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On generalized Bernstein Kantorovich Schurer type operators and its approximation behaviour

Nand Kishor Jha[a], Nadeem Rao[b,*]

[a]*Department of Mathematics, Chandigarh University, Mohali-140413, Punjab, India*
[b]*Department of Mathematics, University Center for Research and Development, Chandigarh University, Mohali-140413, Panjab, India*

Abstract. In the present paper, we introduce generalized Bernstein Kantorovich Schurer type operators and its approximation properties. Firstly, we calculate the some estimates for these operators. Further, we study the uniform convergence and order of approximation in terms of Korovkin type theorem and modulus of continuity for the space of univariate continuous functions and bivariate continuous functions in their sections. In continuation, local and global approximation properties are studied in terms of first and second order modulus of smoothness, Peetre's K-functional and weight functions in various functional spaces.

1. Introduction

Indeed, approximation theory serves as a versatile tool across numerous disciplines, offering methods to represent intricate functions with simpler ones. Its impact extends from mathematics to engineering, encompassing computational science, data analysis, and computer graphics. In computational realms, approximation theory aids in describing geometric shapes and tackling differential equations, crucial for numerical analysis and efficient algorithm design.

In applied mathematics, approximation theory contributes significantly to control theory, where concepts like control points and control nets are pivotal in studying parametric curves and surfaces, essential for designing control systems in engineering applications ([1], [2]). With the surge of artificial intelligence, data science, and machine learning, approximation theory has found fresh applications. Techniques rooted in approximation theory are instrumental in the development of algorithms for data analysis, pattern recognition, and predictive modeling, forming the basis for constructing models that approximate intricate relationships within datasets.

Furthermore, in domains like computer graphics and computer algebra systems, approximation theory is indispensable. It enables the representation of curves and surfaces using simpler mathematical constructs, facilitating tasks such as rendering realistic images and efficiently solving symbolic equations. Beyond these fields, many scientists in medical sciences and other areas are also leveraging the principles

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* Corresponding author: Nadeem Rao
Email addresses: nandkishorjha1982@gmail.com (Nand Kishor Jha), nadeemrao1990@gmail.com (Nadeem Rao)
ORCID iDs: https://orcid.org/0009-0008-5297-0486 (Nand Kishor Jha), https://orcid.org/0009-0008-5297-0486 (Nadeem Rao)

of approximation theory to advance their research ([3], [4], [5]).

The interdisciplinary nature and wide-ranging applications of approximation theory underscore its importance as a foundational concept in modern science and engineering.

The first sequence of operators to support the above application part was introduced by Bernstein [7]. Although, his motive was to provide a short and elegant proof of Weierstrass theorem of approximation with the assistance of binomial distribution as:

$$B_l(g;u) = \sum_{\nu=0}^{l} \binom{l}{\nu} u^{\nu}(1-u)^{l-\nu} g\left(\frac{\nu}{l}\right), \quad u \in (0,1), \tag{1}$$

where g belongs to $C(0,1)$. He proved that these operators approximate uniformly on $(0,1)$ to every continuous function $g \in C[0,1]$. The Bernstein operators have been one of the most extensively examined positive linear operators in the area of approximation theory. However, these operator are not applicable for discontinuous functions.
Further, to achieve flexibility in approximation properties of Bernstein operators given by (1), Schurer [6] constructed a new sequence of Bernstein operators [7] which is denoted as $B_{l+p} : C[0,1+p] \to C[0,1+p]$ and defined by:

$$B_{l+p}(g;u) = \sum_{\nu=0}^{l+p} g\left(\frac{\nu}{l}\right)\binom{l+p}{\nu} u^{\nu}(1-u)^{l+p-\nu}, u \in [0,1+p], \tag{2}$$

where $p \in \mathbb{N} \cup \{0\}$ and $g \in C[0,1+p]$. But these sequences of operations given in (2) are restricted to $C[0,1+p]$.

Over the past decade, many generalizations as well as modifications of Bernstein and Kantorovich operators are presented by several authors and researchers, e.g., Alotaibi et al. ([8], [9]), Mursaleen et al. ([10] - [13]), Mohiuddine et al. ([14], [15]), Aslan et al. ([16], [17]), Nasiruzzaman. et al. ([18], [19]), Ayman Mursaleen et al. [20], [21]), Özger et al. ([22], [23]), Acu et al. ([24], [25]), Rao et al. ([26] - [28]), and Rani et al. [29] etc.

Recently, Usta ([30]) presented a new sequence of Bernstein operators for the function g, which are continuous and defined on $(0,1)$ with $u \in (0,1)$ as follows:

$$\tilde{P}_l^*(g;u) = \frac{1}{l}\sum_{\nu=0}^{l} \binom{l}{\nu} (\nu - lu)^2 u^{\nu-1}(1-u)^{l-\nu-1} g\left(\frac{\nu}{l}\right), \quad l \in \mathbb{N}. \tag{3}$$

Remark 1.1. *These operators given in* (3) *are restricted for the space of continuous functions only.*

In addition of above literature and to discuss approximation properties for lebesgue integrable functions, we define generalized Bernstein Kantorovich Schurer type operators as follows $H_{l+p}^* : L_B(0,1) \to L_B(0,1)$, (where $L_B(0,1)$ denotes the space of bounded and Lebesgue measurable functions):

$$H_{l+p}^*(g;u) = \sum_{\nu=0}^{l+p} Q_{l+p,\nu}(u) \int_{\frac{\nu}{l+p+1}}^{\frac{\nu+1}{l+p+1}} g(t)dt, \tag{4}$$

where

$$Q_{l+p,\nu}(u) = \frac{(l+p+1)}{l+p}\binom{l+p}{\nu}(\nu-(l+p)u)^2 u^{\nu-1}(1-u)^{l+p-\nu-1}.$$

Remark 1.2. *For any $g, h \in C(0,1)$ and $a_1, a_2 \in \mathbb{R}$, we have*

$$\begin{aligned} H^*_{l+p}(a_1 g + a_2 h; u) &= \sum_{\nu=0}^{l+p} Q_{l+p,\nu}(u) \int_{\frac{\nu}{l+p+1}}^{\frac{\nu+1}{l+p+1}} (a_1 g + a_2 h)(t)dt \\ &= a_1 \sum_{\nu=0}^{l+p} Q_{l+p,\nu}(u) \int_{\frac{\nu}{l+p+1}}^{\frac{\nu+1}{l+p+1}} g(t)dt + a_2 \sum_{\nu=0}^{l+p} Q_{l+p,\nu}(u) \int_{\frac{\nu}{l+p+1}}^{\frac{\nu+1}{l+p+1}} h(t)dt \\ &= a_1 H^*_{l+p}(g; u) + a_2 H^*_{l+p}(h; u). \end{aligned}$$

Which implies that the operator $H^*_{l+p}(.;.)$ is linear operator.

Remark 1.3. *Also for any $g \geq 0$, we must have $H^*_{l+p}(g; u) \geq 0$, which shows that the sequence of operators are positive.*

The structure of our research work is organized as: Section 1 compute some estimates for the operators 4 in terms of test functions and central moments. In section 2, we study the uniform convergence theorem and approximation order via of Korovkin theorem and first order modulus of continuity for the space of univariate continuous functions and bivariate continuous functions respectively. In section 3, we discuss the local and global approximation results using first and second order modulus of continuity, Peetre's K-functional in several functional spaces.

To discuss the existence and convergence of operators (4), we Consider $e_i(t) = t^i$, $i = 0, 1, 2, 3$. Then, in the following Lemmas (2.1) and (2.2) we estimate the operators introduced in terms of central moments and test functions.

2. Basic estimates

Lemma 2.1. *For operators $H^*_{l+p}(.;.)$ defined by (4), the following identities are as follows:*

$$\begin{aligned} H^*_{l+p}(e_0; u) &= 1, \\ H^*_{l+p}(e_1; u) &= \left(\frac{l+p-2}{l+p+1}\right) u + \frac{3}{2(l+p+1)}, \\ H^*_{l+p}(e_2; u) &= \left(\frac{(l+p)^2 - 7(l+p) + 6}{(l+p+1)^2}\right) u^2 + \left(\frac{6(l+p) - 8}{(l+p+1)^2}\right) u + \frac{7}{3(l+p+1)^2}. \end{aligned}$$

Proof. From the result of Lemma 1 of [30] and the operators (4), we can easily prove above results of Lemma 2.1. □

Lemma 2.2. *Let $\psi^i_u(t) = (t-u)^i$, $i = 0, 1, 2$. Then, we have the central moments of generalized Bernstein Kantorovich Schurer type operators (4) as follows:*

$$\begin{aligned} H^*_{l+p}((t-u)^0; u) &= 1, \\ H^*_{l+p}((t-u)^1; u) &= \frac{3}{l+p+1}\left(\frac{1}{2} - u\right) \quad := \psi_{l+p}, \qquad (Say) \\ H^*_{l+p}((t-u)^2; u) &= \frac{1}{(l+p+1)^2}\left\{(11 - 3(l+p)u^2 + (3(l+p) - 11)u + \frac{7}{3}\right\} \\ &:= A^*_{l+p}. \qquad (Say) \end{aligned}$$

Proof. On account of Lemma 2.1 and linearity properties, we can easily prove Lemma 2.2. □

Lemma 2.3. *Let $g \in C_B(0,1)$. Then, $\|H^*_{l+p}(g)\| \leq \|g\|$.*

Proof. In the light of Lemma 2.1 and norm defined for $C_B(0,1)$, we can easily prove the result. □

3. Rapidity of convergence and order of Approximation

Definition 3.1. *Let $g \in C(0,1)$. Then, the modulus of continuity is defined as:*

$$\omega(g;\tilde{\eta}) = \sup_{|u_1-u_2|\leq\tilde{\eta}} |g(u_1) - g(u_2)|, \qquad u_1, u_2 \in (0,1).$$

Theorem 3.2. *Let $H^*_{l+p}(.;.)$ be given in (4). Then, $\forall\, g \in C_B(0,1) \cap E$, $H^*_{l+p}(.;.) \rightrightarrows g$ on each compact subset of $(0,1)$, where $\rightrightarrows$ symbol denotes uniform convergence and $E = \left\{g : u \geq 0, \dfrac{g(u)}{1+u^2} \text{is convergent for } u \longrightarrow \infty\right\}$.*

Proof. Using Korovkin result which implies the convergence uniformly operators which are positive and linear, it is adequate to see that

$$\lim_{l\to\infty} H^*_{l+p}(t^i;u) = u^i,\ i = 0,1,2,$$

uniformly on $(0,1)$. In the view of Lemma 2.1, we can arrive at the desired result. □

In the view of Shisha et al. [31], one can show that the order of approximation via Ditzian-Totik modulus of continuity.

Theorem 3.3. *Let $g \in C_B(0,1)$. Then, operators $H^*_{l+p}(.;.)$ given in (4), we have*

$$|H^*_{l+p}(g;u) - g(u)| \leq 2\omega(g;\tilde{\eta}),$$

where $\tilde{\eta} = \sqrt{A^*_{l+p}}$.

Theorem 3.4. *(See[31]) Suppose that $L : C[c,d] \longrightarrow B[c,d]$ be the positive linear operator and consider γ_u be a function defined by*

$$\beta_u(y) = |y-u|, (u,y) \in [c,d]\times[c,d].$$

If $g \in C_B([c,d])$, for $u \in [c,d]$ and $\delta > 0$. Then, the operator L verifies the following results:

$$|(Lg)(u) - g(u)| \leq |g(u)||(Le_0)(u) - L|(Le_0)(u) + \tilde{\eta}^{-1}\sqrt{(Le_0)(u)(L\gamma_u^2(u))}\omega_g(\tilde{\eta}).$$

Theorem 3.5. *Let $g \in C_B(0,1)$. Then, for the operator $H^*_{l+p}(.;.)$ presented by (4), we have*

$$|H^*_{l+p}(g;u) - g(u)| \leq 2\omega(g;\tilde{\eta}),\ \text{where } \tilde{\eta} = \sqrt{A^*_{l+p};u}.$$

Proof. In view of Lemma 2.1, 2.2 and Theorem 3.2, we have

$$\left|H^*_{l+p}(g;u) - g(u)\right| \leq \left\{1 + \tilde{\eta}^{-1}\sqrt{A^*_{l+p}}\right\}\omega(g;\tilde{\eta}),$$

which prove the Theorem 3.5 choosing $\tilde{\eta} = \sqrt{A^*_{l+p}}$. □

4. Direct Results

Here, we recall a functional space as: $C_B(0,1)$, where $C_B(0,1)$ denotes a space of continuous and bounded functions and Peetre's K-functional is as:

$$K_2(g,\tilde{\eta}) = \inf_{h\in C_B^2(0,1)} \left\{ \|g-h\|_{C_B(0,1)} + \tilde{\eta}\|h''\|_{C_B^2(0,1)} \right\},$$

where $C_B^2(0,1) = \{h \in C_B(0,1) : h', h'' \in C_B(0,1)\}$ endowed with $\|g\| = \sup_{0<u<1} |g(u)|$. Further, we call second order Ditzian-Totik modulus of continuity is as:

$$\omega_2(g;\sqrt{\tilde{\eta}}) = \sup_{0<v\leq\sqrt{\tilde{\eta}}} \sup_{u\in(0,1)} |g(u+2v) - 2g(u+v) + g(u)|.$$

We also have a relation from [32] page no. 177, Theorem 2.4 as follows:

$$K_2(g;\tilde{\eta}) \leq \tilde{C}\omega_2(g;\sqrt{\tilde{\eta}}), \tag{5}$$

where $\tilde{C}$ is an absolute constant. Next, in order to discuss the approximation result, we consider the auxiliary sequence of operator as:

$$\widehat{H}^*_{l+p}(g;u) = H^*_{l+p}(g;u) + g(u) - g\left(\left(\frac{l+p-2}{l+p+1}\right)u + \frac{3}{2(l+p+1)}\right), \tag{6}$$

where $g \in C_B(0,1)$, $u \geq 0$.

Lemma 4.1. *Let $g \in C_B^2(0,1)$. Then, for all $u \geq 0$, one has*

$$|\widehat{H}^*_{l+p}(g;u) - g(u)| \leq \xi_{l+p}(u)\|g''\|,$$

where

$$\begin{aligned}\xi_{l+p}(u) &= \left(\frac{(l+p)^3 - 15(l+p)^2 - 38(l+p) - 24}{(l+p+1)^3}\right)u^3 + \left(\frac{\frac{27}{2}(l+p)^2 - \frac{117}{2}(l+p) + 45}{(l+1)^3}\right)u^2 \\ &\quad + \left(\frac{\frac{43}{2}(l+p) - 25}{(l+p+1)^3}\right)u + \frac{15}{4(l+p+1)^3}.\end{aligned}$$

Proof. For the auxiliary operators are given in the Definition (6), we have

$$\widehat{H}^*_{l+p}(1;u) = 1,\ \widehat{H}^*_{l+p}(\eta_1;u) = 0 \text{ and } |\widehat{H}^*_{l+p}(g;u)| \leq 3\|g\|. \tag{7}$$

In view of Taylor's expansion and $g \in C_B^2(0,1)$, we have

$$g(t) = g(u) + (t-u)g'(u) + \int_u^t (t-w)g''(w)dw. \tag{8}$$

Operating (6) both the side in above equation, we have

$$\widehat{H}^*_{l+p}(g;u) - g(u) = g'(u)\widehat{H}^*_{l+p}(t-u;u) + \widehat{H}^*_{l+p}\left(\int_u^t (t-w)g''(w)dw;u\right).$$

From (6) and (7), we get

$$\begin{aligned}\widehat{H}^*_{l+p}(g;u) - g(u) &= \widehat{H}^*_{l+p}\left(\int_u^t (t-w)(g)''(w)dw;u\right) \\ &= H^*_{l+p}\left(\int_u^t (t-w)g''(w)dw;u\right)\end{aligned}$$

$$-\int_u^{\left(\frac{l+p-2}{l+p+1}\right)u+\frac{3}{2(l+p+1)}}\left(\left(\frac{l+p-2}{l+p+1}\right)u+\frac{3}{2(l+p+1)}-w\right)g^{''}(w)dw. \tag{9}$$

Since,

$$\left|\int_u^t (t-w)g^{''}(w)dw\right| \leq (t-u)^2\|g^{''}\|. \tag{10}$$

Then, we get

$$\left|\int_u^{\left(\frac{l+p-2}{l+p+1}\right)u+\frac{3}{2(l+p+1)}}\left(\left(\frac{l+p-2}{l+p+1}\right)u+\frac{3}{2(l+p+1)}-w\right)g^{''}(w)dw\right|$$

$$\leq \left(\left(\frac{l+p-2}{l+p+1}\right)u+\frac{3}{2(l+p+1)}-u\right)^2\|g^{''}\|. \tag{11}$$

Applying (10) and (11) in (9), we obtain

$$\begin{aligned}\left|H^*_{l+p}(g;u)-g(u)\right| &\leq \left\{H^*_{l+p}((t-u)^2;u)+\left(\frac{l+p-2}{l+p+1}\right)u+\frac{3}{2(l+p+1)}\right\}\|g^{''}\| \\ &= \xi_{l+p}(u)\|g^{''}\|,\end{aligned}$$

We arrive the required result. □

Theorem 4.2. *For $g \in C_B^2(0,1)$. Then, there exist a constant $C > 0$ such that*

$$|H^*_{l+p}(g;u)-g(u)| \leq C\omega_2(g;\sqrt{\xi_{l+p}})+\omega(g;H^*_{l+p}(\xi_{l+p};u)),$$

where $\xi_{l+p}(u)$ is defined by the Lemma 4.1.

Proof. For $g \in C_B^2(0,1)$, $g \in C_B(0,1)$ and in account of the definition of $\widehat{H}^*_{l+p}(.;.)$, we have

$$|H^*_{l+p}(g;u)-g(u)| \leq |\widehat{H}^*_{l+p}(g-h;u)|+|(g-h)(u)|+|\widehat{H}^*_{l+p}(h;u)-h(u)|+\left|g\left(\left(\frac{l+p-2}{l+p+1}\right)u+\frac{3}{2(l+p+1)}\right)-h(u)\right|.$$

In the direction of of Lemma 4.1 and inequalities in (7), we yield

$$\begin{aligned}\left|H^*_{l+p}(g;u)-g(u)\right| &\leq 4\|g-h\|+|H^*_{l+p}(g;u)-g(u)|+\left|g\left(\left(\frac{l+p-2}{l+p+1}\right)u+\frac{3}{2(l+p+1)}\right)-g(u)\right| \\ &\leq 4\|g-h\|+\xi_{l+p}(u)\|h^{''}\|+\omega(g;H^*_{l+p}(\xi_{l+p};u)).\end{aligned}$$

In view of Peetre's K-functional

$$\left|H^*_{l+p}(g;u)-g(u)\right| \leq C\omega_2\left(g;\sqrt{\xi_{l+p}(u)}\right)+\omega(g;H^*_{l+p}(\xi_{l+p};u)),$$

we arrive the required result. □

Here we recall Lipschitz-type space here [33] as: Consider $\rho_1 > 0$ and $\rho_2 > 0$, are two fixed real values. $Lip_M^{\rho_1\rho_2}(\gamma) := \left\{g \in C_B(0,1) : |g(t)-h(u)| \leq M\frac{|t-u|^\gamma}{(t+\rho_1 u+\rho_2 u^2)^{\gamma/2}} : u,t \in (0,\infty)\right\}$, $M > 0$ is a constant and $0 < \gamma \leq 1$.

Theorem 4.3. *For $g \in Lip_M^{\rho_1,\rho_2}(\gamma)$ and $u \in (0,\infty)$. Then, the operators defined by (4), one has*

$$\left|H^*_{l+p}(g;u) - g(u)\right| \leq M\left(\frac{\eta_{l+p}(u)}{\rho_1 u + \rho_2 u^2}\right)^{\frac{\gamma}{2}}, \tag{12}$$

where $\gamma \in (0,1)$ and $\eta_{l+p}(u) = H^*_{l+p}(\xi^2_{l+p};u)$.

Proof. For $\gamma = 1$ and $u \in (0,1)$, we get

$$\left|H^*_{l+p}(g;u) - g(u)\right| \leq H^*_{l+p}(|g(t) - h(u);u) \leq MH^*_{l+p}\left(\frac{|t-u|}{(t+\rho_1 u + \rho_2 u^2)^{1/2}};u\right).$$

Therefore

$$\frac{1}{t+\rho_1 u + \rho_2 u^2} < \frac{1}{(\rho_1 u + \rho_2 u^2)},$$

for all $u \in (0,1)$, one has

$$\begin{aligned}\left|H^*_{l+p}(g;u) - g(u)\right| &\leq \frac{M}{(\rho_1 u + \rho_2 u^2)^{1/2}}\left(H^*_{l+p}(t-u)^2;u\right)^{1/2}\\ &\leq M\left(\frac{\eta_{l+p}(u)}{\rho_1 u + \rho_2 u^2}\right)^{1/2}.\end{aligned}$$

In view of Hölder's inequality, the Theorem 4.3 now holds for $\gamma = 1$ and $\gamma \in (0,\infty)$. with $q_1 = 2/\gamma$ and $q_2 = 2/2-\gamma$, one has

$$\begin{aligned}\left|H^*_{l+p}(g;u) - g(u)\right| &\leq \left(H^*_{l+p}(|g(t) - h(u)|^{\gamma/2};u)\right)^{\gamma/2}\\ &\leq MH^*_{l+p}\left(\frac{|t-u|^2}{t+\rho_1 u + \rho_2 u^2};u\right)^{\gamma/2}.\end{aligned}$$

Since, $\dfrac{1}{t+\rho_1 u + \rho_2 u^2} < \dfrac{1}{\rho_1 u + \rho_2 u^2}$ for all $u \in (0,\infty)$, we get

$$|H^*_{l+p}(g;u) - g(u)| \leq M\left(\frac{H^*_{l+p}(|t-u|^2;u)}{\rho_1 u + \rho_2 u^2}\right)^{\gamma/2} \leq M\left(\frac{\eta_{l+p}(u)}{\rho_1 u + \rho_2 u^2}\right)^2.$$

We arrive the required result.

Now, we recall r^{th} term order Lipschitz-type maximal function suggested by Lenze [34] as follows:

$$\widetilde{\omega}(g;u) = \sup_{t\neq u,\ t\in(0,1)} \frac{|g(t) - g(u)|}{|t-u|^r}, \quad r \in (0,1), \tag{13}$$

and $u \in (0,1)$. □

Theorem 4.4. *Let $g \in C_B(0,1)$ and $r \in (0,1)$. Then, for all $u \in (0,1)$, one has*

$$\left|H^*_{l+p}(g;u) - g(u)\right| \leq \widetilde{\omega}_r(g;u)(\eta_{l+p}(u))^{\gamma/2}.$$

Proof. We know that

$$\left|H^*_{l+p}(g;u) - g(u)\right| \leq H^*_{l+p}\left(|g(t) - g(u)|; u\right).$$

From equation (13), one has

$$\left|H^*_{l+p}(g;u) - g(u)\right| \leq \widetilde{\omega}_r\left((g;u)(H^*_{l+p}|t-u|^r; u)\right).$$

By Hölder's inequality with $q_1 = 2/r$ and $q_2 = 2/2 - r$, we have

$$\left|H^*_{l+p}(g;u) - g(u)\right| \leq \widetilde{\omega}_r(g;u)\left(H^*_{l+p}|t-u|^2; u\right)^{r/2}.$$

Which completes the required result. □

5. Approximation Properties Globally

Let $\nu(u) = 1 + u^2, 0 < u < 1$ is a weight function. Then, $B_{\nu(u)}(0,1) = \{g(u) : |g(u)| \leq \tilde{M}_g(\nu(u))$, here $\tilde{M}_g$ is a constant based on g and $C_{\nu(u)}(0,1)$ represents the space of continuous function in $B_{\nu(u)}(0,1)$ equipped with $\|g(u)\|_\nu = \sup\limits_{u\in(0,1)} \frac{|g(u)|}{\nu(u)}$ and $C^{\tilde{k}}_{\nu(u)}(0,1) = \{g \in C_{\nu(u)}(0,1) : \lim\limits_{u\to\infty} \frac{g(u)}{\nu(u)} = \tilde{k}$, where constant $\tilde{k}$ is depending on $g\}$.

Now, the first and second Ditzian-Totik modulus of smoothness for the function g are given by

$$\omega_b(g,\tilde{\eta}) = \sup_{0\leq t\leq\tilde{\eta}} \sup_{u,u+tb(u)\in(0,1)} \{|g(u+tb(u)) - g(u)|\}$$

and

$$\omega_2^b(g,\tilde{\eta}) = \sup_{0\leq t\leq\tilde{\eta}} \sup_{u,u+tb(u)\in(0,1)} \{|g(u+tb(u)) - 2g(u) + g(u-tb(u))|\},$$

respectively, where $b(u) = \{u(1-u)\}^{\frac{1}{2}}$ and $\tilde{\eta} > 0$. Suppose that

$$K_{2,b(u)}(g;\tilde{\eta}) = \inf_{f\in\mathcal{W}^2(b)} \left\{\|g-f\| + \tilde{\eta}\|b^2 f''\| : f \in C_B^2(0,1)\right\},$$

be the corresponding K-functional, where

$$\mathcal{W}^2(b) = \{f \in C_B^2(0,1) : f' \in AC^{loc}(0,1),\ \|b^2 f''\| \leq \infty\} \tag{14}$$

Here $AC^{loc}(0,1)$ denotes the set of all locally continuous functions defined on $(0,1)$. Also It is clear from [32] that there exist a real constant $C > 0$ such that

$$C^{-1}\omega_2^b(g,\sqrt{\tilde{\eta}}) \leq K_{2,b(u)}(g;\tilde{\eta}) \leq C\omega_2^b(g,\sqrt{\tilde{\eta}}). \tag{15}$$

Now we establish a global approximation for the defined operators (4).

Theorem 5.1. *For $g \in C_B(0,1)$ and $u \in (0,1)$ there exists $C > 0$*

$$|H^*_{l+p}(g;u) - g(u)| \leq C\omega_2^b\left(g, \frac{\tau_{l+p}(u)}{2b(u)}\right) + \omega_b\left(g, \frac{\psi_{l+p}(u)}{b(u)}\right), \tag{16}$$

where $\tau_{l+p}(u) = \{\mu_{l+p}(u) + \psi^2_{l+p}(u)\}^{\frac{1}{2}}$, with $\mu_{l+p}(u) = \sqrt{A^*_{l+p}}$ and $b(u) = \{u(1-u)\}^{\frac{1}{2}}$.

Proof. For $g \in C_B(0,1)$, already we define the auxiliary operator in (6) such as:

$$\widehat{H}^*_{l+p}(g;u) = H^*_{l+p}(g;u) + g(u) - g\left(\left(\frac{l+p-2}{l+p+1}\right)u + \frac{3}{2(l+p+1)}\right).$$

Let $x = \lambda u + (1-\lambda)s$, where $\lambda \in [0,1]$. As b^2 is concave on $(0,1)$, we must have the condition $b^2(x) \geq \lambda b^2(u) + (1-\lambda)b^2(s)$. Then, we have

$$\frac{|s-x|}{b^2(x)} \leq \frac{\lambda|u-s|}{\lambda b^2(u) + (1-\lambda)b^2(s)} \leq \frac{|s-u|}{b^2(u)}.$$

Also, in view of Lemma (2.3), for the operators (16), we have

$$\begin{aligned} |\widehat{H}^*_{l+p}(g;u) - g(u)| &\leq |\widehat{H}^*_{l+p}(g-f;u)| + |\widehat{H}^*_{l+p}(f;u) - f(u)| + |(g(u) - f(u)| \\ &\leq 4\|g-f\| + |\widehat{H}^*_{l+p}(f;u) - f(u)|. \end{aligned} \tag{17}$$

In view of Taylor's formula, we have

$$\begin{aligned} |\widehat{H}^*_{l+p}(g;u) - g(u)| &\leq H^*_{l+p}\Big(\int_u^s |s-x||f''(x)|dx;u\Big) + \Big|\int_u^{u+\psi_{l+p}(u)} |u+\psi_{l+p}(u)-x||f''(x)|dx\Big| \\ &\leq \|b^2 f''\|H^*_{l+p}\Big(\Big|\int_u^s \frac{|s-x|}{b^2(x)}dx\Big|;u\Big) + \|b^2 f''\|\Big|\int_u^{u+\psi_{l+p}(u)} \frac{|u+\psi_{l+p}(u)-x|}{b^2(x)}dx\Big| \\ &\leq b^{-2}(u)\|b^2 f''\|H^*_{l+p}((|s-u|)^2;u) + b^{-2}(u)\|b^2 f''\|\psi^2_{l+p}(u) \\ &\leq b^{-2}(u)\|b^2 f''\|[\mu_{l+p}(u) + \psi^2_{l+p}(u)]. \end{aligned}$$

By using the above inequality, (17) yields

$$|\widehat{H}^*_{l+p}(g;u) - g(u)| \leq 4\|g-f\| + b^{-2}(u)\|b^2 f''\|[\mu_{l+p}(u) + \psi^2_{l+p}(u)].$$

Now by using (17) and taking the infimum over all $g \in \mathcal{W}^2(b)$, we have

$$|\widehat{H}^*_{l+p}(g;u) - g(u)| \leq C\omega^b_2\Big(g, \frac{\sqrt{\mu_{l+p}(u) + \psi^2_{l+p}(u)}}{2b(u)}\Big).$$

But, In view of definition of first order Ditzian-Totik modulus of smoothness, one has

$$|g(u+\psi_{l+p}(u)) - g(u)| = \Big|g\Big(u + b(u)\frac{\psi_{l+p}(u)}{b(u)}\Big) - g(u)\Big| \leq \omega_b\Big(g, \frac{\psi_{l+p}(u)}{b(u)}\Big).$$

Hence, we get

$$|\widehat{H}^*_{l+p}(g;u) - g(u)| \leq C\omega^b_2\Big(g, \frac{\sqrt{\mu_{l+p}(u) + \psi^2_{l+p}(u)}}{2b(u)}\Big) + \omega_b\Big(g, \frac{\psi_{l+p}(u)}{b(u)}\Big),$$

which is the required result. □

Theorem 5.2. *([35], [36]) Suppose that the sequence of positive linear operators $(L_n)_{n\geq 1}$ acting from $C_\nu(0,1)$ to $B_\nu(0,1)$ satisfies the conditions*

$$\lim_{n\to\infty} \|L_n(e_i;.) - e_i\|_\nu = 0, \ \ where\ i = 0,1,2,$$

then, for $g \in C^{\tilde{k}}_\nu(0,1)$, we have

$$\lim_{n\to\infty} \|L_n g - g\|_\nu = 0.$$

Remark 5.3. *Throughout the paper, we consider test function as $e_i(t) = t^i$, $i = 0,1,2$.*

6. Bi-variate generalized Bernstein Kantorovich Schurer operators

Take $T^2 = \{(u_1, u_2) : 0 < u_1 < 1, 0 < u_2 < 1\}$ and $C(T^2)$ is the class of all continuous function on T^2 equipped with norm $\|f\|_{C(T^2)} = \sup_{(u_1,u_2)\in T^2} |f(u_1, u_2)|$. Then, for all $g \in C(T^2)$ and $l_1 + p, l_2 + p \in \mathbb{N}$, we established a bivariate sequence as:

$$H^*_{l_1+p,l_2+p}(g; u_1, u_2) = \sum_{\nu_1=0}^{l+p}\sum_{\nu_2=0}^{l+p} Q_{l_1+p,l_2+p,\nu_1,\nu_2}(u_1, u_2) \int_{\frac{\nu_1}{l_1+p+1}}^{\frac{\nu_1+1}{l_1+p+1}} \int_{\frac{\nu_2}{l_2+p+1}}^{\frac{\nu_2+1}{l_2+p+1}} g(t_1, t_2)dt_1dt_2, \tag{18}$$

where

$$\begin{aligned} Q_{l_1+p,l_2+p,\nu_1,\nu_2}(u_1, u_2) &= Q_{l_1+p,\nu_1}(u_1)Q_{l_2+p,\nu_2}(u_2), \quad \text{and} \\ Q_{l_i+p,\nu_i}(u_i) &= \frac{(l_i + p + 1)}{l_i + p}\begin{pmatrix} l_i + p \\ \nu_i \end{pmatrix}(\nu_i - (l_i + p)u)^2 u^{\nu_i-1}(1 - u)^{l_i+p-\nu_i-1}, \\ \text{for } i &= 1, 2. \end{aligned}$$

Lemma 6.1. *Let $e_{\nu_1,\nu_2}(u_1, u_2) = u_1^{\nu_1}u_2^{\nu_2}$. Then, for the operator (18), we get*

$$\begin{aligned} H^*_{l_1+p,l_2+p}(e_{0,0}; u_1, u_2) &= 1, \\ H^*_{l_1+p,l_2+p}(e_{1,0}; u_1, u_2) &= \left(\frac{l_1 + p - 2}{l_1 + p + 1}\right)u_1 + \frac{3}{2(l_1 + p + 1)}, \\ H^*_{l_1+p,l_2+p}(e_{0,1}; u_1, u_2) &= \left(\frac{l_2 + p - 2}{l_2 + p + 1}\right)u_2 + \frac{3}{2(l_2 + p + 1)}, \\ H^*_{l_1+p,l_2+p}(e_{2,0}; u_1, u_2) &= \left(\frac{(l_1 + p)^2 - 7(l_1 + p) + 6}{(l_1 + p + 1)^2}\right){u_1}^2 + \left(\frac{6(l_1 + p) - 8}{(l_1 + p + 1)^2}\right)u_1 + \frac{7}{3(l_1 + p + 1)^2}, \\ H^*_{l_1+p,l_2+p}(e_{0,2}; u_1, u_2) &= \left(\frac{(l_2 + p)^2 - 7(l_2 + p) + 6}{(l_2 + p + 1)^2}\right){u_2}^2 + \left(\frac{6(l_2 + p) - 8}{(l_2 + p + 1)^2}\right)u_2 + \frac{7}{3(l_2 + p + 1)^2}, \\ H^*_{l_1+p,l_2+p}(e_{3,0}; u_1, u_2) &= \left(\frac{(l_1 + p)^3 - 15(l_1 + p)^2 - 38(l_1 + p) - 24}{(l_1 + p + 1)^3}\right){u_1}^3 + \left(\frac{\frac{27}{2}(l_1 + p)^2 - \frac{117}{2}(l_1 + p) + 45}{(l_1 + 1)^3}\right){u_1}^2 \\ &\quad + \left(\frac{\frac{43}{2}(l_1 + p) - 25}{(l_1 + p + 1)^3}\right)u_1 + \frac{15}{4(l_1 + p + 1)^3}, \\ H^*_{l_1+p,l_2+p}(e_{0,3}; u_1, u_2) &= \left(\frac{(l_2 + p)^3 - 15(l_2 + p)^2 - 38(l_2 + p) - 24}{(l_2 + p + 1)^3}\right){u_2}^3 + \left(\frac{\frac{27}{2}(l_2 + p)^2 - \frac{117}{2}(l_2 + p) + 45}{(l_2 + 1)^3}\right){u_2}^2 \\ &\quad + \left(\frac{\frac{43}{2}(l_2 + p) - 25}{(l_2 + p + 1)^3}\right)u_2 + \frac{15}{4(l_2 + p + 1)^3}. \end{aligned}$$

Proof. From (2.1) and linearity, property we get

$$\begin{aligned} H^*_{l_1+p,l_2+p}(e_{0,0}; u_1, u_2) &= H^*_{l_1+p,l_2+p}(e_0; u_1, u_2)H^*_{l_1+p,l_2+p}(e_0; u_1, u_2), \\ H^*_{l_1+p,l_2+p}(e_{1,0}; u_1, u_2) &= H^*_{l_1+p,l_2+p}(e_1; u_1, u_2)H^*_{l_1+p,l_2+p}(e_0; u_1, u_2), \\ H^*_{l_1+p,l_2+p}(e_{0,1}; u_1, u_2) &= H^*_{l_1+p,l_2+p}(e_0; u_1, u_2)H^*_{l_1+p,l_2+p}(e_1; u_1, u_2), \\ H^*_{l_1+p,l_2+p}(e_{2,0}; u_1, u_2) &= H^*_{l_1+p,l_2+p}(e_2; u_1, u_2)H^*_{l_1+p,l_2+p}(e_0; u_1, u_2), \\ H^*_{l_1+p,l_2+p}(e_{0,2}; u_1, u_2) &= H^*_{l_1+p,l_2+p}(e_0; u_1, u_2)H^*_{l_1+p,l_2+p}(e_2; u_1, u_2), \\ H^*_{l_1+p,l_2+p}(e_{3,0}; u_1, u_2) &= H^*_{l_1+p,l_2+p}(e_3; u_1, u_2)H^*_{l_1+p,l_2+p}(e_0; u_1, u_2), \\ H^*_{l_1+p,l_2+p}(e_{0,3}; u_1, u_2) &= H^*_{l_1+p,l_2+p}(e_0; u_1, u_2)H^*_{l_1+p,l_2+p}(e_3; u_1, u_2). \end{aligned}$$

□

7. Degree of Convergence

For $g \in C(\mathcal{T}^2)$ the modulus of continuity of the second order is

$$\omega\left(g;\delta_{l_1},\eta_{l_2}\right)=\sup_{(u_1,u_2)\in T^2}\{\mid g(t,s)-g(u_1,u_2)\mid:(t,s),(u_1,u_2)\in\mathcal{T}^2\},$$

with $\mid t-u_1\mid\leq\eta_{l_1}$, $\mid s-u_2\mid\leq\eta_{l_2}$ and $\eta>0$, given by the partial modulus of continuity as follows:

$$\omega_1\left(g;\eta\right)=\sup_{0\leq u_2\leq\infty}\ \sup_{|x_1-x_2|\leq\eta}\{\mid g(x_1,u_2)-g(x_2,u_2)\mid\},$$

$$\omega_2\left(g;\eta\right)=\sup_{0\leq u_1\leq\infty}\ \sup_{|u_1-u_2|\leq\eta}\{\mid g(u_1,u_1)-g(u_1,u_2)\mid\}.$$

Theorem 7.1. *For any $g\in C(\mathcal{T}^2)$, we have*

$$\mid H^*_{l_1+p,l_2+p}(g;u_1,u_2)-g(u_1,u_2)\mid\leq 2\Big(\omega_1(g;\delta_{u_1,l_1})+\omega_2(g;\delta_{l_2,u_2})\Big).$$

Proof. In order to proof of above Theorem 7.1, generally, we use the well-known Cauchy-Schwartz inequality. Then, we have

$$\begin{aligned}
\mid H^*_{l_1+p,l_2+p}(g;u_1,u_2) &- g(u_1,u_2)\mid\leq H^*_{l_1+p,l_2+p}\left(\mid g(t,s)-g(u_1,u_2)\mid;u_1,u_2\right)\\
&\leq H^*_{l_1+p,l_2+p}\left(\mid g(t,s)-g(u_1,s)\mid;u_1,u_2\right)+H^*_{l_1+p,l_2+p}\left(\mid g(u_1,s)-g(u_1,u_2)\mid;u_1,u_2\right)\\
&\leq H^*_{l_1+p,l_2+p}\left(\omega_1(g;\mid t-u_1\mid);u_1,u_2\right)+H^*_{l_1+p,l_2+p}\left(\omega_2(g;\mid s-u_2\mid);u_1,u_2\right)\\
&\leq \omega_1(g;\delta_{l_1})\left(1+\delta_{l_1}^{-1}H^*_{l_1+p,l_2+p}(\mid t-u_1\mid;u_1,u_2)\right)+\omega_2(g;\delta_{l_2})\left(1+\delta_{l_2}^{-1}H^*_{l_1+p,l_2+p}(\mid s-u_2\mid;u_1,u_2)\right)\\
&\leq \omega_1(g;\delta_{l_1})\left(1+\frac{1}{\delta_{l_1}}\sqrt{H^*_{l_1+p,l_2+p}((t-u_1)^2;u_1,u_2)}\right)\\
&+ \omega_2(g;\delta_{l_2})\left(1+\frac{1}{\delta_{l_2}}\sqrt{H^*_{l_1+p,l_2+p}((s-u_2)^2;u_1,u_2)}\right).
\end{aligned}$$

If we choose $\delta^2_{l_1}=\delta^2_{l_1,u_1}=H^*_{l_1+p,l_2+p}((t-u_1)^2;u_1,u_2)$ and $\delta^2_{l_2}=\delta^2_{l_2,u_2}=H^*_{l_1+p,l_2+p}((s-u_2)^2;u_1,u_2)$, then we get achieve our results. □

Now, we analyse the convergence in terms of the Lipschitz class for bivariate functions. Maximal Lipschitz function space on $E\times E\subset\mathcal{T}^2$, for $M>0$ and $\zeta,\zeta\in(0,1)$ is given by

$$\begin{aligned}
\mathcal{L}_{\zeta,\zeta}(E\times E) &= \Big\{g:\sup(1+t)^{\zeta}(1+s)^{\zeta}\left(g_{\zeta,\zeta}(t,s)-g_{\zeta,\zeta}(u_1,u_2)\right)\\
&\leq M\frac{1}{(1+u_1)^{\zeta}}\frac{1}{(1+u_2)^{\zeta}}\Big\},
\end{aligned}$$

where g is taken as continuous and bounded function on $\mathcal{T}^2$ and

$$g_{\zeta,\zeta}(t,s)-g_{\zeta,\zeta}(u_1,u_2)=\frac{\mid g(t,s)-g(u_1,u_2)\mid}{\mid t-u_1\mid^{\zeta}\mid s-u_2\mid^{\zeta}};\quad (t,s),(u_1,u_2)\in\mathcal{T}^2. \tag{19}$$

Theorem 7.2. *For $g\in\mathcal{L}_{\zeta,\zeta}(E\times E)$. Then, for any $\zeta,\zeta\in(0,1)$, there exists $M>0$ such that*

$$\begin{aligned}
\mid H^*_{l_1+p,l_2+p}(g;u_1,u_2)-g(u_1,u_2)\mid &\leq M\Big\{\Big((d(u_1,E))^{\zeta}+\left(\delta^2_{l_1,u_1}\right)^{\frac{\zeta}{2}}\Big)\\
&\times \Big((d(u_2,E))^{\zeta}+\left(\delta^2_{l_2,u_2}\right)^{\frac{\zeta}{2}}\Big)\\
&+ \Big(d((u_1,E))^{\zeta}\,(d(u_2,E))^{\zeta}\Big)\Big\},
\end{aligned}$$

where δ_{l_1,u_1} and δ_{l_2,u_2} defined by above Theorem 7.1.

Proof. Consider $\mid u_1 - x_0 \mid = d(u_1, E)$ and $\mid u_2 - y_0 \mid = d(u_2, E)$, for any $(u_1, u_2) \in \mathcal{T}^2$, and $(x_0, y_0) \in E \times E$. Let $d(u_1, E) = \inf\{\mid u_1 - u_2 \mid : u_2 \in E\}$. Thus, we write

$$\mid g(t,s) - g(u_1,u_2) \mid \leq M \mid g(t,s) - g(x_0,y_0) \mid + \mid g(x_0,y_0) - g(u_1,u_2) \mid . \tag{20}$$

Apply $H^*_{l_1+p,l_2+p}(.;.,.)$, we have

$$\begin{aligned} \mid H^*_{l_1+p,l_2+p}(g;u_1,u_2) &- g(u_1,u_2) \mid \\ &\leq H^*_{l_1+p,l_2+p}\left(\mid g(u_1,u_2) - g(x_0,y_0) \mid + \mid g(x_0,y_0) - g(u_1,u_2) \mid\right) \\ &\leq MH^*_{l_1+p,l_2+p}\left(\mid t - x_0 \mid^{\zeta} \mid s - y_0 \mid^{\zeta}; u_1, u_2\right) \\ &+ M \mid u_1 - x_0 \mid^{\zeta} \mid u_2 - y_0 \mid^{\zeta} . \end{aligned}$$

For every $A, B \geq 0$ and $\zeta \in (0,1)$, by using inequality $(A+B)^{\zeta} \leq A^{\zeta} + B^{\zeta}$, therefore,

$$\mid t - x_0 \mid^{\zeta} \leq \mid t - u_1 \mid^{\zeta} + \mid u_1 - x_0 \mid^{\zeta},$$

$$\mid s - y_0 \mid^{\zeta} \leq \mid s - u_2 \mid^{\zeta} + \mid u_2 - y_0 \mid^{\zeta} .$$

Thus,

$$\begin{aligned} \mid H^*_{l_1+p,l_2+p}(g;u_1,u_2) &- g(u_1,u_2) \mid \\ &\leq MH^*_{l_1+p,l_2+p}\left(\mid t - u_1 \mid^{\zeta} \mid s - u_2 \mid^{\zeta}; u_1, u_2\right) \\ &+ M \mid u_1 - x_0 \mid^{\zeta} H^*_{l_1+p,l_2+p}\left(\mid s - u_2 \mid^{\zeta}; u_1, u_2\right) \\ &+ M \mid u_2 - y_0 \mid^{\zeta} H^*_{l_1+p,l_2+p}\left(\mid t - u_1 \mid^{\zeta}; u_1, u_2\right) \\ &+ 2M \mid u_1 - x_0 \mid^{\zeta} \mid u_2 - y_0 \mid^{\zeta} H^*_{l_1+p,l_2+p}\left(\mu_{0,0}; u_1, u_2\right). \end{aligned}$$

Apply Hölders inequality on $H^*_{l_1+p,l_2+p}(.;.,.)$, we get

$$\begin{aligned} H^*_{l_1+p,l_2+p}\left(\mid t - u_1 \mid^{\zeta} \mid s - u_2 \mid^{\zeta}; u_1, u_2\right) &= \mathcal{U}^{\lambda_1}_{l_1,k}\left(\mid t - u_1 \mid^{\zeta}; u_1, u_2\right) \\ &\times \mathcal{V}^{\lambda_2}_{l_2,l}\left(\mid s - u_2 \mid^{\zeta}; u_1, u_2\right) \\ &\leq \left(H^*_{l_1+p,l_2+p}(\mid t - u_1 \mid^2; u_1, u_2)\right)^{\frac{\zeta}{2}} \\ &\times \left(H^*_{l_1+p,l_2+p}(\mu_{0,0}; u_1, u_2)\right)^{\frac{2-\zeta}{2}} \\ &\times \left(H^*_{l_1+p,l_2+p}(\mid s - u_2 \mid^2; u_1, u_2)\right)^{\frac{\zeta}{2}} \\ &\times \left(H^*_{l_1+p,l_2+p}(\mu_{0,0}; u_1, u_2)\right)^{\frac{2-\zeta}{2}} . \end{aligned}$$

Thus, we can obtain

$$\begin{aligned} \mid H^*_{l_1+p,l_2+p}(g;u_1,u_2) - g(u_1,u_2) \mid &\leq M\left(\delta^2_{l_1,u_1}\right)^{\frac{\zeta}{2}}\left(\delta^2_{l_2,u_2}\right)^{\frac{\zeta}{2}} \\ &+ 2M\left(d(u_1,E)\right)^{\zeta}\left(d(u_2,E)\right)^{\zeta} \\ &+ M\left(d(u_1,E)\right)^{\zeta}\left(\delta^2_{l_2,u_2}\right)^{\frac{\zeta}{2}} \\ &+ L\left(d(u_2,E)\right)^{\zeta}\left(\delta^2_{l_1,u_1}\right)^{\frac{\zeta}{2}} . \end{aligned}$$

Which completes the proof. □

8. Conclusion

In this study, we introduce generalized Bernstein Kantorovich Schurer type operators and estimate some lemmas to support approximation results in subsequent sections in terms of test function and central moments. Next, the convergence results and approximation rate in the sense of Korovkin theorem and classical modulus of continuity are studied. Further, we investigate direct results of approximation via Peetre's K-functional, modulus of continuity of second order, Lipschitz space of functions and Lipschitz type r^{th} order maximal function. In the last section, we discuss global approximation results and convergence results in terms of statistical approximation.

Conflict of interest

The authors declared that they have no conflict of interest.

Data availability statement

Data sharing not applicable

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