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Characterizations of $\{1,3\}$-Bohemian inverses of structured matrices

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Abstract. This paper presents $\{1,3\}$-Bohemian inverses of a certain type of structured $\{-1,0,1\}$-matrices, particularly full and well-settled matrices. It begins by characterizing the rank-one Bohemian matrices for the population $\mathbb{P} = \{-1,0,1\}$. Characterizations of the $\{3\}$ and $\{1,3\}$-Bohemian inverses are presented for arbitrary population over the set $\{-1,0,1\}$. Furthermore, explicit formulas are provided to enumerate the $\{1,3\}$-Bohemian inverses of these matrices when the population is exactly $\{-1,0,1\}$. Moreover, corresponding results for $\{3\}$-inverses are obtained.

1. Introduction and preliminaries

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ denote a field of real and complex numbers. The set of all m by n matrices over $\mathbb{K}$ are denoted by $\mathbb{K}^{m\times n}$. Often, the notation A_{mn} is used to indicate an $m \times n$ matrix A. A matrix $A \in \mathbb{K}^{m\times n}$ is designated as a Bohemian matrix if all its entries are from a fixed (typically finite and discrete) set, known as the population, and denoted as $\mathbb{P}$. For instance, Bernoulli matrices [15], Hadamard matrices [8], Mandelbrot matrices [3], Metzler matrices [2] etc. are Bohemian matrices. Properties of matrices that are simultaneously upper Hessenberg and Toeplitz Bohemians were considered in [5]. The complete set of p by q Bohemian matrices over a population $\mathbb{P}$ will be denoted as $\mathbb{P}^{p\times q}$. The term Bohemian matrix is a mnemonic for the "BOunded HEight Matrix of INtegers". Although not under the same name, the study of this matrix type has a much longer history. These matrices are frequently used as illustrations while studying linear systems, solving the system and determining its eigenvalues. Low-dimension Bohemian matrices are simple, but there are still many unanswered questions. For instance, the number of singular 6×6 matrices containing entries from the set $\{-1,0,+1\}$ is still unknown. The complexity arises from the number of such matrices, even though the computation for a single matrix is straightforward.

In this area, matrices with integer elements are studied in [12] and [13], among other sources. Bohemian matrices find applications in signal processing [9], error correcting codes when working with Hadamard matrices [8], as well as in investigating sign properties of Metzler matrices appearing in sign-pattern matrices ([2],[7]), etc.

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Besides, if $A \in \mathbb{P}^{n\times n}$ is Bohemian, A^{-1} is not always Bohemian with respect to the same population $\mathbb{P}$. The matrices for which the inverse is also Bohemian for the same population are known as Rhapsodic. Therefore, it is natural to ask which matrices are Rhapsodic. Trivially, if the population $\mathbb{P}$ is a subfield of the field $\mathbb{K}$, then all $n \times n$ nonsingular matrices with population $\mathbb{P}$ are rhapsodic. Mandelbrot matrices and Unimodular matrices-that is, $n \times n$ Bohemian matrices with population $\mathbb{Z}$ with a determinant ± 1 appear as another common class of this kind of matrices. This rhapsodic behaviour of inverses is studied in [14] with weaker constraint, when A^{-1} is similar to a Bohemian matrix. Based on Barrett, Butler, and Hall's conjecture (refer to [1]), in [10], the authors investigated the inversion of equimodular matrices, defined as square matrices with all entries having identical modulus and which may be Bohemian.

It is universally known that the generalized inverses are defined for rectangular matrices. For $A \in \mathbb{K}^{m\times n}$, the four Penrose equations with unknown $X \in \mathbb{K}^{n\times m}$ are restated from [11]

$$(1)\ AXA = A, \quad (2)\ XAX = X, \quad (3)\ (AX)^* = AX, \quad (4)\ (XA)^* = XA.$$

If X satisfies the (i), $(j), \ldots$ equations, then it is denoted as the $\{i, j, \ldots\}$-inverse of A, and the notation for the set of all $\{i, j, ...\}$-inverses of A is $A\{i, j, \ldots\}$. Often, an element of the set $A\{i, j, \ldots\}$ is denoted as $A^{(i,j,\ldots)}$. In particular, for $A \in \mathbb{K}^{m\times n}$, the $\{1, 2, 3, 4\}$-inverse of A is the well-known Moore-Penrose inverse, which is unique and denoted as $A^\dagger$ [11]. The generalized inverses satisfying equation (1) of the Penrose equations are called the inner inverses.

Following the results, a natural step will be to study the rhapsodic behaviour of generalized inverses. The results in this direction can be seen in [6] and [4]. Recently, in [4], the inner inverses of Bohemian matrices are studied. For $A \in \mathbb{P}^{m\times n}$, $X \in \mathbb{K}^{n\times m}$ is said to be an inner Bohemian inverse of A if $X \in A\{1\} \cap \mathbb{P}^{n\times m}$ [4]. Even though $A\{1\}$ is non-empty for all $A \in \mathbb{K}^{m\times n}$, the set of inner Bohemian inverses can be empty for some matrices. An instance is given in [4]. Some of the works done in [4] include the following. The structured matrices full and well-settled matrices are defined for the population $\mathbb{P} = \{-1, 0, 1\}$. The set $A\{1\}$ is characterized, and the cardinality of Bohemian inner inverses was obtained for the introduced structured matrices. The full matrices are rank-one Bohemian matrices with population $\mathbb{P} = \{-1, 0, 1\}$.
Moreover, since the full matrices are rank-one, we classified all the rank-one Bohemian matrices with population $\mathbb{P} = \{-1, 0, 1\}$, as stated in Theorem 3.1. Given Theorem 3.1, we conclude that finding inner inverses for the full matrices is enough to find the inner inverses of any rank-one Bohemian matrix with population $\mathbb{P} = \{-1, 0, 1\}$.
Analogous to the definition of inner Bohemian inverse, $X \in \mathbb{K}^{n\times m}$ is said to be $\{1, 3\}$-Bohemian inverse of $A \in \mathbb{P}^{m\times n}$, if $X \in A\{1, 3\} \cap \mathbb{P}^{n\times m}$. Similarly, for $\{3\}$, $\{4\}$ and $\{1, 4\}$-Bohemian inverse. The notation for the set of all $\{i, j, ...\}$-Bohemian inverses of A is $A_{\mathbb{P}}\{i, j, \ldots\}$. To study $\{1, 3\}$-inverses of Bohemian matrices, we defined another set of full matrices using the one described by [4] to get the $\{1, 3\}$-inverses for all rank-one Bohemian matrices with population $\mathbb{P} = \{-1, 0, 1\}$ as given in Section 3. It is known that the set $A\{1, 3\}$ is non-empty for all $A \in \mathbb{K}^{m\times n}$ besides the set $A_{\mathbb{P}}\{1, 3\}$ can be empty. For instance, for $\mathbb{P} = \{-1, 0, 1\}$ and $A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \end{pmatrix}$, $A_{\mathbb{P}}\{1\}$ is non-empty (refer to [4]), while

$$A\{1, 3\} = \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \\ x_5 & x_6 \end{pmatrix} : x_1 + x_3 = -(x_2 + x_4) = \frac{1}{2}, x_5, x_6 \in \mathbb{K} \right\}$$

and hence $A_{\mathbb{P}}\{1, 3\}$ is an empty set.

In addition, we investigated the $\{1, 3\}$-inverses of all rank-one Bohemian matrices and found the cardinality of the set $A_{\mathbb{P}}\{1, 3\}$ for the full matrices.

Furthermore, we will see some properties of $\{3\}$ and $\{4\}$-inverses. Before that, we introduce some notations similar to notations in [4] which will be followed throughout this manuscript. By $A = (B \ \ C) \in \mathbb{K}^{m\times(n_1+n_2)}$, we mean A is a partitioned matrix such that $B \in \mathbb{K}^{m\times n_1}$ and $C \in \mathbb{K}^{m\times n_2}$. Similarly, by $A = \begin{pmatrix} B \\ C \end{pmatrix} \in \mathbb{K}^{(m_1+m_2)\times n}$, means A is partitioned into the blocks $B \in \mathbb{K}^{m_1\times n}$ and $C \in \mathbb{K}^{m_2\times n}$. Additionally, we denote

$0, (1_{mn})$, and (-1_{mn}) for the zero matrix, matrix with all entries as 1 and the matrix with all entries -1 respectively. Furthermore, to refer simultaneously the matrices $(1_{mn_1} \mid -1_{mn_2})$ and $(-1_{mn_1} \mid 1_{mn_2})$, we use the notation $(\pm 1_{mn_1} \mid \mp 1_{mn_2})$.
Moreover for $\mathfrak{S} \subseteq \mathbb{K}^{m\times n}$ and matrices U, V of suitable order, the notation $\#\mathfrak{S}$ represents the number of matrices which belong to the set $\mathfrak{S}$, while $U\mathfrak{S}V$ is a notation for the set

$$U\mathfrak{S}V := \{USV | S \in \mathfrak{S}\}.$$

Similarly the notations $U\mathfrak{S}$ and $\mathfrak{S}V$ are used. For $\lambda \in \mathbb{K}$, $\lambda\mathfrak{S}$ is a notation for

$$\lambda\mathfrak{S} := \{\lambda S |\ S \in \mathfrak{S}\},$$

and

$$\mathfrak{S}^{\mathrm{T}} := \{S^{\mathrm{T}} |\ S \in \mathfrak{S}\},$$

where S^{T} denotes the transpose of the matrix S.

First, we restate the notion of full and well-settled matrices defined in [4].

Definition 1.1. [4] *A matrix $A \in \mathbb{K}^{m\times n}$ is known as full if it takes any of the subsequent forms*

1. *Type I full matrix:* $(\pm 1_{mn})$.
2. *Type II full matrix:* $(\pm 1_{mn_1} | \mp 1_{mn_2})$.
3. *Type III full matrix:* $(\pm 1_{mn_1} | 0_{mn_2})$.
4. *Type IV full matrix:* $(\pm 1_{mn_1} | \mp 1_{mn_2} | 0_{mn_3})$.

Definition 1.2. [4] *A matrix $A \in \mathbb{K}^{m\times n}$ is known as well-settled if, upon multiplication by appropriate permutation matrices P and Q, PAQ takes the general form*

$$PAQ = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_s \end{pmatrix},$$

where each $A_i \in \mathbb{K}^{p_i\times q_i}$ is full. Also, A is called pure well-settled if all A_i's are of the same type; otherwise, it is called mixed well-settled.

2. Motivation

Further, we will state some properties of {3} and {4}-inverses which are consequences of their definitions.

Lemma 2.1. *Let $A \in \mathbb{K}^{m\times n}$, and $U \in \mathbb{K}^{m\times m}$, $V \in \mathbb{K}^{n\times n}$ be unitary. It holds that*

1. $(\lambda A)\{3\} = \frac{1}{\lambda}(A\{3\})$, *where* $0 \neq \lambda \in \mathbb{K}$.
2. $(UAV)\{3\} = V^*(A\{3\})U^*$.
3. $(A\{3\})^* = A^*\{4\}$ *and* $(A\{4\})^* = A^*\{3\}$.

Proof. 1. $X \in (\lambda A)\{3\} \iff (\lambda AX)^* = \lambda AX \iff \lambda X \in A\{3\} \iff X \in \frac{1}{\lambda}(A\{3\})$.

2. Let $X \in V(UAV)\{3\}U$ then $\exists Y \in (UAV)\{3\}$ such that $X = VYU$. Thus $(UAVY)^* = UAVY$. Hence,

$$(AX)^* = (AVYU)^* = (AVYU)^*U^*U = (UAVYU)^*U = U^*(UAVY)^*U = U^*UAVYU = AX.$$

Conversely if $X \in A\{3\}$, then $(AX)^* = AX$, which implies

$$(UAV(V^*XU^*))^* = (UAXU^*)^* = U(AX)^*U^* = UAXU^* = UAVV^*XU^*.$$

Therefore, $V^*XU^* \in (UAV)\{3\}$ and $X \in V(UAV)\{3\}U$.

3. $X \in (A\{3\})^* \iff X = Y^*$ with $Y \in A\{3\} \iff X = Y^*$ and $(AY)^* = AY \iff X = Y^*$ and $Y^*A^* = AY \iff X = Y^*$ and $(Y^*A^*)^* = (AY)^* = Y^*A^* \iff Y^* \in A^*\{4\}$. Hence $(A\{3\})^* = A^*\{4\}$. The other parts follow similarly.
□

Lemma 2.2. *Let* $A = (B_{mn_1}|0_{mn_2}) \in \mathbb{K}^{m\times(n_1+n_2)}$, *then*

1. $A\{3\} = \left\{\begin{pmatrix} X \\ Y \end{pmatrix} \mid X_{n_1m} \in B_{mn_1}\{3\}, Y_{n_2m} \in \mathbb{K}^{n_2\times m}\right\}$.
2. $A\{4\} = \left\{\begin{pmatrix} X \\ Y \end{pmatrix} \mid X_{n_1m} \in B_{mn_1}\{4\}, YB = 0\right\}$.

Lemma 2.3. *Let* $A = \begin{pmatrix} B_{m_1n} \\ 0_{m_2n} \end{pmatrix} \in \mathbb{K}^{(m_1+m_2)\times n}$, *then*

1. $A\{3\} = \left\{\begin{pmatrix} X & Y \end{pmatrix} \mid X_{nm_1} \in B_{m_1n}\{3\}, BY = 0\right\}$.
2. $A\{4\} = \left\{\begin{pmatrix} X & Y \end{pmatrix} \mid X_{nm_1} \in B_{m_1n}\{4\}, Y_{m_2n} \in \mathbb{K}^{m_2\times n}\right\}$.

Note that while the zero matrix blocks mentioned in lemmas 2.2 and 2.3 can be neglected for finding inner inverses (refer to [4]), this approach is not applicable for $\{3\}$ and $\{4\}$-inverses. The aforementioned lemmas prompt us to find another set of full matrices to investigate $\{3\}$ and $\{4\}$-inverses. Furthermore, to study $\{1,3\}$ and $\{1,4\}$-inverses, it is adequate to focus on $\{1,3\}$ inverses as per Lemma 2.1(3).

The results arising from current research are emphasized as follows.

(1) The set of rank-one Bohemian matrices over the population $\mathbb{P} = \{-1, 0, 1\}$ is characterized, and the set of full matrices is introduced.

(2) Representations and properties of $\{1,3\}$-inverses of rank-one Bohemian matrices over $\mathbb{P}$ are studied, as well as the cardinalities of $\{1,3\}$-Bohemian inverses on full matrices.

(3) The set of $\{1,3\}$-Bohemian inverses of well-settled matrices of arbitrary rank is investigated.

The framework of this paper is as follows. Motivation and preliminary results are presented in Section 2. In Section 3, we characterize the rank-one Bohemian matrices for population $\mathbb{P} = \{-1, 0, 1\}$ and define a set of full matrices. The set of $\{1,3\}$-inverses of rank-one Bohemian matrices over $\mathbb{P}$ is studied, as well as the cardinalities of $\{1,3\}$-Bohemian inverses of full matrices. In section 4, we studied the $\{1,3\}$-Bohemian inverses of larger rank matrices, i.e., the well-settled matrices using the results of Section 3. Section 5 gives some concluding remarks and ideas for possible further research.

3. $\{1,3\}$-Bohemian inverses on full matrices

In this section, to begin with, we will define the full matrices required to study $\{1,3\}$-inverses by characterizing the rank-one Bohemian matrices.

The following result gives a representation theorem for the rank-one Bohemian matrices, which plays a vital role in characterizing the rank-one Bohemian matrices.

Theorem 3.1. *Let* $\mathbb{P} = \{-1, 0, 1\}$. *Any rank-one* $A \in \mathbb{P}^{m\times n}$ *can be represented as* $A = P_1DA'P_2$, *where* P_1, P_2 *are appropriate permutation matrices,* $D \in \mathbb{P}^{m\times m}$ *is an invertible diagonal matrix, and in the case A has no zero rows,* $A' \in \mathbb{P}^{m\times n}$ *is a full matrix; otherwise if A has k zero rows then* $A' = \begin{pmatrix} B \\ 0 \end{pmatrix}$, *where* $B \in \mathbb{P}^{(m-k)\times n}$ *is a full matrix.*

Proof. Since A has rank-one, it can be written as $A = xy^{\mathrm{T}}$, where $x \in \mathbb{K}^{m\times 1}$ and $y \in \mathbb{K}^{n\times 1}$. Also, since A is Bohemian, there exist vectors x and y such that they are Bohemian vectors (vectors having entries from the set $\mathbb{P}$). Also, without loss of generality, suppose A has k zero rows where $k \in \{0, 1, \ldots, m-1\}$. Now,

$$A = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix} = I_m P \begin{pmatrix} x_{a_1} \\ \vdots \\ x_{a_{m-k}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix}$$

$$= P \begin{pmatrix} x_{a_1} & & & & & \\ & \ddots & & & & \\ & & x_{a_{m-k}} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix}$$

$$= PD \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & & \vdots \\ y_1 & y_2 & \cdots & y_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

where P is a permutation matrix such that the zero rows of A are permuted to the last k rows of A. Hence,

$$P \begin{pmatrix} x_{a_1} \\ \vdots \\ x_{a_{m-k}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix},$$

and

$$D = \begin{pmatrix} x_{a_1} & & & & & \\ & \ddots & & & & \\ & & x_{a_{m-k}} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

Clearly, by construction D is an invertible diagonal matrix. Since $y_i \in \mathbb{P}$, now by utilizing the permutation matrices to swap columns of the matrix, say $Y = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & & \vdots \\ y_1 & y_2 & \cdots & y_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ can be written as $Y = \begin{pmatrix} B \\ 0 \end{pmatrix} Q_1$, where $B \in \mathbb{P}^{(m-k)\times n}$ is a full matrix (refer to Definition 1.1) and Q_1 is a suitable permutation matrix. In the case A has no zero rows, Y can be reduced to BQ_1 such that B is a full matrix. $\square$

In [4], only {1}-inverses of type I and type II full matrices were examined. However, to find {1, 3}-inverses, we must also consider full matrices of type I and type II, as well as transposes of full matrices of type II, III, and IV as per Lemma 2.1(3). Thus, we may restrict our study for {1, 3}-inverses to the subsequent cases:

1. (1_{mn}).
2. $(1_{mn_1} | -1_{mn_2})$.
3. $\begin{pmatrix} 1_{m_1 n} \\ -1_{m_2 n} \end{pmatrix}$.
4. $\begin{pmatrix} 1_{m_1 n} \\ 0_{m_2 n} \end{pmatrix}$.
5. $\begin{pmatrix} 1_{m_1 n} \\ -1_{m_2 n} \\ 0_{m_3 n} \end{pmatrix}$.
6. $\begin{pmatrix} B_{m_1 n} \\ 0_{m_2 n} \end{pmatrix}$ where $B = (1_{m_1 n_1} | -1_{m_1 n_2})$.

In the forthcoming results, we will characterize the {3} and {1, 3}-inverses of full matrices and the cardinalities of the set of all {3} and {1, 3}-Bohemian inverses of full matrices. Further it should be noted that here $\binom{n}{r}$ is the binomial coefficient. In current research, a binomial coefficient is considered as zero whenever it is not defined.

Theorem 3.2. *If* $A = (1_{mn}) \in \mathbb{K}^{m\times n}$, *then it follows*

1. $A\{3\} = \{X \in \mathbb{K}^{n\times m} : c_1 = c_2 = \cdots = c_m\}$.
2. $A\{1, 3\} = \left\{X \in \mathbb{K}^{n\times m} : c_1 = c_2 = \cdots = c_m = \frac{1}{m}\right\}$,

where c_i *denotes the sum of entries of ith column of* X.

Proof. Proof for the first part is trivial. For the second part, it holds since $A\{1, 3\} = A\{1\} \cap A\{3\}$. Hence

$$A\{1, 3\} = \{X \in \mathbb{K}^{n\times m} : \Sigma(X) = 1, c_1 = c_2 = \cdots = c_m\},$$

where $\Sigma(X)$ denotes the sum of all elements in the matrix X. □

The following two corollaries give the number of {3} and {1, 3}-Bohemian inverses of the full matrix (1_{mn}) for different populations $\mathbb{P}$.

Corollary 3.3. *For* $A = (1_{mn}) \in \mathbb{K}^{m\times n}$, $\mathbb{P} \subseteq \mathbb{K}$, *assume* $1 \in \mathbb{P} \subset \mathbb{N}$.

1. *If* $0 \in \mathbb{P}$, *then* $\#(A_{\mathbb{P}}\{1, 3\}) = \begin{cases} n, & \text{if } m = 1 \\ 0, & \text{otherwise} \end{cases}$.
2. *If* $0 \notin \mathbb{P}$, *then* $\#(A_{\mathbb{P}}\{1, 3\}) = \begin{cases} 1, & \text{if } m = n = 1 \\ 0, & \text{otherwise} \end{cases}$.

Proof.
1. Since $\mathbb{P} \subset \mathbb{N}$ and $0 \in \mathbb{P}$, $A_{\mathbb{P}}\{1, 3\} \neq \emptyset$ if and only if $m = 1$ and $c_1 = 1$. That is, if $X \in A_{\mathbb{P}}\{1, 3\}$, then X is a column vector and $\Sigma(X) = 1$. So, there are n possibilities.
2. By the proof of part 1, $m = n = 1$. Hence $A_{\mathbb{P}}\{1, 3\} \neq \emptyset$ if and only if $mn = 1$.

The proof is complete. □

Corollary 3.4. *If* $A = (1_{mn})$ *and* $\mathbb{P} = \{-1, 0, 1\}$.

1. $\#(A_{\mathbb{P}}\{3\}) = \displaystyle\sum_{k=-n}^{n} \left(\sum_{s_1=0}^{n} \binom{n}{s_1} \binom{n - s_1}{s_1 + k} \right)^m$.

2. $\#(A_{\mathbb{P}}\{1,3\}) = \begin{cases} \sum_{s=0}^{n} \binom{n}{s}\binom{n-s}{s+1}, & \text{if } m = 1 \\ 0, & \text{otherwise} \end{cases}.$

Proof. 1. For a specified matrix X, the notations $N^+(X)$ and $N^-(X)$ will be used to denote the number of elements in X equal to 1 and equal to -1, respectively. Consider $X \in A\{3\}$. Let c_i denote the sum of ith column of X. If $c_i = k \; \forall i \in \{1, 2, \ldots, m\}$, clearly $\Sigma(X) = mk$ and $k \in [-n, n]$. Let X_i denote the ith column of X. For $c_1 = k$ and $N^+(X_1) - N^-(X_1) = k$, if $N^-(X_1) = s_1$ then $N^+(X_1) = k + s_1$. The number of choices for the first column of X such that $c_1 = k$ is

$$\binom{n}{s_1}\binom{n-s_1}{k+s_1}.$$

Hence, the cardinality of the set $A_{\mathbb{P}}\{3\}$ is

$$\sum_{k=-n}^{n} \left(\sum_{s_1=0}^{n} \binom{n}{s_1}\binom{n-s_1}{k+s_1} \right)^m .$$

2. For $k = 1$ and $m = 1$ in the preceding part, we obtain the cardinality of the set $A_{\mathbb{P}}\{1,3\}$.
□

Remark 3.5. *Note that for $\mathbb{P} = \{-1, 0, 1\}$ and $A = (1_{mn})$, $\#A_{\mathbb{P}}\{1,3\} = \#A_{\mathbb{P}}\{1\}$ if and only if $A_{\mathbb{P}}\{1,3\}$ is nonempty. In this case, the sets $A_{\mathbb{P}}\{1\}$ and $A_{\mathbb{P}}\{1,3\}$ are identical, however the sets $A\{1\}$ and $A\{1,3\}$ are different.*

Next, we will characterize the $\{1,3\}$-inverses of full matrix $(1_{mn_1} | -1_{mn_2})$. Subsequently, in the following corollary, the number of $\{3\}$ and $\{1,3\}$-Bohemian inverses are found for the same matrix.

Theorem 3.6. *If $A = (1_{mn_1} | -1_{mn_2})$ satisfies $n_1, n_2 \neq 0$ and $n_1 + n_2 = n$, then it follows*

1. $A\{3\} = \left\{ X = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} \in \mathbb{K}^{(n_1+n_2)\times m} : \alpha_1 = \cdots = \alpha_m \right\}.$
2. $A\{1,3\} = \left\{ X = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} \in \mathbb{K}^{(n_1+n_2)\times m} : \alpha_1 = \cdots = \alpha_m = \frac{1}{m} \right\},$

where $\alpha_i := c_i^1 - c_i^2$, $i \in \{1, 2, \ldots, m\}$ and c_i^j represents the sum of elements of ith column in X^j.

Corollary 3.7. *For $A = (1_{mn_1} | -1_{mn_2})$ such that $n_1, n_2 \neq 0$ and $n_1 + n_2 = n$. If $\mathbb{P} = \{-1, 0, 1\}$*

1. $\#(A_{\mathbb{P}}\{1,3\}) = \begin{cases} F_1, & \text{if } m = 1 \\ 0, & \text{otherwise}; \end{cases}$
2. $\#(A_{\mathbb{P}}\{3\}) = \sum_{k=-n}^{n} (F_k)^m,$

where $F_k = \sum_{r_2=0}^{n_2} \sum_{s_2=0}^{n_2} \sum_{s_1=0}^{\lfloor \frac{n_1+n_2-k}{2} - r_2 \rfloor} \binom{n_1}{s_1}\binom{n_2}{s_2}\binom{n_2-s_2}{r_2}\binom{n_1-s_1}{r_2-s_2+s_1+k}$.

Proof. 1. The set is nonempty for $m = 1$, since $\frac{1}{m} \in \mathbb{N}$. So, X is a column vector, and we need $\Sigma(X^1) - \Sigma(X^2) = 1$; hence, the proof follows from [4, Corollary 2 (2)] for $m = 1$.

2. Let X_1^1 and X_1^2 denote the first column of X^1 and first column of X^2 respectively. First, we will find the number of ways $\alpha_1 = k$, i.e. $\Sigma(X_1^1) - \Sigma(X_1^2) = k$. Let $r_i = N^+(X^i)$ and $s_i = N^-(X^i)$. Hence $r_1 - s_1 - (r_2 - s_2) = k$. Also, $r_i + s_i \leq n_i$ whence $r_1 = r_2 - s_2 + s_1 + k$ and $s_1 \leq n_1 - r_1$. In conclusion $s_1 \leq \frac{n_1+n_2-k}{2} - r_2$. Now, the proof follows from [4, Corollary 2], and the number of ways $\alpha_1 = k$ is as follows:

$$F_k = \sum_{r_2=0}^{n_2} \sum_{s_2=0}^{n_2} \sum_{s_1=0}^{\lfloor \frac{n_1+n_2-k}{2} - r_2 \rfloor} \binom{n_1}{s_1}\binom{n_2}{s_2}\binom{n_2-s_2}{r_2}\binom{n_1-s_1}{k+s_1-s_2+r_2}.$$

Hence, the proposed cardinality is verified.

The proof is complete. $\square$

Remark 3.8. *Notice that for* $\mathbb{P} = \{-1, 0, 1\}$ *and* $A = (1_{mn_1} | -1_{mn_2})$, $\#A_{\mathbb{P}}\{1,3\} = \#A_{\mathbb{P}}\{1\}$ *if and only if* $A_{\mathbb{P}}\{1,3\}$ *is nonempty. In this case, the sets* $A_{\mathbb{P}}\{1\}$ *and* $A_{\mathbb{P}}\{1,3\}$ *are also identical, but the sets* $A\{1\}$ *and* $A\{1,3\}$ *are distinct.*

Theorem 3.9 accentuates $\{1,3\}$-inverses of the full matrix $\begin{pmatrix} 1_{m_1 n} \\ -1_{m_2 n} \end{pmatrix}$. In the subsequent corollary, the number of $\{3\}$ and $\{1,3\}$-Bohemian inverses are obtained for the same matrix.

Theorem 3.9. *Let* $A = \begin{pmatrix} 1_{m_1 n} \\ -1_{m_2 n} \end{pmatrix}$ *be such that* $m_1, m_2 \neq 0$ *and* $m_1 + m_2 = m$. *Then the next statements hold:*

1. $A\{3\} = \{X = \begin{pmatrix} X^1 & X^2 \end{pmatrix} \in \mathbb{K}^{n \times (m_1+m_2)} : c_i = c_j = -c_k = -c_l \ \forall i, j \in \{1, \ldots, m_1\}$ *and* $k, l \in \{m_1 + 1, \ldots, m\}\}$.
2. $A\{1,3\} = \{X = \begin{pmatrix} X^1 & X^2 \end{pmatrix} \in \mathbb{K}^{n \times (m_1+m_2)} : c_i = c_j = -c_k = -c_l = \frac{1}{m_1+m_2} \ \forall i, j \in \{1, \ldots, m_1\}$ *and* $k, l \in \{m_1 + 1, \ldots, m\}\}$.

where c_i *represents the sum of elements of ith column in X.*

Proof. 1. The following is observable:

$$AX = \begin{pmatrix} 1_{m_1 n} \\ -1_{m_2 n} \end{pmatrix} \begin{pmatrix} X^1 & X^2 \end{pmatrix} = \begin{pmatrix} 1_{m_1 n} X^1 & 1_{m_1 n} X^2 \\ -1_{m_2 n} X^1 & -1_{m_2 n} X^2 \end{pmatrix},$$
$$(AX)^* = \begin{pmatrix} (1_{m_1 n} X^1)^* & (-1_{m_2 n} X^1)^* \\ (1_{m_1 n} X^2)^* & (-1_{m_2 n} X^2)^* \end{pmatrix}.$$

Hence $(AX)^* = AX$ gives $X^1 \in (1_{m_1 n})\{3\}$, $X^2 \in (1_{m_2 n})\{3\}$ and $1_{m_1 n} X^2 = (-1_{m_2 n} X^1)^*$. This proves (1).

2. Note that $X \in A\{1,3\}$ is equivalent to $X \in A\{1\} \cap A\{3\}$. It is known that $X \in \begin{pmatrix} 1_{m_1 n} \\ -1_{m_2 n} \end{pmatrix}\{1\}$ exists if and only if $\Sigma(X^1) - \Sigma(X^2) = 1$ by Lemma 1 and Theorem 2 of [4]. For $X \in A\{1,3\}$, the identity $\Sigma(X^1) - \Sigma(X^2) = 1$ implies
$$\sum_{i=1}^{m_1} c_i - \sum_{j=m_1+1}^{m} c_j = 1.$$
In conclusion $c_i = \frac{1}{m_1+m_2}$ for $i \in \{1, \ldots, m_1\}$, which proves (2).
$\square$

Corollary 3.10. *For* $A = \begin{pmatrix} 1_{m_1 n} \\ -1_{m_2 n} \end{pmatrix}$ *such that* $m_1, m_2 \neq 0$ *and* $m_1 + m_2 = m$. *If* $\mathbb{P} = \{-1, 0, 1\}$

1. $\#(A_{\mathbb{P}}\{1,3\}) = 0$.
2. $\#(A_{\mathbb{P}}\{3\}) = \displaystyle\sum_{k=-n}^{n} \left(\sum_{s=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{s} \binom{n-s}{s+k} \right)^m.$

Proof. 1. For $\frac{1}{m_1+m_2} \in \mathbb{N}$, $m_1 + m_2 = 1$ where $m_1, m_2 \in \mathbb{N}$. There are no such possibilities, hence the result.
2. Let c_i represents the sum of elements of ith column in X. Let $c_i = k$, $\forall i \in \{1, 2, \ldots, m_1\}$, hence $c_j = -k$, $\forall j \in \{m_1 + 1, \ldots, m\}$. Clearly, $k \in [-n, n]$. To begin with, we will find the number of ways $c_1 = k$. Let X_{11} denote the first column of X. Let $s = N^-(X_{11})$, hence $N^+(X_{11}) = s + k$. Clearly, $s + s + k \leq n$ which gives $s \leq \frac{n-k}{2}$. Hence, the number of ways $c_1 = k$ is
$$S_1 = \sum_{s=0}^{\lfloor \frac{n-k}{2} \rfloor} \binom{n}{s} \binom{n-s}{s+k}.$$

Similarly, the number of ways $c_i = -k$ for some fixed i is:

$$S_2 = \sum_{s=0}^{\lfloor \frac{n-k}{2} \rfloor} \binom{n}{s}\binom{n-s}{s+k}.$$

It is easy to see that $S_1 = S_2$ and $\#(A_{\mathbb{P}}\{3\}) = \sum_{k=-n}^{n}(S_1)^{m_1}(S_2)^{m_2} = \sum_{k=-n}^{n}(S_1)^{m}$. This proves (2). □

Remark 3.11. *Note that by Corollary 3.4 and 3.10, for* $A = (1_{mn})$ *and* $B = \begin{pmatrix} 1_{m_1 n} \\ -1_{m_2 n} \end{pmatrix}$*,* $\#A_{\mathbb{P}}\{3\} = \#B_{\mathbb{P}}\{3\}$*, where* $m_1 + m_2 = m$*, yet the elements of the sets are different and the sets are distinct. For instance, let* $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}$*. Let* $X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix}$*,* $Y = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}$ *and* $Z = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}$*. By Theorem 3.2 and 3.9,* $X \in A_{\mathbb{P}}\{3\}$*,* $Y \in B_{\mathbb{P}}\{3\}$ *and* $Z \in A_{\mathbb{P}}\{3\} \cap B_{\mathbb{P}}\{3\}$ *while* $X, Y \notin A_{\mathbb{P}}\{3\} \cap B_{\mathbb{P}}\{3\}$*.*

Introducing some notations with the assumption that $\mathbb{P} = \{-1, 0, 1\}$:

1. $G_1(n,k) := \#\left\{X \in \mathbb{P}^{n\times 1} | \ \Sigma(X) = k\right\}$,
2. $G_2(n_1, n_2, k) := \#\left\{X = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} \in \mathbb{P}^{(n_1+n_2)\times 1} | \ \Sigma(X^1) - \Sigma(X^2) = k\right\}$,

where $n_1 + n_2 = n$ and $k \in [-n, n]$.

Lemma 3.12. *For* $X \in \mathbb{K}^{n\times 1}$*,* $\mathbb{P} \subseteq \mathbb{K}$*, such that* $\mathbb{P} = \{-1, 0, 1\}$ *then*

1. $G_1(n,k) = \sum_{s_1=0}^{\lfloor \frac{n-k}{2} \rfloor} \binom{n}{s_1}\binom{n-s_1}{k+s_1}$.
2. $G_2(n_1, n_2, k) = \sum_{r_2=0}^{n_2}\sum_{s_2=0}^{n_2}\sum_{s_1=0}^{\lfloor \frac{n_1+n_2-k}{2} - r_2 \rfloor} \binom{n_1}{s_1}\binom{n_2}{s_2}\binom{n_2-s_2}{r_2}\binom{n_1-s_1}{r_2-s_2+s_1+k}$.

The $\{1,3\}$-inverses of the full matrix $\begin{pmatrix} 1_{m_1 n} \\ 0_{m_2 n} \end{pmatrix}$ are described by the following theorem. The number of $\{3\}$ and $\{1,3\}$-Bohemian inverses for the same matrix are then obtained in the subsequent corollary.

Theorem 3.13. *If* $A = \begin{pmatrix} 1_{m_1 n} \\ 0_{m_2 n} \end{pmatrix}$ *such that* $m_1 + m_2 = m$*.*

1. $A\{3\} = \{X = \begin{pmatrix} X^1 & X^2 \end{pmatrix} \in \mathbb{K}^{n\times(m_1+m_2)} : c_i = c_j, c_k = 0 \ \forall i, j \in \{1, 2, \ldots, m_1\}, k \in \{m_1+1, \ldots, m\}\}$
2. $A\{1,3\} = \{X = \begin{pmatrix} X^1 & X^2 \end{pmatrix} \in \mathbb{K}^{n\times(m_1+m_2)} : c_i = c_j = \frac{1}{m_1}, c_k = 0$ $\forall i, j \in \{1, 2, \ldots, m_1\}, k \in \{m_1+1, \ldots, m\}\}$.

where c_i *represents the sum of elements of ith column in X.*

Proof. The proof follows from Lemma 2.3 and Theorem 3.2. □

Corollary 3.14. *For* $A = \begin{pmatrix} 1_{m_1 n} \\ 0_{m_2 n} \end{pmatrix}$ *such that* $m_1 + m_2 = m$*. If* $\mathbb{P} = \{-1, 0, 1\}$*.*

1. $\#(A_{\mathbb{P}}\{1,3\}) = \begin{cases} \#((1_{m_1 n})\{1,3\})S, & \textit{if } m_1 = 1, \\ 0 & \textit{otherwise}, \end{cases}$

2. $\#(A_{\mathbb{P}}\{3\}) = \#((1_{m_1 n})\{3\})S$,

where $S = \left(\sum_{s=0}^{\frac{n}{2}} \binom{n}{s}\binom{n-s}{s}\right)^{m_2}$.

Theorem 3.15. *If* $A = \begin{pmatrix} 1_{m_1 n} \\ -1_{m_2 n} \\ 0_{m_3 n} \end{pmatrix}$ *such that* $m_1, m_2, m_3 \neq 0$ *and* $m_1 + m_2 + m_3 = m$.

1. $A\{3\} = \{X = \begin{pmatrix} X^1 & X^2 & X^3 \end{pmatrix} \in \mathbb{K}^{n\times(m_1+m_2+m_3)} : c_i = c_j = -c_k = -c_l, c_t = 0,$
   $\forall i, j \in \{1, 2, \ldots, m_1\};\ k, l \in \{m_1 + 1, \ldots, m_1 + m_2\}$ *and* $t \in \{m_1 + m_2 + 1, \ldots, m\}\}$,
2. $A\{1, 3\} = \{X = \begin{pmatrix} X^1 & X^2 & X^3 \end{pmatrix} \in \mathbb{K}^{n\times(m_1+m_2+m_3)} : c_i = c_j = -c_k = -c_l = \frac{1}{m_1+m_2}, c_t = 0$
   $\forall i, j \in \{1, 2, \ldots, m_1\};\ k, l \in \{m_1 + 1, \ldots, m_2\}$ *and* $t \in \{m_2 + 1, \ldots, m\}\}$.

where c_i *represents the sum of elements of ith column in* X.

Proof. The proof follows from Lemma 2.3 and Theorem 3.9. □

Corollary 3.16. *For* $A = \begin{pmatrix} 1_{m_1 n} \\ -1_{m_2 n} \\ 0_{m_3 n} \end{pmatrix}$ *such that* $m_1 + m_2 + m_3 = m$. *If* $\mathbb{P} = \{-1, 0, 1\}$.

1. $\#(A_{\mathbb{P}}\{1, 3\}) = \#\left(\begin{pmatrix} 1_{m_1 n} \\ -1_{m_2 n} \end{pmatrix}_{\mathbb{P}} \{1, 3\}\right) S = 0$,
2. $\#(A_{\mathbb{P}}\{3\}) = \#\left(\begin{pmatrix} 1_{m_1 n} \\ -1_{m_2 n} \end{pmatrix}_{\mathbb{P}} \{3\}\right) S$,

where $S = \left(\sum_{s=0}^{\frac{n}{2}} \binom{n}{s}\binom{n-s}{s}\right)^{m_3}$.

Theorem 3.17. *If* $A = \begin{pmatrix} B_{m_1 n} \\ 0_{m_2 n} \end{pmatrix}$ *such that* $m_1 + m_2 = m$ *and* $B = (1_{m_1 n_1} | -1_{m_1 n_2})$ *such that* $n_1 + n_2 = n$.

1. $A\{3\} = \{X = \begin{pmatrix} X^1 & X^2 \end{pmatrix} \in \mathbb{K}^{n\times(m_1+m_2)} : X^1 \in B\{3\}, \alpha_k = 0\ \forall k \in \{m_1 + 1, \ldots, m\}\}$,
2. $A\{1, 3\} = \{X = \begin{pmatrix} X^1 & X^2 \end{pmatrix} \in \mathbb{K}^{n\times(m_1+m_2)} : X^1 \in B\{1, 3\}, \alpha_k = 0\ \forall k \in \{m_1 + 1, \ldots, m\}\}$,

where $X^j = \begin{pmatrix} X^{j1} \\ X^{j2} \end{pmatrix} \in \mathbb{K}^{(n_1+n_2)\times m_j}$, $\alpha_i = c_i^{j1} - c_i^{j2}$ *and* c_i^k *represents the sum of elements of ith column in* X^k *for suitable* $k, i \in \{1, 2, \ldots, m\}$ *and* $j \in \{1, 2\}$.

Proof. The proof follows from Lemma 2.3 and Theorem 3.6. □

Corollary 3.18. *For* $A = \begin{pmatrix} B_{m_1 n} \\ 0_{m_2 n} \end{pmatrix}$ *such that* $m_1 + m_2 = m$ *and* $B = (1_{m_1 n_1} | -1_{m_1 n_2})$ *such that* $n_1 + n_2 = n$.
In the case $\mathbb{P} = \{-1, 0, 1\}$ *it follows*

1. $\#(A_{\mathbb{P}}\{1, 3\}) = \begin{cases} \#(B_{\mathbb{P}}\{1, 3\})T & \text{if } m_1 = 1 \\ 0 & \text{otherwise;} \end{cases}$
2. $\#(A_{\mathbb{P}}\{3\}) = \#(B_{\mathbb{P}}\{3\})T$.

where $T = (G_2(n_1, n_2, k = 0))^{m_2}$.

In conclusion, for population $\mathbb{P} = \{-1, 0, 1\}$ we note that inner Bohemian inverse exists for all full matrices(refer to [4]) hence it exists for all rank-one Bohemian matrices as well, which is not the case for $\{1, 3\}$-Bohemian inverses. For instance, by considering the above results, $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ doesn't have a $\{1, 3\}$-Bohemian inverse. Hence it is concluded that the type of full matrices for which the set $A_{\mathbb{P}}\{1, 3\}$ is non-empty are the following:

1. (1_{1n});
2. $(1_{1n_1} | -1_{1n_2})$;
3. $\begin{pmatrix} 1_{1n} \\ 0_{m_2 n} \end{pmatrix}$;
4. $\begin{pmatrix} B_{1n} \\ 0_{m_2 n} \end{pmatrix}$ where $B = (1_{1n_1} | -1_{1n_2})$.

Following are the cardinality tables for some of the matrices using the formulas obtained in this section and by [4].

Table 1: Total cases of $n \times (n-1)$ full matrices of type I over $\mathbb{P} = \{-1, 0, 1\}$.

$n \times (n-1)$	#(Inner Bohemians)	#({3}-Bohemians)	#(Bohemians)
2×1	2	3	9
3×2	126	45	729
4×3	69576	5157	531441
5×4	363985680	4775301	3486784401
6×5	17812283544870	35684631553	205891132094649
7×6	806,9792,560,277,356,314	2213589290816289	109418989131512359209
8×7	33,609,055,109,399,933,461,665,528	1147756729871238625995	523347633027360537213511521

Table 2: Case of full matrices of type II with $\mathbb{P} = \{-1, 0, 1\}$ size $n \times ((n-1) + (n-2))$.

$n \times ((n-1) + (n-2))$	#(Inner Bohemians)	#({3}-Bohemians)	#(Bohemians)
$3 \times (2+1)$	2907	831	19683
$4 \times (3+2)$	363985680	16688953	3486784401
$5 \times (4+3)$	4024604728349450	23857704965727	50031545098999707
$6 \times (5+4)$	3800557141293418496841798	2491911937232990527809	58149737003040059690390169

4. {1, 3}-Bohemian inverse of well-settled matrices

Current section analyses $\{1, 3\}$-Bohemian inverses for a wider class of Bohemian matrices, particularly for well-settled matrices as defined in [4]. To begin with, we give the following auxiliary lemma, which establishes conditions for several matrix equalities by using the full matrices.

Lemma 4.1. *If $A \in \mathbb{K}^{n \times r}$ and $B \in \mathbb{K}^{s \times m}$, it follows*

1. *If $((1_{mn})A_{nr})^* = 1_{rs}B_{sm}$ then $a_1 = a_2 = \cdots = a_r = b_1 = b_2 = \cdots = b_m$.*
2. *$((1_{mn})A_{nr})^* = (1_{rs_1} | -1_{rs_2})B_{sm}$ then*
   $a_1 = a_2 = \cdots = a_r = \beta_1 = \beta_2 = \cdots = \beta_m$.
3. *If $((1_{mn})A_{nr})^* = \begin{pmatrix} 1_{r_1 s} \\ 0_{r_2 s} \end{pmatrix} \begin{pmatrix} B_{1s \times m_1} & B_{2s \times m_2} \end{pmatrix}$ then*
   $a_1 = a_2 = \cdots = a_{r_1} = b_1 = b_2 = \cdots = b_m$, $a_{r_1+1} = a_{r_1+2} = \cdots = a_r = 0$.
4. *If $((1_{mn})A_{nr})^* = \begin{pmatrix} V_{r_1 s} \\ 0_{r_2 s} \end{pmatrix} B_{sm}$, where $V_{r_1 s} = (1_{r_1 s_1} | -1_{r_1 s_2})$ such that $s_1 + s_2 = s$ and $r_1 + r_2 = r$ then $a_1 = a_2 = \cdots = a_{r_1} = \beta_1 = \beta_2 = \cdots = \beta_m$, $a_{r_1+1} = a_{r_1+2} = \cdots = a_r = 0$.*

5. *If* $\left((1_{mn_1}|-1_{mn_2})\begin{pmatrix}A_{n_1r}\\A_{n_2r}\end{pmatrix}\right)^* = (1_{rs_1}|-1_{rs_2})\begin{pmatrix}B_{s_1m}\\B_{s_2m}\end{pmatrix}$ *then* $\alpha_1 = \alpha_2 = \cdots = \alpha_r = \beta_1 = \beta_2 = \cdots = \beta_m$.
6. *If* $\left((1_{mn_1}|-1_{mn_2})\begin{pmatrix}A_{n_1r}\\A_{n_2r}\end{pmatrix}\right)^* = \begin{pmatrix}1_{r_1s}\\0_{r_2s}\end{pmatrix}\begin{pmatrix}B_{1s\times m_1} & B_{2s\times m_2}\end{pmatrix}$ *then* $\alpha_1 = \alpha_2 = \cdots = \alpha_{r_1} = b_1 = b_2 = \cdots = b_m$, $\alpha_{r_1+1} = \alpha_{r_1+2} = \cdots = \alpha_r = 0$.
7. *If* $((1_{mn_1|-1_{mn_2}})A_{nr})^* = \begin{pmatrix}V_{r_1s}\\0_{r_2s}\end{pmatrix}B_{sm}$, *where* $V_{r_1s} = (1_{r_1s_1}|-1_{r_1s_2})$ *such that* $s_1+s_2 = s$ *and* $r_1+r_2 = r$ *then* $\alpha_1 = \alpha_2 = \cdots = \alpha_{r_1} = \beta_1 = \beta_2 = \cdots = \beta_m$, $\alpha_{r_1+1} = \alpha_{r_1+2} = \cdots = \alpha_r = 0$.
8. *If* $\left(\begin{pmatrix}1_{m_1n}\\0_{m_2n}\end{pmatrix}A_{nr}\right)^* = \begin{pmatrix}1_{r_1s}\\0_{r_2s}\end{pmatrix}\begin{pmatrix}B_{1s\times m_1} & B_{2s\times m_2}\end{pmatrix}$ *then* $a_1 = a_2 = \cdots = a_{r_1} = b_1 = b_2 = \cdots = b_{m_1}$, $a_{r_1+1} = a_{r_1+2} = \cdots = a_r = 0 = b_{m_1+1} = b_{m_1+2} = \cdots = b_m$.
9. *If* $\left(\begin{pmatrix}1_{m_1n}\\0_{m_2n}\end{pmatrix}A_{nr}\right)^* = \begin{pmatrix}V_{r_1s}\\0_{r_2s}\end{pmatrix}B_{sm}$, *where* $V_{r_1s} = (1_{r_1s_1}|-1_{r_1s_2})$ *such that* $s_1+s_2 = s$ *and* $r_1+r_2 = r$ *then* $a_1 = a_2 = \cdots = a_{r_1} = \beta_1 = \beta_2 = \cdots = \beta_{m_1}$, $a_{r_1+1} = a_{r_1+2} = \cdots = a_r = 0 = \beta_{m_1+1} = \beta_{m_1+2} = \cdots = \beta_m$.
10. *If* $\left(\begin{pmatrix}V_{m_1n}\\0_{m_2n}\end{pmatrix}A_{nr}\right)^* = \begin{pmatrix}V_{r_1s}\\0_{r_2s}\end{pmatrix}B_{sm}$, *where* $V_{r_1s} = (1_{r_1s_1}|-1_{r_1s_2})$ *such that* $m_1+m_2 = m$, $s_1+s_2 = s$ *and* $r_1+r_2 = r$ *then* $\alpha_1 = \alpha_2 = \cdots = \alpha_{r_1} = \beta_1 = \beta_2 = \cdots = \beta_{m_1}$, $\alpha_{r_1+1} = \alpha_{r_1+2} = \cdots = \alpha_r = 0 = \beta_{m_1+1} = \beta_{m_1+2} = \cdots = \beta_m$,

where a_i *and* b_i *represents the sum of elements in ith columns of the matrix A and B respectively while* α_i *and* β_i *are defined for ith column of partitioned matrices* $A = \begin{pmatrix}A^1\\A^2\end{pmatrix}$ *and* $B = \begin{pmatrix}B^1\\B^2\end{pmatrix}$ *respectively as* $a_i^1 - a_i^2$ *and* $b_i^1 - b_i^2$ *where* a_i^j *and* b_i^j *represents the sum of entries of ith column of* A^j *and* B^j *respectively.*

Further, we analyze the $\{1,3\},\{3\}$-inverses of pure well-settled matrices in the next four theorems. For the following results, we define some notations. The notation c_i^{jk} will denote the sum of ith column of $X^{jk} \in \mathbb{K}^{q_j\times p_k}$. The notation α_i^{jk} is defined for ith column of partitioned matrix $X^{jk} = \begin{pmatrix}Y^1\\Y^2\end{pmatrix}$ and $\alpha_i^{jk} := c_i^1 - c_i^2$ where c_i^l represents the sum of ith column of Y^l.

Theorem 4.2 ($\{1,3\},\{3\}$-inverses of ($\pm 1_{mn}$)-pure well-settled matrices). *Let* $A = \begin{pmatrix}A_1 & 0 & 0\\0 & \ddots & 0\\0 & 0 & A_s\end{pmatrix}$, $A_i = \epsilon_i 1_{p_iq_i} \in \mathbb{K}^{p_i\times q_i}$, *where* $\epsilon_i \in \{-1,1\}$, *then*

1. $A\{3\} = \left\{X = \begin{pmatrix}X^{11} & \dots & X^{1s}\\ \vdots & \ddots & \vdots\\ X^{s1} & \dots & X^{ss}\end{pmatrix} \in \mathbb{K}^{n\times m} : X^{ii} \in (\epsilon_i 1_{p_iq_i})\{3\}, \epsilon_j c_i^{jk} = \epsilon_k c_i^{kj}, c_i^{jk} = c_l^{jk}\ \forall\ j \neq k, j > k, \forall i,l\right\}$, *where* $X^{ij} \in \mathbb{K}^{q_i\times p_j}$.
2. $A\{1,3\} = \left\{X = \begin{pmatrix}X^{11} & \dots & X^{1s}\\ \vdots & \ddots & \vdots\\ X^{s1} & \dots & X^{ss}\end{pmatrix} \in \mathbb{K}^{n\times m} : X^{ii} \in (\epsilon_i 1_{p_iq_i})\{1,3\}, c_k^{ij} = 0\ \forall\ i \neq j \forall k\right\}$, *where* $X^{ij} \in \mathbb{K}^{q_i\times p_j}$.

Proof. The proof follows easily by Lemma 4.1(1). □

Theorem 4.3 ($\{1,3\},\{3\}$-inverses of ($\pm 1_{mn_1}|\mp 1_{mn_2}$)-pure well-settled matrices.). *Let* $A = \begin{pmatrix}A_1 & 0 & 0\\0 & \ddots & 0\\0 & 0 & A_s\end{pmatrix}$, $A_i = \epsilon_i V_{p_iq_i} \in \mathbb{K}^{p_i\times q_i}$, *where* $\epsilon_i \in \{-1,1\}$ *and* $V_{p_i,q_i} = (1_{p_iq_{i1}}|-1_{p_iq_{i2}})$ *then*

1. $A\{3\} = \left\{ X = \begin{pmatrix} X^{11} & \dots & X^{1s} \\ \vdots & \ddots & \vdots \\ X^{s1} & \dots & X^{ss} \end{pmatrix} \in \mathbb{K}^{n\times m} : X^{ii} \in (\epsilon_i V_{p_i q_i})\{3\}, \epsilon_j \alpha_i^{jk} = \epsilon_k \alpha_i^{kj}, \alpha_i^{jk} = \alpha_l^{jk}\ \forall\ j \neq k, j > k \right\}$,

2. $A\{1,3\} = \left\{ X = \begin{pmatrix} X^{11} & \dots & X^{1s} \\ \vdots & \ddots & \vdots \\ X^{s1} & \dots & X^{ss} \end{pmatrix} \in \mathbb{K}^{n\times m} : X^{ii} \in (\epsilon_i V_{p_i q_i})\{1,3\}, \alpha_k^{ij} = 0\ \forall k,\ \forall\ i \neq j \right\}$,

where $X^{ij} \in \mathbb{K}^{q_i \times p_j}$ *and* $X^{ij} = \begin{pmatrix} X^{i_1 j} \\ X^{i_2 j} \end{pmatrix}$ *such that* $q_{i_1} + q_{i_2} = q_i$.

Proof. The proof follows by Lemma 4.1(5). □

Theorem 4.4 ($\{1,3\}, \{3\}$-inverses of $\begin{pmatrix} \pm 1_{m_1 n} \\ 0 \end{pmatrix}$-pure well-settled matrices.). *Let* $A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_s \end{pmatrix}$, $A_i = \epsilon_i U_{p_i q_i} \in \mathbb{K}^{p_i \times q_i}$, *where* $U_{p_i q_i} = \begin{pmatrix} 1_{p_{i_1} q_i} \\ 0_{p_{i_2} q_i} \end{pmatrix}$ *then*

1. $A\{3\} = \left\{ X = \begin{pmatrix} X^{11} & \dots & X^{1s} \\ \vdots & \ddots & \vdots \\ X^{s1} & \dots & X^{ss} \end{pmatrix} \in \mathbb{K}^{n\times m} : X^{ii} \in (A_i)\{3\}, \right.$,
$\left. c_k^{ij_2} = c_l^{ji_2} = 0, \epsilon_j c_k^{ji_1} = \epsilon_i c_k^{ij_1}, c_k^{ji_1} = c_l^{ji_1}, c_k^{ij_1} = c_l^{ij_1}\ \forall\ i \neq j, i > j \forall k, l \right\}$

2. $A\{1,3\} = \left\{ X = \begin{pmatrix} X^{11} & \dots & X^{1s} \\ \vdots & \ddots & \vdots \\ X^{s1} & \dots & X^{ss} \end{pmatrix} \in \mathbb{K}^{n\times m} : X^{ii} \in (A_i)\{1,3\}, c_i^{jk} = 0\ \forall\ j \neq k \forall i \right\}$.

where $X^{ij} = (X^{ij_1} | X^{ij_2})$.

Proof. It follows from Lemma 4.1(8). □

Theorem 4.5 ($\{1,3\}, \{3\}$-inverses of $\begin{pmatrix} \pm 1_{m_1 n_1} & \mp 1_{m_1 n_2} \\ 0 & 0 \end{pmatrix}$-pure well-settled matrices.). *Let* $A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_s \end{pmatrix}$, $A_i = \epsilon_i \begin{pmatrix} B_i \\ 0 \end{pmatrix} \in \mathbb{K}^{(p_{i_1}+p_{i_2}) \times q_i}$ *such that* $m_1 + m_2 = m$ *and* $B_i = (1_{p_{i_1} q_{i_1}} | -1_{p_{i_1} q_{i_2}})$ *such that* $p_{i_1} + p_{i_2} = p_i$, $q_{i_1} + q_{i_2} = q_i$ *and* $\mathbb{P} \subseteq \mathbb{K}$. *Then*

1. $A\{3\} = \left\{ X = \begin{pmatrix} X^{11} & \dots & X^{1s} \\ \vdots & \ddots & \vdots \\ X^{s1} & \dots & X^{ss} \end{pmatrix} \in \mathbb{P}^{n\times m} : X^{ii} \in (A_i)\{3\}, \right.$
$\left. \alpha_k^{ij_2} = \alpha_l^{ji_2} = 0, \epsilon_j \alpha_k^{ji_1} = \epsilon_i \alpha_k^{ij_1}, \alpha_k^{ji_1} = \alpha_l^{ji_1}, \alpha_k^{ij_1} = \alpha_l^{ij_1}\ \forall\ i \neq j, i > j \forall k, l \right\}$

2. $A\{1,3\} = \left\{ X = \begin{pmatrix} X^{11} & \dots & X^{1s} \\ \vdots & \ddots & \vdots \\ X^{s1} & \dots & X^{ss} \end{pmatrix} \in \mathbb{P}^{n\times m} : X^{ii} \in (A_i)\{1,3\}, \alpha_k^{ij} = 0\ \forall\ i \neq j \forall k \right\}$.

where $X^{ij} = (X^{ij_1} | X^{ij_2}) = \begin{pmatrix} X^{i_1 j_1} & X^{i_1 j_2} \\ X^{i_2 j_1} & X^{i_2 j_1} \end{pmatrix}$. *Also,* $\alpha_k^{ij_1} := c_k^{i_1 j_1} - c_k^{i_2 j_1}$, $\alpha_k^{ji_1} := c_k^{j_1 i_1} - c_k^{j_2 i_1}$, $\alpha_k^{ij_2} := c_k^{i_1 j_2} - c_k^{i_2 j_2}$ *and* $\alpha_k^{ji_2} := c_k^{j_1 i_2} - c_k^{j_2 i_2}$.

Proof. It follows from Lemma 4.1(10). □

In the subsequent result, we will give the cardinalities of the $\{1,3\}$-Bohemian inverses for the pure, well-settled matrices.

Theorem 4.6. *Let* $A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_s \end{pmatrix}$*, and* $\mathbb{P} = \{0, \pm 1\}$ *then*

1. *If* $A_i = \epsilon_i 1_{p_i q_i} \in \mathbb{K}^{p_i \times q_i}$*, where* $\epsilon_i \in \{-1, 1\}$*, then*

$$\#A_{\mathbb{P}}\{1,3\} = \begin{cases} \prod_{i=1}^{s} \#(A_i)_{\mathbb{P}}\{1,3\} \prod_{i \in \{1,\dots,s\}} (G_1(q_i, 0))^{s-1} & \text{if } p_i = 1 \forall i, \\ 0 & \text{otherwise.} \end{cases}$$

2. *If* $A_i = \epsilon_i V_{p_i q_i} \in \mathbb{K}^{p_i \times q_i}$*, where* $\epsilon_i \in \{-1, 1\}$ *and* $V_{p_i, q_i} = (1_{p_i q_{i_1}} | - 1_{p_i q_{i_2}})$*, then*

$$\#A_{\mathbb{P}}\{1,3\} = \begin{cases} \prod_{i=1}^{s} \#(A_i)_{\mathbb{P}}\{1,3\} \prod_{i \in \{1,\dots,s\}} (G_2(q_{i_1}, q_{i_2}, 0))^{s-1} & \text{if } p_i = 1 \forall i, \\ 0 & \text{otherwise.} \end{cases}$$

3. *If* $A_i = \epsilon_i U_{p_i q_i} \in \mathbb{K}^{p_i \times q_i}$*, where* $U_{p_i q_i} = \begin{pmatrix} 1_{p_{i_1} q_i} \\ 0_{p_{i_2} q_i} \end{pmatrix}$ *then*

$$\#A_{\mathbb{P}}\{1,3\} = \begin{cases} \prod_{i=1}^{s} \#(A_i)_{\mathbb{P}}\{1,3\} \prod_{i \in \{1,\dots,s\}} (G_1(q_i, 0))^{s-1} & \text{if } p_{i_1} = 1 \forall i, \\ 0 & \text{otherwise.} \end{cases}$$

4. *If* $A_i = \epsilon_i \begin{pmatrix} B_i \\ 0 \end{pmatrix} \in \mathbb{K}^{(p_{i_1} + p_{i_2}) \times q_i}$ *and* $B_i = (1_{p_{i_1} q_{i_1}} | -1_{p_{i_1} q_{i_2}})$ *such that* $p_{i_1} + p_{i_2} = p_i$ *and* $q_{i_1} + q_{i_2} = q_i$ *then*

$$\#A_{\mathbb{P}}\{1,3\} = \begin{cases} \prod_{i=1}^{s} \#(A_i)_{\mathbb{P}}\{1,3\} \prod_{i \in \{1,\dots,s\}} (G_2(q_{i_1}, q_{i_2}, 0))^{s-1} & \text{if } p_{i_1} = 1 \forall i, \\ 0 & \text{otherwise.} \end{cases}$$

Proof.

1. Let $X \in A\{1,3\}$, according to Theorem 4.2, $X^{ii} \in A_i\{1,3\}$ hence by Corollary 3.4, $p_i = 1$ for all $i \in \{1, 2, \dots, s\}$ whence all X^{ij} are of the form of column vector. Number of ways the diagonal blocks of X can be written are

$$\prod_{i=1}^{s} \#(A_i)_{\mathbb{P}}\{1,3\}.$$

For $i \neq j$, number of ways $c_k^{ij} = 0$ for $j \in \{1, 2, \dots, s\}$ is $(G_1(q_1, 0))^{s-1}$. As a consequence, the outcome is straightforward to determine.

2. By Theorem 4.3 and Corollary 3.7, for the diagonal blocks, the proof is the same as the previous part. Again, the X^{ij} are column vectors. Besides that for any fixed i and j such that $i \neq j$, X^{ij} is partitioned as $X^{ij} = \begin{pmatrix} X^{i_1 j} \\ X^{i_2 j} \end{pmatrix}$ such that $q_{i_1} + q_{i_2} = q_i$. Then, the proof follows straightforwardly.
3. The proof is similar to that of the first part.
4. The proof follows similarly as for the second part.

□

In Theorem 4.7, we analyze mixed well-settled matrices.

Theorem 4.7 ($\{1,3\}$-Bohemian inverses of mixed well-settled matrices.). *Let* $A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_s \end{pmatrix}$, $A_i \in \mathbb{K}^{p_i \times q_i}$ *are full-matrices, and* $\mathbb{P} \subseteq \mathbb{K}$. *Possible choices for* A_i *are*
$\pm 1_{p_i q_i}$, $\pm(1_{p_i q_{i1}} | -1_{p_i q_{i2}})$, $\pm \begin{pmatrix} 1_{p_{i_1} q_i} \\ 0 \end{pmatrix}$, $\pm \begin{pmatrix} B_{p_{i_1} q_i} \\ 0_{p_{i_2} q_i} \end{pmatrix}$,
where $B = (1_{p_{i_1} q_{i_1}} | -1_{p_{i_1} q_{i_2}})$. *Then*

1. $A\{1,3\} = \left\{ X = \begin{pmatrix} X^{11} & \cdots & X^{1s} \\ \vdots & \ddots & \vdots \\ X^{s1} & \cdots & X^{ss} \end{pmatrix} \in \mathbb{P}^{n \times m} : X^{ii} \in (A_i)\{1,3\}, (A_i X^{ij})^* = A_j X^{ji} \right\}$.

For $i \neq j$, $i > j$, X^{ij} *and* X^{ji} *satisfies*

1. *If* $A_i = \pm 1_{p_i q_i}, A_j = \pm 1_{p_j q_j}$ *or* $\pm \begin{pmatrix} 1_{p_{j_1} q_j} \\ 0 \end{pmatrix}$; $A_i = \pm \begin{pmatrix} 1_{p_{i_1} q_i} \\ 0 \end{pmatrix}, A_j = \pm \begin{pmatrix} 1_{p_{j_1} q_j} \\ 0 \end{pmatrix}$,
   then $c_k^{ij} = 0\ \forall k, c_k^{ji} = 0\ \forall k$.
2. *If* $A_i = \pm 1_{p_i q_i}, A_j = \pm(1_{p_j q_{j_1}} | -1_{p_j q_{j_2}})$ *or* $\pm \begin{pmatrix} B_{p_{j_1} q_j} \\ 0_{p_{j_2} q_j} \end{pmatrix}$; $A_i = \pm \begin{pmatrix} 1_{p_{i_1} q_i} \\ 0 \end{pmatrix}, A_j = \pm \begin{pmatrix} B_{p_{j_1} q_j} \\ 0_{p_{j_2} q_j} \end{pmatrix}$
   where $B = (1_{p_{j_1} q_{j_1}} | -1_{p_{j_1} q_{j_2}})$ *then* $c_k^{ij} = 0 \forall k, \alpha_k^{ji} = 0\ \forall k$.
3. *If* $A_i = \pm(1_{p_i q_{i_1}} | -1_{p_i q_{i_2}}), A_j = \pm(1_{p_j q_{j_1}} | -1_{p_j q_{j_2}})$ *or* $\pm \begin{pmatrix} B_{p_{j_1} q_j} \\ 0_{p_{j_2} q_j} \end{pmatrix}$; $A_i = \pm \begin{pmatrix} B_{p_{i_1} q_i} \\ 0_{p_{i_2} q_i} \end{pmatrix}, A_j = \pm \begin{pmatrix} B_{p_{j_1} q_j} \\ 0_{p_{j_2} q_j} \end{pmatrix}$
   where $B = (1_{p_{j_1} q_{j_1}} | -1_{p_{j_1} q_{j_2}})$ *then* $\alpha_k^{ij} = 0 \forall k, \alpha_k^{ji} = 0\ \forall k$.
4. *If* $A_i = \pm(1_{p_i q_{i_1}} | -1_{p_i q_{i_2}}), A_j = \pm \begin{pmatrix} 1_{p_{j_1} q_j} \\ 0 \end{pmatrix}$ *then* $\alpha_k^{ij} = 0 \forall k, c_k^{ji} = 0\ \forall k$.

Proof. The proof is trivial for block diagonal elements X^{ii}. *We derive various cases by using the equations* $AXA = A$ *and* $(AX)^* = AX$ *and equating the individual elements. For the condition of inner inverses, we rely on the outcomes acquired for inner Bohemian inverses.* □

The following result will give the cardinality of $\{1,3\}$-Bohemian inverses of mixed well-settled matrices.

Theorem 4.8 ($\{1,3\}$-Bohemian inverses of mixed well-settled matrices.). *Let* $A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_s \end{pmatrix}$, $A_i \in \mathbb{K}^{p_i \times q_i}$ *are full-matrices, and* $\mathbb{P} \subseteq \mathbb{K}$. *Let* $I, J, K, L \subset \mathbb{N}$ *with* $I \cup J \cup K \cup L = \{1, 2, ..., s\}$ *and* I, J, K, L *are disjoint sets, such that*

$$A_i = \begin{cases} \pm 1_{p_i q_i} & \text{if } i \in I \\ \pm(1_{p_i q_{i_1}} | -1_{p_i q_{i_2}}) & \text{if } i \in J \\ \pm \begin{pmatrix} 1_{p_{i_1} q_i} \\ 0 \end{pmatrix} & \text{if } i \in K \\ \pm \begin{pmatrix} B_{p_{i_1} q_i} \\ 0_{p_{i_2} q_i} \end{pmatrix} \text{ where } B = (1_{p_{i_1} q_{i_1}} | -1_{p_{i_1} q_{i_2}}) & \text{if } i \in L. \end{cases}$$

Then the cardinality $\#(A_{\mathbb{P}}\{1,3\})$ *for* $\mathbb{P} = \{-1, 0, 1\}$ *is*

1. *In the case $p_i = 1$ whenever $i \in I \cup J$ and $p_{i_1} = 1$ whenever $i \in K \cup L$ it follows*

$$\#(A_{\mathbb{P}}\{1,3\}) = \prod_{i \in I \cup J \cup K \cup L} \#(A_i)_{\mathbb{P}}\{1,3\} \prod_{\substack{i \in I, j \in I \cup K, \\ \text{or } i,j \in K, \\ i \neq j, i > j}} G_1(q_i,0)G_1(q_j,0) \prod_{\substack{i \in I, j \in J \cup L, \\ \text{or } i \in K, j \in L, i > j}} G_1(q_i,0)G_2(q_{j_1},q_{j_2},0)$$
$$\prod_{\substack{i \in J, j \in J \cup L, \\ \text{or } i,j \in L, \\ i \neq j, i > j}} G_2(q_{i_1},q_{i_2},0)G_2(q_{j_1},q_{j_2},0) \prod_{\substack{i \in J, j \in K, \\ i \neq j, i > j}} G_2(q_{i_1},q_{i_2},0)G_1(q_j,0).$$

2. *In the case $p_i \neq 1$ for some $i \in I \cup J$ or $p_{i_1} \neq 1$ for some $i \in K \cup L$ it follows $\#(A_{\mathbb{P}}\{1,3\}) = 0$.*

Proof. The first product contributes to the number of possible diagonal blocks, i.e. the X^{ii} case. It is obtained trivially. For X^{ij}, $i \neq j$, it follows from the cases obtained in Theorem 4.7. □

5. Concluding remarks

Current research studies $\{1,3\}$-Bohemian inverses of certain types of structured $\{-1,0,1\}$-matrices, especially full and well-settled matrices. The set of rank-one Bohemian matrices over the population $\mathbb{P} = \{-1,0,1\}$ is characterized and the set of full matrices is introduced. In addition, $\{1,3\}$-inverses of rank-one Bohemian matrices over $\mathbb{P}$ are studied, as well as the cardinalities of $\{1,3\}$-Bohemian inverses on full matrices. The set of $\{1,3\}$-Bohemian inverses of arbitrary rank well-settled matrices is investigated.

Further research plan is based on representations and properties of outer generalized Bohemian matrices. It is interesting to establish correlation between inner and outer Bohemian inverses over different populations and various matrix structures.

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