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New inequalities for some quadratic forms and related results

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Abstract. We prove some new Jensen type inequalities for the Berezin symbol of self-adjoint operators and some class of positive operators on the reproducing kernel Hilbert space $\mathcal{H}(\Omega)$ over some set Ω. Recall that the Berezin symbol $\tilde{A}$ of operator A on $\mathcal{H}(\Omega)$ is defined by the following special type of quadratic form: $\tilde{A}(\lambda) := \langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle, \lambda \in \Omega$, where k_λ is the reproducing kernel of the space $\mathcal{H}(\Omega)$, i.e., $f(\lambda) = \langle f, \hat{k}_\lambda\rangle$ for all $f \in \mathcal{H}(\Omega)$ and $\lambda \in \Omega$; $\hat{k}_\lambda := \frac{k_\lambda}{\|k_\lambda\|_{\mathcal{H}(\Omega)}}$ is the normalized reproducing kernel.

1. Introduction

In this paper, we prove some new Jensen type inequalities for some restricted quadratic form which are the Berezin symbol of self-adjoint operators and some positive operators acting on the reproducing kernel Hilbert spaces. Recall that the reproducing kernel Hilbert space (shortly RKHS) is the Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of complex-valued functions on some Ω such that the evaluation functionals $\varphi_\lambda(f) = f(\lambda)$, $\lambda \in \Omega$, are continuous on $\mathcal{H}$ and for every $\lambda \in \Omega$ there exist a function $f_\lambda \in \mathcal{H}$ such that $f_\lambda(\lambda) \neq 0$ or, equivalently, there is no $\lambda_0 \in \Omega$ such that $f(\lambda_0) = 0$ for all $f \in \mathcal{H}$. Then by the Riesz representation theorem for each $\lambda \in \Omega$ there exists a unique function $k_\lambda \in \mathcal{H}$ which is called the reproducing kernel of the space $\mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda\rangle$ for all $f \in \mathcal{H}$. The function $\hat{k}_\lambda := \frac{k_\lambda}{\|k_\lambda\|}, \lambda \in \Omega$, is called the normalized reproducing kernel of $\mathcal{H}$. The prototypical RKHSs are the Hardy space $H^2(\mathbb{D})$, the Bergman space $L^2_a(\mathbb{D})$, the Dirichlet space $\mathcal{D}^2(\mathbb{D})$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc and the Fock space $\mathcal{F}(\mathbb{C})$. A detailed presentation of the theory of reproducing kernels and RKHSs is given, for instance in Aronzajn [1, 23]. Reproducing kernels play important role in many branches of pure and applied mathematics including frame theory, wavelets, signals, fractals theories (see for instance, Jorgensen's book [12] and its references).

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In the present paper, we give new applications of reproducing kernels method and convex functions. Namely, we define the so-called Berezin symbol (transform) of operators on the RKHS and prove new important Jensen type inequalities for Berezin symbols of self-adjoint operators. In particular, we obtain some upper bounds for the Berezin number (or Berezin radius) of such type operators.

Note that for a bounded linear operator A on $\mathcal{H}$ (i.e., for $A \in \mathcal{B}(\mathcal{H})$ its Berezin symbol $\tilde{A}$ is defined on Ω by (see Berezin [6])

$$\tilde{A}(\lambda) := \langle A\hat{k}_\lambda(z), \hat{k}_\lambda(z)\rangle,\ \lambda \in \Omega,$$

In other words, Berezin symbol $\tilde{A}$ is the function on Ω defined by restriction of the quadratic form $\langle Ax, x\rangle$ with $x \in \mathcal{H}$ to the subset of all normalized reproducing kernels of the unit sphere in $\mathcal{H}$. It is clear from the Cauchy-Schwarz inequality that $\tilde{A}$ is the bounded function on Ω whose values lie in the numerical range of the operator A. So, the Berezin number $ber\,(A)$ and the Berezin set $Ber\,(A)$ of operator A are defined respectively by

$$ber(A) := \sup_{\lambda\in\Omega} \left|\tilde{A}(\lambda)\right|,$$

and

$$Ber(A) := Range(\tilde{A}) = \{\tilde{A}(\lambda) : \lambda \in \Omega\}.$$

It is obvious that $ber(A) \leq w(A) \leq \|A\|$ and $Ber(A) \subset W(A)$, where $w(A)$ denotes the numerical radius and $W(A)$ is the numerical range of operator A.

Moreover, the Berezin number of an operator A satisfies the following properties:

1. $ber(A) = ber(A^*)$.
2. $ber\,(\alpha A) = |\alpha| ber(A)$ for all $\alpha \in \mathbb{C}$.
3. $ber(A + B) \leq ber(A) + ber(B)$ for all $A, B \in \mathcal{B}(\mathcal{H})$.

Notice that, in general the Berezin number does not define a norm. However, if $\mathcal{H}$ is a reproducing kernel Hilbert space of analytic functions (for instance on the unit disc $\mathbb{D}$), then $ber(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H}(\mathbb{D}))$ (see [19, 24, 25]). The Berezin symbol has been studied in detail for Toeplitz and Hankel operators on Hardy and Bergman spaces. For example, the Berezin symbol $\tilde{T}_\varphi$ on the Toeplitz operator $T_\varphi(\varphi \in L^\infty(\delta\mathbb{D}))$ on $(\mathcal{H}^2(\mathcal{D})$ coincides with harmonic extension $\tilde{\varphi}$ of function φ into the unit disc $\mathbb{D}$; in particular, if $\varphi \in H^\infty(\mathbb{D})$, i.e., if the symbol function φ is a bonded analytic function on $\mathbb{D}$, then $\tilde{T}_\varphi = \varphi$. Also it is well known that the Toeplitz operator on the Bergman space $L_a^2(\mathbb{D})$ is compact if and only if its Berezin symbol $\tilde{T}_\varphi$ vanishes on the boundary $\partial\mathbb{D}$, i.e., if $\lim_{\lambda\to\mathfrak{z}} \tilde{T}_\varphi(\lambda) = 0$ for all $\mathfrak{z} \in \partial\mathbb{D}$ (see Axler and Zheng [3]). A one more nice property of the Berezin symbol is the following:

If $\tilde{A} = \tilde{B}$, then $A = B$. Therefore, the Berzein symbol uniquely determines the operator. The Berezin symbol, Berezin set and Berezin number have been investigated by many authors over the years, a few of them are [4, 5, 7–9, 13–18, 20–22, 26–30])

Now, for any operators $A \in \mathcal{B}(\mathcal{H})$, we define the following so-called Berezin norm of A:

$$\|A\|_{Ber} := \sup_{\lambda\in\Omega} \|A\hat{k}_\lambda\|.$$

Clearly

$$ber(A) \leq \|A\|_{Ber} \leq \|A\| \tag{1}$$

Since the family $\{k_\lambda : \lambda \in \Omega\}$ is complete in $\mathcal{H}$, it is elementary to verify that $\|A\|_{Ber} = 0$ if and only if $A = 0$. Then it is easy to verify that $\|A\|_{Ber}$ share the properties $(i) - (iii)$ with $ber(A)$, and hence $\|\cdot\|_{Ber}$ is the norm in $\mathcal{B}(\mathcal{H})$. Note that since

$$ber(A) \leq w(A)\|A\|_{Ber} \leq \|A\|,$$

and

$$\|A\|_{Ber} \leq \|A\|,$$

the inequality involving $ber(A) \leq \|A\|_{Ber}$ is in general better than inequality involving $ber(A) \leq \|A\|$. To the inequalities involving numerical radius $w(A)$ and an inequalities

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|,$$

are devoted many papers, see for instance, [2] and references therein.

In this paper, we prove Berezin symbols inequalities involving the Berezin number and the Berezin norms of operators. We mainly will focus to the classical Jersen type inequalities for Berezin symbols of some self-adjoint and positive operators.

2. Reverses of the Jensen inequality for Berezin symbols of functions of self-adjoint operators

Let A be a self-adjoint bounded linear operator on a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. The Gelfand map establishes a $*$-isometrically isomorphism Φ between the set $C(S_p(A))$ of all continous functions defined on the spectrum of A, denoted $S_p(A)$ and the C^*- algebra $C^*(A)$ generated by A and the identity operator I on H as follows: for any $f, g \in C(S_p(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

1. $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;
2. $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;
3. $\|\Phi(f)\| = \|f\| := \sup_{t\in S_p(A)} |f(t)|$;
4. $\Phi(f_0) = I$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in S_p(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(S_p(A)),$$

and we call it the continuous functional calculus for a self-adjoint operator A.

If A is a self-adjoint operator and f is a real valued continous function on $S_p(A)$, then $f(t) \geq 0$ for every $t \in S_p(A)$ implies that $f(A) \geq 0$ i.e., $f(A)$ is positive operator on H. Moreover, if both f and g are real valued functions on $S_p(A)$ then the following important property holds:

$$f(t) \geq g(t) \tag{2}$$

for any $t \in S_p(A)$ implies that $f(A) \geq g(A)$ in the operator order of $\mathcal{B}(H)$.

2.1. An operator version of the Dragomir-Ionescu inequality

The following result gives an operator version of the Dragomir-Ionescu inequality (see [2, 11]).

Theorem 2.1. *Let $J \subset (-\infty, \infty)$ be an interval and $f : J \to \mathbb{R}$, where $\mathbb{R} = (-\infty, \infty)$, be a convex and differentiable function on $\dot{J}$ (the interior of J) whose derivative f' is continous on $\dot{J}$. If A is a self-adjoint operator on the reproducing kernel Hilbert space $\mathcal{H}(\Omega)$ over some suitable set Ω of the complex plane $\mathbb{C}$ with $S_p(A) \subseteq [m, M] \subset \dot{J}$, then*

$$(0 \leq)\widetilde{f(A)}(\lambda) - f\left(\tilde{A}(\lambda)\right) \leq \widetilde{f'(A)A}(\lambda) - \tilde{A}(\lambda)\widetilde{f'(A)}(\lambda)$$

for all $\lambda \in \Omega$.

Proof. By considering that f is convex and differentiable, we have that

$$f(t) - f(s) \leq f'(t)(t-s) \text{ for all } t, s \in [m, M].$$

Now, if we put in this inequality $s = \tilde{A}(\lambda) = \langle A\hat{k}_\lambda, \hat{k}_\lambda \rangle \in [m, M]$ for any $\lambda \in \Omega$. Since $S_p(A) \subseteq [m, M]$, then we have

$$f(t) - f(\tilde{A}(\lambda)) \leq f'(t)(t - \tilde{A}(\lambda)) \text{ for any } t \in [m, M], \ any\ \lambda \in \Omega. \tag{3}$$

If we fix $\lambda \in \Omega$ in (3) and use the property (2) then we obtain

$$\langle [f(A) - f(\tilde{A}(\lambda))I_{\mathcal{H}}]\hat{k}_\lambda, \hat{k}_\lambda \rangle \leq \langle f'(A)(A - \tilde{A}(\lambda)I_{\mathcal{H}})\hat{k}_\lambda, \hat{k}_\lambda \rangle$$

which is clearly equivalent to the required inequality in the theorem. □

The following concrete cases are of interest.

2.2. Example

(a) Let A be a positive definite operator on the reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$. Then we have the following inequality:

$$(0 \leq)\ \ln(\tilde{A}(\lambda)) - \widetilde{\ln(A)}(\lambda) \leq \tilde{A}(\lambda)\widetilde{A^{-1}}(\lambda) - 1 \tag{4}$$

for every $\lambda \in \Omega$.

(b) If A is a self-adjoint operator on $\mathcal{H}$, then we have the inequality:

$$(0 \leq)\widetilde{\exp(A)}(\lambda) - \exp(\tilde{A}(\lambda)) \leq A\,\widetilde{\exp(A)}(\lambda) - \tilde{A}(\lambda)\widetilde{\exp(A)}(\lambda),$$

for each $\lambda \in \Omega$.

(c) If $p \geq 1$ and A is a positive operator on $\mathcal{H}$, then

$$(0 \leq)\widetilde{A^p}(\lambda) - A(\lambda)^p \leq p[\widetilde{A^p}(\lambda) - \tilde{A}(\lambda)\widetilde{A^{p-1}}(\lambda)], \tag{5}$$

for each $\lambda \in \Omega$. If A is positive definite, then the inequality (5) also holds for $p < 0$.

If $0 < p < 1$ and A is positive definite, then the reverse inequality also holds

$$\widetilde{A^p}(\lambda) - A(\lambda)^p \geq p[\widetilde{A^p}(\lambda) - \tilde{A}(\lambda)\widetilde{A^{p-1}(\lambda)}] \geq 0,$$

for each $\lambda \in \Omega$ (we recall that the Berezin symbol is, in general not multiplicative, i.e., $\widetilde{AB} \neq \tilde{A}\tilde{B}$).

2.3. Further reverse inequalities

Note that in applications it would be more useful to find upper bounds for the quantity

$$\widetilde{f(A)}(\lambda) - f(\tilde{A}(\lambda)), \lambda \in \Omega,$$

that are in terms of the spectrum margins m, M and of the function f. Our first result in this direction is the following.

Theorem 2.2. *Let J be an interval in $\mathbb{R}$ and $f : J \to \mathbb{R}$ be a convex and differentiable function on $\dot{J}$ whose derivative f' is continuous on $\dot{J}$. If A is a self-adjoint operator on the reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ with $S_p(A) \subseteq [m, M] \subset \dot{J}$. Then*

$$(0 \leq)\widetilde{f(A)}(\lambda) - f(\tilde{A}(\lambda)) \leq \begin{cases} \frac{1}{2}(M-m)[\|f'(A)\hat{k}_\lambda\|^2 - \widetilde{f'(A)}^2(\lambda)]^{\frac{1}{2}} \\ \frac{1}{2}(f'(M) - f'(m))[\|A\hat{k}_\lambda\|^2 - \tilde{A}(\lambda)^2]^{\frac{1}{2}} \end{cases} \tag{6}$$

$$\leq \frac{1}{4}(M-m)(f'(M) - f'(m))$$

for any $\lambda \in \Omega$. We also have the inequality

$$
\begin{aligned}
&(0 \leq)\widetilde{f(A)}(\lambda) - f(\tilde{A}(\lambda)) \leq \frac{1}{4}(M-m)(f'(M)-f'(m)) \\
&- \begin{cases} [\langle M\hat{k}_\lambda - A\hat{k}_\lambda, A\hat{k}_\lambda - m\hat{k}_\lambda\rangle\langle f'(M)\hat{k}_\lambda - f'(A)\hat{k}_\lambda, f'(A)\hat{k}_\lambda - f'(m)\hat{k}_\lambda\rangle]^{\frac{1}{2}}, \\ \left|\tilde{A}(\lambda) - \frac{M+m}{2}\right| \left|\widetilde{f'(A)}(\lambda) - \frac{f'(M)+f'(m)}{2}\right| \end{cases} \\
&\leq \frac{1}{4}(M-m)(f'(M)-f'(m)),
\end{aligned} \tag{7}
$$

for all $\lambda \in \Omega$. Moreover, if $m > 0$ and $f'(m) > 0$, then we also have

$$
(0 \leq)\widetilde{f(A)}(\lambda) - f(\tilde{A}(\lambda)) \leq \begin{cases} \frac{1}{4}\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}}\tilde{A}(\lambda)\widetilde{f'(A)}(\lambda), \\ (\sqrt{M}-\sqrt{m})(\sqrt{f'(M)}-\sqrt{f'(m)})[\tilde{A}(\lambda)\widetilde{f'(A)}(\lambda)]^{\frac{1}{2}} \end{cases}. \tag{8}
$$

Proof. Indeed, for the proof we will use the following Grüss type inequality (see [10]). Let A be a self-adjoint operator on the reproducing kernel Hilbert space $(\mathcal{H}(\Omega); \langle\cdot,\cdot\rangle)$ and assume that $S_p(A) \subseteq [m, M]$ for some scalars $m < M$. If $h, g \in C[m, M]$ and $\gamma := \min_{t\in[m,M]} h(t)$ and $\Gamma := \max_{t\in[m,M]} h(t)$ then

$$
\left|\widetilde{h(A)g(A)}(\lambda) - \widetilde{h(A)}(\lambda)\widetilde{g(A)}(\lambda)\right| \leq \frac{1}{2}(\Gamma-\gamma)\left[\|g(A)\hat{k}_\lambda\|^2 - \widetilde{g(A)}(\lambda)^2\right]^{\frac{1}{2}} \left(\leq \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta)\right) \tag{9}
$$

for each $\lambda \in \Omega$, where $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$. Therefore, we can assert that

$$
\widetilde{Af'(A)}(\lambda) - \tilde{A}(\lambda)\widetilde{f'(A)}(\lambda) \leq \frac{1}{2}(M-m)\left[\|f'(A)\tilde{k}_\lambda\|^2 - \widetilde{f'(A(\lambda))}^2\right]^{\frac{1}{2}} \leq \frac{1}{4}(M-m)(f'(M)-f'(m)) \tag{10}
$$

and

$$
\widetilde{Af'(A)}(\lambda) - \tilde{A}(\lambda)\widetilde{f'(A)}(\lambda) \leq \frac{1}{2}(f'(M)-f'(m))[\|A\hat{k}_\lambda\|^2 - A(\lambda)^2]^{\frac{1}{2}} \leq \frac{1}{4}(M-m)(f'(M)-f'(m)) \tag{11}
$$

for all $\lambda \in \Omega$, which together with inequality in (2) provide the desired result (6).

The following inequality is immediate from the result of the paper [2].

$$
\begin{aligned}
&\left|\widetilde{h(A)g(A)}(\lambda) - \widetilde{h(A)}(\lambda)\widetilde{g(A)}(\lambda)\right| \\
&\leq \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta) \\
&- \begin{cases} [\langle \Gamma\hat{k}_\lambda - h(A)\hat{k}_\lambda, f(A)\hat{k}_\lambda - \gamma\hat{k}_\lambda\rangle\langle \Delta\hat{k}_\lambda - g(A)\hat{k}_\lambda, g(A)\hat{k}_\lambda - \delta\hat{k}_\lambda\rangle]^{\frac{1}{2}}, \\ |\widetilde{h(A)}(\lambda) - \frac{\Gamma+\gamma}{2}||\widetilde{g(A)}(\lambda) - \frac{\Delta+\delta}{2}|, \end{cases}
\end{aligned}
$$

for all $\lambda \in \Omega$. Then we can state that

$$
\begin{aligned}
&\widetilde{Af'(A)}(\lambda) - \tilde{A}(\lambda)\widetilde{f'(A)}(\lambda) \\
&\leq \frac{1}{4}(M-m)(f'(M)-f'(m)) \\
&- \begin{cases} [\langle M\hat{k}_\lambda - A\hat{k}_\lambda, A\hat{k}_\lambda - m\hat{k}_\lambda\rangle\langle f'(M)\hat{k}_\lambda - f'(A)\hat{k}_\lambda, f'(A)\hat{k}_\lambda - f'(m)\hat{k}_\lambda\rangle]^{\frac{1}{2}} \\ |\tilde{A}(\lambda) - \frac{M+m}{2}||\widetilde{f'(A)}(\lambda) - \frac{f'(M)+f'(m)}{2}| \end{cases}
\end{aligned}
$$

for each $\lambda \in \Omega$, which together with inequality in Theorem 2.1 provide the desired result (7). In order to prove inequality (8), we make use the inequality.

$$
\begin{aligned}
&\widetilde{Af'(A)}(\lambda) - \tilde{A}(\lambda)\widetilde{f'(A)}(\lambda) \\
&\leq \begin{cases} \frac{1}{4}\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}}\tilde{A}(\lambda)\widetilde{f'(A)}(\lambda), \\ (\sqrt{M}-\sqrt{m})(\sqrt{f'(M)}-\sqrt{f'(m)})[\tilde{A}(\lambda)\widetilde{f'(A)}(\lambda)]^{\frac{1}{2}} \end{cases}
\end{aligned}
$$

for each $\lambda \in \Omega$, which can be obtained by using a results in [2]. This inequality together with (6) provide the desired result (8). The theorem is proven. □

The following is immediate from Theorem 2.2.

Corollary 2.3. *We have*

$$ber(f(A)) \leq f(ber(A)) + \frac{1}{4}(M-m)(f'(M) - f'(m))$$

for any increasing function f in Theorem 2.2.

2.4. Some special inequalities

1. Consider the convex function $f : (0, \infty) \to \mathbb{R}$, $f(x) = -\ln x$. On utilizing the inequality (6), then for any positive definite operator A on the Hilbert space $\mathcal{H}(\Omega)$, we have that

$$\begin{aligned}
&(0 \leq) \ln(\tilde{A}(\lambda)) - \widetilde{\ln(A)}(\lambda) \\
&\leq \begin{cases} \frac{1}{2}(M-m)[\|A^{-1}\hat{k}_{(\lambda)}\|^2 - \widetilde{A^{-1}}(\lambda)^2]^{\frac{1}{2}} \\ \frac{1}{2}\frac{M-m}{mM}[\|A\hat{k}_{(\lambda)}\|^2 - \widetilde{A}(\lambda)^2]^{\frac{1}{2}}. \end{cases} \\
&\left(\leq \frac{1}{4}\frac{(M-m)^2}{mM}\right)
\end{aligned} \tag{12}$$

for any $\lambda \in \Omega$.

The following is immediate from (8).

Corollary 2.4. *We have:*

$$\begin{aligned}
&\sup_{\lambda\in\Omega}(\widetilde{f(A)}(\lambda) - f(\tilde{A}(\lambda)) \\
&\leq \begin{cases} \frac{1}{4}\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}} ber(A)ber(f'(A)), \\ \left(\sqrt{M} - \sqrt{m})(\sqrt{f'(M)} - \sqrt{f'(m)}\right)(ber(A)ber(f'(A)))^{\frac{1}{2}} \end{cases}.
\end{aligned}$$

The next corollary is obtained from inequality (6).

Corollary 2.5. *We have*

$$\sup_{\lambda\in\Omega}(\widetilde{f(A)}(\lambda) - f(\tilde{A}(\lambda)) \leq \frac{1}{2}(f'(M) - f'(m))[\|A\|_{Ber} - \inf_{\lambda\in\Omega}\tilde{A}^2(\lambda)]^{\frac{1}{2}}.$$

However, if we use the inequality (7), then we get the following as well:

$$\begin{aligned}
&(0 \leq) \ln(\tilde{A}(\lambda)) - \widetilde{\ln A}(\lambda) \leq \frac{1}{4}\frac{(M-m)^2}{mM} \\
&- \begin{cases} [\langle M\hat{k}_\lambda - A\hat{k}_\lambda, A\hat{k}_\lambda - m\hat{k}_\lambda\rangle\langle M^{-1}\hat{k}_\lambda - A^{-1}\hat{k}_\lambda, A^{-1}\hat{k}_\lambda - m^{-1}\hat{k}_\lambda\rangle]^{\frac{1}{2}}, \\ |\tilde{A}(\lambda) - \frac{M+m}{2}\|\widetilde{A^{-1}}(\lambda) - \frac{M+m}{2mM}| \end{cases} \\
&\left(\leq \frac{1}{4}\frac{(M-m)^2}{mM}\right)
\end{aligned} \tag{13}$$

for all $\lambda \in \Omega$. Thus, we have from (13) by taking supremum that

$$\sup_{\lambda\in\Omega}|\ln(\tilde{A}(\lambda))| \leq ber(\ln(A)) + \frac{1}{4}\frac{(M-m)^2}{mM}.$$

2. Consider the convex function $f:(0,\infty)\to(\infty,-\infty)$, $f(x)=x\ln x$. By applying the inequality (6), then for any positive definite operator A on $\mathcal{H}(\Omega)$, we obtain the inequality

$$\begin{aligned}&(0\le)\widetilde{A\ln(A)}(\lambda)-\tilde{A}(\lambda)\ln(\tilde{A}(\lambda))\\&\le\begin{cases}\frac{1}{2}(M-m)[\|\ln(eA)\hat{k}_{(\lambda)}\|^2-\widetilde{\ln(eA)}(\lambda)^2]^{\frac{1}{2}}\\ \ln\sqrt{\frac{M}{m}}[\|A\hat{k}_{(\lambda)}\|^2-\tilde{A}(\lambda)^2]^{\frac{1}{2}}\end{cases}\end{aligned}$$

$$\left(\le\frac{1}{2}(M-m)\ln\sqrt{\frac{M}{m}}\right)$$

for any $\lambda\in\Omega$. If we now use inequality (7), then we have the following inequality as well

$$\begin{aligned}&(0\le)\widetilde{A\ln(A)}(\lambda)-\tilde{A}(\lambda)\ln(\tilde{A}(\lambda))\le\frac{1}{2}(M-m)\ln\sqrt{\frac{M}{m}}\\&-\begin{cases}[\langle M\hat{k}_{(\lambda)}-A\hat{k}_{(\lambda)},A\hat{k}_{(\lambda)}-m\hat{k}_{(\lambda)}\rangle\langle\ln(M)\hat{k}_{(\lambda)}-\ln(A)\hat{k}_{(\lambda)},\ln(A)\hat{k}_{(\lambda)}-\ln(m)\hat{k}_{(\lambda)}\rangle]^{\frac{1}{2}}\\ |\tilde{A}(\lambda)-\frac{M+m}{2}||\widetilde{\ln(A)}(\lambda)-\ln\sqrt{mM}|\end{cases}\end{aligned}\tag{14}$$

$$\left(\le\frac{1}{2}(M-m)\ln\sqrt{\frac{M}{m}}\right)$$

for every $\lambda\in\Omega$, which clearly implies that

$$ber(\ln(A))\le ber(A)\sup_{\lambda\in\Omega}|\ln(\tilde{A}(\lambda))|+\frac{1}{2}(M-m)\ln\sqrt{\frac{M}{m}}.$$

3. Consider now the following convex function $f:\mathbb{R}\to(0,\infty)$, $f(x)=\exp(\alpha x)$ with $\alpha>0$. By using inequalities (6)-(8) for $f(x)=\exp(\alpha x)$ and for a self-adjoint operator A, then we have

$$\begin{aligned}&(0\le)\widetilde{\exp(\alpha A)}(\lambda)-\exp(\alpha\tilde{A}(\lambda))\\&\le\begin{cases}\frac{1}{2}\alpha(M-m)[\|\exp(\alpha A)\hat{k}_{\lambda}\|^2-\widetilde{\exp(\alpha A)}(\lambda)^2]^{\frac{1}{2}}\\ \frac{1}{2}\alpha(\exp(\alpha M))-\exp(\alpha m)[\|A\hat{k}_{\lambda}\|^2-\tilde{A}(\lambda)^2]^{\frac{1}{2}}\end{cases}\quad\left(\le\frac{1}{4}\alpha(M-m)(\exp(\alpha M)-\exp(\alpha m)\right),\end{aligned}\tag{15}$$

and

$$\begin{aligned}&(0\le)\widetilde{\exp(\alpha A)}(\lambda)-\exp(\alpha\tilde{A}(\lambda))\le\frac{1}{4}\alpha(M-m)(\exp(\alpha M)-\exp(\alpha m))\\&-\alpha\begin{cases}[\langle M\hat{k}_{\lambda}-A\hat{k}_{\lambda},A\hat{k}_{\lambda}-m\hat{k}_{\lambda}\rangle]^{\frac{1}{2}}[\langle\exp(\alpha M)\hat{k}_{\lambda}-\exp(\alpha A)\hat{k}_{\lambda},\exp(\alpha A)\hat{k}_{\lambda}-\exp(\alpha m)\hat{k}_{\lambda}\rangle]^{\frac{1}{2}},\\ |\tilde{A}(\lambda)-\frac{M+m}{2}||\widetilde{\exp(\alpha A)}(\lambda)-\frac{\exp(\alpha M)+\exp(\alpha m)}{2}|\end{cases}\end{aligned}\tag{16}$$

$$(\le\frac{1}{4}\alpha(M-m)(\exp(\alpha M)-\exp(\alpha m)),$$

and

$$\begin{aligned}&(0\le)\widetilde{\exp(\alpha A)}(\lambda)-\exp(\alpha\tilde{A}(\lambda))\\&\le\alpha\begin{cases}\frac{1}{4}\frac{(M-m)(\exp(\alpha M)-\exp(\alpha m))}{\sqrt{Mm}\exp\left[\frac{\alpha(M+m)}{2}\right]}\tilde{A}(\lambda)\widetilde{\exp(\alpha A)}(\lambda),\\ (\sqrt{M}-\sqrt{m})\left(\exp(\frac{\alpha M}{2})-\exp(\frac{\alpha m}{2})\right)[\tilde{A}(\lambda)\widetilde{\exp(\alpha A)}(\lambda)]^{\frac{1}{2}}\end{cases}\end{aligned}\tag{17}$$

for all $\lambda\in\Omega$, respectively.

Now, we consider the convex function $f : \mathbb{R} \to (0, \infty)$, $f(x) = \exp(-\beta x)$ with $\beta > 0$. If we apply the inequalities (6) and (7) for $f(x)) = \exp(-\beta x)$ and for a self-adjoint operator A, then we get the following results

$$(0 \leq)\widetilde{\exp(-\beta A)}(\lambda) - \exp(-\beta \tilde{A}(\lambda)) \leq \beta \begin{cases} \frac{1}{2}(M-m)\left[\|\exp(-\beta A)\hat{k}_\lambda\|^2 - \widetilde{\exp(-\beta A)}(\lambda)^2\right]^{\frac{1}{2}} \\ \frac{1}{2}(\exp(-\beta m) - \exp(-\beta M)\left[\|A\hat{k}_\lambda\|^2 - \tilde{A}(\lambda)^2\right]^{\frac{1}{2}} \end{cases} \tag{18}$$

$$\left(\leq \frac{1}{4}\beta(M-m)(\exp(-\beta m) - \exp(-\beta M)\right)$$

and

$$(0 \leq)\widetilde{\exp(-\beta A)}(\lambda) - \exp(-\beta \tilde{A}(\lambda)) \leq \frac{1}{4}\beta(M-m)(\exp(-\beta m) - \exp(-\beta M) - \beta \begin{cases} \left[\langle M\hat{k}_\lambda - A\hat{k}_\lambda, A\hat{k}_\lambda - m\hat{k}_\lambda\rangle\right]^{\frac{1}{2}} \left[\langle \exp(-\beta M)\hat{k}_\lambda - \exp(-\beta A)\hat{k}_\lambda, \exp(-\beta A)\hat{k}_\lambda - \exp(-\beta m)\hat{k}_\lambda\rangle\right]^{\frac{1}{2}} \\ |\tilde{A}(\lambda) - \frac{M+m}{2}\|\widetilde{\exp(-\beta A)}(\lambda) - \frac{\exp(-\beta M)+\exp(-\beta m)}{2}| \end{cases} \tag{19}$$

$$(\leq \frac{1}{4}\beta(M-m)(\exp(-\beta m) - \exp(-\beta M))$$

for all $\lambda \in \Omega$, respectively.

4. Finally, let us consider the convex function $f : [0, \infty) \to [0, \infty)$, $f(x) = x^p$ with $p \geq 1$. Then by using inequalities (6) and (7) for the positive operator A we get the following inequalities

$$(0 \leq)\langle A^p\hat{k}_\lambda, \hat{k}_\lambda\rangle - \langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle^p \leq p \times \begin{cases} \frac{1}{2}(M-m)[\|A^{p-1}\hat{k}_\lambda\|^2 - \langle A^{p-1}\hat{k}_\lambda, \hat{k}_\lambda\rangle^2]^{\frac{1}{2}} \\ \frac{1}{2}(M^{p-1} - m^{p-1})[\|A\hat{k}_\lambda\|^2 - \langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle^2]^{\frac{1}{2}} \end{cases} \left(\leq \frac{1}{4}p(M-m)(M^{p-1} - m^{p-1})\right) \tag{20}$$

and

$$(0 \leq)\langle A^p\hat{k}_\lambda, \hat{k}_\lambda\rangle - \langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle^p \leq \frac{1}{4}p(M-m)(M^{p-1} - m^{p-1}) - p \begin{cases} [\langle M\hat{k}_\lambda - A\hat{k}_\lambda, A\hat{k}_\lambda - m\hat{k}_\lambda\rangle\langle M^{p-1}\hat{k}_\lambda - A^{p-1}\hat{k}_\lambda, A^{p-1}\hat{k}_\lambda - m^{p-1}\hat{k}_\lambda\rangle]^{\frac{1}{2}} \\ |\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle - \frac{M+m}{2}\|\langle A^{p-1}\hat{k}_\lambda, \hat{k}_\lambda\rangle - \frac{M^{p-1}+m^{p-1}}{2}| \end{cases} \tag{21}$$

$$\left(\leq \frac{1}{4}p(M-m)(M^{p-1} - m^{p-1})\right)$$

for all $\lambda \in \Omega$, respectively.

If the operator A is positive definite ($m > 0$) then, by using (7) we have that

$$(0 \leq)\langle A^p\hat{k}_\lambda, \hat{k}_\lambda\rangle - \langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle^p \leq p \begin{cases} \frac{1}{4}\frac{(M-m)(M^{p-1}-m^{p-1})}{M^{\frac{p}{2}}m^{\frac{p}{2}}}\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle A^{p-1}\hat{k}_\lambda, \hat{k}_\lambda\rangle \\ (\sqrt{M} - \sqrt{m})(M^{\frac{p-1}{2}} - m^{\frac{p-1}{2}})[\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle, \langle A^{p-1}\hat{k}_\lambda, \hat{k}_\lambda\rangle]^{\frac{1}{2}} \end{cases} \tag{22}$$

for all $\lambda \in \Omega$.

Now, it follows from (22) that

$$\sup_{\lambda \in \Omega}\left(\widetilde{A^p}(\lambda) - \tilde{A}(\lambda)^p\right) \leq p(\sqrt{M} - \sqrt{m})(M^{\frac{p-1}{2}} - m^{\frac{p-1}{2}})(ber(A) \cdot ber(A^{p-1}))^{\frac{1}{2}}.$$

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