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Beta type generalization of complex q-Baskakov-Schurer-Szász-Stancu operators in compact disks

Larisa Cheregi[a], Ioana-Alexandra Şomîtcă[a], M. Mursaleen[b,c,*]

[a] *Department of Mathematics,Technical University of Cluj-Napoca, 400027 Cluj-Napoca, Romania*
[b] *Department of Mathematical Sciences, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602105, Tamilnadu, India*
[c] *Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan*

Abstract. In this paper, we construct a Beta-type generalization of the complex q-Baskakov-Schurer-Szász-Stancu operators in compact disks. We present a modified Beta operator as a robust method for approximating functions on compact disks. The versatility of the operator comes from its ability to manage complex variables and adjust to various weight functions, making it an adaptable tool that can cater to a diverse array of applications.

1. Introduction

During the past decade, researchers have been actively exploring the application of approximation operators, which include q-analogues and integral type operators, in the field of approximation theory (see [1], [3], [4], [6], [12]). The information about the approximation properties of complex q-polynomials that are associated with analytic functions on compact disks can be found in [5]–[8]. Yüksel introduced a linear positive operator of q-Baskakov-Schurer-Szasz type and its Stancu generalization in [15]. Gupta discussed the q-analogue of the complex q-Baskakov-Szász-Stancu operators in [8]. Cheregi [4] has studied in this direction q-Baskakov-Schurer-Szasz-Stancu on compact disks. In the actual study, we build upon these works and present a variant of Beta-type generalization of complex q-Baskakov-Schurer-Szasz-Stancu operators in compact disks. This paper deals with modified Beta operators [9]. For each integer $k \geq 0$, $q > 0$, the q-integer $[k]_q$ and q-factorial $[k]_q!$ are defined by

$$[k]_q := \begin{cases} \frac{1-q^k}{1-q}, & q \in \mathbb{R}^* \setminus \{1\} \\ k, & q = 1. \end{cases}, \tag{1}$$

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* Corresponding author: M. Mursaleen
Email addresses: larisa.Cheregi@math.utcluj.ro (Larisa Cheregi), somitca.i.ioana@campus.usv.ro (Ioana-Alexandra Şomîtcă), mursaleenm@gmail.com (M. Mursaleen)
ORCID iDs: https://orcid.org/0000-0001-6174-1142 (Larisa Cheregi), https://orcid.org/0000-0001-8124-6691 (Ioana-Alexandra Şomîtcă), https://orcid.org/0000-0003-4128-0427 (M. Mursaleen)

for $k \in \mathbb{N}$ and $[0]_q = 0$, the q-factorial is define in the following:

$$[k]_q! := \begin{cases} [1]_q[2]_q \cdots [k]_q, & \forall k \in \mathbb{N}^* \\ 1, & k = 0. \end{cases}, \tag{2}$$

and respectively

$$(1+z)_q^k := \begin{cases} \prod_{i=0}^{n-1}(1+q^i z), & \forall k \in \mathbb{N}^* \\ 1, & k = 0. \end{cases} \tag{3}$$

For integers $0 \le k \le n$, the q-binomial coefficient is defined as:

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} \tag{4}$$

For fixed $q > 1$, we denote the q-derivative $D_q f(z)$ of f by

$$D_q f(z) := \begin{cases} \frac{f(qz)-f(z)}{(q-1)z}, & z \neq 0 \\ f^{'}(0), & z = 0, \end{cases} \tag{5}$$

also $D_q^0 f := f$ and $D_q^n f := D_q(D_q^{n-1} f)$, $\forall n \in \mathbb{N}$.

For the q-exponential function, we have 2 forms:

If $|q| < 1$ and $|z| < \frac{1}{1-q}$ the q-exponential function $e_q(z)$ was defined by Jackson:

$$e_q(z) := \sum_{k=0}^{\infty} \frac{z^k}{[k]_q!} = \frac{1}{(1-(1-q)z)_q^{\infty}} \tag{6}$$

If $|q| > 1$,$e_q(z)$ is an entire function and

$$e_q(z) = \prod_{j=0}^{\infty} \left(1 + (q-1)\frac{z}{q^{j+1}}\right) \tag{7}$$

To acquire another q-exponential function, we have to invert the base in (6)

$$E_q(z) = e_{\frac{1}{q}}(z) = \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}} z^k}{[k]_q!} = (1-(1-q)z)_q^{\infty}, |q| < 1 \tag{8}$$

We immediately attain from (7) that

$$E_q(z) = \prod_{j=0}^{\infty} (1 + (1-q)z \cdot q^j), 0 < |q| < 1 \tag{9}$$

The q-improper integral used in the present paper is outlined

$$\int_0^{\frac{\infty}{A}} f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} f\left(\frac{q^n}{A}\right) \frac{q^n}{A}, A > 0 \tag{10}$$

provided that the sum converges absolutely.

We consider an analytic function in a disk of radius R and center O.

Let D_R be a disk $D_R = \{z \in \mathbb{C} \| z| < R\}$ in the complex plane $\mathbb{C}$.
Denote by $H(D_R)$ the space of all analytic functions on D_R, $f : [R,\infty) \cup \bar{D_{\mathbb{R}}} \to \mathbb{C}$ continuous in $(R,\infty) \cup \bar{D_R}$.
For $f \in H(D_R)$, we may write $f(z) = \sum_{m=0}^{\infty} c_m z^m$, $\forall z \in D_R$.
Let $p, k \in \mathbb{N}$, $n \in \mathbb{N} \setminus \{0\}$, and f be a continuous function of real value in the interval $[0,\infty)$, $0 < q < 1$ [7].

$$B^{(\alpha,\beta)}_{n,p,q}(f)(x) = \frac{[n+p-1]_q}{[n+p]_q} \sum_{k=0}^{\infty} b^k_{n,p}(x) \int_0^{\infty} s^k_{n,p}(t) f\left(\frac{[n+p]_q t + \alpha}{[n+p]_q + \beta}\right) d_q t \tag{11}$$

where

$$b^k_{n,p}(x) = \binom{n+p+k}{k}_q q^{k^2} \frac{x^k}{(1+x)^{n+p+k+1}}, \tag{12}$$

separately

$$s^k_{n,p}(x) = \frac{([n+p+1]_q t)^k}{[k]_q!} e_q^{-[n+p+1]_q t}, \tag{13}$$

2. Original results

Lemma 2.1. *We define* $T^{(\alpha,\beta}_{n,p,k}(z) := B^{(\alpha,\beta)}_{n,p,q}(e_k)(z)$ *and* $\mathbb{N}^0$ *denotes the set of all non-negatives integers. For all* $n, p, k \in \mathbb{N}^0$, $0 \le \alpha \le \beta$ *and* $z \in \mathbb{C}$, *we have the following recurrence formula*

$$T^{(\alpha,\beta)}_{n,p,k+1}(z) = \frac{z(1+z)}{[n+p+1]_q} D_q T^{(\alpha,\beta)}_{n,p,k}(z) + \frac{[n+p+1]_q z + k + 1}{[n+p+1]_q} T^{(\alpha,\beta)}_{n,p,k}(z) \tag{14}$$

Proof. [Proof of Lemma 2.1] We denote $e_k(z) = z^k$ and write

$$z(1+z) D_q T^{(\alpha,\beta)}_{n,p,k}(z) = \frac{[n+p-1]_q}{[n+p]_q} \sum_{k=1}^{\infty} z(1+z) D_q b^k_{n,p}(z) \int_0^{\infty} s^k_{n,p}(t) t^k dt$$

$$= \frac{[n+p-1]_q}{[n+p]_q} \sum_{k=1}^{\infty} (k - [n+p+1]_q z) b^k_{n,p}(z) \int_0^{\infty} s^k_{n,p}(t) t^k dt$$

$$= \frac{[n+p-1]_q}{[n+p]_q} \sum_{k=1}^{\infty} b^k_{n,p}(z) \int_0^{\infty} [(k - [n+p+1]_q t) + ([n+p+1]_q t - [n+p+1]_q z)] s^k_{n,p}(t) t^k dt$$

$$= \frac{[n+p-1]_q}{[n+p]_q} \sum_{k=1}^{\infty} b^k_{n,p}(z) \int_0^{\infty} D_q s^k_{n,p}(t) t^{k+1} dt + [n+p+1]_q T^{(\alpha,\beta)}_{n,p,k+1}(z) - [n+p+1]_q z T^{(\alpha,\beta)}_{n,p,k}(z)$$

$$z\,(1+z)\,D_qT^{(\alpha,\beta)}_{n,p,k}(z) = -(k+1)T^{(\alpha,\beta)}_{n,p,k}(z) - [n+p+1]_q zT^{(\alpha,\beta)}_{n,p,k}(z) + [n+p+1]_q T^{(\alpha,\beta)}_{n,p,k+1}(z)$$

$$[n+p+1]_q T^{(\alpha,\beta)}_{n,p,k+1}(z) = z\,(1+z)\,D_qT^{(\alpha,\beta)}_{n,p,k}(z) + (k+1+[n+p+1]_q z)T^{(\alpha,\beta)}_{n,p,k}(z)$$

$$T^{(\alpha,\beta)}_{n,p,k+1}(z) = \frac{z\,(1+z)}{[n+p+1]_q}D_qT^{(\alpha,\beta)}_{n,p,k}(z) + \frac{[n+p+1]_q z+k+1}{[n+p+1]_q}T^{(\alpha,\beta)}_{n,p,k}(z).$$

□

So the desired result is obtained for $z \in \mathbb{C}$.

Lemma 2.2. *Let $0 \le \alpha \le \beta$, denoting $B^{(0,0)}_{n,p,q}(e_j)(z)$ by $B_{n,p,q}(e_j)(z)]$, for all $n, p, k \in \mathbb{N}^0$, we have the following recursive relation for the images of monomials e_k under $B^{(\alpha,\beta)}_{n,p,q}$ in terms of $B_{n,p,q}(e_j)$, $j = 0, 1, \cdots, k$:*

$$T^{(\alpha,\beta)}_{n,p,k}(z) = \sum_{j=0}^{k}\binom{k}{j}_q \frac{[n+p+1]_q^j\alpha^{k-j}}{([n+p+1]_q+\beta)^k}\cdot B_{n,p,q}(e_j)(z)] \tag{15}$$

Proof. For $k = 0$, equality (15) holds. Let it be true for $k = m$, specifically

$$T^{(\alpha,\beta)}_{n,p,m}(z) = \sum_{j=0}^{m}\binom{m}{j}_q \frac{[n+p+1]_q^j\alpha^{m-j}}{([n+p+1]_q+\beta)^m}\cdot B_{n,p,q}(e_j)(z)] \tag{16}$$

Applying (16), we have

$$\begin{aligned} T^{(\alpha,\beta)}_{n,p,m+1}(z) =& \frac{z\,(1+z)}{[n+p+1]_q}\sum_{j=0}^{m}\binom{m}{j}_q \frac{[n+p+1]_q^j\alpha^{m-j}}{([n+p+1]_q+\beta)^m}D_qB_{n,p,q}(e_j)(z) \\ &+ \frac{[n+p+1]_q z+k+1}{[n+p+1]_q}\sum_{j=0}^{m}\binom{m}{j}_q \frac{[n+p+1]_q^j\alpha^{m-j}}{([n+p+1]_q+\beta)^m}B_{n,p,q}(e_j)(z) \\ &= \sum_{j=0}^{m}\binom{m}{j}_q \frac{[n+p+1]_q^{j+1}\alpha^{m-j}}{([n+p+1]_q+\beta)^{m+1}}\times \\ &\left[\frac{z\,(1+z)}{[n+p+1]_q}D_qB_{n,p,q}(e_j)(z) + \frac{[n+p+1]_q z+k+1}{[n+p+1]_q}B_{n,p,q}(e_j)(z)\right] \end{aligned}$$

From recurrence relation for the complex q-Beta-Baskakov-Szasz-Schurer-Stancu operator, it succeeds that

$$\begin{aligned} T^{(\alpha,\beta)}_{n,p,m+1}(z) =& \sum_{j=1}^{m}\binom{m}{j-1}_q \frac{[n+p+1]_q^j\alpha^{m-j+1}}{([n+p+1]_q+\beta)^{m+1}}B_{n,p,q}(e_j)(z) \quad + \sum_{j=0}^{m}\binom{m}{j}_q \frac{[n+p+1]_q^j\alpha^{m-j+1}}{([n+p+1]_q+\beta)^{m+1}}\cdot B_{n,p,q}(e_j)(z) \\ &= \sum_{j=0}^{m+1}\binom{m+1}{j}_q \frac{[n+p+1]_q^j\alpha^{m-j+1}}{([n+p+1]_q+\beta)^{m+1}}B_{n,p,q}(e_j)(z) \end{aligned}$$

which proves the lemma. □

3. Approximation of complex q-Baskakov-Szasz-Schurer-Stancu operators

In the succeeding part, we give quantitative estimates concerning approximation with the following theorem.

Theorem 3.1. *For $1 < R < \infty$ and $f : [R, \infty) \cup \bar{D_R} \to \mathbb{C}$ continuous and bounded in $[0, \infty)$ and analytic in D_R, specifically $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$, $\forall k = 0, 1, \cdots$ Let $0 \le \alpha \le \beta$ and $1 < r < R$ be arbitrary but fixed. Then for all $|z| \le r$ and $q > 1$,$n \in \mathbb{N}$, we have*

$$|B_{n,p,q}^{(\alpha,\beta)}(f)(z) - f(z)| \le \frac{[n+p+1]_q(\beta+1)+\beta}{[n+p+1]_q([n+p+1]_q+\beta)} \tag{17}$$

where

$$C_r = 2(r+2)\sum_{k=1}^{\infty} |c_k|(k+2)!r^{k-1} < \infty$$

Proof. By Lemma 2.2 we acquire

$$T_{n,p,k}^{(\alpha,\beta)}(e_k)(z) - e_k(z) = \sum_{j=0}^{k-1} \binom{k}{j}_q \frac{[n+p+1]_q^j \alpha^{k-j}}{([n+p+1]_q+\beta)^k} \cdot \left(B_{n,p,q}(e_j)(z) - e_j(z)\right) + \sum_{j=0}^{k-1} \binom{k}{j}_q \frac{[n+p+1]_q^j \alpha^{k-j}}{([n+p+1]_q+\beta)^k} \cdot e_j(z)$$
$$+ \frac{[n+p+1]_q^k}{([n+p+1]_q+\beta)^k} B_{n,p,q}(e_k)(z) - e_k(z)$$

Using Lemma 2.1, one can receive

$$T_{n,p,k}^{(\alpha,\beta)}(z) - z^k = \frac{z(1+z)}{[n+p+1]_q} D_q\left(T_{n,p,k-1}^{(\alpha,\beta)}(z) - z^{k-1}\right) + \frac{[n+p+1]_q z + k}{[n+p+1]_q}\left(T_{n,p,k-1}^{(\alpha,\beta)}(z) - z^{k-1}\right)$$
$$+ \frac{(2k-1)+(k-1)r}{[n+p+1]_q} z^{k-1}$$

for all $z \in \mathbb{C}$, $k, n, p \in \mathbb{N}$. Now for $1 \le r \le R$ we denote the norm $\|\cdot\|$ in $C(\bar{D_R})$ where $\bar{D_R} = \{z \in \mathbb{C} : |z| \le r\}$. In the closed unit disk we have the inequality $|D_q P_k(z)| \le \frac{k}{r}\|P_k\|_r$, for all $|z| \le r$, where $P_k(z)$ is a polynomial of degree $\le k$.
Thus, from the above recurrence relation, we obtain

$$|T_{n,p,k}^{(\alpha,\beta)}(z) - z^k| \le \frac{r(1+r)}{[n+p+1]_q}\left(\frac{k-1}{r}\right)\|T_{n,p,k-1}^{(\alpha,\beta)}\|_r + \frac{[n+p+1]_q r + k}{[n+p+1]_q}|T_{n,p,k-1}^{(\alpha,\beta)}(z) - z^{k-1}| + \frac{k(r+2)}{[n+p+1]_q} r^{k-1}$$

which, using the notation $\gamma = r+2$ suggests

$$|T_{n,p,k}^{(\alpha,\beta)}(z) - z^k| \le \left(r + \frac{(2+r)k}{[n+p+1]_q}\right)\|T_{n,p,k-1}^{(\alpha,\beta)}\|_r + \frac{k}{[n+p+1]_q}(2+r)r^{k-1}$$
$$= \left(r + \frac{\gamma k}{[n+p+1]_q}\right)\|T_{n,p,k-1}^{(\alpha,\beta)}\|_r + \frac{k}{[n+p+1]_q}\gamma r^{k-1}$$

Using induction with respect to k, $n+p+1 \ge \gamma$, the results are

$$|T_{n,p,k}^{(\alpha,\beta)}(z) - z^k| \le \frac{\gamma(k+2)!}{[n+p+1]_q} r^{k-1}, \forall k \ge 1 \tag{18}$$

The above recurrence becomes

$$\left|T^{(\alpha,\beta)}_{n,p,k+1}(z) - z^{k+1}\right| \leq \left(r + \frac{\gamma(k+1)}{[n+p+1]_q}\right)\frac{(k+2)!}{[n+p+1]_q}\gamma r^{k-1} + \frac{\gamma(k+1)}{[n+p+1]_q}r^k$$
$$\leq \left(r + \frac{\gamma(k+1)}{[n+p+1]_q}\right)(k+2)! + r(k+1)$$

Since $n+p+1 \geq \gamma$

$$|T^{(\alpha,\beta)}_{n,p,k+1}(z) - z^{k+1}| \leq (r+k+1)(k+2)! + r(k+1) \leq (k+3)!r$$

clearly valid for all $k \geq 1$ and fixed $r \geq 1$.
Using (18) we have

$$\|T^{(\alpha,\beta)}_{n,p,k}(e_k) - e_k\|_r \leq \sum_{j=0}^{k-1}\binom{k}{j}_q \frac{[n+p+1]_q^j\alpha^{k-j}}{([n+p+1]_q+\beta)^k}\cdot\|B_{n,p,q}(e_j)-e_j\|_r \quad + \sum_{j=0}^{k-1}\binom{k}{j}_q \frac{[n+p+1]_q^j\alpha^{k-j}}{([n+p+1]_q+\beta)^k}\cdot r^j$$
$$+ \frac{[n+p+1]_q^k}{([n+p+1]_q+\beta)^k}\|B_{n,p,q}(e_k)-e_k\|_r + \left(1 - \frac{[n+p+1]_q^k}{([n+p+1]_q+\beta)^k}\right)\cdot r^k$$
$$\leq \frac{([n+p+1]_q+\alpha)^k}{([n+p+1]_q+\beta)^k}\cdot\frac{\gamma(k+2)!}{[n+p+1]_q}\cdot r^{k-1} \quad + r^k\left[\frac{([n+p+1]_q+\alpha)^k}{([n+p+1]_q+\beta)^k} - \frac{[n+p+1]_q^k}{([n+p+1]_q+\beta)^k}\right]$$
$$+ \frac{[n+p+1]_q^k}{([n+p+1]_q+\beta)^k}\cdot\frac{\gamma(k+2)!}{[n+p+1]_q}\cdot r^{k-1} + \left(1 - \frac{[n+p+1]_q^k}{([n+p+1]_q+\beta)^k}\right)\cdot r^k$$
$$\leq \frac{2\gamma(k+2)!}{[n+p+1]_q}\cdot r^{k-1} + 2r^k\cdot\frac{k\beta}{[n+p+1]_q+\beta}$$
$$\leq \frac{([n+p+1]_q)(\beta+1)+\beta}{([n+p+1]_q)([n+p+1]_q+\beta)}\cdot 2(k+2)!\cdot\gamma\cdot r^{k-1}$$

utilizing the inequality $1 - \prod_{j=1}^{k} x_j \leq \sum_{j=1}^{k}(1-x_j)$ where $0 \leq x_j \leq 1$ and $j = 1, \cdots, k$.

Now we write $B^{(\alpha,\beta)}_{n,p,q}(f)(z) = \sum_{k=1}^{\infty} c_k T^{(\alpha,\beta)}_{n,p,k}(z)$ which implies

$$\left|B^{(\alpha,\beta)}_{n,p,q}(f)(z) - f(z)\right| \leq \sum_{k=1}^{\infty}|c_k|\cdot\left|T^{(\alpha,\beta)}_{n,p,k}(z) - z^k\right|$$
$$\leq \sum_{k=1}^{\infty}|c_k|\cdot 2\gamma\frac{([n+p+1]_q)(\beta+1)+\beta}{([n+p+1]_q)([n+p+1]_q+\beta)}\cdot(k+2)!\cdot r^{k-1} \tag{19}$$
$$= 2(r+2)\cdot\frac{([n+p+1]_q)(\beta+1)+\beta}{([n+p+1]_q)([n+p+1]_q+\beta)}\sum_{k=1}^{\infty}|c_k|\cdot(k+2)!\cdot r^{k-1}$$

for all $1 \leq r \leq R$ and $z \leq r$. we denote

$$C_r = 2(r+2)\sum_{k=1}^{\infty}|c_k|(k+2)!r^{k-1} < \infty$$

Hence, the demonstration has been concluded. □

Lemma 3.2. *For all $|z| \leq r$, $r \geq 1$, $q > 1$, $n, p, k \in \mathbb{N}$ with $n + p \geq r + 1$ we have $|T^{(\alpha,\beta)}_{n,p,k}(e_k)(z)| \leq 4r^k(k + 2)!$.*

Proof.

$$\|T^{(\alpha,\beta)}_{n,p,k}(e_k)\|_r \leq \|T^{(\alpha,\beta)}_{n,p,k}(e_k) - e_k\|_r + \|e_k\|_r \tag{20}$$

which by relationship (18) implies

$$\|T^{(\alpha,\beta)}_{n,p,k}(e_k)\|_r \leq \frac{(r+2)(k+2)!}{[n+p+1]_q} \cdot r^{k-1} + r^k = \frac{(r+2)(k+2)!}{r([n+p+1]_q)} r^k + r^k$$

$$\|T^{(\alpha,\beta)}_{n,p,k}(e_k)\|_r \leq r^k \left(1 + \frac{2}{r}\right)(k+2)! + r^k \leq 3r^k(k+2)! + r^k \leq 4r^k(k+2)!$$

for all $r \geq 1$, $n, p, k \in \mathbb{N}$ and $n + p \geq r + 1$. □

Theorem 3.3. *Let $0 \leq \alpha \leq \beta$, $1 < r < R$ and $q > 1$. Assuming the conditions of the Theorem 3.1, for all $|z| \leq r$ and $n, p \in \mathbb{N}$, the Voronovskaja-type result*

$$\begin{aligned} &\left| B^{(\alpha,\beta)}_{n,p,q}(f)(z) - f(z) - \frac{\alpha + 1 - \beta(z-1)}{[n+p+1]_q} f'(z) - \frac{z(z+2)}{2[n+p+1]_q} f''(z) \right| \\ &\qquad \leq \frac{T_r(f)}{([n+p+1]_q)^2} + \frac{2N_r(f)}{[n+p+1]_q} + \frac{H_r(f)}{([n+p+1]_q + \beta)^2} \end{aligned} \tag{21}$$

where

$$T_r(f) = M_{1,r}(f) + M_{2,r}(f) + M_{3,r}(f)$$

with

$$M_{1,r}(f) = 2r \sum_{k=2}^{\infty} |c_k|(k-1)^2(k^2 - 2k + 2)(r+1)^{k+1} < \infty$$

$$M_{2,r}(f) = \frac{1}{2} \sum_{k=2}^{\infty} |c_k|[(k-1)^2(k-2)^2(r+1)^{k+1}] < \infty$$

$$M_{3,r}(f) = (r+2) \sum_{k=2}^{\infty} |c_k|(k-1)(k+3)!(r+1)^{k+1} < \infty$$

and

$$N_r(f) = \sum_{k=n+p+3}^{\infty} |c_k|(r+2)(k+2)!r^{k-1} < \infty$$

$$H_r(f) = \sum_{k=0}^{\infty} |c_k|[(k-1)(2\alpha^2 + \beta^2 + \alpha\beta) + (r+2)(\alpha+\beta) + \beta(\beta^2 + \alpha + 1)](k+2)!kr^k < \infty$$

Proof. For all $z \in D_R$, let we consider

$$B_{n,p,q}^{(\alpha,\beta)}(f)(z) - f(z) - \frac{\alpha+1-\beta(z-1)}{[n+p+1]_q}f'(z) - \frac{z(z+2)}{2[n+p+1]_q}f''(z) = B_{n,p,q}(f)(z) - f(z) - \frac{1}{[n+p+1]_q}f'(z) - \frac{z(z+2)}{2[n+p+1]_q}f''(z) + B_{n,p,q}^{(\alpha,\beta)}(f)(z) - B_{n,p,q}(f)(z) - \frac{\alpha-\beta(z-1)}{[n+p+1]_q}f'(z)$$

Using the fact that $f(z) = \sum_{k=0}^{\infty} c_k z^k$ in second term we get

$$\left|B_{n,p,q}^{(\alpha,\beta)}(f)(z) - f(z) - \frac{\alpha+1-\beta(z-1)}{[n+p+1]_q}f'(z) - \frac{z(z+2)}{2[n+p+1]_q}f''(z)\right| \le \left|B_{n,p,q}(f)(z) - f(z) - \frac{1}{[n+p+1]_q}f'(z) - \frac{z(z+2)}{2[n+p+1]_q}f''(z)\right| + \left|\sum_{k=0}^{\infty}|c_k|\left(B_{n,p,q}^{(\alpha,\beta)}(e_k)(z) - B_{n,p,q}(e_k)(z) - \frac{\alpha-\beta(z-1)}{[n+p+1]_q}kz^{k-1}\right)\right|$$

To estimate the first sum we write $B_{n,p,q}(f)(z) = \sum_{k=0}^{\infty}|c_k|B_{n,p,q}(e_k)(z)$.

Thus,

$$\left|B_{n,p,q}(f)(z) - f(z) - \frac{1}{[n+p+1]_q}f'(z) - \frac{z(z+2)}{2[n+p+1]_q}f''(z)\right| \le \sum_{k=0}^{\infty}|c_k|\left|B_{n,p,q}(e_k)(z) - e_k(z) - \frac{1}{[n+p+1]_q}kz^{k-1} - \frac{z(z+2)}{2[n+p+1]_q}k(k-1)z^{k-2}\right|$$

$$\left|B_{n,p,q}(f)(z) - f(z) - \frac{1}{[n+p+1]_q}f'(z) - \frac{z(z+2)}{2[n+p+1]_q}f''(z)\right| \le \sum_{k=0}^{\infty}|c_k|\left|B_{n,p,q}(e_k)(z) - e_k(z) - \frac{[k(k-1)z+2k(2-k)]z^{k-1}}{2[n+p+1]_q}\right|$$

By Lemma 2.1, for all $n,p,k \in \mathbb{N}$ and $z \in \mathbb{C}$ we have

$$B_{n,p,q}(e_{k+1})(z) = \frac{z(1+z)}{[n+p+1]_q}D_q B_{n,p,q}(e_k)(z) + \frac{[n+p+1]_q z + k + 1}{[n+p+1]_q}B_{n,p,q}(e_k)(z)$$

Now we denote

$$O_{n,k}(z) = B_{n,p,q}(e_k)(z) - e_k(z) - \frac{[k(k-1)z+2k(2-k)]z^{k-1}}{2[n+p+1]_q} \tag{22}$$

$O_{n,k}(z)$ is a polynomial of degree less than or equal to k. By simple computation and the use of the above recurrence relation, we are led to:

$$O_{n,k}(z) = \frac{z(1+z)}{[n+p+1]_q}D_q O_{n,k-1}(z) + \frac{[n+p+1]_q z + k}{[n+p+1]_q}O_{n,k-1}(z) + E_{n,k}(z)$$

where

$$E_{n,k}(z) = \frac{2z^{k-2}}{[n+p+1]_q^2}(k-1)^2(k-2)(k^2-2k+2) + \frac{z^{k-1}(k-1)^2(k-2)^2(z+1)}{2[n+p+1]_q^2}$$

for all $k \geq 2$, $n, p \in \mathbb{N}$ and $|z| \leq r$. Using theorem 3.1 we have the estimate

$$\left|B_{n,p,q}(e_k)(z) - e_k(z)\right| \leq \frac{(r+2)(k+2)!}{[n+p+1]_q} r^{k-1}$$

It follows

$$\left|O_{n,k}(z)\right| \leq \frac{r(r+1)}{[n+p+1]_q}\left|D_q O_{n,k-1}(z)\right| + \left(r + \frac{k}{[n+p+1]_q}\right)\left|O_{n,k-1}(z)\right| + \left|E_{n,k}(z)\right|$$

$O_{n,k-1}(z)$ is a polynomial of degree $\leq k-1$, so we get

$$\left|D_q O_{n,k-1}(z)\right| \leq \frac{k-1}{r}\|O_{n,k-1}(z)\|_r$$

From (22)

$$\begin{aligned}
\left|D_q O_{n,k-1}(z)\right| &\leq \frac{k-1}{r}\left\{\|B_{n,p,q}(e_{k+1}) - e_{k+1}\|_r + \left\|\frac{[(k-1)(k-2)e_1 + 2(k-1)(3-k)]e_{k-2}}{2[n+p+1]_q}\right\|\right\} \\
&\leq \frac{k-1}{r}\left[\frac{(r+2)(k+2)!}{[n+p+1]_q} r^{k-2} + \frac{[(k-1)(k-2)r + 2(k-1)(3-k)]r^{k-2}}{2[n+p+1]_q}\right] \\
&\leq \frac{k-1}{r} \cdot \frac{r^{k-2}}{[n+p+1]_q} \cdot [(r+2)(k+2)! + (k-1)(k-2)(r+2)] \\
&= \frac{(k-1)r^{k-2}(r+2)\,[(k+2)! + (k-1)(k-2)]}{r[n+p+1]_q} \\
&\leq \frac{r^{k-3}(r+2)(k+3)!}{[n+p+1]_q}
\end{aligned}$$

Thus, we get

$$\frac{r(1+r)}{[n+p+1]_q}\left|D_q O_{n,k-1}(z)\right| \leq \frac{r^{k-2}(r+1)(r+2)(k+3)!}{[n+p+1]_q^2}$$

and

$$\left|O_{n,k}(z)\right| \leq \frac{r^{k-2}(r+1)(r+2)(k+3)!}{[n+p+1]_q^2} + \left(r + \frac{k}{[n+p+1]_q}\right)\left|O_{n,k-1}(z)\right| + \left|E_{n,k}(z)\right|$$

for all $k \geq 2$, $n, p \in \mathbb{N}$, $n + p > r + 1$. For $k \leq n$, $n + p > r + 1$ and $|z| \leq r$ using $r + \frac{k}{[n+p+1]_q} \leq r + 1$ we get

$$\left|O_{n,k}(z)\right| \leq \frac{r^{k-2}(r+1)(r+2)(k+3)!}{[n+p+1]_q^2} + (r+1)\left|O_{n,k-1}(z)\right| + \left|E_{n,k}(z)\right|$$

with

$$\left|E_{n,k}(z)\right| \leq \frac{2r^{k-2}}{[n+p+1]_q^2}(k-1)^2(k-2)(k^2-2k+2) + \frac{r^{k-1}(k-1)^2(k-2)^2(r+1)}{2[n+p+1]_q^2}$$

where $\left|O_{n,0}(z)\right| = \left|O_{n,1}(z)\right| = 0$. For any $z \in \mathbb{C}$ and $2 \leq k \leq n+p+1$ we obtain

$$\begin{aligned}
\left|O_{n,k}(z)\right| &\leq \frac{2(r+1)^k}{[n+p+1]_q^2}\sum_{j=2}^{k}(j-1)^2(j-2)(j^2-2j+2) \\
&\quad + \frac{r(r+1)^{k+1}}{2[n+p+1]_q^2}\sum_{j=2}^{k}(j-1)^2(j-2)^2 \\
&\quad + \frac{(r+1)^{k+1}(r+2)}{[n+p+1]_q^2}\sum_{j=2}^{k}(j+3)! \\
&\leq \frac{2(k-1)^2 r(r+1)^{k+1}}{[n+p+1]_q}\left\{(k-2)(k^2-2k+2)+\frac{(k-2)^2}{4r}\right\} + \frac{(k-1)(r+2)(r+1)^{k+1}}{[n+p+1]_q}(k+3)!
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left|B_{n,p,q}(f)(z) - f(z) - \frac{1}{[n+p+1]_q}f'(z) - \frac{z(z+2)}{2[n+p+1]_q}f''(z)\right| \leq \sum_{k=2}^{n+p+2}|c_k|\left|O_{n,k}(z)\right| + \sum_{k=n+p+3}^{\infty}|c_k|\left|O_{n,k}(z)\right| \\
&\leq \frac{1}{[n+p+1]_q^2}\sum_{k=2}^{\infty}2|c_k|(k-1)^2 r(r+1)^{k+1}\left\{(k-2)(k^2-2k+2)+\frac{(k-2)^2}{4r}\right\} \\
&+\frac{1}{[n+p+1]_q^2}\sum_{k=2}^{\infty}(k-1)(r+2)(r+1)^{k+1}(k+3)! + \sum_{k=n+p+3}^{\infty}|c_k|\left|B_{n,p,q}(e_k)(z)-e_k(z)\right| - \frac{[k(k-1)z-2k(k-2)]z^{k-1}}{2[n+p+1]_q} \\
&\leq \frac{1}{[n+p+1]_q^2}\sum_{k=2}^{\infty}|c_k|(k-1)^2 r(r+1)^{k+1}\left\{(k-2)(k^2-2k+2)+\frac{(k-2)^2}{4r}\right\} \\
&+\frac{1}{[n+p+1]_q^2}\sum_{k=2}^{\infty}(k-1)(r+2)(r+1)^{k+1}(k+3)! + \sum_{k=n+p+3}^{\infty}|c_k|\left[\frac{(r+2)(k+2)!}{[n+p+1]_q}r^{k-1} + \frac{k^2(r+2)r^{k-1}}{2[n+p+1]_q}\right] \\
&\leq \frac{1}{[n+p+1]_q^2}\sum_{k=2}^{\infty}2|c_k|(k-1)^2 r(r+1)^{k+1}\left\{(k-2)(k^2-2k+2)+\frac{(k-2)^2}{4r}\right\} \\
&+\frac{1}{[n+p+1]_q^2}\sum_{k=2}^{\infty}(k-1)(r+2)(r+1)^{k+1}(k+3)! + \sum_{k=n+p+3}^{\infty}|c_k|\frac{(r+2)(k+2)!}{[n+p+1]_q}r^{k-1} \leq \frac{T_r(f)}{([n+p+1]_q)^2} \\
&+\frac{2}{[n+p+1]_q}N_r(f)
\end{aligned} \tag{23}$$

where

$$T_r(f) = M_{1,r}(f) + M_{2,r}(f) + M_{3,r}(f)$$

with

$$M_{1,r}(f) = 2r\sum_{k=2}^{\infty}|c_k|(k-1)^2(k^2-2k+2)(r+1)^{k+1} < \infty$$

$$M_{2,r}(f) = \frac{1}{2}\sum_{k=2}^{\infty}|c_k|[(k-1)^2(k-2)^2(r+1)^{k+1}] < \infty$$

$$M_{3,r}(f) = (r+2)\sum_{k=2}^{\infty}|c_k|(k-1)(k+3)!(r+1)^{k+1} < \infty$$

and

$$N_r(f) = \sum_{k=n+p+3}^{\infty} |c_k|(r+2)(k+2)!r^{k-1} < \infty$$

Next, to estimate the second sum, using Lemma 2.2, we rewrite as follows

$$\begin{aligned}
B_{n,p,q}^{(\alpha,\beta)}(e_k)(z) - B_{n,p,q}(e_k)(z) + \frac{\beta(z-1)-\alpha}{[n+p+1]_q}kz^{k-1} &= \sum_{j=0}^{k-1} \binom{k}{j}_q \frac{[n+p+1]_q^j \alpha^{k-j}}{([n+p+1]_q+\beta)^k} \cdot B_{n,p,q}(e_j)(z) \\
&\quad - \left[1 - \left(\frac{[n+p+1]_q}{[n+p+1]_q+\beta}\right)^k\right] \cdot B_{n,p,q}(e_k)(z) \\
&\quad + \frac{\beta(z-1)-\alpha}{[n+p+1]_q+\beta}kz^{k-1} \\
&= \sum_{j=0}^{k-2} \binom{k}{j}_q \frac{[n+p+1]_q^j \alpha^{k-j}}{([n+p+1]_q+\beta)^k} \cdot B_{n,p,q}(e_j)(z) \\
&\quad + \frac{k\alpha[n+p+1]_q^{k-1}}{([n+p+1]_q+\beta)^k} B_{n,p,q}(e_{k-1})(z) \\
&\quad - \sum_{j=0}^{k-1} \binom{k}{j}_q \frac{[n+p+1]_q^j \beta^{k-j}}{([n+p+1]_q+\beta)^k} \cdot B_{n,p,q}(e_k)(z) \\
&\quad + \frac{\beta(z-1)-\alpha}{[n+p+1]_q+\beta}kz^{k-1} \\
&= \sum_{j=0}^{k-2} \binom{k}{j}_q \frac{[n+p+1]_q^j \alpha^{k-j}}{([n+p+1]_q+\beta)^k} B_{n,p,q}(e_j)(z) \\
&\quad + \frac{k\alpha[n+p+1]_q^{k-1}}{([n+p+1]_q+\beta)^k} \left(B_{n,p,q}(e_{k-1})(z) - z^{k-1}\right) \\
&\quad - \sum_{j=0}^{k-2} \binom{k}{j}_q \frac{[n+p+1]_q^j \beta^{k-j}}{([n+p+1]_q+\beta)^k} B_{n,p,q}(e_k)(z) \\
&\quad - \frac{k\beta[n+p+1]_q^{k-1}}{([n+p+1]_q+\beta)^k} \left(B_{n,p,q}(e_k)(z) - z^k\right) \\
&\quad + \frac{k\alpha}{[n+p+1]_q+\beta} z^{k-1} \left(\frac{[n+p+1]_q^{k-1}}{([n+p+1]_q+\beta)^{k-1}} - 1\right) \\
&\quad + \frac{k\beta}{[n+p+1]_q+\beta} z^k \left(1 - \frac{[n+p+1]_q^{k-1}}{([n+p+1]_q+\beta)^{k-1}}\right) \\
&\quad - \frac{\beta(\alpha+\beta+\beta z+[n+p+1]_q)}{[n+p+1]_q([n+p+1]_q+\beta)}.
\end{aligned}$$

Using Lemma 3.2 and the following inequality

$$1 - \left(\frac{[n+p+1]_q}{[n+p+1]_q+\beta}\right)^k \le \sum_{j=1}^{k} \left(1 - \frac{[n+p+1]_q}{[n+p+1]_q+\beta}\right) = \frac{k\beta}{[n+p+1]_q+\beta} \tag{24}$$

We get

$$\left|\sum_{j=0}^{k-2}\binom{k}{j}_q \frac{[n+p+1]_q^j\alpha^{k-j}}{([n+p+1]_q+\beta)^k}B_{n,p,q}(e_j)(z)\right| \le \sum_{j=0}^{k-2}\binom{k}{j}_q \frac{[n+p+1]_q^j\alpha^{k-j}}{([n+p+1]_q+\beta)^k}\cdot|B_{n,p,q}(e_j)(z)|$$
$$= \sum_{j=0}^{k-2}\frac{k(k-1)}{(k-j)(k-1-j)}\binom{k-2}{j}_q \frac{[n+p+1]_q^j\alpha^{k-j}}{([n+p+1]_q+\beta)^k}\cdot|B_{n,p,q}(e_j)(z)|$$
$$\le \frac{k(k-1)}{2}\cdot\frac{\alpha^2}{([n+p+1]_q+\beta)^2}\cdot 4k!r^{k-2}\sum_{j=0}^{k-2}\binom{k-2}{j}_q\frac{[n+p+1]_q^j\alpha^{k-j-2}}{([n+p+1]_q+\beta)^{k-2}}$$
$$\le \frac{2k(k-1)k!r^{k-2}\alpha^2}{([n+p+1]_q+\beta)^2}.$$

We obtain

$$\left|B_{n,p,q}^{(\alpha,\beta)}(e_k)(z)-B_{n,p,q}(e_k)(z)+\frac{\beta(z-1)-\alpha}{[n+p+1]_q+\beta}kz^{k-1}\right| \le \frac{2k(k-1)k!r^{k-2}\alpha^2}{([n+p+1]_q+\beta)^2}+\frac{k(r+2)\alpha}{([n+p+1]_q+\beta)^2}(k+1)!r^{k-2}$$
$$+\frac{2k(k-1)\beta^2}{([n+p+1]_q+\beta)^2}k!r^{k-2}+\frac{k(k-1)\alpha\beta}{([n+p+1]_q+\beta)^2}r^{k-1}$$
$$+\frac{k\beta(r+2)(k+2)!}{([n+p+1]_q+\beta)^2}r^{k-1}+\frac{k(k-1)\beta^2}{([n+p+1]_q+\beta)^2}r^k$$
$$+\frac{\beta(\alpha+\beta+\beta r+[n+p+1]_q)kr^{k-1}}{([n+p+1]_q+\beta)^2}.$$

which led to

$$\left|B_{n,p,q}^{(\alpha,\beta)}(e_k)(z)-B_{n,p,q}(e_k)(z)+\frac{\beta(z-1)-\alpha}{[n+p+1]_q+\beta}kz^{k-1}\right| \le \frac{H_r(f)}{([n+p+1]_q+\beta)^2}$$

$$H_r(f)=\sum_{k=0}^{\infty}|c_k|[(k-1)(2\alpha^2+\beta^2+\alpha\beta)+(r+2)(\alpha+\beta)+\beta(\beta^2+\alpha+1)](k+2)!kr^k<\infty$$

which combined with (23) establishes the intended outcome. □

Theorem 3.4. *Suppose that $q>1, 0\le\alpha\le\beta$ and the hypotheses on f in the statement of Theorem 3.1 hold, let $1<r<R$ be fixed. Then, for all $n,p\in\mathbb{N}$, $n+p>r+1$ and $|z|\le r$ we have*

$$\|B_{n,p,q}^{(\alpha,\beta)}(f)-f\|_r \sim \frac{1}{[n+p+1]_q} \tag{25}$$

where the constants in the equivalence depend only on f,α,β and r, if f is not a polynomial of degree ≤ 0.

Proof. For all $n,p\in\mathbb{N}$ and $|z|\le r$ we have

$$B_{n,p,q}^{(\alpha,\beta)}(f)(z)-f(z)=\frac{\alpha-\beta+1-\beta z}{[n+p+1]_q}f'(z)+\frac{z(z+2)}{2[n+p+1]_q}f''(z)$$
$$+\frac{1}{[n+p+1]_q^2}\left[[n+p+1]_q^2\left[B_{n,p,q}^{(\alpha,\beta)}(f)(z)-f(z)\right]\right.$$
$$\left.+\frac{\beta z-\beta-\alpha-1}{[n+p+1]_q}f'(z)-\frac{z(z+2)}{2[n+p+1]_q}f''(z)\right].$$

By utilizing the subsequent inequality

$$\|F + G\|_r \geq |\|F\|_r - \|G\|_r| \geq \|F\|_r - \|G\|_r$$

results

$$\|B_{n,p,q}^{(\alpha,\beta)}(f) - f\|_r \geq \frac{1}{[n+p+1]_q} \cdot \|(\alpha + 1 - \beta e_1 + \beta)f' + \frac{e_1\,(e_1+2)}{2} f''(z)\|_r -$$
$$\frac{1}{[n+p+1]_q^2}[n+p+1]_q^2 \cdot \cdot \left\| B_{n,p,q}^{(\alpha,\beta)}(f) - f + \frac{\beta e_1 - \beta - \alpha - 1}{[n+p+1]_q} f' - \frac{e_1\,(e_1+2)}{2} f''(z) \right\|_r$$

Taking into account the hypotheses on f, since f is not a polynomial of degree ≤ 1 in D_R, we can write

$$\left\| (\alpha + 1 - \beta e_1 + \beta)f' + \frac{e_1\,(e_1+2)}{2} f''(z) \right\|_r > 0$$

Now, by Theorem 3.3, it follows that

$$[n+p+1]_q^2 \left\| B_{n,p,q}^{(\alpha,\beta)}(f) - f + \frac{\beta e_1 - (\alpha+\beta+1)}{[n+p+1]_q} f' - \frac{e_1\,(e_1+2)}{2} f''(z) \right\|_r \leq C$$

where $C > 0$ is an independent constant.

Since $\frac{1}{[n+p+1]_q} \to 0$ as $n \to \infty$, there exists an index n_0 depending on f, α, β, q and r such that for all $n \geq n_0$ we have

$$\left\| (\alpha + \beta + 1 - \beta e_1)f' + \frac{e_1\,(e_1+2)}{2} f''(z) \right\|_r - \frac{1}{[n+p+1]_q}[n+p+1]_q^2 \cdot \|B_{n,p,q}^{(\alpha,\beta)}(f) - f$$
$$+ \frac{\beta e_1 - (\alpha+\beta+1)}{[n+p+1]_q + \beta} f' - \frac{e_1\,(e_1+2)}{2[n+p+1]_q} f''(z)\|_r \geq \frac{1}{2}\|(\alpha + \beta + 1 - \beta e_1)f' + \frac{e_1\,(e_1+2)}{2} f''(z)\|_r,$$

which implies :

$$\|B_{n,p,q}^{(\alpha,\beta)}(f) - f\|_r \geq \frac{1}{2[n+p+1]_q} \left\| (\alpha\beta + +1 - \beta e_1)f' + \frac{e_1\,(e_1+2)}{2} f''(z) \right\|_r$$

for all $n \geq n_0$. For $1 \leq n \leq n_0 - 1$ we get

$$\|B_{n,p,q}^{(\alpha,\beta)}(f) - f\|_r \geq \frac{M_{r,n}(f)}{[n+p+1]_q} \tag{26}$$

with

$$M_{r,n}(f) = [n+p+1]_q \|B_{n,p,q}^{(\alpha,\beta)}(f) - f\|_r > 0$$

Finally, we obtain

$$\|B_{n,p,q}^{(\alpha,\beta)}(f) - f\|_r \geq \frac{M_r^{\alpha,\beta}(f)}{[n+p+1]_q} \tag{27}$$

with

$$M_r^{\alpha,\beta}(f) = min\{M_{r,1}(f), \cdots, M_{r,n_0-1}(f), \frac{1}{2}\|(\alpha + \beta - \beta e_1) \cdot f' + \frac{e_1\,(e_1+2)}{2} \cdot f''\|_r\}$$

which combined with Theorem 3.1, we get the desired conclusion. □

Theorem 3.5. *Suppose that $q > 1, 0 \leq \alpha \leq \beta$ and the hypotheses on f in the statement of Theorem 3.1 hold, let $1 < r < r_1 < R$ be fixed. Then, for all $n, p, u \in \mathbb{N}$, $n + p > r + 2$ and $|z| \leq r$ we have*

$$\| \left(B_{n,p,q}^{(\alpha,\beta)}(f)\right)^{(u)} - f^{(u)} \|_r \sim \frac{1}{[n+p+1]_q} \tag{28}$$

where the constants in the equivalence depend only on $f, q, u, r_1, \alpha, \beta$ and r, if f is not a polynomial of degree $\leq u - 1$.

Proof. [Proof of Theorem 3.5] Denote by Γ the circle of radius r_1 and center O with $1 \leq r < r_1 < R$. Since $|z| \leq r$ and $\gamma \in \Gamma$, we have $|\gamma - z| \geq r_1 - r$ and from Cauchy's formulas we obtain for all $|z| \leq r$ and $n, p \in \mathbb{N}, n + p > r + 2$ that

$$\begin{aligned} &\left(B_{n,p,q}^{(\alpha,\beta)}(f)(z)\right)^{(u)} - f^{(u)}(z) \leq \frac{u!}{2\pi} \left| \int_\Gamma \frac{B_{n,p,q}^{(\alpha,\beta)}(f)(\gamma) - f(\gamma)}{(\gamma - z)^{u+1}} d\gamma \right| \\ &\leq \frac{M_{r_1}^{(\alpha,\beta)}(f)}{[n+p+1]_q} \cdot \frac{u!}{2\pi} \cdot \frac{2\pi r_1}{(r_1 - r)^{u+1}} = \frac{M_{r_1}^{(\alpha,\beta)}(f)}{[n+p+1]_q} \cdot \frac{u! r_1}{(r_1 - r)^{u+1}} \end{aligned} \tag{29}$$

this verifies one of the inequalities in the equivalence.

From Cauchy's formula, we get

$$\left(B_{n,p,q}^{(\alpha,\beta)}(f)(z)\right)^{(u)} - f^{(u)}(z) = \frac{u!}{2\pi i} \int_\Gamma \frac{B_{n,p,q}^{(\alpha,\beta)}(f)(\gamma) - f(\gamma)}{(\gamma - z)^{u+1}} d\gamma. \tag{30}$$

For all $\gamma \in \Gamma$ and $p, n \in \mathbb{N}$ we have

$$\begin{aligned} B_{n,p,q}^{(\alpha,\beta)}(f)(\gamma) - f(\gamma) =& \frac{1}{[n+p+1]_q} \left[(\alpha + 1 - \beta\gamma) f'(\gamma) + \frac{\gamma(\gamma+2)}{2} f''(\gamma) \right] \\ &+ \frac{1}{[n+p+1]_q} [n+p+1]_q^2 \Bigg(B_{n,p,q}^{(\alpha,\beta)}(f)(\gamma) - f(\gamma) \\ &- \frac{(\alpha + 1 - \beta\gamma)}{[n+p+1]_q + \beta} f'(\gamma) - \frac{\gamma(\gamma+2)}{2[n+p+1]_q} f''(\gamma) \Bigg). \end{aligned} \tag{31}$$

We apply Cauchy's formula and obtain

$$\begin{aligned} B_{n,p,q}^{(\alpha,\beta)}(f)(z)^{(u)} - f^{(u)}(z) =& \frac{1}{[n+p+1]_q} \left[(\alpha + 1 - \beta z) f'(z) + \frac{z(z+2)}{2} f''(z) \right]^{(u)} \\ &+ \frac{1}{[n+p+1]_q} \Bigg[\frac{u!}{2\pi i} \int_\Gamma \Bigg(\frac{[n+p+1]_q^2}{(\gamma - z)^{u+1}} B_{n,p,q}^{(\alpha,\beta)}(f)(\gamma) \\ &- f(\gamma) + \frac{\beta\gamma - \alpha - 1}{[n+p+1]_q} f'(\gamma) - \frac{\gamma(\gamma+2)}{2[n+p+1]_q} f''(\gamma) \Bigg) d\gamma \Bigg]. \end{aligned} \tag{32}$$

Passing to the norm $\| \cdot \|_r$ we obtain

$$\begin{aligned} \|B_{n,p,q}^{(\alpha,\beta)}(f)^{(u)} - f^{(u)}\|_r \geq& \frac{1}{[n+p+1]_q} \left\| \left[(\alpha + 1 - \beta e_1) f' + \frac{e_1(e_1+2)}{2} f''(z) \right]^{(u)} \right\|_r \\ &- \frac{1}{[n+p+1]_q} \Bigg\| \frac{u!}{2\pi i} \int_\Gamma \frac{[n+p+1]_q^2}{(\gamma - z)^{u+1}} \Bigg(B_{n,p,q}^{(\alpha,\beta)}(f)(\gamma) - f(\gamma) \\ &+ \frac{\beta\gamma - \alpha - 1}{[n+p+1]_q + \beta} f'(\gamma) - \frac{\gamma(\gamma+2)}{2[n+p+1]_q} f''(\gamma) \Bigg) d\gamma \Bigg\|_r \end{aligned} \tag{33}$$

and from Theorem 3.3, for all $p, n \in \mathbb{N}$ we have

$$\begin{aligned} &\|\frac{u!}{2\pi i}\int_{\Gamma}\frac{[n+p+1]_q^2}{(\gamma-z)^{u+1}}(B_{n,p,q}^{(\alpha,\beta)}(f)(\gamma)-f(\gamma)+\frac{\beta\gamma-\alpha-1}{[n+p+1]_q+\beta}f'(\gamma)-\frac{\gamma(\gamma+2)}{2[n+p+1]_q}f''(\gamma))d\gamma\|_r \\ &\qquad \leq \frac{u!}{2\pi}\cdot\frac{2\pi r_1}{(r_1-r)^{u+1}}\sum_{j=1}^{5}M_{r_1,j}^{(\alpha,\beta)}(f) \end{aligned} \tag{34}$$

Since f is not a polynomial of degree ≤ 0 in D_R, it can be written

$$\left\|\left[(\alpha+\beta+1-\beta e_1)f'+\frac{e_1(e_1+2)}{2[n+p+1]_q}f''\right]^{(u)}\right\|_r > 0 \tag{35}$$

(see [6]). The remainder of the proof can be obtained similarly as in the proof of theorem 3.4. □

4. Conclusions

We constructed a variant of the Beta-type generalization of the complex q-Baskakov-Schurer-Szász-Stancu operators in compact disks and studied several approximation properties. The modified Beta operator that we proposed is a powerful tool for approximating functions on compact disks. Its ability to handle complex variables and adapt to different weight functions make it a versatile tool for a wide range of applications.

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All the authors contributed equally and significantly in writing this paper.

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