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The best approximation of algebraic polynomials in exponentially weighted Orlicz spaces

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Abstract. In this paper, we study the best approximation of algebraic polynomials in exponentially weighted Orlicz spaces. By using H*ö*lder inequality, Minkowski inequality, de la vall*é*e Poussin mean and related analysis skills, we obtain two results about best approximation of algebraic polynomials and corresponding convergence theorems.

1. Introduction

Algebraic polynomials are simple function types, and we often consider using them to approximate complex functions. There are fruitful results about polynomial approximation in continuous function space and L_p spaces(such as references [1]-[10]). However, scholars research and obtain relatively few results in Orlicz spaces. Orlicz spaces are natural extension of the ideas contained in D. Hilbert's pioneering work. Due to the needs of study integral equation theory, starting from the Hilbert spaces L_2 and combining with the Lebesgue spaces $L_p(p \geq 1)$, using arbitrary convex functions satisfying conditions such as $M(0) = 0$, $M(u) > 0(u > 0)$ replaces the functions $\varphi(u) = u^p$ which determine the spaces $L_p[a, b]$ $(\|f\|_p = (\int_a^b |f(x)|^p dx)^{\frac{1}{p}} < \infty)$. Therefore, $L_p[a, b]$ spaces expand and elevate to Orlicz spaces, which were introduced by Polish mathematician W.Orlicz in 1932.

Letting $w(x) = e^{-Q(x)}$ is exponential weight function, here Q is nonnegative and even function on R. In this paper, we consider $w(x) = e^{-Q(x)}$ belongs to the set $F(C^2+)$ ($w(x) \in F(C^2+)$ detailed meaning can be found in Definition 1.1).

Let

$$T(x) =: \frac{xQ'(x)}{Q(x)}, x \neq 0. \tag{1.1}$$

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If T is bounded, then $w(x)$ is called a Freud-type weight function;
If T is unbounded, then $w(x)$ is called an Erdös-type weight function.

By reference[11], for $x > 0$, the positive root $a_x = a_x(w)$ of equation

$$x = \frac{2}{\pi}\int_0^1 \frac{a_x u Q'(a_x u)}{(1-u^2)^{\frac{1}{2}}}\mathrm{d}u$$

is called the Mhaskar-Rakhmanov-Saff number(MRS number).

Definition 1.1[11] When Q is a even and continuous function defined on $R \to [0,\infty)$ and satisfies the following five conditions, we say that $w(x) = e^{-Q(x)} \in F(C^2+)$.

(1) $Q'(x)$ is a continuous function on R and $Q(0) = 0$;
(2) $Q''(x)$ exists and is positive on $R\backslash\{0\}$;
(3) $\lim\limits_{x\to\infty} Q(x) = \infty$;
(4) The function T defined in (1.1) is quasi-increasing in $(0,\infty)$ (i.e. There is $C > 0$ such that $T(x) \le CT(y)$ for $0 < x < y$). And there is $\Lambda \in R$, such that

$$T(x) \ge \Lambda > 1, x \in R\backslash\{0\};$$

(5) There exists $C_1 > 0$, such that

$$\frac{Q''(x)}{|Q'(x)|} \le C_1 \frac{|Q'(x)|}{Q(x)}, x \in R;$$

There also exists a compact subinterval $J \subset R (0 \in J)$ and $C_2 > 0$, such that

$$\frac{Q''(x)}{|Q'(x)|} \ge C_2 \frac{|Q'(x)|}{Q(x)}, x \in R\backslash J.$$

Let $Q \in C^3(R)$, $\lambda > 0$. If there exist $C > 0, K > 0$,

$$\left|\frac{Q'''(x)}{Q''(x)}\right| \le C\left|\frac{Q''(x)}{Q'(x)}\right|, \quad \frac{|Q'(x)|}{Q^{\lambda}(x)} \le C$$

hold for all $|x| \ge K$, then we say $w \in F_\lambda(C^3+)$.

By reference [11], we know that $F_\lambda(C^3+) \subset F(C^2+)$.

Let $w(x) \in F(C^2+)$. Due to $Q'(x)$ is positive and increasing on $(0,\infty)$, the following equations

$$\lim_{x\to\infty} a_x = \infty, \lim_{x\to 0^+} a_x = 0; \ \lim_{x\to\infty}\frac{a_x}{x} = 0, \lim_{x\to 0^+}\frac{a_x}{x} = \infty$$

hold.

Let $w(x) \in F(C^2+)$. For $x > 0$, we put

$$\sigma(t) =: inf\{a_x : \frac{a_x}{x} \le t\}.$$

Because $\frac{a_x}{x}$ is monotonically decreasing, there is a unique $x > 0$ such that

$$t = \frac{a_x}{x}, \ \sigma(t) = a_x.$$

Hence

$$\lim_{t\to 0^+}\sigma(t) = \infty.$$

We also let

$$\Phi_t(x) =: \sqrt{\left|1 - \frac{|x|}{\sigma(t)}\right|} + \frac{1}{\sqrt{T(\sigma(t))}}, \ \ (x \in R).$$

In this paper, $M(u)$ and $N(v)$ are used to represent complementary N functions. The definition of the N functions are as follows,

Definition 1.2 A real valued function $M(u)$ defined on R and satisfying the following properties is called the N function

(1) $M(u)$ is even and continuous convex function;

(2) $M(u) > 0$ for $u > 0$, and $M(0) = 0$.

(3) $\lim\limits_{u\to 0} \frac{M(u)}{u} = 0, \lim\limits_{u\to\infty} \frac{M(u)}{u} = \infty$.

For details on the properties of N functions, please refer to reference [12].

In this paper, $L_M^*(R)$ represent the Orlicz spaces generated by the $M(u)$ defined on R. For $f \in L_M^*(R)$, the Orlicz norm is

$$\|f\|_M = \sup_{\rho(v;N)\le 1} \left| \int_R f(x)v(x)\mathrm{d}x \right|,$$

where $\rho(v;N) = \int_R N(v(x))\mathrm{d}x$ represents the module of $v(x)$ with respect to $N(v)$.

By reference [12], the Orlicz norm can be also calculated by

$$\|f\|_M = \inf_{\alpha>0} \frac{1}{\alpha}(1 + \int_R M(\alpha f(x))\mathrm{d}x).$$

According to reference [12], it is known that weighted Orlicz spaces are

$$L_{M,w}^*(R) = \{f : \exists K > 0, \int_R M(Kf(x)w(x))\mathrm{d}x < \infty\},$$

and

$$\|f\|_{M,w} = \sup_{\rho(v;N)\le 1} \left| \int_R f(x)w(x)v(x)\mathrm{d}x \right|.$$

Let $\{p_n\}$ be orthogonal polynomials concerning a weight $w(x)$, i.e.

$$\int_R p_n(x)p_m(x)w^2(x)\mathrm{d}x = \delta_{m,n},$$

and $\delta_{m,n} = 0$ for $n \ne m$.

For $f \in L_{M,w}^*(R)$, define the partial sum of Fourier series as

$$S_n(f)(x) =: \sum_{k=0}^{n-1} c_k(f)p_k(x),$$

where

$$c_k(f) =: \int_R f(t)p_k(t)w^2(t)\mathrm{d}t.$$

According to [13][14], the partial sum of Fourier series can also given by

$$S_m(f)(x) =: \int_R K_m(x,t)f(t)w^2(t)\mathrm{d}t,$$

where

$$K_m(x,t) = \sum_{k=0}^{m-1} p_k(x)p_k(t).$$

The de la Vallée Poussin mean $V_n(f)$ of f is defined by

$$V_n(f)(x) =: \frac{1}{n} \sum_{j=n+1}^{2n} S_j(f)(x).$$

Definition 1.3 For $f \in L^*_{M,w}(R)$, let

$$\Delta_t(f)(x) =: f(x+\frac{t}{2}) - f(x-\frac{t}{2}).$$

Similar to the definition in reference [15], we define the continuous modulus in weighted Orlicz spaces as follows.

If w is Freud-type, then

$$\omega(f,t)_{M,w(R)} =: \sup_{0<h\le t} \|\Delta_h(f)\|_{M,w(|x|\le\sigma(t))} + \inf_{c\in R} \|f-c\|_{M,w(|x|\ge\sigma(t))};$$

If w is Erd$\ddot{o}$s-type, then

$$\omega(f,t)_{M,w(R)} =: \sup_{0<h\le t} \|\Delta_{h\Phi_t(x)}(f)\|_{M,w(|x|\le\sigma(2t))} + \inf_{c\in R} \|f-c\|_{M,w(|x|\ge\sigma(4t))}.$$

For $f \in L^*_{M,w}(R)$, $p \in \mathbf{P}_n$ ($\mathbf{P}_n$ is the set of polynomials of order not exceeding n), we write that

$$E_n(f)_{M,w(R)} =: \inf \|f-p\|_{M,w(R)}$$

is the best algebraic polynomial approximation of f in $L^*_{M,w}(R)$.

Attention In this paper, C is used to represent constant, but C can represent different numberns in different places.

2. Related lemmas

Lemma 2.1[13][15] For $x > 0$, we have

$$\frac{x\sqrt{T(a_x)}}{a_x} \asymp Q'(a_x),$$

i.e. there exist constants C_1, C_2 such that

$$C_1\frac{x\sqrt{T(a_x)}}{a_x} \le Q'(a_x) \le C_2\frac{x\sqrt{T(a_x)}}{a_x}.$$

Lemma 2.2 For every absolutely continuous function g with $g(0) = 0$, and $g' \in L^*_{M,w}(R)$, we have

$$\|Q'g\|_{M,w(R)} \le C\|g'\|_{M,w(R)}.$$

Proof According to the result of Theorem 6 in reference [15], for $1 \le p \le \infty$, we have

$$\|Q'gw\|_{L_p(R)} \le C\|wg'\|_{L_p(R)}.$$

For $p = 1$, we have

$$\int_R |Q'(x)g(x)w(x)|dx \le C\int_R |g'(x)w(x)|dx.$$

So

$$\sup_{\rho(v;N)\le 1}\int_R |Q'(x)g(x)w(x)v(x)|dx \le C\sup_{\rho(v;N)\le 1}\int_R |g'(x)w(x)v(x)|dx.$$

Therefore,

$$\begin{aligned}\|Q'g\|_{M,w(R)} &= \sup_{\rho(v;N)\le 1}\left|\int_R Q'(x)g(x)w(x)v(x)dx\right| \\ &\le \sup_{\rho(v;N)\le 1}\int_R |Q'(x)g(x)w(x)v(x)|dx \\ &\le C\sup_{\rho(v;N)\le 1}\left|\int_R |g'(x)w(x)v(x)|dx\right| \\ &= C\|g'\|_{M,w(R)}.\end{aligned}$$

Thus, we get Lemma 2.2.

Lemma 2.3[15] When x and y satisfy $|x| \leq \sigma(2t)$ and $|x-y| \leq t\frac{\Phi_t(x)}{2}$, we have

$$w(x) \asymp w(y),$$

i.e. there exist C_1, C_2 such that

$$C_1 w(y) \leq w(x) \leq C_2 w(y).$$

Lemma 2.4 For $w \in F(C^2+), p \in \mathbf{P}_n$, we have

$$\|p\|_{M,w(R)} \leq C\|p\|_{M,w(|x|<a_n)}.$$

Proof According to Lemma 2 in reference [14], for $w \in F(C^2+)$ and $0 < p \leq +\infty$, we have

$$\|pw\|_{L_p(R)} \leq C\|pw\|_{L_p(|x|<a_n)}.$$

Using the above inequality for $p = 1$ and combining it with Hölder inequality in Orlicz spaces(cf.[12, P.78]), we have

$$\|p\|_{M,w(R)} = \sup_{\rho(v;N)\leq 1}\left|\int_R p(x)w(x)v(x)\mathrm{d}x\right| \leq C\int_R |p(x)w(x)|\mathrm{d}x = C\|pw\|_{L_1(R)}$$
$$\leq C\|pw\|_{L_1(|x|<a_n)} = C\int_{|x|<a_n} |p(x)w(x)|\mathrm{d}x \leq C\|p\|_{M,w(|x|<a_n)}\|1\|_{N(|x|<a_n)} \leq C\|p\|_{M,w(|x|<a_n)}.$$

Lemma 2.5[14] Let $w \in F(C^2+)$, $wf \in L_\infty(R)$. There exists a positive constant $C = C(w)$ such that

$$\|V_n(f)\frac{w}{T^{\frac{1}{4}}}\|_{L_\infty(R)} \leq C\|fw\|_{L_\infty(R)}.$$

Lemma 2.6[14] Let $w \in F(C^2+)$ and $T(a_n) \leq C(\frac{n}{a_n})^{\frac{2}{3}}$ $(C > 0)$. For $T^{\frac{1}{4}}fw \in L_\infty(R)$, there exists a positive constant $C = C(w)$ such that

$$\|V_n(f)w\|_{L_\infty(R)} \leq C\|T^{\frac{1}{4}}fw\|_{L_\infty(R)}.$$

Lemma 2.7 Let $w \in F(C^2+)$ and $T(a_n) \leq C(\frac{n}{a_n})^{\frac{2}{3}}$ $(C > 0)$.

For $T^{\frac{1}{4}}f \in L^*_{M,w}(R)$, there exists a positive constant $C = C(w)$ such that

$$\|V_n(f)\|_{M,w(R)} \leq C\|T^{\frac{1}{4}}f\|_{M,w(R)}; \tag{2.1}$$

For $f \in L^*_{M,w}(R)$, there exists a positive constant $C = C(w)$ such that

$$\|V_n(f)\frac{1}{T^{\frac{1}{4}}}\|_{M,w(R)} \leq C\|f\|_{M,w(R)}. \tag{2.2}$$

Proof Since $K_m(x,t) = K_m(t,x)$, we see $\int_R S_m(f)(x)g(x)w^2(x)\mathrm{d}x = \int_R f(x)S_m(g)(x)w^2(x)\mathrm{d}x$, and hence

$$\int_R V_n(f)(x)g(x)w^2(x)\mathrm{d}x = \int_R f(x)V_n(g)(x)w^2(x)\mathrm{d}x.$$

Using Lemma 2.5, we have

$$\begin{aligned}
\|V_n(f)\|_{M,w(R)} &= \sup_{\rho(v;N)\le 1}\left|\int_R V_n(f)(x)w(x)v(x)\mathrm{d}x\right| \\
&= \sup_{\rho(v;N)\le 1}\sup_{\|gw\|_{L_\infty}\le 1}\left|\int_R V_n(f)(x)g(x)w^2(x)v(x)\mathrm{d}x\right| \\
&= \sup_{\rho(v;N)\le 1}\sup_{\|gw\|_{L_\infty}\le 1}\left|\int_R f(x)V_n(g)(x)w^2(x)v(x)\mathrm{d}x\right| \\
&= \sup_{\rho(v;N)\le 1}\sup_{\|gw\|_{L_\infty}\le 1}\left|\int_R (T^{\frac14}(x)f(x)w(x)v(x))(\frac{1}{T^{\frac14}(x)}V_n(g)(x)w(x))\mathrm{d}x\right| \\
&\le \sup_{\rho(v;N)\le 1}\sup_{\|gw\|_{L_\infty}\le 1}\|\frac{1}{T^{\frac14}}V_n(g)w\|_{L_\infty}\left|\int_R T^{\frac14}(x)f(x)w(x)v(x)\mathrm{d}x\right| \\
&\le C\sup_{\rho(v;N)\le 1}\sup_{\|gw\|_{L_\infty}\le 1}\|gw\|_{L_\infty}\left|\int_R T^{\frac14}(x)f(x)w(x)v(x)\mathrm{d}x\right| \\
&\le C\sup_{\rho(v;N)\le 1}\left|\int_R T^{\frac14}(x)f(x)w(x)v(x)\mathrm{d}x\right| \\
&= C\|T^{\frac14}f\|_{M,w(R)}.
\end{aligned}$$

Thus, inequation (2.1) is established. Nextly, we prove inequation (2.2). According to Lemma 2.6, we have

$$\begin{aligned}
\|V_n(f)\frac{1}{T^{\frac14}}\|_{M,w(R)} &= \sup_{\rho(v;N)\le 1}\left|\int_R V_n(f)(x)\frac{1}{T^{\frac14}(x)}w(x)v(x)\mathrm{d}x\right| \\
&= \sup_{\rho(v;N)\le 1}\sup_{\|gw\|_{L_\infty}\le 1}\left|\int_R V_n(f)(x)\frac{g(x)}{T^{\frac14}(x)}w^2(x)v(x)\mathrm{d}x\right| \\
&= \sup_{\rho(v;N)\le 1}\sup_{\|gw\|_{L_\infty}\le 1}\left|\int_R f(x)V_n(\frac{g}{T^{\frac14}})(x)w^2(x)v(x)\mathrm{d}x\right| \\
&\le \sup_{\rho(v;N)\le 1}\sup_{\|gw\|_{L_\infty}\le 1}\|wV_n(\frac{g}{T^{\frac14}})\|_{L_\infty}\left|\int_R f(x)w(x)v(x)\mathrm{d}x\right| \\
&\le C\sup_{\rho(v;N)\le 1}\sup_{\|gw\|_{L_\infty}\le 1}\|gw\|_{L_\infty}\left|\int_R f(x)w(x)v(x)\mathrm{d}x\right| \\
&\le C\sup_{\rho(v;N)\le 1}\left|\int_R f(x)w(x)v(x)\mathrm{d}x\right| \\
&= C\|f\|_{M,w(R)}.
\end{aligned}$$

Thus, inequation (2.2) is established.

Lemma 2.8 Let $w \in F(C^2+)$ and $T(a_n) \le C(\frac{n}{a_n})^{\frac23}$ $(C > 0)$. For $f \in L^*_{M,w}(R)$, there exists a positive constant $C = C(w)$ such that

$$\|V_n(f)\|_{M,w(R)} \le CT^{\frac14}(a_n)\|f\|_{M,w(R)}.$$

Proof Based on the properties of T function and combining Lemma 2.4 and Lemma 2.7, we have

$$\begin{aligned}
\|V_n(f)\|_{M,w(R)} &\le C\|V_n(f)\|_{M,w[-a_n,a_n]} \\
&= C\|T^{\frac14}\frac{V_n(f)}{T^{\frac14}}\|_{M,w[-a_n,a_n]} \\
&\le CT^{\frac14}(a_n)\|f\|_{M,w(R)}.
\end{aligned}$$

Lemma 2.9[16][17] There exists $n_0 \in N^+$ such that for $f \in L^*_{M,w}(R)$ and $n \geq n_0$,

$$E_n(f)_{M,w(R)} \leq C\omega(f; \frac{a_n}{n})_{M,w(R)}$$

holds.

3. Theorems

Theorem 3.1 Let $w \in F(C^2+)$. For absolutely continuous function f with $f' \in L^*_{M,w}(R)$ and every $n \in N^+$, we have

$$E_n(f)_{M,w(R)} \leq C\frac{a_n}{n}\|f'\|_{M,w(R)}.$$

Proof Because the proof of the Freud-type is similar to the Erdös-type, we will only discuss the latter.

For $|x| \leq \sigma(2t) < \sigma(t)$, we have $\Phi_t(x) \leq 2$. According to Lemma 2.3 and the Minkowski inequality, for every h with $0 < h \leq t$, we have

$$\begin{aligned}
\|\Delta_{h\Phi_t(x)}(f)\|_{M,w(|x|\leq\sigma(2t))} &= \sup_{\rho(v;N)\leq 1}\left|\int_{-\sigma(2t)}^{\sigma(2t)} w(x)\{f(x+h\frac{\Phi_t(x)}{2}) - f(x-h\frac{\Phi_t(x)}{2})\}v(x)\mathrm{d}x\right| \\
&\leq \sup_{\rho(v;N)\leq 1}\left|\int_{-\sigma(2t)}^{\sigma(2t)} w(x)\int_{x-h\frac{\Phi_t(x)}{2}}^{x+h\frac{\Phi_t(x)}{2}} |f'(y)|\mathrm{d}y v(x)\mathrm{d}x\right| \\
&\leq \sup_{\rho(v;N)\leq 1}\left|\int_{-\sigma(2t)}^{\sigma(2t)} Cw(y)\int_{x-h\frac{\Phi_t(x)}{2}}^{x+h\frac{\Phi_t(x)}{2}} |f'(y)|\mathrm{d}y v(x)\mathrm{d}x\right| \\
&\leq C\sup_{\rho(v;N)\leq 1}\left|\int_{-\infty}^{\infty}\int_{-h}^{h} w(x+s)f'(x+s)\mathrm{d}s v(x)\mathrm{d}x\right| \\
&\leq C\|\int_{-h}^{h} w(\cdot+s)f'(\cdot+s)\mathrm{d}s\|_{M(R)} \\
&\leq C\int_{-h}^{h}\|w(\cdot+s)f'(\cdot+s)\|_{M(R)}\mathrm{d}s \\
&\leq C2t\|f'\|_{M,w(R)} \\
&\leq Ct\|f'\|_{M,w(R)}.
\end{aligned}$$

And then,

$$\sup_{0<h\leq t}\|\Delta_{h\Phi_t(x)}(f)\|_{M,w(|x|\leq\sigma(2t))} \leq Ct\|f'\|_{M,w(R)}.$$

If we put $4t = \frac{a_v}{v}$, by Lemma 2.1, we have

$$Q'(\sigma(4t)) = Q'(a_v) > C\frac{v\sqrt{T(a_v)}}{a_v} = C\frac{\sqrt{T(\sigma(4t))}}{4t} \geq C\frac{\sqrt{\Lambda}}{4t} \geq \frac{1}{4Ct}.$$

Owing to the monotonicity of $Q'(x)$, we have $Q'(|x|) > Q'(\sigma(4t))$ for $|x| > \sigma(4t)$. And combining with Lemma

2.2, we get

$$\begin{aligned}\inf_{c\in R}\|f-c\|_{M,w(|x|>\sigma(4t))} \leq&\|Q'\frac{w}{Q'}(f-f(0))\|_{M,w(|x|>\sigma(4t))}\\ \leq&\frac{1}{Q'(\sigma(4t))}\|Q'w(f-f(0))\|_{M(|x|>\sigma(4t))}\\ \leq&4Ct\|Q'w(f-f(0))\|_{M(|x|>\sigma(4t))}\\ \leq&4Ct\|w(f-f(0))'\|_{M(|x|>\sigma(4t))}\\ =&Ct\|f'\|_{M,w(|x|>\sigma(4t))}\\ \leq&Ct\|f'\|_{M,w(R)}.\end{aligned}$$

Therefore, according to the definition of continuous module, we have

$$\omega(f;t)_{M,w(R)} \leq Ct\|f'\|_{M,w(R)}.$$

Take $t=\frac{a_n}{n}$. If there exists $n_0 \in N^+$, for $n \geq n_0$, then Lemma 2.9 shows

$$E_n(f)_{M,w(R)} \leq C\omega(f;\frac{a_n}{n})_{M,w(R)} \leq C\frac{a_n}{n}\|f'\|_{M,w(R)}.$$

i.e.

$$E_n(f)_{M,w(R)} \leq C\frac{a_n}{n}\|f'\|_{M,w(R)}.$$

Let $n < n_0$.

For $|x| \geq 1$, according to $\lim_{|x|\to+\infty}|Q'(x)| = \infty$ and combining with Lemma 2.2, we have

$$\|f-f(0)\|_{M,w(|x|\geq1)} \leq C\|f'\|_{M,w(|x|\geq1)};$$

and for $|x| \leq 1$, using Minkowski inequality, we have

$$\|f-f(0)\|_{M,w(|x|\leq1)} \leq C\|f'\|_{M,w(|x|\leq1)}.$$

Based on the above, it can be concluded that

$$\|f-f(0)\|_{M,w(R)} \leq C\|f'\|_{M,w(R)}.$$

Therefore,

$$E_n(f)_{M,w(R)} \leq E_0(f)_{M,w(R)} \leq \|f-f(0)\|_{M,w(R)} \leq C\|f'\|_{M,w(R)} \leq C\frac{a_n}{n}\|f'\|_{M,w(R)}$$

holds, where $C \geq C\max\{\frac{n}{a_n}; 1 \leq n \leq n_0\}$.

Finally, combining $n \geq n_0$ and $n < n_0$, we get Theorem 3.1.

Theorem 3.2 Let $w \in F(C^2+)$ and $T(a_n) \leq C(\frac{n}{a_n})^{\frac{2}{3}}$ $(C > 0)$.

For $f \in L^*_{M,w}(R)$, there exists a positive constant $C = C(w)$ such that

$$\|(f-V_n(f))\frac{1}{T^{\frac{1}{4}}}\|_{M,w(R)} \leq CE_n(f)_{M,w(R)}; \tag{3.1}$$

$$\|f-V_n(f)\|_{M,w(R)} \leq CT^{\frac{1}{4}}(a_n)E_n(f)_{M,w(R)}. \tag{3.2}$$

For $T^{\frac{1}{4}}f \in L^*_{M,w}(R)$, there exists a positive constant $C = C(w)$ such that

$$\|f-V_n(f)\|_{M,w(R)} \leq CE_n(f)_{M,T^{\frac{1}{4}}w(R)}. \tag{3.3}$$

Proof Since $V_n(p) = p$ for every $p \in \mathbf{P}_n$, we have $V_n(f) = p + V_n(f - p)$. Therefore, according to the properties of the T function and combining with inequation (2.2), we have

$$\begin{aligned}
\|(f - V_n(f))\frac{1}{T^{\frac{1}{4}}}\|_{M,w(R)} &= \|(f - p - V_n(f - p))\frac{w}{T^{\frac{1}{4}}}\|_{M(R)} \\
&\leq \|(f - p)\frac{1}{T^{\frac{1}{4}}}\|_{M,w(R)} + \|V_n(f - p)\frac{1}{T^{\frac{1}{4}}}\|_{M,w(R)} \\
&\leq \|(f - p)\frac{1}{T^{\frac{1}{4}}}\|_{M,w(R)} + C\|f - p\|_{M,w(R)} \\
&\leq C\|f - p\|_{M,w(R)}.
\end{aligned}$$

Let $\varepsilon > 0$. According to the definition of the infimum, there exists $p \in \mathbf{P}_n$ such that

$$\|f - p\|_{M,w(R)} < E_n(f)_{M,w(R)} + \varepsilon.$$

So

$$\|(f - V_n(f))\frac{1}{T^{\frac{1}{4}}}\|_{M,w(R)} < CE_n(f)_{M,w(R)} + \varepsilon.$$

For the arbitrariness of ε, we have

$$\|(f - V_n(f))\frac{1}{T^{\frac{1}{4}}}\|_{M,w(R)} \leq CE_n(f)_{M,w(R)}.$$

Thus, inequation (3.1) is established.

Secondly, we prove inequation (3.2). By Lemma 2.8, we have

$$\begin{aligned}
\|f - V_n(f)\|_{M,w(R)} &\leq \|f - p\|_{M,w(R)} + \|V_n(f - p)\|_{M,w(R)} \\
&\leq \|f - p\|_{M,w(R)} + CT^{\frac{1}{4}}(a_n)\|f - p\|_{M,w(R)} \\
&\leq CT^{\frac{1}{4}}(a_n)\|f - p\|_{M,w(R)}.
\end{aligned}$$

Let $\varepsilon' > 0$. According to the definition of the infimum, there exists $p \in \mathbf{P}_n$ such that

$$CT^{\frac{1}{4}}(a_n)\|f - p\|_{M,w(R)} < CT^{\frac{1}{4}}(a_n)E_n(f)_{M,w(R)} + \varepsilon'.$$

Because ε' is arbitrary, we get inequation (3.2).

Finally, we prove inequation (3.3). It is similar to the proofs of (3.1) and (3.2). Let $\varepsilon'' > 0$. For the arbitrariness of ε'' and inequation (2.1), there exists $p \in \mathbf{P}_n$ such that

$$\begin{aligned}
\|f - V_n(f)\|_{M,w(R)} &\leq \|f - p\|_{M,w(R)} + \|V_n(f - p)\|_{M,w(R)} \\
&\leq \|f - p\|_{M,w(R)} + C\|T^{\frac{1}{4}}(f - p)\|_{M,w(R)} \\
&\leq \|T^{\frac{1}{4}}(f - p)\|_{M,w(R)} + C\|T^{\frac{1}{4}}(f - p)\|_{M,w(R)} \\
&\leq C\|T^{\frac{1}{4}}(f - p)\|_{M,w(R)} \\
&< CE_n(f)_{M,T^{\frac{1}{4}}w(R)} + \varepsilon'' \\
&\leq CE_n(f)_{M,T^{\frac{1}{4}}w(R)}.
\end{aligned}$$

Theorem 3.3 Let $w \in F(C^2+)$. For $f \in L^*_{M,w}(R)$, we have

$$\lim_{n\to\infty} \|(f - V_n(f))\frac{1}{T^{\frac{1}{4}}}\|_{M,w(R)} = 0.$$

Proof According to inequation (3.1) of Theorem 3.2 and combining the property of best approximation, i.e. $\lim_{n\to\infty} E_n(f)_{M,w(R)} = 0$, thus we get Theorem 3.3.

Theorem 3.3 is a convergence theorem for algebraic polynomial approximation. $wT^{\frac{1}{4}}$ may not belong to $F(C^2+)$, so the similar result of Theorem 3.3 cannot be directly used for (3.3). To overcome this difficulty, reference [18] introduced a modified weight function. Let $w \in F_\lambda(C^3+)$ with $(0 < \lambda \le \frac{3}{2})$. Afterwards, we can establish a new weight function $w^* \in F(C^2+)$ which satisfies $wT^{\frac{1}{4}} \asymp w^*, a_n \asymp a_n^*, T \asymp T^*$. Therefore, we obtain $E_n(f)_{M,T^{\frac{1}{4}}w} \le CE_n(f)_{M,w^*}$.

Theorem 3.4 Let $w \in F_\lambda(C^3+)$ $(0 < \lambda \le \frac{3}{2})$. For $T^{\frac{1}{4}}f \in L^*_{M,w}(R)$, we have

$$\lim_{n\to\infty} \|f - V_n(f)\|_{M,w(R)} = 0.$$

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