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Euler I-convergence and its associated sequence spaces

Mohammad Imran Idrisi[a,*], Simranjeet Kour[b], Nadeem Rao[c]

[a]*Mathematics Division, School of Advanced Sciences, Vellore Institute of Technology, Chennai Campus , India*
[b]*Department of Computer Science, University Institute of Engineering, Chandigarh University, Mohali, Punjab, India*
[c]*Department of Mathematics, University Centre for Research and Development, Chandigarh University, Mohali, Punjab, India*

Abstract. Present work is an investigation of some new sequence spaces $c_0^I(e^r)$, $c^I(e^r)$, $\ell_\infty^I(e^r)$ and $\ell_\infty(e^r)$ as a domain of triangle Euler matrix via ideal convergence over an admissible ideal of $\mathbb{N}$. Also, defining some algebraic, topological properties and inclusion relations on these spaces.

1. Introduction

A linear space ω is a sequence space as its elements are sequences from $\mathbb{N}$ to $\mathbb{R}$ or $\mathbb{C}$ and the norm defined on this space is $\|x\| = sup\|x_i\|$, where the sets $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ have their usual meanings. Throughout the paper, the notations ℓ_∞, c, and c_0 will represent the spaces of all sequences which are bounded, convergent, and convergent to zero (null sequences), respectively. An ideal I is defined to be a family of a non-empty set X i.e $I \subseteq 2^{\mathbf{X}}$ if $I_1, I_2 \in I$ implies that their union is in I i.e $I_1 \cup I_2 \in I$, and $I_1 \in I, I_2 \subseteq I_1$ implies that $I_2 \in I$. whereas a filter is a family of sets $F \subseteq 2^{\mathbf{X}}$ if and only if $\emptyset \notin F$, $F_1, F_2 \in F$ implies that their intersection is in F i.e $F_1 \cap F_2 \in F$ and $F_1 \subseteq F_2$ implies that $F_2 \in F$. If $I \neq \emptyset$ and $X \notin I$ then I is said to be non-trivial, admissible if and only if $\{\{x\} : x \in X\} \subseteq I$ and maximal if there is no ideal $J \neq I$ that contains I. For every I to be a non-trivial ideal there must corresponds a filter $F = F(I) = \{Y : X - Y \in I\}$.
After an extensive research about usual convergence of sequences in point set topology with respect to usual metrics, the conception of Ideal convergence or I-convergence came into existence by well known author Kostyrko et al.[16]. Ideal convergence is a gereralization of statistical convergence which was introduced by well known authors Fast and Steinhaus [9, 19].Young researchers or scholars are suggested to go through deep analysis about the concept of usual convergence and Ideal convergence as both the concepts are independent. There are many sequences that are convergent but may not I-convergent.

A large number of research work has been surfaced in the field of ideal convergent sequence spaces by many researchers, for further details one may refer[5, 6, 10–12].

Furter, Ideal convergence likned with summability theory by Šalát et al.[20, 21] and develop some new ideas from the perspective of sequence spaces. To know more about this concept one may refer to [13–15]

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* Corresponding author: Mohammad Imran Idrisi
Email addresses: mhdimranidrisi@gmail.com (Mohammad Imran Idrisi), simranjeetkaur@gmail.com (Simranjeet Kour), nadeemrao1990@gmail.com (Nadeem Rao)
ORCID iDs: https://orcid.org/0000-0002-0631-6386 (Mohammad Imran Idrisi), https://orcid.org/0009-0007-8856-1505 (Simranjeet Kour), https://orcid.org/0000-0002-5681-9563 (Nadeem Rao)

Let, $T = (t_{nk})$ be an infinite matrix of real or complex numbers, where $n, k \in \mathbb{N}$, the sequence defined as

$$T_n(x) = \sum_{k=0}^{\infty} t_{nk} x_k, \quad \text{for each } n \in \mathbb{N}. \tag{1}$$

which is a T-transform of the sequence $x = (x_k) \in \omega$ by a matrix T also assuming that the right of series (1) converges for each $n \in \mathbb{N}$. The mapping of convergent sequences into another convergent sequences is given in Kojima-Schur Theorem[8] which is a necessary and sufficient conditions as follows

(i) $\sum_{k=1}^{\infty} |t_{nk}| \leq N$ for every $n > m$;

(ii) $\lim_{n\to\infty} t_{nk} = \beta_k$ for every fixed k;

(iii) $\sum_{k=1}^{\infty} t_{nk} = T_n \to \beta$ as $n \to \infty$.

Also, if $t_{nk} = 0$ for $k > n$ and $t_{nn} \neq 0$ for all $n \in \mathbb{N}$ then $T = t_{nk}$. is known to be trangle matrix which has a unique inverse T^{-1} for $|T| \neq 0$ and T^{-1} is a triangle matrix. Moreover, let the matrix domain $\lambda_T := \{x = (x_k) \in \omega : T(x) \in \lambda\}$...(1.2), for every sequence $x = (x_k) \in \lambda$ which is also a sequense space. λ_T is a BK-space normed by $\|x\|_{\lambda_T} = \|T(x)\|_\lambda$ for $x \in \lambda_T$[2] only if λ is a BK-space and T is triangle matrix. A number of research papers have been published on this idea and its generalization[3, 4, 17, 18] which motivated us to investigate some of the new sequence spaces by Euler transfornfation of a sequence x_k.

Recalling, the Euler matrix of order r, $E^r = (e^r_{nk})$ is an infinite matrix defined as,

$$e^r_{nk} = \begin{cases} \binom{n}{k}(1-r)^{n-k} r^k, & \text{if } 0 \leq k \leq n, \\ 0, & k > n \end{cases}$$

where all subscripts in $\mathbb{N}$. The Euler matrix has been used in the analysis of sequence spaces and also over ℓ_p-space it is contemplated as bounded linear operator. Recently, by using the Euler matrix of order r, B. Altay, F. Basar, M. Mursaleen [1] has investigated e^r_p and e^r_∞ as sequence spaces with all sequences whose Euler-transformation of the sequence $x = (x_k)$ are in ℓ_p and ℓ_∞ which are also sequence spaces respectively i.e

$$\lambda(E^r) = \left\{ x = (x_k) \in \omega : \sum_n \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k \in \lambda \right\}$$

for $\lambda \in \{\ell_p, \ell_\infty\}$. Throughout the paper c^I_0, c^I and ℓ^I_∞, will be representing the sequence spaces of all sequences which are null, convergent and bounded via an ideal I.

In this paper, by usnig Euler matrix of order r and via an ideal convergence,we investigated $c^I_0(e^r)$, $c^I(e^r)$, $\ell^I_\infty(e^r)$ and $\ell_\infty(e^r)$ as sequences spaces with all sequences whose E^r-transformation of $x = (x_k)$ are in c^I_0, c^I, ℓ^I_∞ and ℓ_∞, respectively. We define the sequence $e^n(x)$ as Euler-tranformation of a sequence $x = (x_k)$ as follows:

$$e^n x = \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k$$

In order to define main results, we recall some useful definitions and lemmas related to this investigation

Definition 1.1 ([19]). If $E = \{s \in E : s \leq n\} \subset \mathbb{N}$, then the natural density of the set E is defined as

$$d(E) = \lim_{n\to\infty} \frac{1}{n} |E| \text{ exists}$$

where, $|E|$ is the cardinality of pre-defined set E.

Definition 1.2 ([9]). A sequence $x = (x_k)$ is statistically converges to a number $\zeta \in \mathbb{R}$ if, for every $\varepsilon > 0$ however small, $d(\{k \in \mathbb{N} : |x_k - \zeta| \geq \varepsilon\}) = 0$. and represented as $st-\lim x_k = \zeta$. In case $\zeta = 0$, then the sequence $x = (x_k)$ is said to be st-null.

Definition 1.3 ([20]). A sequence $x = (x_k)$ is said to be I-Cauchy if, for every $\varepsilon > 0$ however small, $\exists$ a number $m = m(\varepsilon)$ such that the set $\{k \in \mathbb{N} : |x_k - x_m| \geq \varepsilon\}$ belongs to an ideal I.

Definition 1.4 ([16]). A sequence $x = (x_k)$ is said to be I-convergent to a number $\zeta \in \mathbb{R}$ if, for every $\varepsilon > 0$ however small, the set $\{k \in \mathbb{N} : |x_k - \zeta| \geq \varepsilon\}$ belongs to an ideal I and represented as $I-\lim x_k = \zeta$. In case $\zeta = 0$, then (x_k) is said to be I-null.

Definition 1.5 ([10]). A sequence $x = (x_k)$ is said to be I-bounded if there exists a positive real number $M > 0$ however large, such that, the set $\{k \in \mathbb{N} : |x_k| > M\}$ belongs to an ideal I.

Definition 1.6 ([20]). Let there exists two sequences $x = (x_k)$ and $y = (y_k)$. We say that $x_k = y_k$ for almost all k relative to I if the set $\{k \in \mathbb{N} : x_k \neq y_k\}$ belongs to an ideal I.

Definition 1.7 ([20]). A sequence space S is said to be normal or solid, if the Cauchy product $(\alpha_k x_k)$ belongs to S, whenever $(x_k) \in S$ and for any sequence of scalars (α_k) with the condition $|\alpha_k| < 1$, for every $k \in \mathbb{N}$.

Definition 1.8 ([20]). Let $S = \{s_i \in \mathbb{N} : s_1 < s_2 < \cdots\} \subseteq \mathbb{N}$ and K be a sequence space. A S-step space of K is a sequence space

$$\lambda_S^K = \{(x_{s_i}) \in \omega : (x_s) \in K\}.$$

A canonical pre-image of a sequence $(x_{s_i}) \in \lambda_S^K$ is a sequence $(y_s) \in \omega$ defined as follows:

$$y_s = \begin{cases} x_s, & \text{if } s \in S, \\ 0, & \text{otherwise.} \end{cases}$$

A canonical pre-image of a step space λ_S^K is a set of canonical pre-images of all elements in λ_S^K, i.e., y is in canonical pre-image of λ_S^K iff y is canonical pre-image of some element $x \in \lambda_S^K$.

Definition 1.9 ([20]). A sequence space S is said to be monotone, if it contains the canonical pre-images of its step space.

Lemma 1.1 ([20]). Every solid space $\Longrightarrow$ monotone space.

Lemma 1.2 ([21]). Let $K_1 \in \mathcal{F}(I)$ and $K_2 \subseteq \mathbb{N}$. If $K_2 \notin I$, then $K_1 \cap K_2 \notin I$.

2. Main results

In this section, we investigated some new sequence spaces $c_0^I(e^r)$, $c^I(e^r)$, $\ell_\infty^I(e^r)$ and $\ell_\infty(e^r)$ defined by Euler transformation $e^n(x)$ of a sequence $x = (x_k)$ over an admissible ideal I of subsets of $\mathbb{N}$ and study some algebraic, topological properties and prove some inclusion relations on these spaces.

$$\begin{aligned} c_0^I(e^r) &:= \{x = (x_k) \in \omega : \{n \in \mathbb{N} : |e^n(x)| \geq \varepsilon\} \in I\}, \\ c^I(e^r) &:= \{x = (x_k) \in \omega : \{n \in \mathbb{N} : |e^n(x) - \zeta| \geq \varepsilon, \text{ for some } \zeta \in \mathbb{R}\} \in I\}, \\ \ell_\infty^I(e^r) &:= \{x = (x_k) \in \omega : \exists M > 0 \text{ s.t } \{n \in \mathbb{N} : |e^n(x)| \geq M\} \in I\}, \\ \ell_\infty(e^r) &:= \{x = (x_k) \in \omega : \sup_n |e^n(x)| < \infty\}. \end{aligned}$$

Also,

$$m_0^I(e^r) := c_0^I(e^r) \cap \ell_\infty(e^r) \quad \text{and} \quad m^I(e^r) := c^I(e^r) \cap \ell_\infty(e^r).$$

Sequence spaces $c_0^I(e^r)$, $c^I(e^r)$, $\ell_\infty^I(e^r)$, $m^I(e^r)$, and $m_0^I(e^r)$ can be redefined as follows:

$$c_0^I(e^r) = (c_0^I)_{e^r}, c^I(e^r) = (c^I)_{e^r}, \ell_\infty^I(e^r) = (\ell_\infty^I)_{e^r}$$

$$m^I(e^r) = (m^I)_{e^r} \quad \text{and} \quad m_0^I(e^r) = (m_0^I)_{e^r}.$$

Definition 2.1. A sequence $x = (x_k)$ is said to be Euler I-Cauchy if for each $\varepsilon > 0$, however small, there exists a positive integer $m_{(\varepsilon)} \in \mathbb{N}$ s.t

$$\{n \in \mathbb{N} : |e^n(x) - e^m(x)| \geq \varepsilon\}$$

belongs to I, where $I \subseteq \mathbb{N}$ be an admissible ideal.

Example 2.1. Define a class of finite subsets of $\mathbb{N}$ i.e $I^f = \{N \subseteq \mathbb{N} : N \text{ is finite}\}$ is an admissible ideal in $\mathbb{N}$ and $c^{I^f}(e^r) = e_c^r$.

Example 2.2. Let, S_{e^r} will denote the the space of all Euler statistically convergent sequences i.e

$$S_{e^r} := \left\{x = (x_k) : d\Big(\{n \in \mathbb{N} : |e^n(x) - \zeta| \geq \varepsilon\}\Big) = 0, \text{ for any real } \zeta\right\}.$$

We define $I^d = \{N \subseteq \mathbb{N} : d(N) = 0\}$ a non trivial ideal that imples that $c^{I^d}(e^r) = S_{e^r}$, where $d(N)$ represents natural density of the set N.

Example 2.3. Every usual Euler convergent sequence converges Euler statistically but the converse may not be true. To prove this result we consider a sequence $x = (x_k)$ defined as follows:

$$e^n(x) = \begin{cases} 1, & \text{if } n \text{ is a square,} \\ 0, & \text{otherwise.} \end{cases}$$

That is $e^n(x) = \{1, 0, 0, 1, 0, 0, 0, 0, 1, 0, \ldots\}$ and taking the limit $\zeta = 0$. Then we have the inclusion

$$\{n \in \mathbb{N} : |e^n(x) - \zeta| \geq \varepsilon\} \subset \{1, 4, 9, 16, \ldots, r^2, (r+1)^2 \ldots\} \qquad ......\star$$

Since, Natural density of the set on right of $(\star)$ is zero i.e the set of squares of natural numbers, so as a result we get,

$$d(\{n \in \mathbb{N} : |e^n(x) - \zeta| \geq \varepsilon\}) = 0.$$

This implies that, the sequence Euler statistically convergent $(x_k) \in S_{e^r}$, but the sequence is not usual Euler convergent $(x_k) \notin C_{e^r}$.

Theorem 2.1. The spaces $c^I(e^r)$, $c_0^I(e^r)$, $\ell_\infty^I(e^r)$, $m_0^I(e^r)$, and $m^I(e^r)$ are linear spaces over the real numbers $\mathbb{R}$.

Proof. Let $x = (x_k)$, $y = (y_k) \in c^I(e^r)$ be two arbitrary sequences and α_1, α_2 are scalars. Now, since $x, y \in c^I(e^r)$, then for given $\varepsilon > 0$, there exist $\zeta_1, \zeta_2 \in \mathbb{R}$, such that

$$\left\{n \in \mathbb{N} : |e^n(x) - \zeta_1| \geq \frac{\varepsilon}{2}\right\} \in I \quad \text{and} \quad \left\{n \in \mathbb{N} : \left|e^n(y) - \zeta_2\right| \geq \frac{\varepsilon}{2}\right\} \in I.$$

Now, let

$$A_1 = \left\{n \in \mathbb{N} : |e^n(x) - \zeta_1| < \frac{\varepsilon}{2|\alpha_1|}\right\} \in \mathcal{F}(I),$$

$$A_2 = \left\{n \in \mathbb{N} : \left|e^n(y) - \zeta_2\right| < \frac{\varepsilon}{2|\alpha_2|}\right\} \in \mathcal{F}(I)$$

be such that $A_1^c, A_2^c \in I$. Then

$$\begin{aligned} A_3 &= \left\{n \in \mathbb{N} : \left|e^n(\alpha_1 x + \alpha_2 y) - (\alpha_1\zeta_1 + \alpha_2\zeta_2)\right| < \varepsilon\right\} \\ &\supseteq \left\{\left\{n \in \mathbb{N} : |e^n(x) - \zeta_1| < \frac{\varepsilon}{2|\alpha_1|}\right\} \cap \left\{n \in \mathbb{N} : \left|e^n(y) - \zeta_2\right| < \frac{\varepsilon}{2|\alpha_2|}\right\}\right\}. \end{aligned} \tag{2}$$

As a result we get here, the right side of above equation (2) belongs to the filter $\mathcal{F}(I)$ associated with I over the set of natural numbers which implies that its complement set always belongs to Ideal I and hence we get $(\alpha_1 x + \alpha_2 y) \in c^I(e^r)$. $\implies c^I(e^r)$ is linear space over $\mathbb{R}$.
The proof for the remaining spaces $c_0^I(e^r)$, $\ell_\infty^I(e^r)$, $m_0^I(e^r)$, and $m^I(e^r)$ can be prove by following the similar way as above.
■

Theorem 2.2. Spaces $\lambda(e^r)$ are normed spaces with respect to the sup-norm as follows:

$$||x||_{\lambda(e^r)} = \sup_n |e^n(x)|, \quad \text{where } \lambda \in \left\{c^I, c_0^I, \ell_\infty^I, \ell_\infty\right\}. \tag{3}$$

Theorem 2.3. A sequence $x = (x_k)$ is said to be Euler I-convergent if and only if for every $\varepsilon > 0$, $\exists$ $m = m(\varepsilon) \in \mathbb{N}$, such that

$$\{n \in \mathbb{N} : |e^n(x) - e^m(x)| < \varepsilon\} \in \mathcal{F}(I). \tag{4}$$

Proof. Let, the sequence $x = (x_k)$ is Euler I-convergent to some number $\zeta \in \mathbb{R}$, then for a given $\varepsilon > 0$ however small, we have

$$A_\varepsilon = \left\{n \in \mathbb{N} : |e^n(x) - \zeta| < \frac{\varepsilon}{2}\right\} \in \mathcal{F}(I).$$

Fix an integer $m = m(\varepsilon) \in A_\varepsilon$. Then we have

$$|e^n(x) - e^m(x)| \leq |e^n(x) - \zeta| + |\zeta - e^m(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n \in A_\varepsilon$. Hence (4) holds.

Conversely, suppose that (4) holds for all $\varepsilon > 0$. Then

$$B_\varepsilon = \{n \in \mathbb{N} : e^n(x) \in [e^n(x) - \varepsilon, e^n(x) + \varepsilon]\} \in \mathcal{F}(I), \text{ for all } \varepsilon > 0.$$

Let $J_\varepsilon = [e^n(x) - \varepsilon, e^n(x) + \varepsilon]$. Fixing $\varepsilon > 0$, we have $B_\varepsilon \in \mathcal{F}(I)$ and $B_{\frac{\varepsilon}{2}} \in \mathcal{F}(I)$. Hence $B_\varepsilon \cap B_{\frac{\varepsilon}{2}} \in \mathcal{F}(I)$. provided

$$J = J_\varepsilon \cap J_{\frac{\varepsilon}{2}} \neq \emptyset,$$

which implies that,

$$\{n \in \mathbb{N} : e^n(x) \in J\} \in \mathcal{F}(I)$$

and hence

$$\text{diam }(J) \leq \frac{1}{2} \text{ diam }(J_\varepsilon),$$

where, diam(J) represents length of interval J and by induction we get sequence of closed intervals as follows: $J_\varepsilon = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots$ s.t

$$\text{diam }(I_n) \leq \frac{1}{2} \text{ diam }(I_{n-1}), \text{ for } n = (2, 3, \dots)$$

as a result we get,

$$\{n \in \mathbb{N} : e^n(x) \in I_n\} \in \mathcal{F}(I).$$

Which implies that $\exists\, \zeta \in \bigcap_{n\in\mathbb{N}} I_n$ and it is a routine work to verify that $\zeta = I\!-\lim e^n(x)$ showing that $x = (x_k)$ is Euler I-convergent. Provided that result follows. ■

Theorem 2.4. The inclusions $\ell_\infty^I(e^r) \supset c^I(e^r) \supset c_0^I(e^r)$ hold and are strict.

Proof. The inclusion $c^I(e^r) \supset c_0^I(e^r)$ is obviously true. We will show only its strictness for this we take a sequence $x = (x_k)$ s.t $e^n(x) = 2$ which implies that $e^n(x)$ belongs to c^I but not belongs to c_0^I.
Moreover, let $x = (x_k) \in c^I(e^r)$ then there exists a real number ζ s.t $I\text{-}\lim e^n(x) = \zeta$ i.e

$$\{n \in \mathbb{N} : |e^n(x) - \zeta| \geq \varepsilon\} \in I.$$

We have

$$|e^n(x)| = |e^n(x) - \zeta + \zeta| \leq |e^n(x) - \zeta| + |\zeta|.$$

From the above result we can say that the sequence (x_k) must be an element of $\ell_\infty^I(e^r)$.
Also to show strictness of inclusion i.e $\ell_\infty^I(e^r) \supset c^I(e^r)$ we consider an example:

Example 2.4. Define a sequence $x = (x_k)$ such that

$$e^n(x) = \begin{cases} 1, & \text{if } n \text{ is odd non-square,} \\ 0, & \text{if } n \text{ is even non-square} \\ \sqrt{n}, & \text{if } n \text{ is square,} \end{cases}$$

$\implies$ sequence $e^n(x)$ belongs to ℓ_∞^I, but $e^n(x)$ not belongs to c^I provided the sequence $x \in \ell_\infty^I(e^r) \backslash c^I(e^r)$.

As a result, we get that the inclusion $\ell_\infty^I(e^r) \supset c^I(e^r) \supset c_0^I(e^r)$ follows strictly. ∎

Example 2.5. Every Euler bounded sequence is Euler I-bounded but the converse may not be true. This follows from the following example as; let's define a sequence $x = (x_k)$ s.t

$$e^n(x) = \begin{cases} \frac{n^2}{n+1}, & \text{for prime n,} \\ 0, & \text{otherwise.} \end{cases}$$

which proves that $e^n(x)$ is not bounded sequence but the set $\{n \in \mathbb{N} : |e^n(x)| \geq 1\}$ belongs to ideal. Hence the sequence (x_k) is Euler I-bounded.

Theorem 2.5. The sequence spaces:

(a) $c^I(e^r)$ and $\ell_\infty(e^r)$, do not contain each other but overlap only.
(b) $c_0^I(e^r)$ and $\ell_\infty(e^r)$, do not contain each other but overlap only.

Proof.

(a) We shall prove the spaces $c^I(e^r)$ and $\ell_\infty(e^r)$ are not disjoint spaces for this we consider a sequence $x = (x_k)$ s.t $e^n(x) = \frac{1}{n}$ for n belongs to $\mathbb{N}$ then $x \in c^I(e^r)$ and $x \in \ell_\infty(e^r)$ both. Moreover,we define a sequence $x = (x_k)$ s.t

$$e^n(x) = \begin{cases} \sqrt{n}, & n \text{ is a square,} \\ 0, & \text{otherwise.} \end{cases}$$

so that, $x \in c^I(e^r)$ but $x \notin \ell_\infty(e^r)$. Moreover, we again define a sequence $x = (x_k)$ s.t

$$e^n(x) = \begin{cases} n, & n \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

then as a result we get, $(x) \in \ell_\infty(e^r)$ but $x \notin c^I(e^r)$.
The other part can be established following same technique. ∎

Theorem 2.6. The spaces defined as:

$$m_0^I(e^r) := c_0^I(e^r) \cap \ell_\infty(e^r) \quad \text{and} \quad m^I(e^r) := c^I(e^r) \cap \ell_\infty(e^r).$$

are closed in $\ell_\infty(e^r)$ as a subspace.

Proof. We consider a Cauchy sequence $(x_k^{(i)})$ in $m^I(e^r) \subset \ell_\infty(e^r)$. Then $(x_k^{(i)})$ converges to a point in $\ell_\infty(e^r)$ and $\lim_{i\to\infty} e^{n(i)}(x) = e^n(x)$. Let $I - \lim e^{n(i)}(x) = \zeta_i$ for every $i \in \mathbb{N}$. Then we only need to show that

(a) Sequence (ζ_i) converges to ζ;
(b) The limit, $I - \lim e^n(x) = \zeta$ exists.

(a) As $(x_k^{(i)})$ is a Cauchy sequence then for each $\varepsilon > 0$ however small, there always exists a positive integer $m \in \mathbb{N}$ s.t

$$\left|e^{n(i)}(x) - e^{n(j)}(x)\right| < \frac{\varepsilon}{3}, \text{ for all } i, j \geq m. \tag{5}$$

Now, consider two sets A_i and A_j in an ideal I defined as:

$$A_i = \left\{n \in \mathbb{N} : |e^{n(i)}(x) - \zeta_i| \geq \frac{\varepsilon}{3}\right\} \tag{6}$$

and

$$A_j = \left\{n \in \mathbb{N} : |e^{n(j)}(x) - \zeta_j| \geq \frac{\varepsilon}{3}\right\}. \tag{7}$$

Moreover, let $i, j \geq m$ and $n \notin A_i \cap A_j$ then we get,

$$|\zeta_i - \zeta_j| \leq |e^{n(i)}(x) - \zeta_i| + |e^{n(j)}(x) - \zeta_j| + |e^{n(i)}(x) - e^{n(j)}(x)| < \varepsilon \text{ by (5), (6), and(7).}$$

Thus (ζ_i) is a Cauchy sequence and thus convergens to $\zeta \in \mathbb{R}$ i.e $\lim_{i\to\infty} \zeta_i = \zeta$.

(b) Let $\delta > 0$ however small, be given, then we have a positive integer n_0 s.t

$$|\zeta_i - \zeta| < \frac{\delta}{3}, \text{ for every } i > n_0. \tag{8}$$

Which implies that $(x_k^{(i)}) \to x_k$ as $i \to \infty$. Thus

$$|e^{n(i)}(x) - e^n(x)| < \frac{\delta}{3}, \text{ for every } i > n_0. \tag{9}$$

Since $(e^{n(j)}(x))$ is I-convergent to a real no. ζ_j then $\exists\, \mathcal{E} \in I$ s.t for every $n \notin \mathcal{E}$ so we get,

$$|e^{n(j)}(x) - \zeta_j| < \frac{\delta}{3}. \tag{10}$$

Moreover, let $j > n_0$ then $\forall\, n \notin \mathcal{E}$, we get by (8), (9), and (10) s.t

$$|e^n(x) - \zeta| \leq |e^n(x) - e^{n(j)}(x)| + |e^{n(j)}(x) - \zeta_j| + |\zeta_j - \zeta| < \delta.$$

Therefore, (x_k) is Euler I-convergent to a real no. ζ. Thus $m^I(e^r)$ is closed in $\ell_\infty(e^r)$ as a subspace. Following the same way the proof of other parts can be established, hence can be omitted. ∎

Theorem 2.7. Sequence spaces $c^I(e^r)$, $c_0^I(e^r)$, and $\ell_\infty^I(e^r)$ are BK-spaces with respect to the sup-norm as follows:

$$\|x\|_{\lambda(e^r)} = \sup_n |e^n(x)|, \quad \text{where } \lambda \in \left\{c^I, c_0^I, \ell_\infty^I, \ell_\infty\right\}. \tag{11}$$

Proof. It is known that the spaces c^I, c_0^I, and ℓ_∞^I are BK-spaces. Moreover,(1.2) satisfies and the Euler matrix is a triangle matrix. Now, by considering all these three facts and also by Theorem of Wilansky [22], we have concluded that spaces are BK-spaces and hence the proof is completed. ■

In the view of Theorem 2.6 and since the inclusions $m^I(e^r) \subset \ell_\infty(e^r)$ and $m_0^I(e^r) \subset \ell_\infty(e^r)$ are strict, we formulate the following result without proof:

Theorem 2.8. Spaces $m^I(e^r)$ and $m_0^I(e^r)$ are nowhere dense in $\ell_\infty(e^r)$ as a subset.

Theorem 2.9. Spaces $c_0^I(e^r)$ and $m_0^I(e^r)$ are monotone and solid respectively.

Proof. First we shall prove the result only for $c_0^I(e^r)$.
Let $x = (x_k) \in c_0^I(e^r)$. For $\varepsilon > 0$ however small, we have

$$\{n \in \mathbb{N} : |e^n(x)| \geq \varepsilon\} \in I \tag{12}$$

Let $a = (a_k)$ be a scalar sequence satisfies $|a| \leq 1 \ \forall \ k \in \mathbb{N}$ then we have

$$|e^n(ax)| = |ae^n(x)| \leq |a|\,|e^n(x)| \leq |e^n(x)|\,, \forall n \in \mathbb{N}. \tag{13}$$

From the above (12), (13) equations, we conclude that:

$$\{n \in \mathbb{N} : |e^n(ax)| \geq \varepsilon\} \subseteq \{n \in \mathbb{N} : |e^n(x)| \geq \varepsilon\} \in I$$

which implies that te set,

$$\{n \in \mathbb{N} : |e^n(\alpha x)| \geq \varepsilon\} \text{ belongs to ideal I.}$$

Therefore as a result we get the sequence $(ax_k) \in c_0^I(e^r)$.
$\Longrightarrow$ Space $c_0^I(e^r)$ is a solid space.
Also, as we know that Every solid space is Monotone(1.1) $\Longrightarrow$ the space $c_0^I(e^r)$ is Monotone space. ■

Corollary. *If the ideal I is neither maximal nor $I = I_f$, then the sequence spaces $c^I(e^r)$ and $m^I(e^r)$ are neither solid nor monotone.*

Proof. We shall prove the result by introducing an example as follows: ■

Example 2.6. Let $I = I_f$ and let $S = \{n \in \mathbb{N} : n \text{ is an odd integer}\}$. Consider the S-step space S_K of K as:

$$S_K = \{(x_k) \in \omega : (x_k) \in S\}.$$

Now, defining a sequence $(y_k) \in S_K$ s.t

$$e^n(y) = \begin{cases} e^n(x), & \text{if } n \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, we consider a sequence (x_k) defined as $e^n(x) = 3 \ \forall \ n \in \mathbb{N}$ then the sequence $(x_k) \in E(e^r)$, but its S-step space preimage does not belongs to $E(e^r)$, where $E = c^I$ and m^I.
In this way, as a result we find $E(e^r)$ are not monotone and by following the lemma(1.1) spaces $E(e^r)$ are not solid.

Theorem 2.10. Let, for a sequence $x = (x_k)$ and a non-trivial admissible ideal I in $\mathbb{N}$ if there exists a sequence $y = (y_k) \in c^I(e^r)$ s.t $e^n(x) = e^n(y)$ for almost all n relative to I, then $x \in c^I(e^r)$.

Proof. Suppose that $e^n(x) = e^n(y)$ for almost all n relative to I. i.e,

$$\{n \in \mathbb{N} : e^n(x) \neq e^n(y)\} \in I.$$

Consider the sequence (y_k) is Euler I-convergent to ζ then for every $\varepsilon > 0$ however small, we have the following set belongs to ideal I i.e,

$$\{n \in \mathbb{N} : |e^n(y) - \zeta| \geq \varepsilon\} \in I.$$

Since, we have considered I as an admissible ideal of set of natural numbers so we have the result by following the below inclusion:

$$\{n \in \mathbb{N} : |e^n(x) - \zeta| \geq \varepsilon\} \subseteq \{n \in \mathbb{N} : e^n(x) \neq e^n(y)\} \cup \{n \in \mathbb{N} : |e^n(y) - \zeta| \geq \varepsilon\}.$$

■

Declarations

Competing interests The authors declare that they have no competing interests.

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