displays the WE of the concomtiant, $Z_{[r,n,1,1]}$, of the rth sequential order statistic, $Y_{r,n,1,1}$, in Sarmanov copula, where $m_1 = m_2 ... = m_{n-1} = \kappa = 1$, see [29]. The computations are carried out by using MATHEMATICA ver.12. The following properties can be extracted from Tables 1 and 2:

- Generally, we have $N^{(\omega)}(Z_{[r,n]}) := N^{(\omega)}(Z_{[r,n]};\rho) \le N^{(\omega)}(Z^*_{[r:n]};\rho^*) := N^{(\omega)}(Z^*_{[r:n]})$, $\rho = \rho^*$ at $r \le \frac{n+1}{2}$, and $N^{(\omega)}(Z_{[r,n]};\rho) \ge N^{(\omega)}(Z^*_{[r:n]};\rho^*)$ at $r \ge \frac{n+1}{2}$.
- The value of $N^{(\omega)}(Z_{[r,n]};\rho)$ and $N^{(\omega)}(Z^*_{[r:n]};-\rho^*)$ increases as the value of r increases. In contrast, the value of $N^{(\omega)}(Z_{[r,n]};-\rho)$ and $N^{(\omega)}(Z^*_{[r:n]};\rho^*)$ decreases as the value of r increases.
- Generally, $N^{(\omega)}(Z_{[r,n]};-\rho) = N^{(\omega)}(Z_{[n-r+1,n]};\rho)$, and $N^{(\omega)}(Z^*_{[r:n]};-\rho^*) = N^{(\omega)}(Z^*_{[n-r+1:n]};\rho^*)$.

- The value of $N^{(\omega)}(Z_{[r,n]};-\rho) = N^{(\omega)}(Z_{[r,n]};\rho)$ at $r = \frac{n+1}{2}$, and $N^{(\omega)}(Z^*_{[r:n]};-\rho^*) = N^{(\omega)}(Z^*_{[r:n]};\rho^*) = 0$, at $r = \frac{n+1}{2}$.
- The value of $N^{(\omega)}(Z_{[r,n,1,1]};\rho)$ ($Z_{[r,n,1,1]}$ is rth SOS) increases as the value of r increases for ($\rho = \nu > 0$). In contrast, the value of $N^{(\omega)}(Z_{[r,n,1,1]};\rho)$ decreases as the value of r increases for($\rho = \nu < 0$).

Table 1: $N^{(\omega)}(Z_{[r,n]};\rho)$ and $N^{(\omega)}(Z^*_{[r:n]};\rho^*)$ in the Sarmanov and FGM copulas

$N^{(\omega)}(Z_{[r,n]};\rho)$						$N^{(\omega)}(Z^*_{[r:n]};\rho^*)$					
n	**r**	$\nu = 0.2$ $\rho = 0.2$	$\nu = -0.2$ $\rho = -0.2$	$\nu = 0.3$ $\rho = 0.3$	$\nu = -0.3$ $\rho = -0.3$	**n**	**r**	$\lambda = 0.6$ $\rho^* = 0.2$	$\lambda = -0.6$ $\rho^* = -0.2$	$\lambda = 0.9$ $\rho^* = 0.3$	$\lambda = -0.9$ $\rho^* = -0.3$
5	1	-0.0805294	0.0537157	-0.131916	0.0711481	5	1	0.052004	-0.0791182	0.0648656	-0.127289
5	2	-0.0362364	0.0293815	-0.0561734	0.0401953	5	2	0.029852	-0.0365457	0.0419717	-0.0571103
5	3	-0.000164193	-0.000164193	-0.000837677	-0.000837677	5	3	0	0	0	0
5	4	0.0293815	-0.0362364	0.0401953	-0.0561734	5	4	-0.0365457	0.029852	-0.0571103	0.0419717
5	5	0.0537157	-0.0805294	0.0711481	-0.131916	5	5	-0.0791182	0.052004	-0.127289	0.0648656
7	1	-0.0929843	0.0589709	-0.15445	0.0771057	7	1	0.0561715	-0.0906482	0.0667842	-0.146745
7	2	-0.0571103	0.0419717	-0.0906482	0.0561715	7	2	0.0419717	-0.0571103	0.0561715	-0.0906482
7	3	-0.0264743	0.0223857	-0.0406067	0.030419	7	3	0.0230642	-0.0268227	0.0330676	-0.0415485
7	4	-0.000223705	-0.000223705	-0.00114299	-0.00114299	7	4	0	0	0	0
7	5	0.0223857	-0.0264743	0.030419	-0.0406067	7	5	-0.0268227	0.0230642	-0.0415485	0.0330676
7	6	0.0419717	-0.0571103	0.0561715	-0.0906482	7	6	-0.0571103	0.0419717	-0.0906482	0.0561715
7	7	0.0589709	-0.0929843	0.0771057	-0.15445	7	7	-0.0906482	0.0561715	-0.146745	0.0667842
9	1	-0.100796	0.0620182	-0.168926	0.0804768	9	1	0.0583824	-0.0977317	0.0670384	-0.158779
9	2	-0.0706917	0.0489644	-0.114027	0.0647657	9	2	0.0482542	-0.0701451	0.0621149	-0.112267
9	3	-0.0442169	0.0343585	-0.0689975	0.0460416	9	3	0.0349386	-0.0445948	0.0482542	-0.0701451
9	4	-0.0208874	0.0180366	-0.031978	0.0242549	9	4	0.0187694	-0.0211728	0.0271933	-0.0326109
9	5	-0.000266397	-0.000266397	-0.00136244	-0.00136244	9	5	0	0	0	0
9	6	0.0180366	-0.0208874	0.0242549	-0.031978	9	6	-0.0211728	0.0187694	-0.0326109	0.0271933
9	7	0.0343585	-0.0442169	0.0460416	-0.0689975	9	7	-0.0445948	0.0349386	-0.0701451	0.0482542
9	8	0.0489644	-0.0706917	0.0647657	-0.114027	9	8	-0.0701451	0.0482542	-0.112267	0.0621149
9	9	0.0620182	-0.100796	0.0804768	-0.168926	9	9	-0.0977317	0.0583824	-0.158779	0.0670384

Table 2: $N^{(\omega)}(Z_{[r,n,1,1]};\rho)$ in SAR(ν) copula

n	**r**	$\rho = \nu = 0.2$	$\rho = \nu = -0.2$	$\rho = \nu = 0.3$	$\rho = \nu = -0.3$
5	1	-0.100796	0.0620182	-0.168926	0.0804768
5	2	-0.0672196	0.0472565	-0.107982	0.0626934
5	3	-0.0336798	0.0275186	-0.0519846	0.0372313
5	4	0.00131241	-0.00173134	0.0011532	-0.00325082
5	5	0.0428106	-0.0583617	0.0578873	-0.0931077
7	1	-0.110064	0.0654141	-0.18647	0.0841328
7	2	-0.0855241	0.0557444	-0.140452	0.0729163
7	3	-0.0612572	0.0440931	-0.0975182	0.0583759
7	4	-0.0369554	0.0296362	-0.0571796	0.0396479
7	5	-0.0120468	0.0107	-0.0186348	0.0140989
7	6	0.0147692	-0.0167853	0.0198971	-0.0257021
7	7	0.0478843	-0.0682395	0.0639566	-0.11004
9	1	-0.115383	0.0672634	-0.196725	0.0860617
9	2	-0.0960151	0.0600804	-0.159723	0.0779949
9	3	-0.0768849	0.0518234	-0.124773	0.0679417
9	4	-0.0578804	0.0422192	-0.0916733	0.0557707
9	5	-0.0388331	0.0308132	-0.0601683	0.0409563
9	6	-0.0194654	0.0168069	-0.02987	0.0222607
9	7	0.000739645	-0.00135563	-6.50073E-05	-0.00306784
9	8	0.0229514	-0.0273641	0.0308569	-0.0420018
9	9	0.0510979	-0.0749602	0.0677453	-0.121784

Weighted extropy for Sarmanov and FGM families

Theorem 3.9. *If $Z_{[r,n,\underline{m},\kappa]}$ is the CGOS, then from* (4) *and* (10) *the WEX is given by*

$$\begin{aligned}
\xi^{(\omega)}(Z_{[r,n,\underline{m},\kappa]}) &= \left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\xi^{(\omega)}(Z)+\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\xi^{(\omega)}(V_1)\\
&+ \left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\xi^{(\omega)}(V_2)-\left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)E(Zf_{V_1}(Z))\\
&- \left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)E(Zf_{V_2}(Z))-\frac{3}{2}\left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\\
&\times \left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)E(Zf_{V_3}(Z)),
\end{aligned}$$

where $\xi^{(\omega)}(Z)=\frac{-1}{2}\int_{-\infty}^{\infty}zf_Z^2(z)dz$ *and* $\xi^{(\omega)}(V_i)=\frac{-1}{2}\int_{-\infty}^{\infty}zf_{V_i}^2(z)dz$ *are the WEX measures of the RVs Z and* V_i, $i=1,2$, *respectively, and* $E(Zf_{V_i}(Z))=\int_{-\infty}^{\infty}zf_Z(z)f_{V_i}(z)dz,\ i=1,2,3.$

Proof. *By using* (4) *and* (10), *then the WEX is given by*

$$\begin{aligned}
\xi^{(\omega)}(Z_{[r,n,\underline{m},\kappa]}) &= \frac{-1}{2}\int_{-\infty}^{\infty}zf^2_{[r,n,\underline{m},\kappa]}(z)dz\\
&= \frac{-1}{2}\int_{-\infty}^{\infty}z\Bigg[\left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)f_Z(z)+\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)f_{V_1}(z)\\
&+ 5\Psi^{(\underline{m},\kappa)}_{r,n:2}f_{V_2}(z)\Bigg]^2dz\\
&= \left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\xi^{(\omega)}(z)+\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\xi^{(\omega)}(V_1)\\
&+ \left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\xi^{(\omega)}(V_2)-\left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)E(Zf_{V_1}(Z))\\
&- \left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)E(Zf_{V_2}(Z))-\frac{3}{2}\left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\\
&\times \left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)E(Zf_{V_3}(Z)).
\end{aligned}$$

This completes the proof. □

Remark 3.10. *Assume that Y and Z are jointly distributed as SAR-EW. Then, the WEX of CGOS is given by*

$$\begin{aligned}
\xi^{(\omega)}_{EW}(Z_{[r,n,\underline{m},\kappa]}) &= \left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\xi^{(\omega)}_{EW}(Z)+\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\xi^{(\omega)}_{EW}(V_1)\\
&+ \left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\xi^{(\omega)}_{EW}(V_2)-\left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\\
&\times E_{EW}(Zf_{V_1}(Z))-\left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)E_{EW}(Zf_{V_2}(Z))\\
&- \frac{3}{2}\left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)E_{EW}(Zf_{V_3}(Z)),
\end{aligned}$$

where

$$\xi^{(\omega)}_{EW}(Z)=\frac{-1}{2}\int_0^{\infty}z\Big[(\tau_2h(z;\varepsilon_2))\exp(-\tau_2H(z;\varepsilon_2))\Big]^2dz,$$

$$\xi^{(\omega)}_{EW}(V_i)=\frac{-1}{2}\int_0^{\infty}\left(i+1\right)^2z\left(\Big[1-\exp(-\tau_2H(z;\varepsilon_2))\Big]^i\Big[(\tau_2h(z;\varepsilon_2))\exp(-\tau_2H(z;\varepsilon_2))\Big]\right)^2dz,$$

and

$$E_{EW}(Zf_{V_i}(Z))=\int_0^{\infty}(i+1)z\Big[1-\exp(-\tau_2H(z;\varepsilon_2))\Big]^i\Big[(\tau_2h(z;\varepsilon_2))\exp(-\tau_2H(z;\varepsilon_2))\Big]^2dz.$$

Example 3.11. *Based on Remark 3.10, by choosing $H(z;\varepsilon_2) = z$, and $\tau_2 = \theta_2$, we get a Sarmanov family with exponential margins (denoted by SAR-ED). After simple algebra, we get $\xi^{(\omega)}(Z) = \frac{-1}{8}$, $\xi^{(\omega)}(V_1) = \frac{-13}{72}$, $\xi^{(\omega)}(V_2) = \frac{-87}{400}$, $E(Zf_{V_1}(Z)) = \frac{5}{18}$, $E(Zf_{V_2}(Z)) = \frac{13}{48}$, and $E(Zf_{V_3}(Z)) = \frac{77}{300}$. Then,*

$$\begin{aligned}\xi^{(\omega)}(Z_{[r,n,\underline{m},\kappa]}) &= \frac{-1}{8}\left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2-\frac{13}{72}\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\\ &- \frac{87}{400}\left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2-\frac{5}{18}\left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\\ &- \frac{13}{48}\left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)-\frac{77}{200}\left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\\ &\times \left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right).\end{aligned}$$

Example 3.12. *Based on Remark 3.10, by choosing $H(z;\varepsilon_2) = -\log(1-z)$, and $\tau_2 = 1$ we get the Sarmanov copula, then we have $\xi^{(\omega)}(Z) = \frac{-1}{4}$, $\xi^{(\omega)}(V_1) = \frac{-1}{2}$, $\xi^{(\omega)}(V_2) = \frac{-3}{4}$, $E(Zf_{V_1}(Z)) = \frac{2}{3}$, $E(Zf_{V_2}(Z)) = \frac{3}{4}$, and $E(Zf_{V_3}(Z)) = \frac{4}{5}$. Then,*

$$\begin{aligned}\xi^{(\omega)}(Z_{[r,n,\underline{m},\kappa]}) &= \frac{-1}{4}\left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2-\frac{1}{2}\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\\ &- \frac{3}{4}\left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2-\frac{2}{3}\left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\\ &- \frac{3}{4}\left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)-\frac{6}{5}\left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\\ &\times \left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right).\end{aligned}$$

Theorem 3.13. *Suppose that $Z^*_{[r:n]}$ be the COS based on FGM family, then from* (4) *and* (23)*, the WEX is given by*

$$\xi^{(\omega)}(Z^*_{[r:n]}) = \xi^{(\omega)}(Z)-\frac{1}{2}\left(\frac{\lambda(n-2r+1)}{n+1}\right)^2 H^{(\omega)}(2)-\left(\frac{\lambda(n-2r+1)}{n+1}\right)H^{(\omega)}(1),$$

where $\xi^{(\omega)}(Z) = \frac{-1}{2}\int_{-\infty}^{\infty} zf_Z^2(z)dz$, and $H^{(\omega)}(p) = \int_{-\infty}^{\infty} zf_Z^2(z)\left(1-2F_Z(z)\right)^p dz,\ p = 1, 2$.

Proof. *The WEX of $Z^*_{[r:n]}$ is given by*

$$\begin{aligned}\xi^{(\omega)}(Z^*_{[r:n]}) &= \frac{-1}{2}\int_{-\infty}^{\infty} zf^{*2}_{[r:n]}(z)dz\\ &= \frac{-1}{2}\int_{-\infty}^{\infty} z\left(f_Z(z)\left(1+\frac{\lambda(n-2r+1)}{n+1}(1-2F_Z(z))\right)\right)^2 dz\\ &= \xi^{(\omega)}(Z)-\left(\frac{\lambda(n-2r+1)}{n+1}\right)H^{(\omega)}(1)-\frac{1}{2}\left(\frac{\lambda(n-2r+1)}{n+1}\right)^2 H^{(\omega)}(2).\end{aligned}$$

This completes the proof. □

Example 3.14. *For the FGM copula, we have $\xi^{(\omega)}(Z) = \frac{-1}{4}$, $H^{(\omega)}(1) = \frac{-1}{6}$, and $H^{(\omega)}(2) = \frac{1}{6}$.*
Thus, the WEX of $Z^*_{[r:n]}$ is given by

$$\xi^{(\omega)}(Z^*_{[r:n]}) = \frac{-1}{4}+\frac{1}{6}\left(\frac{\lambda(n-2r+1)}{n+1}\right)-\frac{1}{12}\left(\frac{\lambda(n-2r+1)}{n+1}\right)^2.$$

Table 3 displays the WEX of $Z_{[r,n,\underline{m},\kappa]}$ in SAR(ν) with exponential marginals. From Table 3, the following properties can be extracted:

- Generally, by adopting $\xi^{(\omega)}(Z_{[r,n]}) := \xi^{(\omega)}(Z_{[r,n]};\nu)$, we have $\xi^{(\omega)}(Z_{[r,n]};-\nu) = \xi^{(\omega)}(Z_{[n-r+1,n]};\nu)$.
- The value of $\xi^{(\omega)}(Z_{[r,n]};\nu)$ increases as the value of r increases for $(\nu > 0)$. In contrast, the value of $\xi^{(\omega)}(Z_{[r,n]};\nu)$ decreases as the value of r increases for $(\nu < 0)$.
- The value of $\xi^{(\omega)}(Z_{[r,n]};-\nu) = \xi^{(\omega)}(Z_{[r,n]};\nu)$ at $r = \frac{n+1}{2}$.
- The value of $\xi^{(\omega)}(Z_{[r,n,1,1]};\nu)$ increases as the value of r increases for $(\nu > 0)$. In contrast, the value of $\xi^{(\omega)}(Z_{[r,n,1,1]};\nu)$ decreases as the value of r increases for$(\nu < 0)$.

Table 4 displays a comparison between the WEX of the rth concomitant $Z_{[r,n,\underline{m},\kappa]}$ in the Sarmanov copula and the rth concomitant $Z^*_{[r:n]}$ in the FGM copula. Table 5 displays the WEX of $Z_{[r,n,1,1]}$ in Sarmanov copula. The following properties can be extracted from Tables 4 and 5

- Generally, we have $\xi^{(\omega)}(Z_{[r,n]};\rho) \leq \xi^{(\omega)}(Z^*_{[r:n]};\rho^*)$ at $r \leq \frac{n+1}{2}$, and $\xi^{(\omega)}(Z_{[r,n]};\rho) \geq \xi^{(\omega)}(Z^*_{[r:n]};\rho^*)$ at $r \geq \frac{n+1}{2}$.
- The value of $\xi^{(\omega)}(Z_{[r,n]};\rho)$ and $\xi^{(\omega)}(Z^*_{[r:n]};-\rho^*)$ increases as the value of r increases. In contrast, the value of $\xi^{(\omega)}(Z_{[r,n]};-\rho)$ and $\xi^{(\omega)}(Z^*_{[r:n]};\rho^*)$ decreases as the value of r increases.
- Generally, $\xi^{(\omega)}(Z_{[r,n]};-\rho) = \xi^{(\omega)}(Z_{[n-r+1,n]};\rho)$, and $\xi^{(\omega)}(Z^*_{[r:n]};-\rho^*) = \xi^{\omega}(Z^*_{[n-r+1:n]};\rho^*)$.
- The value of $\xi^{(\omega)}(Z_{[r,n]};-\rho) = \xi^{(\omega)}(Z_{[r,n]};\rho)$ at $r = \frac{n+1}{2}$, and $\xi^{(\omega)}(Z^*_{[r:n]};-\rho^*) = \xi^{(\omega)}(Z^*_{[r:n]};\rho^*) = -0.25$, at $r = \frac{n+1}{2}$.
- The value of $\xi^{(\omega)}(Z_{[r,n,1,1]};\rho)$ increases as the value of r increases for $(\rho = \nu > 0)$. In contrast, the value of $\xi^{(\omega)}(Z_{[r,n,1,1]};\rho)$ decreases as the value of r increases for $(\rho = \nu < 0)$.

Table 3: $\xi^{(\omega)}(Z_{[r,n]};\nu)$ and $\xi^{(\omega)}(Z_{[r,n,1,1]};\nu)$ in SAR(ν) with exponential marginals

		$\xi^{(\omega)}(Z_{[r,n]};\nu)$						$\xi^{(\omega)}(Z_{[r,n,1,1]};\nu)$			
n	**r**	$\nu = 0.2$	$\nu = -0.2$	$\nu = 0.3$	$\nu = -0.3$	**n**	**r**	$\nu = 0.2$	$\nu = -0.2$	$\nu = 0.3$	$\nu = -0.3$
5	1	-0.138374	-0.115873	-0.147084	-0.112808	5	1	-0.14098	-0.113716	-0.151825	-0.109809
5	2	-0.132837	-0.121796	-0.13847	-0.122039	5	2	-0.137159	-0.117913	-0.145232	-0.116217
5	3	-0.127441	-0.127441	-0.13066	-0.13066	5	3	-0.132921	-0.122601	-0.138953	-0.123653
5	4	-0.121796	-0.132837	-0.122039	-0.13847	5	4	-0.127459	-0.127972	-0.130939	-0.131696
5	5	-0.115873	-0.138374	-0.112808	-0.147084	5	5	-0.118519	-0.135497	-0.116746	-0.14231
7	1	-0.140009	-0.114551	-0.150012	-0.110965	7	1	-0.142079	-0.112718	-0.153974	-0.10845
7	2	-0.135833	-0.119167	-0.143125	-0.118125	7	2	-0.139413	-0.115699	-0.148996	-0.112859
7	3	-0.131875	-0.123633	-0.137421	-0.12523	7	3	-0.136672	-0.118932	-0.144613	-0.117996
7	4	-0.127859	-0.127859	-0.131663	-0.131663	7	4	-0.133667	-0.122424	-0.140282	-0.123625
7	5	-0.123633	-0.131875	-0.12523	-0.137421	7	5	-0.130081	-0.126235	-0.135187	-0.129532
7	6	-0.119167	-0.135833	-0.118125	-0.143125	7	6	-0.125284	-0.130613	-0.127911	-0.135765
7	7	-0.114551	-0.140009	-0.110965	-0.150012	7	7	-0.117453	-0.136943	-0.115256	-0.144721
9	1	-0.14098	-0.113716	-0.151825	-0.109809	9	1	-0.142687	-0.112142	-0.155214	-0.107682
9	2	-0.137607	-0.117495	-0.145963	-0.115585	9	2	-0.140625	-0.114451	-0.151162	-0.111001
9	3	-0.134452	-0.121204	-0.141262	-0.12155	9	3	-0.138559	-0.116918	-0.147606	-0.114839
9	4	-0.131334	-0.124758	-0.136897	-0.127203	9	4	-0.136415	-0.11954	-0.144327	-0.119082
9	5	-0.128127	-0.128127	-0.132309	-0.132309	9	5	-0.13409	-0.122323	-0.141035	-0.123608
9	6	-0.124758	-0.131334	-0.127203	-0.136897	9	6	-0.13143	-0.125292	-0.137328	-0.128302
9	7	-0.121204	-0.134452	-0.12155	-0.141262	9	7	-0.128173	-0.128527	-0.132589	-0.133109
9	8	-0.117495	-0.137607	-0.115585	-0.145963	9	8	-0.123793	-0.132291	-0.125729	-0.138279
9	9	-0.113716	-0.14098	-0.109809	-0.151825	9	9	-0.116718	-0.137874	-0.114208	-0.146293

Table 4: $\xi^{(\omega)}(Z_{[r,n]};\rho)$ and $\xi^{(\omega)}(Z^*_{[r:n]};\rho^*)$ in the Sarmanov and FGM copulas

$\xi^{(\omega)}(Z_{[r,n]};\rho)$						$\xi^{(\omega)}(Z^*_{[r:n]};\rho^*)$					
n	**r**	$\nu=0.2$ $\rho=0.2$	$\nu=-0.2$ $\rho=-0.2$	$\nu=0.3$ $\rho=0.3$	$\nu=-0.3$ $\rho=-0.3$	**n**	**r**	$\lambda=0.6$ $\rho^*=0.2$	$\lambda=-0.6$ $\rho^*=-0.2$	$\lambda=0.9$ $\rho^*=0.3$	$\lambda=-0.9$ $\rho^*=-0.3$
5	1	-0.331687	-0.195306	-0.385969	-0.175684	5	1	-0.196667	-0.33	-0.18	-0.38
5	2	-0.286327	-0.220422	-0.306421	-0.208992	5	2	-0.22	-0.286667	-0.2075	-0.3075
5	3	-0.250163	-0.250163	-0.250827	-0.250827	5	3	-0.25	-0.25	-0.25	-0.25
5	4	-0.220422	-0.286327	-0.208992	-0.306421	5	4	-0.286667	-0.22	-0.3075	-0.2075
5	5	-0.195306	-0.331687	-0.175684	-0.385969	5	5	-0.33	-0.196667	-0.38	-0.18
7	1	-0.344722	-0.189722	-0.410664	-0.168789	7	1	-0.191875	-0.341875	-0.175469	-0.400469
7	2	-0.3075	-0.2075	-0.341875	-0.191875	7	2	-0.2075	-0.3075	-0.191875	-0.341875
7	3	-0.2765	-0.2275	-0.290664	-0.219039	7	3	-0.226875	-0.276875	-0.216719	-0.291719
7	4	-0.250222	-0.250222	-0.251125	-0.251125	7	4	-0.25	-0.25	-0.25	-0.25
7	5	-0.2275	-0.2765	-0.219039	-0.290664	7	5	-0.276875	-0.226875	-0.291719	-0.216719
7	6	-0.2075	-0.3075	-0.191875	-0.341875	7	6	-0.3075	-0.2075	-0.341875	-0.191875
7	7	-0.189722	-0.344722	-0.168789	-0.410664	7	7	-0.341875	-0.191875	-0.400469	-0.175469
9	1	-0.352977	-0.18646	-0.42682	-0.164828	9	1	-0.1892	-0.3492	-0.1732	-0.4132
9	2	-0.321443	-0.200221	-0.366526	-0.182402	9	2	-0.2008	-0.3208	-0.1843	-0.3643
9	3	-0.294377	-0.215308	-0.319443	-0.202585	9	3	-0.2148	-0.2948	-0.2008	-0.3208
9	4	-0.270897	-0.231886	-0.281998	-0.225336	9	4	-0.2312	-0.2712	-0.2227	-0.2827
9	5	-0.250264	-0.250264	-0.251339	-0.251339	9	5	-0.25	-0.25	-0.25	-0.25
9	6	-0.231886	-0.270897	-0.225336	-0.281998	9	6	-0.2712	-0.2312	-0.2827	-0.2227
9	7	-0.215308	-0.294377	-0.202585	-0.319443	9	7	-0.2948	-0.2148	-0.3208	-0.2008
9	8	-0.200221	-0.321443	-0.182402	-0.366526	9	8	-0.3208	-0.2008	-0.3643	-0.1843
9	9	-0.18646	-0.352977	-0.164828	-0.42682	9	9	-0.3492	-0.1892	-0.4132	-0.1732

Table 5: $\xi^{(\omega)}(Z_{[r,n,1,1]};\rho)$ in SAR(ν) copula

n	**r**	$\rho=\nu=0.2$	$\rho=\nu=-0.2$	$\rho=\nu=0.3$	$\rho=\nu=-0.3$
5	1	-0.352977	-0.18646	-0.42682	-0.164828
5	2	-0.317865	-0.202007	-0.360102	-0.184708
5	3	-0.283743	-0.222294	-0.30214	-0.211938
5	4	-0.248685	-0.251731	-0.248822	-0.253243
5	5	-0.206685	-0.308807	-0.190338	-0.344571
7	1	-0.362856	-0.182807	-0.446711	-0.160474
7	2	-0.336855	-0.193088	-0.39508	-0.173258
7	3	-0.311725	-0.205266	-0.348987	-0.189242
7	4	-0.287037	-0.220121	-0.307383	-0.209277
7	5	-0.262048	-0.23927	-0.268644	-0.235713
7	6	-0.235181	-0.266789	-0.229827	-0.275714
7	7	-0.201399	-0.318941	-0.183593	-0.362401
9	1	-0.368568	-0.180809	-0.458495	-0.158144
9	2	-0.347886	-0.188486	-0.416387	-0.167474
9	3	-0.327837	-0.197192	-0.377973	-0.178677
9	4	-0.308262	-0.20719	-0.342832	-0.19196
9	5	-0.288927	-0.218909	-0.310403	-0.207812
9	6	-0.269472	-0.233118	-0.279887	-0.227323
9	7	-0.249257	-0.251354	-0.250027	-0.253047
9	8	-0.226916	-0.277391	-0.218503	-0.29206
9	9	-0.198027	-0.325881	-0.179322	-0.374934

3.2. Residual extropy

Theorem 3.15. *Suppose $Z_{[r,n,\underline{m},\kappa]}$ is the CGOS based on SAR(ν), then from* (2) *and* (10) *the RE is given by*

$$\begin{aligned}
\xi_t(Z_{[r,n,\underline{m},\kappa]}) &= \left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\xi_t(Z)+\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\xi_t(V_1)\\
&+ \left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\xi_t(V_2)-\left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\frac{E(f_{V_1}(Z))}{\overline{F}^2_{[r,n,\underline{m},\kappa]}(t)}\\
&- \left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\frac{E(f_{V_2}(Z))}{\overline{F}^2_{[r,n,\underline{m},\kappa]}(t)}-\frac{3}{2}\left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\\
&\times \left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\frac{E(f_{V_3}(Z))}{\overline{F}^2_{[r,n,\underline{m},\kappa]}(t)},
\end{aligned}$$

where $\xi_t(Z)=\frac{-1}{2\overline{F}^2_Z(t)}\int_t^\infty f_Z^2(z)dz$ and $\xi_t(V_i)=\frac{-1}{2\overline{F}^2_{V_i}(t)}\int_t^\infty f_{V_i}^2(z)dz$, $i=1,2$, are the RE measures of the RVs Z and V_i, respectively, and $E(f_{V_i}(Z))=\int_t^\infty f_Z(z)f_{V_i}(z)dz$, $i=1,2,3$.

Proof. We have

$$\begin{aligned}
\xi_t(Z_{[r,n,\underline{m},\kappa]}) &= \frac{-1}{2\overline{F}^2_{[r,n,\underline{m},\kappa]}(t)}\int_t^\infty f^2_{[r,n,\underline{m},\kappa]}(z)dz\\
&= \frac{-1}{2\overline{F}^2_{[r,n,\underline{m},\kappa]}(t)}\int_t^\infty\Bigg[\left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)f_Z(z)+\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)f_{V_1}(z)\\
&+ 5\Psi^{(\underline{m},\kappa)}_{r,n:2}f_{V_2}(z)\Bigg]^2dz\\
&= \left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\xi_t(Z)+\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\xi_t(V_1)\\
&+ \left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\xi_t(V_2)-\left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\frac{E(f_{V_1}(Z))}{\overline{F}^2_{[r,n,\underline{m},\kappa]}(t)}\\
&- \left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\frac{E(f_{V_2}(Z))}{\overline{F}^2_{[r,n,\underline{m},\kappa]}(t)}-\frac{3}{2}\left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\\
&\times \left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\frac{E(f_{V_3}(Z))}{\overline{F}^2_{[r,n,\underline{m},\kappa]}(t)}.
\end{aligned}$$

This completes the proof. □

Remark 3.16. *Assume that Y and Z are SAR-EW. Then, the RE of CGOS is given by*

$$\begin{aligned}
\xi_{tEW}(Z_{[r,n,\underline{m},\kappa]}) &= \left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\xi_{tEW}(Z)+\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\xi_{tEW}(V_1)+\left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\\
&\times \xi_{tEW}(V_2)-\left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\frac{E_{EW}(f_{V_1}(Z))}{\overline{F}^2_{EW[r,n,\underline{m},\kappa]}(t)}\\
&- \left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\frac{E_{EW}(f_{V_2}(Z))}{\overline{F}^2_{EW[r,n,\underline{m},\kappa]}(t)}-\frac{3}{2}\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\\
&\times \left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\frac{E_{EW}(f_{V_3}(Z))}{\overline{F}^2_{EW[r,n,\underline{m},\kappa]}(t)},
\end{aligned}$$

where

$$\xi_{t(EW)}(Z)=\frac{-1}{2\overline{F}^{2}_{Z(EW)}(t)}\int_t^{\infty}\Big[\,(\tau_2 h(z;\varepsilon_2))\exp(-\tau_2 H(z;\varepsilon_2))\,\Big]^2 dz,$$

$$\xi_{t(EW)}(V_i)=\frac{-1}{2\overline{F}^{2}_{V_i(EW)}(t)}\int_t^{\infty}\Big(i+1\Big)^2\Bigg(\Big[\,1-\exp(-\tau_2 H(z;\varepsilon_2))\,\Big]^i\Big[\,(\tau_2 h(z;\varepsilon_2))\exp(-\tau_2 H(z;\varepsilon_2))\,\Big]\Bigg)^2 dz,$$

and

$$E_{EW}(f_{V_i}(Z))=\int_t^{\infty}(i+1)\Big[\,1-\exp(-\tau_2 H(z;\varepsilon_2))\,\Big]^i\Big[\,(\tau_2 h(z;\varepsilon_2))\exp(-\tau_2 H(z;\varepsilon_2))\,\Big]^2 dz.$$

Example 3.17. *Based on Remark 3.16, by choosing $H(z;\varepsilon_2)=-\log(1-t^c)$, and $\tau_2=1$, we get a Sarmanov family with power distribution margins, where DF $F_Y(t)=F_Z(t)=t^c,\ 0\le t\le 1, c>0$. Thus, we get*

$$\xi_t(Z)=\frac{-c^2(1-t^{2c-1})}{2(2c-1)\overline{F}^{2}_{[r,n,\underline{m},\kappa]}(t)},\ \xi_t(V_1)=\frac{-2c^2(1-t^{4c-1})}{(4c-1)\overline{F}^{2}_{[r,n,\underline{m},\kappa]}(t)},\ \xi_t(V_2)=\frac{-9c^2(1-t^{6c-1})}{2(6c-1)\overline{F}^{2}_{[r,n,\underline{m},\kappa]}(t)},$$

$$E(f_{V_1}(Z))=\frac{2c^2(1-t^{3c-1})}{3c-1},\ E(f_{V_2}(Z))=\frac{3c^2(1-t^{4c-1})}{4c-1},\ \text{and}\ E(f_{V_3}(Z))=\frac{4c^2(1-t^{5c-1})}{5c-1}.$$

Then,

$$\begin{aligned}\xi_t(Z_{[r,n,\underline{m},\kappa]}) \;=\; & \frac{1}{\overline{F}^{2}_{[r,n,\underline{m},\kappa]}(t)}\Bigg[\frac{-c^2(1-t^{2c-1})}{2(2c-1)}\Big(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\Big)^2-\frac{2c^2(1-t^{4c-1})}{4c-1}\\
\times\; & \Big(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\Big)^2-\frac{9c^2(1-t^{6c-1})}{2(6c-1)}\Big(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\Big)^2-\frac{2c^2(1-t^{3c-1})}{3c-1}\\
\times\; & \Big(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\Big)\Big(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\Big)-\frac{3c^2(1-t^{4c-1})}{4c-1}\Big(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\Big)\\
\times\; & \Big(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\Big)-\frac{6c^2(1-t^{5c-1})}{5c-1}\Big(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\Big)\Big(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\Big)\Bigg].\end{aligned}$$

Table 6 displays the RE of $Z_{[r,n,\underline{m},\kappa]}$ in Sarmanov with power marginals at $c=2$, $t=0.5$. From Table 6, the following properties can be extracted:

- Generally, $\xi_t(Z_{[r,n]};-\nu)=\xi_t(Z_{[n-r+1,n]};\nu)$.
- The value of $\xi_t(Z_{[r,n]};\nu)$ increases as the value of r increases for $(\nu>0)$. In contrast, the value of $\xi_t(Z_{[r,n]};\nu)$ decreases as the value of r increases for $(\nu<0)$.
- The value of $\xi_t(Z_{[r,n]};-\nu)=\xi_t(Z_{[r,n]};\nu)$ at $r=\frac{n+1}{2}$.
- The value of $\xi_t(Z_{[r,n,1,1]};\nu)$ increases as the value of r increases for $(\nu>0)$.

Table 6: $\xi_t(Z_{[r,n]})$ and $\xi_t(Z_{[r,n,1,1]})$ from SAR(ν) with power marginals at $c = 2$

$\xi_t(Z_{[r,n]})$						$\xi_t(Z_{[r,n,1,1]})$					
n	**r**	$\nu = 0.2$	$\nu = -0.2$	$\nu = 0.3$	$\nu = -0.3$	**n**	**r**	$\nu = 0.2$	$\nu = -0.2$	$\nu = 0.3$	$\nu = -0.3$
5	1	-1.13512	-1.00135	-1.212	-1.00543	5	1	-1.16888	-1.00067	-1.28506	-1.01188
5	2	-1.07026	-1.01009	-1.08548	-1.00474	5	2	-1.11214	-1.00274	-1.16283	-1.00477
5	3	-1.0312	-1.0312	-1.02579	-1.02579	5	3	-1.06561	-1.01066	-1.07651	-1.00577
5	4	-1.01009	-1.07026	-1.00474	-1.08548	5	4	-1.02932	-1.032	-1.02335	-1.02665
5	5	-1.00135	-1.13512	-1.00543	-1.212	5	5	-1.00428	-1.10135	-1.00244	-1.14369
7	1	-1.15537	-1.00075	-1.25513	-1.00912	7	1	-1.18574	-1.00081	-1.32366	-1.01561
7	2	-1.09745	-1.00436	-1.13434	-1.00372	7	2	-1.14115	-1.00114	-1.22273	-1.00836
7	3	-1.05717	-1.01363	-1.0628	-1.00765	7	3	-1.10214	-1.00358	-1.14222	-1.00504
7	4	-1.03036	-1.03036	-1.02456	-1.02456	7	4	-1.06865	-1.00929	-1.08093	-1.00568
7	5	-1.01363	-1.05717	-1.00765	-1.0628	7	5	-1.04069	-1.02098	-1.03777	-1.01428
7	6	-1.00436	-1.09745	-1.00372	-1.13434	7	6	-1.01837	-1.04603	-1.01171	-1.04554
7	7	-1.00075	-1.15537	-1.00912	-1.25513	7	7	-1.00263	-1.11525	-1.00372	-1.17046
9	1	-1.16888	-1.00067	-1.28506	-1.01188	9	1	-1.19584	-1.00099	-1.34741	-1.01801
9	2	-1.11737	-1.00232	-1.17325	-1.00533	9	2	-1.15925	-1.00081	-1.26232	-1.01121
9	3	-1.07859	-1.00721	-1.09862	-1.00458	9	3	-1.12617	-1.00175	-1.19039	-1.00724
9	4	-1.05011	-1.01602	-1.05143	-1.0098	9	4	-1.09656	-1.00413	-1.13098	-1.00535
9	5	-1.02984	-1.02984	-1.02387	-1.02387	9	5	-1.07042	-1.00859	-1.08353	-1.00582
9	6	-1.01602	-1.05011	-1.0098	-1.05143	9	6	-1.04773	-1.01642	-1.04749	-1.01048
9	7	-1.00721	-1.07859	-1.00458	-1.09862	9	7	-1.02854	-1.03033	-1.02233	-1.02442
9	8	-1.00232	-1.11737	-1.00533	-1.17325	9	8	-1.01296	-1.0573	-1.00758	-1.06248
9	9	-1.00067	-1.16888	-1.01188	-1.28506	9	9	-1.00185	-1.12526	-1.00504	-1.19051

3.3. Past extropy

Theorem 3.18. *If $Z_{[r,n,\underline{m},\kappa]}$ is the CGOS based on SAR(ν), then from* (3) *and* (10) *the PE is given by*

$$\begin{aligned}
\xi^{(t)}(Z_{[r,n,\underline{m},\kappa]}) &= \left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\xi^{(t)}(Z)+\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\xi^{(t)}(V_1)\\
&+ \left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\xi^{(t)}(V_2)-\left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\\
&\times \frac{E(f_{V_1}(Z))}{F^2_{[r,n,\underline{m},\kappa]}(t)}-\left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\frac{E(f_{V_2}(Z))}{F^2_{[r,n,\underline{m},\kappa]}(t)}-\frac{3}{2}\left(5\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\\
&\times \left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)\frac{E(f_{V_3}(Z))}{F^2_{[r,n,\underline{m},\kappa]}(t)},
\end{aligned}\tag{24}$$

where $\xi^{(t)}(Z)=\frac{-1}{2F_Z^2(t)}\int_0^t f_Z^2(z)dz$ and $\xi^{(t)}(V_i)=\frac{-1}{2F_{V_i}^2(t)}\int_0^t f_{V_i}^2(z)dz$ are the PE measures of the RVs Z and V_i, $i=1,2$, respectively, and $E(f_{V_i}(Z))=\int_0^t f_Z(z)f_{V_i}(z)dz$, $i=1,2,3$.

Proof. By using (3) and (10), then the PE is given by

$$\begin{aligned}
\xi^{(t)}(Z_{[r,n,\underline{m},\kappa]}) &= \frac{-1}{2F^2_{[r,n,\underline{m},\kappa]}(t)}\int_0^t f^2_{[r,n,\underline{m},\kappa]}(z)dz\\
&= \frac{-1}{2F^2_{[r,n,\underline{m},\kappa]}(t)}\int_0^t\Bigg[\left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)f_Z(z)+\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)f_{V_1}(z)\\
&+ 5\Psi^{(\underline{m},\kappa)}_{r,n:2}f_{V_2}(z)\Bigg]^2dz\\
&= \left(1-3\Psi^{(\underline{m},\kappa)}_{r,n:1}+\frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\xi^{(t)}(Z)+\left(3\Psi^{(\underline{m},\kappa)}_{r,n:1}-\frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\xi^{(t)}(V_1)
\end{aligned}$$

$$+ \left(5\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)^2 \xi^{(t)}(V_2) - \left(1-3\Psi_{r,n:1}^{(\underline{m},\kappa)}+\frac{5}{2}\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)\left(3\Psi_{r,n:1}^{(\underline{m},\kappa)}-\frac{15}{2}\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)\frac{E(f_{V_1}(Z))}{F^2_{[r,n,\underline{m},\kappa]}(t)}$$
$$- \left(1-3\Psi_{r,n:1}^{(\underline{m},\kappa)}+\frac{5}{2}\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)\left(5\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)\frac{E(f_{V_2}(Z))}{F^2_{[r,n,\underline{m},\kappa]}(t)} - \frac{3}{2}\left(5\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)$$
$$\times \left(3\Psi_{r,n:1}^{(\underline{m},\kappa)}-\frac{15}{2}\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)\frac{E(f_{V_3}(Z))}{F^2_{[r,n,\underline{m},\kappa]}(t)}.$$

This completes the proof of the theorem. □

Remark 3.19. *Assume that Y and Z are SAR-EW. Then, the PE of CGOS is given by*

$$\begin{aligned}
\xi^{(t)}_{EW}(Z_{[r,n,\underline{m},\kappa]}) &= \left(1-3\Psi_{r,n:1}^{(\underline{m},\kappa)}+\frac{5}{2}\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)^2 \xi^{(t)}_{EW}(Z) + \left(3\Psi_{r,n:1}^{(\underline{m},\kappa)}-\frac{15}{2}\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)^2 \xi^{(t)}_{EW}(V_1) + \left(5\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)^2\\
&\times\ \xi^{(t)}_{EW}(V_2) - \left(1-3\Psi_{r,n:1}^{(\underline{m},\kappa)}+\frac{5}{2}\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)\left(3\Psi_{r,n:1}^{(\underline{m},\kappa)}-\frac{15}{2}\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)\frac{E_{EW}(f_{V_1}(Z))}{F^2_{EW[r,n,\underline{m},\kappa]}(t)}\\
&-\ \left(1-3\Psi_{r,n:1}^{(\underline{m},\kappa)}+\frac{5}{2}\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)\left(5\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)\frac{E_{EW}(f_{V_2}(Z))}{F^2_{EW[r,n,\underline{m},\kappa]}(t)} - \frac{3}{2}\left(5\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)\\
&\times\ \left(3\Psi_{r,n:1}^{(\underline{m},\kappa)}-\frac{15}{2}\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)\frac{E_{EW}(f_{V_3}(Z))}{F^2_{EW[r,n,\underline{m},\kappa]}(t)},
\end{aligned}$$

where

$$\xi^{(t)}_{EW}(Z) = \frac{-1}{2F^2_{Z(EW)}(t)}\int_0^t \Big[\ (\tau_2 h(z;\varepsilon_2))\exp(-\tau_2 H(z;\varepsilon_2))\ \Big]^2 dz,$$

$$\xi^{(t)}_{EW}(V_i) = \frac{-1}{2F^2_{V_i(EW)}(t)}\int_0^t \left(\ i+1\ \right)^2 \left(\ \Big[\ 1-\exp(-\tau_2 H(z;\varepsilon_2))]^i[(\tau_2 h(z;\varepsilon_2))\exp(-\tau_2 H(z;\varepsilon_2))\ \Big]\ \right)^2 dz,$$

and

$$E_{EW}(f_{V_i}(Z)) = \int_0^t (i+1)\Big[\ 1-\exp(-\tau_2 H(z;\varepsilon_2))\ \Big]^i \Big[\ (\tau_2 h(z;\varepsilon_2))\exp(-\tau_2 H(z;\varepsilon_2))\ \Big]^2 dz.$$

Example 3.20. *Based on Remark 3.19, by choosing $H(z;\varepsilon_2)=z$, and $\tau_2=\theta_2$ we get a SAR-ED. After simple algebra, we get $\xi^{(t)}(Z) = (\frac{\theta_2}{4F^2_{[r,n,\underline{m},\kappa]}(t)}(e^{-2t\theta_2}-1))$, $\xi^{(t)}(V_1) = (\frac{-\theta_2 e^{-4t\theta_2}}{6F^2_{[r,n,\underline{m},\kappa]}(t)}(e^{t\theta_2}-1)^3(3+e^{t\theta_2}))$, $\xi^{(t)}(V_2) = (\frac{-3\theta_2 e^{-6t\theta_2}}{20F^2_{[r,n,\underline{m},\kappa]}(t)}(e^{t\theta_2}-1)^5$ $(e^{t\theta_2}+5))$, $E(f_{V_1}(Z)) = \theta_2(\frac{1+2e^{-3t\theta_2}}{3}-e^{-2t\theta_2})$, $E(f_{V_2}(Z)) = (\frac{\theta_2 e^{-4t\theta_2}}{4}(e^{t\theta_2}-1)^3(3+e^{t\theta_2}))$, and $E(f_{V_3}(Z)) = (\frac{\theta_2 e^{-5t\theta_2}}{5}$ $(e^{t\theta_2}-1)^4(4+e^{t\theta_2}))$. Then,*

$$\begin{aligned}
\xi^{(t)}(Z_{[r,n,\underline{m},\kappa]}) &= \frac{1}{F^2_{[r,n,\underline{m},\kappa]}(t)}\Bigg[\frac{\theta_2}{4}(e^{-2t\theta_2}-1)\left(1-3\Psi_{r,n:1}^{(\underline{m},\kappa)}+\frac{5}{2}\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)^2 - \frac{\theta_2 e^{-4t\theta_2}}{6}\left(\ e^{t\theta_2}-1\ \right)^3\\
&\times\ (3+e^{t\theta_2})\left(3\Psi_{r,n:1}^{(\underline{m},\kappa)}-\frac{15}{2}\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)^2 - \frac{3\theta_2 e^{-6t\theta_2}}{20}\left(\ e^{t\theta_2}-1\ \right)^5\left(\ e^{t\theta_2}+5\ \right)\left(5\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)^2\\
&-\ \theta_2\left(\ \frac{1+2e^{-3t\theta_2}}{3}-e^{-2t\theta_2}\ \right)\left(1-3\Psi_{r,n:1}^{(\underline{m},\kappa)}+\frac{5}{2}\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)\left(3\Psi_{r,n:1}^{(\underline{m},\kappa)}-\frac{15}{2}\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)\\
&-\ \frac{\theta_2 e^{-4t\theta_2}}{4}\left(\ e^{t\theta_2}-1\ \right)^3\left(\ 3+e^{t\theta_2}\ \right)\left(1-3\Psi_{r,n:1}^{(\underline{m},\kappa)}+\frac{5}{2}\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)\left(5\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)\\
&-\ \frac{3\theta_2 e^{-5t\theta_2}}{10}\left(\ e^{t\theta_2}-1\ \right)^4\left(\ 4+e^{t\theta_2}\ \right)\left(5\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)\left(3\Psi_{r,n:1}^{(\underline{m},\kappa)}-\frac{15}{2}\Psi_{r,n:2}^{(\underline{m},\kappa)}\right)\Bigg].
\end{aligned}$$

Table 7 displays the PE of $Z_{[r,n,\underline{m},\kappa]}$ based on SAR-ED, where the mean of each exponential marginal is 1, and $t=0.3$. From Table 7, the following properties can be extracted:

- Generally, $\xi^{(t)}(Z_{[r,n]}; -\nu) = \xi^{(t)}(Z_{[n-r+1,n]}; \nu)$.
- The value of $\xi^{(t)}(Z_{[r,n]}; -\nu) = \xi^{(t)}(Z_{[r,n]}; \nu)$ at $r = \frac{n+1}{2}$.
- In the vast majority of the cases $\xi^{(t)}(Z_{[r,n]}; \nu)$ decreases as the value of r increases for $(\nu > 0)$. In contrast, the value of $\xi^{(t)}(Z_{[r,n]}; \nu)$ increases as the value of r increases for $(\nu < 0)$.
- The value of $\xi^{(t)}(Z_{[r,n,1,1]}; \nu)$ increases as the value of r increases for$(\nu < 0)$.

Table 7: $\xi^{(t)}(Z_{[r,n]})$ and $\xi^{(t)}(Z_{[r,n,1,1]})$ from SAR-ED

$\xi^{(t)}(Z_{[r,n]})$						$\xi^{(t)}(Z_{[r,n,1,1]})$					
n	**r**	$\nu = 0.2$	$\nu = -0.2$	$\nu = 0.3$	$\nu = -0.3$	**n**	**r**	$\nu = 0.2$	$\nu = -0.2$	$\nu = 0.3$	$\nu = -0.3$
5	1	-1.66818	-1.70248	-1.66678	-1.71817	5	1	-1.66824	-1.7108	-1.66818	-1.73395
5	2	-1.66932	-1.68487	-1.66682	-1.68545	5	2	-1.66782	-1.6959	-1.66705	-1.7052
5	3	-1.67419	-1.67419	-1.66966	-1.66966	5	3	-1.66897	-1.68302	-1.66706	-1.6819
5	4	-1.68487	-1.66932	-1.68545	-1.66682	5	4	-1.67407	-1.67338	-1.66935	-1.66869
5	5	-1.70248	-1.66818	-1.71817	-1.66678	5	5	-1.69371	-1.6686	-1.7017	-1.66671
7	1	-1.66813	-1.70748	-1.66726	-1.72761	7	1	-1.66855	-1.71491	-1.6705	-1.74187
7	2	-1.66808	-1.69202	-1.66712	-1.69801	7	2	-1.66772	-1.70348	-1.66667	-1.7196
7	3	-1.6696	-1.68073	-1.66689	-1.67822	7	3	-1.66768	-1.69289	-1.66772	-1.69929
7	4	-1.67348	-1.67348	-1.6687	-1.6687	7	4	-1.66844	-1.68349	-1.66764	-1.68241
7	5	-1.68073	-1.6696	-1.67822	-1.66689	7	5	-1.67084	-1.67581	-1.66698	-1.671
7	6	-1.69202	-1.66808	-1.69801	-1.66712	7	6	-1.67748	-1.67045	-1.67329	-1.66692
7	7	-1.70748	-1.66813	-1.72761	-1.66726	7	7	-1.69717	-1.66816	-1.70797	-1.66671
9	1	-1.66824	-1.7108	-1.66818	-1.73395	9	1	-1.66883	-1.71736	-1.67276	-1.74659
9	2	-1.66776	-1.69727	-1.66699	-1.70776	9	2	-1.66787	-1.70813	-1.66702	-1.72858
9	3	-1.66825	-1.68648	-1.66754	-1.68771	9	3	-1.66754	-1.69935	-1.6671	-1.7115
9	4	-1.66982	-1.67846	-1.66693	-1.67458	9	4	-1.66761	-1.69116	-1.66824	-1.69594
9	5	-1.67304	-1.67304	-1.6682	-1.6682	9	5	-1.66816	-1.68378	-1.66813	-1.68273
9	6	-1.67846	-1.66982	-1.67458	-1.66693	9	6	-1.6696	-1.67744	-1.66707	-1.67294
9	7	-1.68648	-1.66825	-1.68771	-1.66754	9	7	-1.67291	-1.67246	-1.66801	-1.66769
9	8	-1.69727	-1.66776	-1.70776	-1.66699	9	8	-1.68041	-1.66917	-1.67747	-1.66713
9	9	-1.7108	-1.66824	-1.73395	-1.66818	9	9	-1.69968	-1.66799	-1.71262	-1.66668

3.4. Cumulative residual extropy

Theorem 3.21. *The CREX of $Z_{[r,n,\underline{m},\kappa]}$ from SAR(ν) is provided by*

$$\begin{aligned}
\xi^*(Z_{[r,n,\underline{m},\kappa]}) &= \xi^*(Z) - \frac{9}{2}\left(\Psi^{(\underline{m},\kappa)}_{r,n:1}\right)^2 E\Big[\,U^2(1-U)^2\psi(U)\,\Big] - 3\Psi^{(\underline{m},\kappa)}_{r,n:1} E\Big[\,U(1-U)^2\psi(U)\,\Big] - \frac{25}{8} \\
&\times \left(\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2 E\Big[\,U^2(1-U)^2(1-2U)^2\psi(U)\,\Big] + \frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2} E\Big[\,U(1-U)^2(1-2U)\psi(U)\,\Big] \\
&+ \frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:1}\Psi^{(\underline{m},\kappa)}_{r,n:2} E\Big[\,U^2(1-U)^2(1-2U)\psi(U)\,\Big], \qquad (25)
\end{aligned}$$

where $\xi^*(Z)$ is the CREX of Z, U is a uniform RV on $(0,1)$, and $\psi(u) = \frac{1}{f_Y(\Psi(u))}$ is the quantile density function, i.e., $\psi(u) = \frac{d\Psi(u)}{du}$.

Proof. Using (5) and (11), then the CREX is provided by

$$\begin{aligned}
\xi^*(Z_{[r,n,\underline{m},\kappa]}) &= \frac{-1}{2}\int_0^\infty \overline{F}^2_{[r,n,\underline{m},\kappa]}(z)dz \\
&= \frac{-1}{2}\int_0^\infty \overline{F}^2_Z(z)\left[1 + 3\Psi^{(\underline{m},\kappa)}_{r,n:1}F_Z(z) + \frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}F_Z(z)(2F_Z(z)-1)\right]^2 dz
\end{aligned}$$

$$\begin{aligned}
&= \xi^*(Z) - \frac{9}{2}\left(\Psi^{(\underline{m},\kappa)}_{r,n:1}\right)^2 \mathrm{E}\left[U^2(1-U)^2\psi(U)\right] - 3\Psi^{(\underline{m},\kappa)}_{r,n:1}\mathrm{E}\left[U(1-U)^2\psi(U)\right] - \frac{25}{8}\\
&\times \left(\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2 \mathrm{E}\left[U^2(1-U)^2(1-2U)^2\psi(U)\right] + \frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\mathrm{E}\left[U(1-U)^2(1-2U)\psi(U)\right]\\
&+ \frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:1}\Psi^{(\underline{m},\kappa)}_{r,n:2}\mathrm{E}\left[U^2(1-U)^2(1-2U)\psi(U)\right].
\end{aligned}$$

This completes the proof of the theorem. □

Remark 3.22. *Assume that Y and Z are SAR-EW. Then, the CREX of CGOS is given by*

$$\begin{aligned}
\xi^*_{EW}(Z_{[r,n,\underline{m},\kappa]}) &= \xi^*_{EW}(Z) - \frac{9}{2}\left(\Psi^{(\underline{m},\kappa)}_{r,n:1}\right)^2 E_{EW}\left[\frac{F_Z^2(z)\bar{F}_Z^2(z)}{f_Z(z)}\right] - 3\Psi^{(\underline{m},\kappa)}_{r,n:1}E_{EW}\left[\frac{F_Z(z)\bar{F}_Z^2(z)}{f_Z(z)}\right] - \frac{25}{8}\\
&\times \left(\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2 E_{EW}\left[\frac{F_Z^2(z)\bar{F}_Z^2(z)(1-2F_Z(z))^2}{f_Z(z)}\right] + \frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}E_{EW}\left[\frac{F_Z(z)\bar{F}_Z^2(z)}{f_Z(z)}\right]\\
&\times \left(1-2F_Z(z)\right) + \frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:1}\Psi^{(\underline{m},\kappa)}_{r,n:2}E_{EW}\left[\frac{F_Z^2(z)\bar{F}_Z^2(z)(1-2F_Z(z))}{f_Z(z)}\right],
\end{aligned}$$

where

$$\xi^*_{EW}(Z) = \frac{-1}{2}\int_0^\infty \left[\exp(-\tau_2 H(z;\varepsilon_2))\right]^2 dz,$$

$$E_{EW}\left[\frac{F_Z^2(z)\bar{F}_Z^2(z)}{f_Z(z)}\right] = \int_0^\infty \left(\left[1-\exp(-\tau_2 H(z;\varepsilon_2))\right]^2\left[\exp(-\tau_2 H(z;\varepsilon_2))\right]\right)^2 dz,$$

$$E_{EW}\left[\frac{F_Z(z)\bar{F}_Z^2(z)}{f_Z(z)}\right] = \int_0^\infty \left(\left[1-\exp(-\tau_2 H(z;\varepsilon_2))\right]\left[\exp(-\tau_2 H(z;\varepsilon_2))\right]\right)^2 dz,$$

$$\begin{aligned}
E_{EW}\left[\frac{F_Z^2(z)\bar{F}_Z^2(z)(1-2F_Z(z))^2}{f_Z(z)}\right] &= \int_0^\infty\left(\left[1-\exp(-\tau_2 H(z;\varepsilon_2))\right]^2\left[\exp(-\tau_2 H(z;\varepsilon_2))\right]\right)^2\\
&\quad\times\left[2\exp(-\tau_2 H(z;\varepsilon_2))-1\right]^2 dz,
\end{aligned}$$

$$\begin{aligned}
E_{EW}\left[\frac{F_Z(z)\bar{F}_Z^2(z)(1-2F_Z(z))}{f_Z(z)}\right] &= \int_0^\infty\left(\left[1-\exp(-\tau_2 H(z;\varepsilon_2))\right]\left[\exp(-\tau_2 H(z;\varepsilon_2))\right]\right)^2\\
&\quad\times\left[2\exp(-\tau_2 H(z;\varepsilon_2))-1\right] dz,
\end{aligned}$$

and

$$\begin{aligned}
E_{EW}\left[\frac{F_Z^2(z)\bar{F}_Z^2(z)(1-2F_Z(z))}{f_Z(z)}\right] &= \int_0^\infty\left(\left[1-\exp(-\tau_2 H(z;\varepsilon_2))\right]^2\left[\exp(-\tau_2 H(z;\varepsilon_2))\right]\right)^2\\
&\quad\times\left[2\exp(-\tau_2 H(z;\varepsilon_2))-1\right] dz.
\end{aligned}$$

Example 3.23. *Based on Remark 3.22, by choosing $H(z;\varepsilon_2) = -\log(\frac{z}{\theta_2})$, and $\tau_2 = 1$, we get the Sarmanov copula*

$$F_{Y,Z}(y,z) = \frac{yz}{\theta_1\theta_2}\left[1 + 3\nu\left(1-\frac{y}{\theta_1}\right)\left(1-\frac{z}{\theta_2}\right) + 5\nu^2\left(1-\frac{2y}{\theta_1}\right)\left(1-\frac{2z}{\theta_2}\right)\left(1-\frac{y}{\theta_1}\right)\left(1-\frac{z}{\theta_2}\right)\right],$$
$$0 < y < \theta_1, 0 < z < \theta_2.$$

From (25), *we infer*

$$\xi^*(Z_{[r,n,\underline{m},\kappa]}) = \frac{-\theta_2}{6} - \frac{3\theta_2}{20}\left(\Psi^{(\underline{m},\kappa)}_{r,n:1}\right)^2 - \frac{\theta_2}{4}\Psi^{(\underline{m},\kappa)}_{r,n:1} - \frac{5\theta_2}{336}\left(\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2 + \frac{\theta_2}{24}\Psi^{(\underline{m},\kappa)}_{r,n:2}.$$

Example 3.24. *Based on Remark 3.22, by choosing $H(z;\varepsilon_2) = z$, and $\tau_2 = \theta_2$, we get a Sarmanov family with exponential margins. The CREX of $Z_{[r,n,\underline{m},\kappa]}$ is provided by*

$$\xi^*(Z_{[r,n,\underline{m},\kappa]}) = \frac{-1}{4\theta_2} - \frac{3}{8\theta_2}\left(\Psi^{(\underline{m},\kappa)}_{r,n:1}\right)^2 - \frac{1}{2\theta_2}\Psi^{(\underline{m},\kappa)}_{r,n:1} + \frac{5}{96\theta_2}\left(\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2 - \frac{1}{8\theta_2}\Psi^{(\underline{m},\kappa)}_{r,n:2}.$$

3.5. Negative cumulative residual extropy

The NCREX of $Z_{[r,n,\underline{m},\kappa]}$ from SAR(ν) is provided by

$$\begin{aligned}\tilde{\xi}(Z_{[r,n,\underline{m},\kappa]}) &= \frac{1}{2}\int_0^\infty \Big[1 - F_{[r,n,\underline{m},\kappa]}(z)\Big]^2 dz\\
&= \tilde{\xi}(Z) + \frac{9}{2}\left(\Psi^{(\underline{m},\kappa)}_{r,n:1}\right)^2 \mathrm{E}\Big[U^2(1-U)^2\psi(U)\Big] + 3\Psi^{(\underline{m},\kappa)}_{r,n:1}\mathrm{E}\Big[U(1-U)^2\psi(U)\Big] + \frac{25}{8}\\
&\times (\Psi^{(\underline{m},\kappa)}_{r,n:2})^2\mathrm{E}\Big[U^2(1-U)^2(1-2U)^2\psi(U)\Big] - \frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\mathrm{E}\Big[U(1-U)^2(1-2U)\psi(U)\Big]\\
&- \frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:1}\Psi^{(\underline{m},\kappa)}_{r,n:2}\mathrm{E}\Big[U^2(1-U)^2(1-2U)\psi(U)\Big],\end{aligned}$$

where $\tilde{\xi}(Z)$ is the NCREX of Z.

4. Estimating of Cumulative Residual Extropy Based on Concomitants of Generalized Order Statistics and SAR(ν)

In this part, we examine the issue of estimating the CREX for concomitants using the empirical CREX. Let (Y_j, Z_j), $j = 1, 2, ...,$ be a SAR sequence of random sample. Using the relation $-\xi^*(.) = \tilde{\xi}(.)$, where $\xi^*(.)$ and $\tilde{\xi}(.)$ are defined respectively in (1.5) and (1.6), the empirical CREX of $Z_{[r,n,\underline{m},\kappa]}$ can be calculated as follows:

$$\begin{aligned}\hat{\xi}^*(Z_{[r,n,\underline{m},\kappa]}) &= \frac{-1}{2}\int_0^\infty \Big[1 - \hat{F}_{[r,n,\underline{m},\kappa]}(z)\Big]^2 dz\\
&= \frac{-1}{2}\int_0^\infty \left[1 - \left(\hat{F}_Y(y)\left(1 + 3\Psi^{(\underline{m},\kappa)}_{r,n:1}(\hat{F}_Z(z)-1) + \frac{5}{4}\Psi^{(\underline{m},\kappa)}_{r,n:2}(4\hat{F}^2_Z(z) - 6\hat{F}_Z(z) + 2)\right)\right)\right]^2 dz\\
&= \frac{-1}{2}\sum_{j=1}^{n-1}\int_{z_{(j)}}^{z_{(j+1)}}\Bigg[\Big(1-\hat{F}_Z(z)\Big)^2 + 9(\Psi^{(\underline{m},\kappa)}_{r,n:1})^2\hat{F}^2_Z(z)\Big(1-\hat{F}_Z(z)\Big)^2 + 6\Psi^{(\underline{m},\kappa)}_{r,n:1}\hat{F}_Z(z)\\
&\times \Big(1-\hat{F}_Z(z)\Big)^2 + \frac{25}{16}(\Psi^{(\underline{m},\kappa)}_{r,n:2})^2\hat{F}^2_Z(z)\Big(4\hat{F}^2{}_Z(z) - 6\hat{F}_Z(z) + 2\Big)^2 - \frac{5}{2}\Psi^{(\underline{m},\kappa)}_{r,n:2}\hat{F}_Z(z)\\
&+ \Big(1-\hat{F}_Z(z)\Big)\Big(4\hat{F}^2{}_Z(z) - 6\hat{F}_Z(z) + 2\Big) - \frac{15}{2}\Psi^{(\underline{m},\kappa)}_{r,n:1}\Psi^{(\underline{m},\kappa)}_{r,n:2}\hat{F}^2{}_Z(z)\Big(1-\hat{F}_Z(z)\Big)\\
&\times \Big(4\hat{F}^2{}_Z(z) - 6\hat{F}_Z(z) + 2\Big)\Bigg]dz\\
&= \frac{-1}{2}\sum_{j=1}^{n-1}\left[\left(1-\frac{j}{n}\right)^2\Lambda_j + 9\left(\Psi^{(\underline{m},\kappa)}_{r,n:1}\right)^2\left(\frac{j}{n}\left(1-\frac{j}{n}\right)\right)^2 + 6\Psi^{(\underline{m},\kappa)}_{r,n:1}\frac{j}{n}\left(1-\frac{j}{n}\right)^2\right.\\
&+ \frac{25}{8}\left(\Psi^{(\underline{m},\kappa)}_{r,n:2}\right)^2\left(\frac{j}{n}\right)^2\left(2\left(\frac{j}{n}\right)^2 - \frac{3j}{n} + 1\right)^2 - 5\Psi^{(\underline{m},\kappa)}_{r,n:2}\frac{j}{n}\left(1-\frac{j}{n}\right)\\
&\times \left.\left(2\left(\frac{j}{n}\right)^2 - \frac{3j}{n} + 1\right) - 15\Psi^{(\underline{m},\kappa)}_{r,n:1}\Psi^{(\underline{m},\kappa)}_{r,n:2}\left(\frac{j}{n}\right)^2\left(1-\frac{j}{n}\right)\left(2\left(\frac{j}{n}\right)^2 - \frac{3j}{n} + 1\right)\right],\end{aligned}$$

where for any DF $F(.)$, the symbol $\hat{F}(.)$ stands for the empirical DF of $F(.)$ and $\Lambda_j = z_{(j+1)} - z_{(j)}$, $j = 1, 2, ..., n-1$, are the sample spacings arising from ordered random samples of Z_j.

Example 4.1. *Let (Y_i, Z_i), $i = 1, 2, ..., n$, be a random sample from SAR-ED. Then the sample spacings Λ_j are independent RVs. Moreover, Λ_j has the exponential distributed with mean $\frac{1}{\theta_2(n-j)}$, $j = 1, 2, ..., n-1$ (for more details see [16]). According to Pyke [32], the empirical CREX expectation and variance based on $Z_{[r,n]}$ are as follows:*

$$
\begin{aligned}
E\left[\hat{\xi}^*(Z_{[r,n]};\nu)\right] &= \frac{-1}{2\theta_2}\sum_{j=1}^{n-1}\frac{1}{(n-j)}\Bigg[\left(1-\frac{j}{n}\right)^2+9\left(\Delta_{r,n:1}^{(\nu)}\right)^2\left(\frac{j}{n}(1-\frac{j}{n})\right)^2+6\Delta_{r,n:1}^{(\nu)}\frac{j}{n}\left(1-\frac{j}{n}\right)^2\\
&+\frac{25}{8}\left(\Delta_{r,n:2}^{(\nu)}\right)^2\left(\frac{j}{n}\right)^2\left(2\left(\frac{j}{n}\right)^2-\frac{3j}{n}+1\right)^2-5\Delta_{r,n:2}^{(\nu)}\frac{j}{n}\left(2\left(\frac{j}{n}\right)^2-\frac{3j}{n}+1\right)\\
&\times\left(1-\frac{j}{n}\right)-15\Delta_{r,n:1}^{(\nu)}\Delta_{r,n:2}^{(\nu)}\left(\frac{j}{n}\right)^2\left(1-\frac{j}{n}\right)\left(2\left(\frac{j}{n}\right)^2-\frac{3j}{n}+1\right)\Bigg],\\
Var\left[\hat{\xi}^*(Z_{[r,n]};\nu)\right] &= \frac{-1}{4\theta_2^2}\sum_{j=1}^{n-1}\frac{1}{(n-j)^2}\Bigg[\left(1-\frac{j}{n}\right)^2+9\left(\Delta_{r,n:1}^{(\nu)}\right)^2\left(\frac{j}{n}(1-\frac{j}{n})\right)^2+6\Delta_{r,n:1}^{(\nu)}\frac{j}{n}\left(1-\frac{j}{n}\right)^2\\
&+\frac{25}{8}\left(\Delta_{r,n:2}^{(\nu)}\right)^2\left(\frac{j}{n}\right)^2\left(2\left(\frac{j}{n}\right)^2-\frac{3j}{n}+1\right)^2-5\Delta_{r,n:2}^{(\nu)}\frac{j}{n}\left(2\left(\frac{j}{n}\right)^2-\frac{3j}{n}+1\right)\\
&\times\left(1-\frac{j}{n}\right)-15\Delta_{r,n:1}^{(\nu)}\Delta_{r,n:2}^{(\nu)}\left(\frac{j}{n}\right)^2\left(1-\frac{j}{n}\right)\left(2\left(\frac{j}{n}\right)^2-\frac{3j}{n}+1\right)\Bigg]^2,
\end{aligned}
$$

where $\Delta_{r,n:1}^{(\nu)} = \frac{\nu(\,2r-n-1\,)}{n+1}$ and $\Delta_{r,n:2}^{(\nu)} = 2\nu^2\left[1-6\frac{r(n-r+1)}{(n+1)(n+2)}\right]$.

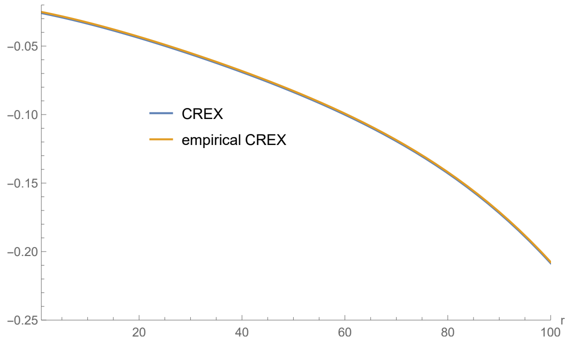
Figure 1 shows the relation between the CREX and its empirical in $Z_{[r,n]}$ based on SAR-ED. We note that, at any value of r the values of CREX are very close to the values of empirical CREX.

5. Real Data Application

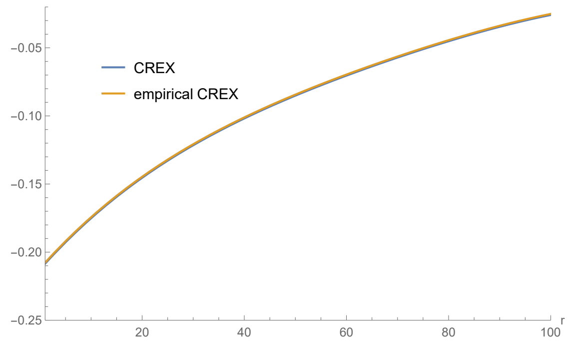
Diabetic nephropathy data: This example aims to examine the WEX and PE of a real-world data set based on the SAR-ED. Using diabetic nephropathy data, a review of medical information is performed, revealing a poor correlation between the two RVs (bivariate data). This data was obtained from Dr. Path Lal's lab database between January 2012 and August 2013. Researchers measured glucose levels in 132 patients with type 2 diabetic nephropathy throughout their lives, from childhood to adulthood, according to Grover et al. [22]. The RV Y represents the mean duration of diabetes, whereas the RV Z represents the mean serum creatinine level (SrCr), based on studies of 19 patients (cf. [20]). Moreover, Table 8 shows estimated values of the WEX and PE for the model SAR-ED (0.0597262, 0.581825) for the concomitant $Z_{[r,n]}$. Also, Figure 2 illustrates statistical visualizations, including Q-Q plot.

Table 8: WEX, and PE of SAR-ED at $\widehat{\theta}_2 = 0.581825$ and $\widehat{\alpha} = 0.52915$

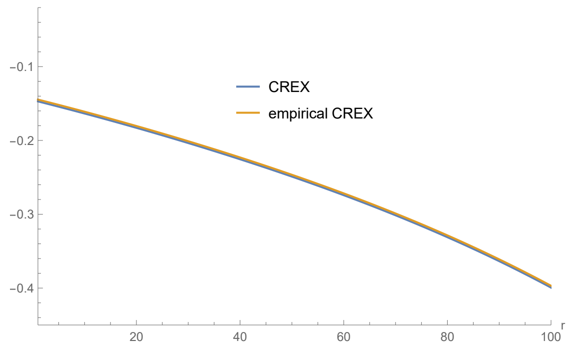
r	1	2	9	10	18	19
$\xi^{(\omega)}(Z_{[r,19]}, 0.52915)$	-0.206866	-0.190498	-0.159881	-0.1566	-0.10939	-0.109682
$\xi^{(0.3)}(Z_{[r,19]}, 0.52915)$	-1.84736	-1.7209	-1.75541	-1.71123	-1.71094	-1.72285



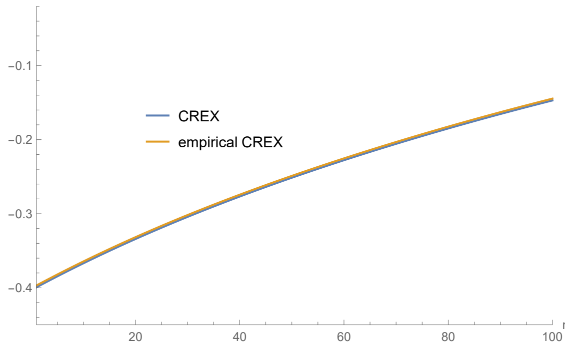
(a) $\nu = 0.5, \theta_2 = 3$



(b) $\nu = -0.5, \theta_2 = 3$

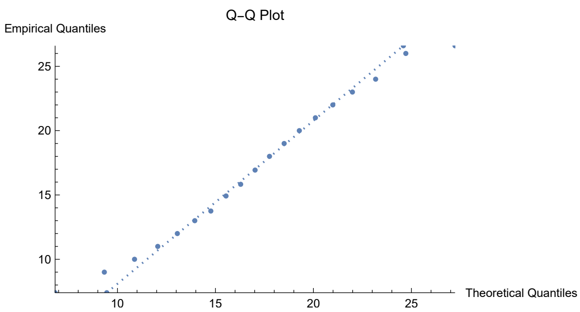


(c) $\nu = 0.25, \theta_2 = 1$

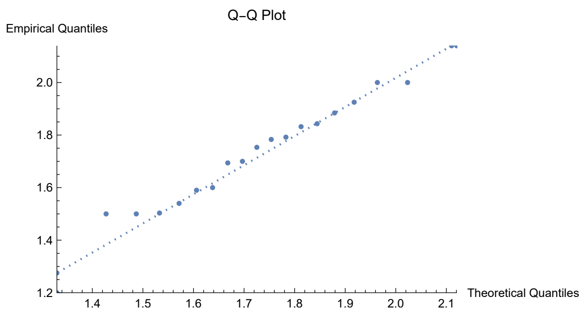


(d) $\nu = -0.25, \theta_2 = 1$

Figure 1: Representation of the CREX and empirical CREX arising from $Z_{[r,100]}$ from SAR-ED



(a) Representation of the mean duration of diabetes



(b) Representation of the mean serum creatinine level

Figure 2: Q-Q plot for diabetic nephropathy data

6. Conclusion and Future Work

The Sarmanov family represents one of the most versatile extensions of the FGM family, retaining many of its key properties, such as radial symmetry and a single parameter. This single-parameter feature is unique to the Sarmanov and FGM families, setting them apart from other families and granting them distinctive characteristics. Moreover, the Sarmanov family provides a broader range of correlations compared to other extensions, rivaling some generalizations of the FGM family, like the Huang-Kotz FGM and iterated FGM.

The findings of this study hold significant theoretical and practical implications for analyzing bivariate data. It explores the statistical properties of the concomitants of generalized order statistics derived from the Sarmanov family of bivariate distributions. Furthermore, it examines and compares various modern information measures for these concomitants within both the Sarmanov and FGM families. These measures include weighted entropy, weighted extropy, past extropy, residual extropy, cumulative residual extropy, and negative cumulative residual extropy. Additionally, the study addresses the estimation of cumulative residual extropy using an empirical approach based on the concomitants of generalized order statistics. Numerical examples are provided to support the theoretical results.

A promising avenue for future research involves studying more complex bivariate data samples using the Sarmanov family of nonidentical bivariate distributions. This direction is expected to yield valuable insights and further advancements in the field.

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