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Wavelet-based approach for approximating Jacobi polynomial via characterized Hausdorff matrix

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Abstract. In the present work, we aim to study the rate of convergence of Jacobi polynomial using characterized Hausdorff matrix. Another important aim of the present work is to estimate the wavelet approximation of Jacobi polynomial using characterized Hausdorff matrix.

1. Introduction

The approximation properties of Fourier series have been extensively investigated by various researchers, including Oskilenker [20], Szegö [23], Zygmund [24], and Moricz ([10, 11]). More recently, this research has been extended to wavelet expansions by the investigators such as Kelly [6] and Mallat [9]. It is crucial to acknowledge that wavelet expansions demonstrate oscillatory characteristics analogous to those observed in Fourier expansions. Consequently, traditional summability techniques are not directly applicable to wavelet expansions due to the nature of the approximation, which involves infinite partial sums. The exploration of Fourier series approximation has been a subject of interest for numerous researchers, and the works of researchers like Oslinker [20], Szegö [23], Zygmund [24], and Moricz [10] have significantly contributed to this field. Recently, attention has shifted towards extending these results to wavelet expansions, with notable studies conducted by researchers such as Kelly [6], Mallat [9], Nigam ([14**?** –18]), Mursaleen and Mukheimer [12], Savas and Mursaleen [21], Agratini [1], Ayman-Mursaleen et al. [3], Nasiruzzaman et al. [13], Gonska [4], Srivastava [19], Kumar et al. [7, 8] etc.
Wavelet approximation has gained prominence due to their ability to capture and represent signals at different scales. They offer advantages over traditional Fourier series in handling non-stationary signals and providing a localized representation of signal features.
It is crucial to recognize similarities and differences between Fourier and wavelet expansions. While both exhibit oscillatory behavior, the direct application of the classical Hausdorff operator already proven effective in Fourier series. The adaptability of traditional techniques to the unique characteristics of wavelet expansions becomes a central concern in advancing the field.
In response to these challenges, this paper deals with the wavelet approximation of of Jacobi polynomial using Hausdorff matrix. The Jacobi polynomial, known for its versatility in various mathematical applications, has been proved to be a suitable tool for addressing the complexities of wavelet approximation.

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The Hausdorff method provides a novel perspective on wavelet approximation. By leveraging Jacobi polynomial, this method aims to overcome the limitations encountered when applying traditional operators in the wavelet domain. The incorporation of Jacobi polynomial offers a tailored approach, aligning with the unique characteristics of wavelet expansion and facilitating more accurate and efficient approximation.

This paper primarily focuses on two key points of investigation:

1. The rapid rate of convergence of Jacobi polynomial using characterized Hausdorff matrix.
2. Wavelet approximation of Jacobi polynomial using characterized Hausdorff matrix.

2. Definitions

2.1. Fourier Series

The trigonometric Fourier series $g(x) = \sum_{\jmath=-\infty}^{\infty} c_\jmath e^{i\jmath x}$ is associated with a periodic real function g of coefficients

$$c_\jmath = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-i\jmath x}dx. \tag{1}$$

In this case, we consider the trigonometric polynomials

$$(s_\varsigma g)(x) = \sum_{\jmath=-\varsigma}^{\varsigma} c_\jmath e^{i\jmath x}, \tag{2}$$

where ς is a non negative integer (2) is called "partial sums" of the Fourier series g.

2.2. Jacobi Polynomial

The normalized Jacobi polynomial is defined as (see [2])

$$R_\varsigma^{(\alpha,\beta)}(\cos\theta) = \frac{P_\varsigma^{(\alpha,\beta)}(\cos\theta)}{P_\varsigma^{(\alpha,\beta)}(1)}, \tag{3}$$

where $P_\varsigma^{(\alpha,\beta)}(1) = \binom{\varsigma+\alpha}{\varsigma} \neq 0$ and $w(a) = (1-a)^\alpha(1+a)^\beta$ for $\alpha > -1,\ \beta > -1$. It is important to note that (2) form a full orthogonal system in the space $L^2([0,\pi]; w)$ such that $\left|R_\varsigma^{(\alpha,\beta)}(\cos\theta)\right| \leq 1$, where $\varsigma \geq 0$ and $\alpha \geq -\frac{1}{2}$.

2.3. Hausdorff Matrix

A Hausdorff matrix (see [5]) $H = (a_{\varsigma,\jmath})$ is an infinite lower triangular matrix with non-zero entries

$$a_{\varsigma,\jmath} = \begin{cases} \binom{\varsigma}{\jmath}\Delta^{\varsigma-\jmath}u_\jmath, & 0 \leq \jmath \leq \varsigma \\ 0, & \jmath > \varsigma, \end{cases} \tag{4}$$

where Δ represents a forward difference operator, denoted by $\Delta u_\varsigma = u_\varsigma - u_{\varsigma-1}$ and $\Delta^{\jmath+1}u_\varsigma = \Delta^\jmath(\Delta u_\varsigma)$. If H is regular, then u_ς is referred to as a moment sequence, which can be represented as $u_\varsigma = \int_0^1 v^\varsigma d\zeta(v)$, where $\zeta(v)$ is referred to as the mass function. $\zeta(v)$ is continous at $v = 0$ and it belongs to $BV[0,1]$ such that $\zeta(0) = 0,\ \zeta(1) = 1$; and for $0 < v < 1$, $\zeta(v) = \frac{[\zeta(v+0)+\zeta(v-0)]}{2}$.
A Hausdorff matrix (4) can also be written as

$$a_{\varsigma,\jmath} = \begin{cases} \binom{\varsigma}{\jmath}\int_0^1 l^\jmath(1-l)^{\varsigma-\jmath}d\gamma(l), & \jmath = 0,1,2,\cdots,\varsigma \\ 0, & \jmath > \varsigma, \end{cases} \tag{5}$$

2.4. Characterizated Hausdorff Matrix

Following [22], we establish a relation of the positivity and monotonicity of the matrix $a_{\varsigma,\jmath}, 0 \leq \jmath \leq \varsigma$; $\varsigma = 0, 1, 2, ...$ for the matrix (5) to its mass function. As defined in [22], $a_{\varsigma,0} = 1$ for $\varsigma = 0$ and for $\varsigma > 0$

$$a_{\varsigma,\jmath} = \begin{cases} \frac{1}{B(1,\varsigma)} \int_0^1 (1-l)^{\varsigma-1} \gamma(l) dl, & \jmath = 0 \\ \frac{1}{\jmath B(\jmath,\varsigma-\jmath+1)} \int_0^1 l^{\jmath-1} (1-l)^{\varsigma-\jmath-1} (\varsigma l - \jmath) \gamma(l) dl, & 0 < \jmath < \varsigma \\ 1 - \frac{1}{B(\varsigma,1)} \int_0^1 l^{\varsigma-1} \gamma(l) dl, & \jmath = \varsigma \\ 0, & \jmath > \varsigma, \end{cases} \tag{6}$$

where $B(m, \varsigma)$ is the beta function.

2.5. Multiresolution Analysis

Let's denote the approximation space at level ξ as ρ_ξ, and the collection $\{\rho_\xi : \xi \in \mathbb{Z}\}$ constitute a multiresolution analysis for the space $L^2(\mathbb{R})$. A scale relation holds true between two consecutive subspaces as $g(\cdot) \in \rho_\xi \Rightarrow g(2\cdot) \in \rho_{\xi+1}$. A size-adjustment function $\phi \in L^2(\mathbb{R})$ such that $\left\{\phi_{i,\xi}(\cdot) = 2^{\frac{\xi}{2}} \phi(2^\xi . - \jmath), \jmath \in \mathbb{Z}\right\}$ constitutes an orthogonal basis for ρ_ξ. Let η_ξ be the orthogonal complement of ρ_ξ in $\rho_{\xi+1}$, expressed as

$$\rho_\xi \oplus \eta_\xi = \rho_{\xi+1}, \tag{7}$$

where η_ξ are commonly known as detail spaces at level ξ. We have $\cdots \rho_{-2} \subset \rho_{-1} \subset \rho_0 \subset \rho_1 \subset \rho_2 \subset \rho_3 \cdots \subset \rho_\xi \subset \rho_{\xi+1} \subset \cdots$, where $\{\rho_\xi\}$ is a multiresolution analysis at level ξ. Thus,

$$\rho_\xi = \rho_0 \oplus \eta_0 \oplus \eta_1 \oplus \cdots \oplus \eta_{\xi-1}$$

and $L^2(\mathbb{R}) = \rho_0 \oplus_{\xi \geq 0} \eta_\xi$. Consider p_ξ as the orthogonal projection of $L^2(\mathbb{R})$ on to ρ_ξ. If $\langle \cdot, \cdot \rangle$ represents the standard inner product in the space $L^2(\mathbb{R})$, then using (7), we have

$$P_{\xi+1} g = P_\xi g + \sum_{\jmath \in \mathbb{Z}} d_{\xi,\jmath} \psi_{\xi,\jmath},$$

where $d_{\xi,\jmath} = \left\langle g, \psi_{\xi,\jmath} \right\rangle$ are the wavelet or the detail coefficients and $g \in L^2(\mathbb{R})$.

3. Known Result

Askey [2] proved the following theorem:

Theorem 3.1. *If $\sum_{\varsigma=0}^{\infty} a_\varsigma$ converges to s, then*

$$u(r, \theta) = \sum_{\varsigma=0}^{\infty} a_\varsigma R_\varsigma^{(\alpha,\beta)}(\cos\theta) r^\varsigma$$

tends to s for $r \to 1$, $\theta = O(1-r)$. If $\alpha > \frac{1}{2}$, then $u(r, \theta)$ tends to s for $r \to 1, \theta \to 0$, without the restriction $\theta = O(1-r)$.

4. Main Results

4.1. Rate of convergence of Jacobi polynomial using characterized Hausdorff matrix

In this section, we prove the following theorems:

Theorem 4.1. (i) *If $\alpha = r$ then $\exists$ a positive constant K_1 such that*

$$\|l_\varsigma^H - s\|_\infty \leq K_1 r^\varsigma$$

(ii) *If $r < \alpha$ then $\exists$ a positive constant K_2 such that*

$$\|l_{\varsigma}^{H} - s\|_{\infty} \leq K_2 r^{\varsigma}$$

(iii) *If $\alpha < r$ then $\exists$ a positive constant K_3 such that*

$$\|l_{\varsigma}^{H} - s\|_{\infty} \leq K_3 r^{\varsigma}$$

Proof. (i) Let $\sum_{\varsigma=0}^{\infty} a_{\varsigma}$ be an infinite series with ς^{th} partial sums

$$s_{\varsigma} = \sum_{v=0}^{\varsigma} a_v \ \ \forall \varsigma > 0.$$

If

$$h_{\varsigma,\tau} = \begin{cases} \frac{1}{B(1,\varsigma)} \int_0^1 (1-l)^{\varsigma-1} \gamma(l) dl R_{\tau+1}^{(\alpha,\beta)}(\cos\theta) r^{\tau+1}, & \tau = 0 \\ \frac{1}{\tau B(\tau,\varsigma-\tau+1)} \int_0^1 l^{\tau-1}(1-l)^{\varsigma-\tau-1}(\varsigma l - \tau)\gamma(l) dl R_{\tau+1}^{(\alpha,\beta)}(\cos\theta) r^{\tau+1}, & 0 < \tau < \varsigma \\ 1 - \frac{1}{B(\varsigma,1)} \int_0^1 l^{\varsigma-1} \gamma(l) dl R_{\tau+1}^{(\alpha,\beta)}(\cos\theta) r^{\tau+1}, & \tau = \varsigma \\ 0, & \tau \geq \varsigma + 1, \end{cases}$$

then using Theorem (3.1), we define

$$\begin{aligned} & l_{\varsigma}^{H}(s_{\varsigma}) \\ =& \frac{1}{B(1,\varsigma)} \int_0^1 (1-l)^{\varsigma-1} \gamma(l) R_{\varsigma+1}^{\alpha,\beta}(\cos\theta) r^{\varsigma+1} s_0 dl \\ &+ \sum_{\jmath=1}^{\varsigma-1} \left(\frac{1}{\jmath B(\jmath, \varsigma - \jmath + 1)} \int_0^l l^{\jmath-1}(1-l)^{\varsigma-\jmath-1}(\varsigma l - \jmath)\gamma(l) R_{\jmath+1}^{\alpha,\beta}(\cos\theta) r^{\jmath+1} dl \right) s_{\jmath} \\ &+ (1 - \frac{1}{B(\varsigma,1)} \int_0^l l^{\varsigma-1} \gamma(l) dl) R_{\varsigma}^{\alpha,\beta}(\cos\theta) r^{\varsigma} dl s_{\varsigma} \end{aligned}$$

Now, we get

$$\begin{aligned} & \|l_{\varsigma}^{H}(s_{\varsigma}) - s\|_{\infty} \\ \leq & \left| \frac{1}{B(1,\varsigma)} \int_0^1 (1-l)^{\varsigma-1} \gamma(l) R_{\varsigma+1}^{\alpha,\beta}(\cos\theta) r^{\varsigma+1} s_0 dl \right| \\ &+ \sum_{\jmath=1}^{\varsigma-1} \left| \frac{1}{\jmath B(\jmath, \varsigma - \jmath + 1)} \int_0^l l^{\jmath-1}(1-l)^{\varsigma-\jmath-1}(\varsigma l - \jmath)\gamma(l) R_{\jmath+1}^{\alpha,\beta}(\cos\theta) r^{\jmath+1} dl \right| \left\| \sum_{v=0}^{\jmath} (s_v - s) \right\|_{\infty} \\ &+ \left| 1 - \frac{1}{B(\varsigma,1)} \int_0^l l^{\varsigma-1} \gamma(l) dl) R_{\varsigma}^{\alpha,\beta}(\cos\theta) r^{\varsigma} dl \right| \left\| (s_{\varsigma} - s) \right\|_{\infty} \\ \leq & |s_0| r^{\varsigma+1} + A \sum_{\jmath=1}^{\varsigma-1} r^{\jmath+1} \left\| \sum_{v=0}^{\jmath} (s_v - s) \right\|_{\infty} + 0 \\ = & |s_0| r^{\varsigma+1} + A \sum_{\jmath=1}^{\varsigma-1} r^{\jmath+1} r^{\jmath} \\ = & |s_0| r^{\varsigma+1} + A \frac{r(r^{2\varsigma} - r^2)}{r^2 - 1} \end{aligned} \tag{8}$$

$$= |s_0|r^{\varsigma+1} + A\frac{r^3(r^{2\varsigma-2}-1)}{r^2-1}$$
$$= |s_0|r^{\varsigma+1} + B(r^{2\varsigma-2}-1)$$
$$\leq |s_0|r^{\varsigma+1} + Br^{2\varsigma-2}$$
$$= |s_0|r^{\varsigma+1} + B\frac{r^{2\varsigma}}{r^2}$$
$$\leq |s_0|r^{\varsigma+1} + B\frac{r^{\varsigma}}{r^2}$$
$$= \left(|s_0|r + \frac{B}{r^2}\right)r^{\varsigma}$$
$$= K_1 r^{\varsigma}$$

□

Proof. (ii) Using (8), we get

$$\begin{aligned}
\|l_{\varsigma}^{H}(s_{\varsigma}) - s\|_{\infty} &\leq |s_0|\alpha^{\varsigma+1} + A\sum_{\jmath=1}^{\varsigma-1} r^{\jmath+1}\alpha^{\jmath} \\
&= |s_0|\alpha^{\varsigma+1} + B\sum_{\jmath=1}^{\varsigma-1}(\alpha r)^{\jmath} \\
&= |s_0|\alpha^{\varsigma+1} + B\frac{(\alpha r)^{\varsigma} - \alpha r}{\alpha r - 1} \\
&\leq |s_0|\alpha^{\varsigma+1} + B\frac{\alpha^{2\varsigma} - \alpha^2}{\alpha^2 - 1} \\
&\leq |s_0|\alpha^{\varsigma+1} + C(\alpha^{2\varsigma} - \alpha^2) \\
&= |s_0|\alpha^{\varsigma+1} + C\alpha^{2\varsigma} \\
&\leq |s_0|\alpha^{\varsigma+1} + C\alpha^{\varsigma} \\
&= (|s_0|\alpha + C)\alpha^{\varsigma} \\
&= K_2\alpha^{\varsigma}, \qquad \text{where } K_2 = |s_0|\alpha + C
\end{aligned}$$

□

Proof. (iii) Using (8), we get

$$\begin{aligned}
\|l_{\varsigma}^{H}(s_{\varsigma}) - s\|_{\infty} &\leq |s_0|r^{\varsigma+1} + A\sum_{\jmath=1}^{\varsigma-1} r^{\jmath+1}\alpha^{\jmath} \\
&= |s_0|r^{\varsigma+1} + B\sum_{\jmath=1}^{\varsigma-1}(\alpha r)^{\jmath} \\
&\leq |s_0|r^{\varsigma+1} + B\frac{(\alpha r)^{\varsigma} - (\alpha r)}{\alpha r - 1} \\
&= |s_0|r^{\varsigma+1} + C((\alpha r)^{\varsigma} - (\alpha r)) \\
&\leq |s_0|r^{\varsigma+1} + C(\alpha r)^{\varsigma} \qquad \alpha \in (0,1) \\
&\leq |s_0|r^{\varsigma+1} + Cr^{2\varsigma} \\
&= (|s_0|r + C)r^{\varsigma}
\end{aligned}$$

$$= K_3 r^{\varsigma}, \qquad \text{where } |s_0| r + C = K_3$$

□

Now, we prove the following theorem:

Theorem 4.2. (i) *If $\alpha = r$ then*

$$\|l_{\varsigma}^{H}(s_{\varsigma}) - s\|_{\infty} = O\left(\frac{\varsigma r^{\varsigma+1} - (\varsigma+1) r^{\varsigma} - r^2 + 2r}{(r-1)^2}\right)$$

(ii) *If $r < \alpha$ then*

$$\|l_{\varsigma}^{H}(s_{\varsigma}) - s\|_{\infty} = O\left(\frac{\varsigma \alpha^{\varsigma+1} - (\varsigma+1) \alpha^{\varsigma} - \alpha^2 + 2\alpha}{(\alpha-1)^2}\right)$$

(iii) *If $\alpha < r$ then*

$$\|l_{\varsigma}^{H}(s_{\varsigma}) - s\|_{\infty} = O\left(\frac{\varsigma r^{\varsigma+1} - (\varsigma+1) r^{\varsigma} - r^2 + 2r}{(r-1)^2}\right)$$

Proof. (i) Using (8), we get

$$\|l_{\varsigma}^{H}(s_{\varsigma}) - s\|_{\infty} = |s_0|\, r^{\varsigma+1} + B \sum_{\jmath=1}^{\varsigma-1} r^{\jmath+1} \left(\sum_{v=0}^{\jmath} r^{v}\right)$$

$$\leq |s_0|\, r^{\varsigma+1} + B \sum_{\jmath=1}^{\varsigma-1} r^{\jmath+1} \left(1 + r + r^2 + \cdots + r^{\jmath}\right)$$

$$= |s_0|\, r^{\varsigma+1} + Br \sum_{\jmath=1}^{\varsigma-1} \left(r^{\jmath} + r^{\jmath+1} + r^{\jmath+2} + \cdots + r^{2\jmath}\right)$$

$$\leq |s_0|\, r^{\varsigma+1} + Br \sum_{\jmath=1}^{\varsigma-1} \left(r^{\jmath} + r^{\jmath} + r^{\jmath} + \cdots + r^{\jmath}\right)$$

$$= |s_0|\, r^{\varsigma+1} + Br \sum_{\jmath=1}^{\varsigma-1} ((\jmath+1) r^{\jmath})$$

$$\leq (|s_0|\, r^{\varsigma+1} + Br) \sum_{\jmath=1}^{\varsigma-1} ((\jmath+1) r^{\jmath})$$

$$= (|s_0|\, r^{\varsigma+1} + Br) \left(\frac{\varsigma r^{\varsigma+1} - (\varsigma+1) r^{\varsigma} - r^2 + 2r}{(r-1)^2}\right)$$

$$= O\left(\frac{\varsigma r^{\varsigma+1} - (\varsigma+1) r^{\varsigma} - r^2 + 2r}{(r-1)^2}\right)$$

□

Proof. (ii) Using (8), we get

$$\|l_{\varsigma}^{H}(s_{\varsigma}) - s\|_{\infty} \leq |s_0|\, r^{\varsigma+1} + A \sum_{\jmath=1}^{\varsigma-1} r^{\jmath+1} \sum_{v=0}^{\jmath} \|s_v - s\|_{\infty}$$

$$= |s_0|\, r^{\varsigma+1} + A \sum_{\jmath=1}^{\varsigma-1} r^{\jmath+1} \left(\sum_{v=0}^{\jmath} C_2 \alpha^v \right), \quad C_2 > 0$$

$$= |s_0|\, r^{\varsigma+1} + Br \sum_{\jmath=1}^{\varsigma-1} r^{\jmath} (1 + \alpha + \cdots + \alpha^{\jmath})$$

$$\leq |s_0|\, r\alpha^{\varsigma} + Br \sum_{\jmath=1}^{\varsigma-1} \alpha^{\jmath} (1 + \alpha + \cdots + \alpha^{\jmath})$$

$$= |s_0|\, r\alpha^{\varsigma} + Br \sum_{\jmath=1}^{\varsigma-1} (\alpha^{\jmath} + \alpha^{\jmath+1} + \cdots + \alpha^{2\jmath})$$

$$= |s_0|\, r\alpha^{\varsigma} + Br \sum_{\jmath=1}^{\varsigma-1} (\jmath + 1)\alpha^{\jmath}$$

$$\leq (|s_0|\, r + Br) \sum_{\jmath=1}^{\varsigma-1} (\jmath + 1)\alpha^{\jmath} \quad (\because r < \alpha)$$

$$\leq (|s_0|\, r + Br) \left(\frac{\varsigma\alpha^{\varsigma+1} - (\varsigma + 1)\alpha^{\varsigma} - \alpha^2 + 2\alpha}{(\alpha - 1)^2} \right)$$

$$= O\left(\frac{\varsigma\alpha^{\varsigma+1} - (\varsigma + 1)\alpha^{\varsigma} - \alpha^2 + 2\alpha}{(\alpha - 1)^2} \right)$$

□

Proof. (iii) Using (8), we get

$$\|l_{\varsigma}^{H}(s_{\varsigma}) - s\|_{\infty} \leq |s_0|\, r^{\varsigma+1} + A \sum_{\jmath=1}^{\varsigma-1} r^{\jmath+1} \sum_{v=0}^{\jmath} \|s_v - s\|_{\infty}$$

$$= |s_0|\, r^{\varsigma+1} + A \sum_{\jmath=1}^{\varsigma-1} r^{\jmath+1} \sum_{v=0}^{\jmath} C_3 \alpha^v, \quad C_3 > 0$$

$$\leq |s_0|\, r^{\varsigma+1} + Br \sum_{\jmath=1}^{\varsigma-1} r^{\jmath} \sum_{v=0}^{\jmath} r^v$$

$$= |s_0|\, r^{\varsigma+1} + Br \sum_{\jmath=1}^{\varsigma-1} r^{\jmath} (1 + r + r^2 + \cdots + r^{\jmath})$$

$$= |s_0|\, r^{\varsigma+1} + Br \sum_{\jmath=1}^{\varsigma-1} (r^{\jmath} + r^{\jmath+1} + \cdots + r^{2\jmath}), \quad r \in [0, 1)$$

$$\leq |s_0|\, r^{\varsigma+1} + Br \sum_{\jmath=1}^{\varsigma-1} (r^{\jmath} + r^{\jmath} + \cdots + r^{\jmath})$$

$$= |s_0|\, r^{\varsigma+1} + Br \sum_{\jmath=1}^{\varsigma-1} (\jmath + 1) r^{\jmath}$$

$$\leq (|s_0|\, r + Br) \sum_{\jmath=1}^{\varsigma-1} (\jmath + 1) r^{\jmath}$$

$$= (|s_0|\, r + Br)\left(\frac{\varsigma r^{\varsigma+1} - (\varsigma+1)r^{\varsigma} - r^2 + 2r}{(r-1)^2}\right)$$
$$= O\left(\frac{\varsigma r^{\varsigma+1} - (\varsigma+1)r^{\varsigma} - r^2 + 2r}{(r-1)^2}\right)$$

□

4.2. Wavelet approximation of Jacobi polynomial using charaterized Hausdorff matrix

Theorem 4.3. *Let P_ξ be the orthogonal projection of the space $L^2(\mathbb{R})$ onto the approximation space V_ξ such that*

$$l_\varsigma^H(P_\xi g) = \frac{\alpha}{\alpha+1}\left(1-(1-r)^{\alpha+1}\right)$$
$$\left[r^{\varsigma+1}P_0 g + A\left(\frac{r^{\varsigma}-r}{r-1}\right)\left\{\jmath P_0 g + \sum_{v=0}^{\xi}\sum_{\jmath\in\mathbb{Z}}\left(\jmath d_{0,\jmath}\psi_{0,\jmath} + (\jmath-1)d_{1,\jmath}\psi_{1,\jmath} + \cdots + d_{\jmath-1,\jmath}\psi_{\jmath-1,\jmath}\right)\right\}\right]$$

Proof. From (8), it follows that

$$l_\varsigma^H(P_\xi g) = \frac{1}{B(1,\varsigma)}\int_0^1 (1-l)^{\varsigma-1}\gamma(l)dl R_{\varsigma+1}^{(\alpha,\beta)}(cos\theta)r^{\varsigma+1}P_0 g$$
$$+\left(\sum_{\jmath=1}^{\varsigma-1}\left(\frac{1}{\jmath B(\jmath,\varsigma-\jmath+1)}\int_0^1 l^{\jmath-1}(1-l)^{\varsigma-\jmath-1}(\varsigma l-\jmath)\gamma(l)dl R_{\jmath+1}^{(\alpha,\beta)}(cos\theta)r^{\jmath+1}\right)\right).$$
$$\left(\sum_{v=1}^{\jmath}P_v g\right) + \sum_{\jmath=\varsigma}^{\varsigma}\left(1-\frac{1}{B(\varsigma-1)}\int_0^1 l^{\varsigma-1}\gamma(l)dl\right)R_\jmath^{(\alpha,\beta)}cos\theta * r^\jmath\left(\sum_{v=1}^{\jmath}P_v g\right)$$

Using mass function $\gamma(l) = \alpha\int_0^s (1-s)^\alpha ds$ in above equation, we get

$$\frac{1}{B(1,\varsigma)}\int_0^1 (1-l)^{\varsigma}\alpha\int_0^s (1-s)^\alpha ds dl r^{\varsigma+1}R_{\jmath+1}^{(\alpha,\beta)}(cos\theta)P_0 g$$
$$+\sum_{\jmath=1}^{\varsigma-1}\left(\frac{1}{\jmath B(\jmath,\varsigma-\jmath+1)}\int_0^1 l^{\jmath-1}(1-l)^{\varsigma-\jmath-1}(\varsigma l-\jmath)\alpha\int_0^s (1-s)^\alpha ds dl r^{\jmath+1}R_{\jmath+1}^{(\alpha,\beta)}(cos\theta)\right)$$
$$\left(\jmath P_0 g + \jmath\sum_{v=0}^{\xi}\sum_{\jmath\in\mathbb{Z}}d_{0,\jmath}\psi_{0,\jmath} + (\jmath-1)\sum_{v=0}^{\xi}\sum_{\jmath\in\mathbb{Z}}d_{1,\jmath}\psi_{1,\jmath} + \cdots + \sum_{v=0}^{\xi}\sum_{\xi\in\mathbb{Z}}d_{\jmath-1,\jmath}\psi_{\jmath-1,\jmath}\right)$$
$$= \frac{1}{B(1,\varsigma)}\int_0^1 (1-l)^{\varsigma}\alpha\frac{1-(1-s)^{\alpha+1}}{\alpha+1}dl r^{\varsigma+1}R_{\jmath+1}^{(\alpha,\beta)}(cos\theta)P_0 g$$
$$+\sum_{\jmath=1}^{\varsigma-1}\left(\frac{1}{\jmath B(\jmath,\varsigma-\jmath+1)}\int_0^1 l^{\jmath-1}(1-l)^{\varsigma-\jmath-1}(\varsigma l-\jmath)\alpha\frac{1-(1-s)^{\alpha+1}}{\alpha+1}dl r^{\jmath+1}R_{\jmath+1}^{(\alpha,\beta)}(cos\theta)\right)$$
$$\left(\jmath P_0 g + \jmath\sum_{v=0}^{\xi}\sum_{\jmath\in\mathbb{Z}}d_{0,\jmath}\psi_{0,\jmath} + (\jmath-1)\sum_{v=0}^{\xi}\sum_{\jmath\in\mathbb{Z}}d_{1,\jmath}\psi_{1,\jmath} + \cdots + \sum_{v=0}^{\xi}\sum_{\jmath\in\mathbb{Z}}d_{\jmath-1,\jmath}\psi_{\jmath-1,\jmath}\right)$$

$$= \frac{1}{B(1,\varsigma)}\alpha\frac{1-(1-s)^{\alpha+1}}{\alpha+1}r^{\varsigma+1}R_{\jmath+1}^{(\alpha,\beta)}(cos\theta)P_0 g\int_0^1 (1-l)^{\varsigma}dl$$

$$+\sum_{\jmath=1}^{\varsigma-1}\left(\frac{\alpha}{\jmath B(\jmath,\varsigma-\jmath+1)}\frac{1-(1-s)^{\alpha+1}}{\alpha+1}r^{\jmath+1}R_{\jmath+1}^{(\alpha,\beta)}(cos\theta)\int_0^1 l^{\jmath-1}(1-l)^{\varsigma-\jmath-1}(\varsigma l-\jmath)dl\right)$$

$$\left(\jmath P_0 g+\jmath\sum_{v=0}^{\xi}\sum_{\jmath\in\mathbb{Z}}d_{0,\jmath}\psi_{0,\jmath}+(\jmath-1)\sum_{v=0}^{\xi}\sum_{\jmath\in\mathbb{Z}}d_{1,\jmath}\psi_{1,\jmath}+\cdots+\sum_{v=0}^{\xi}\sum_{\jmath\in\mathbb{Z}}d_{\jmath-1,\jmath}\psi_{\jmath-1,\jmath}\right)$$

$$=\alpha\frac{1-(1-s)^{\alpha+1}}{\alpha+1}r^{\varsigma+1}R_{\jmath+1}^{(\alpha,\beta)}(cos\theta)P_0 g\frac{1}{B(1,\varsigma)}\int_0^1(1-l)^{\varsigma}dl$$

$$+\alpha\frac{1-(1-s)^{\alpha+1}}{\alpha+1}\left(\sum_{\jmath=1}^{\varsigma-1}r^{\jmath+1}R_{\jmath+1}^{(\alpha,\beta)}(cos\theta)\frac{1}{\jmath B(\jmath,\varsigma-\jmath+1)}\int_0^1 l^{\jmath-1}(1-l)^{\varsigma-\jmath-1}(\varsigma l-\jmath)dl\right)$$

$$\left(\jmath P_0 g+\jmath\sum_{v=0}^{\xi}\sum_{\jmath\in\mathbb{Z}}d_{0,\jmath}\psi_{0,\jmath}+(\jmath-1)\sum_{v=0}^{\xi}\sum_{\jmath\in\mathbb{Z}}d_{1,\jmath}\psi_{1,\jmath}+\cdots+\sum_{v=0}^{\xi}\sum_{\jmath\in\mathbb{Z}}d_{\jmath-1,\jmath}\psi_{\jmath-1,\jmath}\right)$$

$$\leq\alpha\frac{1-(1-s)^{\alpha+1}}{\alpha+1}r^{\varsigma+1}R_{\jmath+1}^{(\alpha,\beta)}(cos\theta)P_0 g+\alpha\frac{1-(1-s)^{\alpha+1}}{\alpha+1}\sum_{\jmath=1}^{\varsigma-1}\left(Ar^{\jmath+1}R_{\jmath+1}^{(\alpha,\beta)}(cos\theta)\right)$$

$$\left(\jmath P_0 g+\jmath\sum_{v=0}^{\xi}\sum_{\jmath\in\mathbb{Z}}d_{0,\jmath}\psi_{0,\jmath}+(\jmath-1)\sum_{v=0}^{\xi}\sum_{\jmath\in\mathbb{Z}}d_{1,\jmath}\psi_{1,\jmath}+\cdots+\sum_{v=0}^{\xi}\sum_{\jmath\in\mathbb{Z}}d_{\jmath-1,\jmath}\psi_{\jmath-1,\jmath}\right)$$

$$\leq\alpha\frac{1-(1-s)^{\alpha+1}}{\alpha+1}r^{\varsigma+1}P_0 g+A\alpha\frac{1-(1-s)^{\alpha+1}}{\alpha+1}\sum_{\jmath=1}^{\varsigma-1}r^{\jmath+1}$$

$$\left(\jmath P_0 g+\jmath\sum_{v=0}^{\xi}\sum_{\jmath\in\mathbb{Z}}d_{0,\jmath}\psi_{0,\jmath}+(\jmath-1)\sum_{v=0}^{\xi}\sum_{\jmath\in\mathbb{Z}}d_{1,\jmath}\psi_{1,\jmath}+\cdots+\sum_{v=0}^{\xi}\sum_{\jmath\in\mathbb{Z}}d_{\jmath-1,\jmath}\psi_{\jmath-1,\jmath}\right)$$

$$\Rightarrow l_{\varsigma}^{H}(P_{\xi}g)=\frac{\alpha}{\alpha+1}\left(1-(1-r)^{\alpha+1}\right)$$

$$\left[r^{\varsigma+1}P_0 g+A\left(\frac{r^{\varsigma}-r}{r-1}\right)\left\{\jmath P_0 g+\sum_{v=0}^{\xi}\sum_{\jmath\in\mathbb{Z}}\left(\jmath d_{0,\jmath}\psi_{0,\jmath}+(\jmath-1)d_{1,\jmath}\psi_{1,\jmath}+\cdots+d_{\jmath-1,\jmath}\psi_{\jmath-1,\jmath}\right)\right\}\right]$$

□

5. Conclusions

The results obtined in Theorems 4.1 and 4.2 give the rate of convergence of Jacobi polynomial by applying chracterized Hausdorff matix while Theorem 4.3 studies wavelet approximation of Jacobi polynomial by applying chracterized Hausdorff matix.

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