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Complete metrizability of the topology of strong Whitney convergence on bornology

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Abstract. The topology of strong Whitney convergence on bornology was introduced by Caserta in [6]. This paper studies the complete metrizability and several other completeness properties of the space of all real-valued continuous functions on a metric space, equipped with the topologies of Whitney and strong Whitney convergence on bornology. The Polishness of these topologies coincides with their complete metrizability.

1. Introduction

For any two metric spaces (X, d) and (Y, ρ), $C(X, Y)$ denotes the set of all continuous functions from X to Y. For $Y = \mathbb{R}$ with the usual metric, the space $C(X, \mathbb{R})$ is denoted by $C(X)$.

A family $\mathcal{B}$ of nonempty subsets of X is called a bornology on X if $\mathcal{B}$ forms a cover of X, is stable under finite union and hereditary under inclusion (see [14]). A base for a bornology $\mathcal{B}$ is any subfamily $\mathcal{B}_0$ of $\mathcal{B}$ such that $\mathcal{B}_0$ is cofinal in $\mathcal{B}$ under set inclusion. If every member of $\mathcal{B}_0$ is closed (compact) in (X, d), then $\mathcal{B}$ is said to have a closed (respectively, compact) base.

The smallest (respectively, largest) bornology on X is the family $\mathcal{F}$ of all finite (respectively, $\mathcal{P}_0(X)$ of all nonempty) subsets of X. The family $\mathcal{K}$ of all nonempty relatively compact subsets of X is another important bornology on X.

Several topologies have been studied on $C(X, Y)$ such as the topology of pointwise convergence τ_p, the topology of uniform convergence τ^u and the topology of uniform convergence on compacta τ_k (see [21]).

For any two metric spaces (X, d), (Y, ρ) and a bornology $\mathcal{B}$, the most commonly used topology on $C(X, Y)$ is the classical topology of uniform convergence on $\mathcal{B}$, denoted by τ_β. A stronger version of τ_β was introduced by Beer and Levi in [2], called the topology of strong uniform convergence on $\mathcal{B}$, denoted by τ^s_β. The topology τ^s_β was further studied in [[3], [5], [16]]. In [6], Caserta generalized the topology τ^s_β to a new topology called the topology of strong Whitney convergence on $\mathcal{B}$, denoted by τ^{sw}_β. The topology τ^w_β of Whitney convergence on $\mathcal{B}$ is a generalization of the well-known topology, the Whitney topology τ^w, introduced by H. Whitney in [24] and further studied in [[13], [17], [11], [19], [18]]. For more details on Whitney topology, see the research monograph [22].

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Several topological properties of τ_β^{sw} and τ_β^w, like metrizability, countability properties, countable tightness, Fréchet, cardinal functions and connectedness have been studied in [9], [7] and [10]. Complete metrizability and Polishness of τ_β and τ_β^s have been studied by L. Holá in [15].

In this paper, we give a characterization of the complete metrizability of τ_β^{sw} and τ_β^w on $C(X)$. Several other completeness properties, such as pseudo-completeness, Čech-completeness, partition-completeness, sieve-completeness and almost Čech-completeness of τ_β^w and τ_β^{sw} on $C(X)$, are studied in this paper. Also, we see that the Polishness of τ_β^{sw} coincides with the complete metrizability. Moreover, the Polishness of τ_β^{sw} is equivalent to the Polishness of τ_β.

2. Preliminaries

All metric spaces are assumed to have at least two points. For any nonempty subset A of X and $\delta > 0$, A^δ denotes the *δ-enlargement* of A defined as $A^\delta = \bigcup_{x\in A} S_\delta(x) = \{x \in X : d(x, A) < \delta\}$, where $S_\delta(x)$ denotes the open ball with center x and radius δ. For other terms and notations, we refer to [[25], [12], [21]].

For a bornology $\mathcal{B}$ and any two metric spaces (X, d) and (Y, ρ), the classical *topology τ_β of uniform convergence on $\mathcal{B}$* for the space $C(X, Y)$ is determined by the uniformity $\Delta_{\mathcal{B}}$ which has basic entourages of the form

$$[B, \epsilon] = \{(f, g) : \rho(f(x), g(x)) < \epsilon \text{ for all } x \in B\} \ (B \in \mathcal{B}, \epsilon > 0).$$

For the bornology $\mathcal{F}$ (respectively, $\mathcal{K}$), $\tau_{\mathcal{F}}$ (respectively, $\tau_{\mathcal{K}}$) is the topology of pointwise convergence τ_p (respectively, the topology of uniform convergence on compacta τ_k). For $\mathcal{B} = \mathcal{P}_0(X)$, τ_β is the topology τ^u of uniform convergence.

The classical *topology τ_β^w of Whitney convergence on $\mathcal{B}$* for the space $C(X, Y)$ is determined by the uniformity $\Delta_{\mathcal{B}}^w$ which has basic entourages of the form

$$[B, \epsilon]^w = \{(f, g) : \rho(f(x), g(x)) < \epsilon(x) \text{ for all } x \in B\} \ (B \in \mathcal{B}, \epsilon \in C^+(X)).$$

Here $C^+(X)$ represents the set of all positive real-valued continuous functions on X. If $\mathcal{B} = \mathcal{P}_0(X)$, then τ_β^w is the Whitney topology τ^w.

The *topology τ_β^s of strong uniform convergence on $\mathcal{B}$* is determined by the uniformity $\Delta_{\mathcal{B}}^s$ which has basic entourages of the form

$$[B, \epsilon]^s = \{(f, g) : \exists\, \delta > 0,\ \rho(f(x), g(x)) < \epsilon \text{ for all } x \in B^\delta\} \ (B \in \mathcal{B}, \epsilon > 0).$$

The *topology τ_β^{sw} of strong Whitney convergence on $\mathcal{B}$* is determined by the uniformity $\Delta_{\mathcal{B}}^{sw}$ which has basic entourages of the form

$$[B, \epsilon]^{sw} = \{(f, g) : \exists\, \delta > 0,\ \rho(f(x), g(x)) < \epsilon(x) \text{ for all } x \in B^\delta\} \ (B \in \mathcal{B}, \epsilon \in C^+(X)).$$

In general, the following relation holds between the above-defined topologies:

$$\tau_\beta \subseteq \tau_\beta^s \subseteq \tau_\beta^{sw} \text{ and } \tau_\beta \subseteq \tau_\beta^w \subseteq \tau_\beta^{sw} \subseteq \tau^w.$$

The concept of a shield was introduced by Beer et al. in [4]. For a nonempty subset A of X, a superset A_1 of A is called a *shield* for A provided that for every closed subset C of X with $C \cap A_1 = \emptyset$, there exists $\delta > 0$ such that $C \cap A^\delta = \emptyset$.

A bornology $\mathcal{B}$ on X is called *shielded from closed sets* if $\mathcal{B}$ contains a shield for each of its members. It is well known that a bornology with a compact base is shielded from closed sets.

Recall that for a bornology $\mathcal{B}$ on a metric space (X, d) with a closed base, $\mathcal{B} \subseteq \mathcal{K}$ if and only if $\mathcal{B}$ has a compact base. From Theorem 2.4 in [9], $\mathcal{B}$ has a compact base if and only if $\tau_\beta = \tau_\beta^s = \tau_\beta^w = \tau_\beta^{sw}$ on $C(X, Y)$ for every metric space (Y, ρ).

3. Complete Metrizability and related completeness properties

In this section, we give a characterization for the complete metrizability of $(C(X), \tau_\beta^{sw})$ and $(C(X), \tau_\beta^{w})$ and study various completeness properties.

A family $\mathfrak{B}$ of nonempty open subsets of a space X is called a *π-base* if every nonempty open subset of X contains at least one member of $\mathfrak{B}$. A space X is called *pseudo-complete* if it has a sequence $\{\mathfrak{B}_n : n \in \mathbb{N}\}$ of π-bases such that whenever $U_n \in \mathfrak{B}_n$ for each n and $\overline{U_{n+1}} \subseteq U_n$, then $\bigcap_{n\in\mathbb{N}} U_n \neq \emptyset$.

Before giving a characterization of complete metrizability for $(C(X), \tau_\beta^{sw})$, from [15] we would like to mention that $(C(X), \tau_\beta)$ is completely metrizable if and only if every nonempty compact set in X belongs to $\mathcal{B}$ and $\mathcal{B}$ has a countable base.

Also, $(C(X), \tau_\beta^{s})$ is completely metrizable if and only if every nonempty compact set in X belongs to $\mathcal{B}$ and $\mathcal{B}$ is shielded from closed sets and has a countable base.

A subset A of a space X is called relatively pseudocompact if $f(A)$ is a bounded subset of $\mathbb{R}$ for all $f \in C(X)$.

In a similar manner to Lemma 3.1 in [15], we can prove the following.

Lemma 3.1. *Let (X, d) be a metric space and $\mathcal{B}$ be a bornology on X with a closed base. If $(C(X), \tau_\beta^{w})((C(X), \tau_\beta^{sw}))$ is of the second Baire category, then every relatively pseudocompact subset of X is contained in $\mathcal{B}$.*

Remark 3.2. *Note that in Lemma 3.1, the condition is necessary for pseudo-completeness of $(C(X), \tau_\beta^{w})((C(X), \tau_\beta^{sw}))$ as well. But the converse need not be true. Consider $X = \mathbb{Q}$, the set of rational numbers, with the usual metric and $\mathcal{B} = \mathcal{K}$. Then every relatively pseudocompact subset of X is in $\mathcal{B}$. But $(C(X), \tau_\beta^{sw}) = (C(X), \tau_\beta^{w}) = C_k(X)$ is not pseudo-complete (in fact, not Baire), by Corollary 4.5 in [20].*

Next, we give a sufficient condition for pseudo-completeness of $(C(X), \tau_\beta^{w})$. Before that we prove a lemma, motivated from Lemma 3.1 in [19].

Lemma 3.3. *Let (X, d) be a metric space and $\mathcal{B}$ be a bornology on X with a closed base. Then for any $f_1, f_2 \in C(X)$, $\phi_1, \phi_2 \in C^+(X)$ and $B_1, B_2 \in \mathcal{B}$ with $B_1 \subseteq B_2$, closure of $[B_2, \phi_2]^w[f_2]$ in $(C(X), \tau_\beta^{w})$ is contained in $[B_1, \phi_1]^w[f_1]$ if and only if $[f_2(x) - \phi_2(x), f_2(x) + \phi_2(x)] \subseteq (f_1(x) - \phi_1(x), f_1(x) + \phi_1(x))$ for every $x \in B_1$.*

Proof. First we prove sufficiency. Let $g \in \overline{[B_2, \phi_2]^w[f_2]}$ and $x \in B_1$. If $g(x) \notin [f_2(x) - \phi_2(x), f_2(x) + \phi_2(x)]$, then $[x, V] = \{h \in C(X) : h(x) \in V\}$ is a neighborhood of g in $(C(X), \tau_\beta^{w})$, where $V = \mathbb{R} \setminus [f_2(x) - \phi_2(x), f_2(x) + \phi_2(x)]$. Since $x \in B_2$, $[x, V] \cap [B_2, \phi_2]^w[f_2] = \emptyset$. This gives a contradiction, hence $g \in [B_1, \phi_1]^w[f_1]$.

For necessity, let $y \in B_1$ and $p \in [f_2(y) - \phi_2(y), f_2(y) + \phi_2(y)]$. Define $g(x) = f_2(x) + \frac{(p - f_2(y))\cdot\phi_2(x)}{\phi_2(y)}$. Observe that $g \in C(X)$ and $g(y) = p$ such that $g \in \overline{[B_2, \phi_2]^w[f_2]}$. As for any neighborhood $[B, \epsilon]^w[g]$ of g in $(C(X), \tau_\beta^{w})$, consider the function $k \in C(X)$, defined by $k(x) = g(x) +$ sign$(f_2(x) - g(x))\cdot\min\{\frac{\epsilon(x)}{2}, |g(x) - f_2(x)|\}$. Then $k \in [B, \epsilon]^w[g] \cap [B_2, \phi_2]^w[f_2]$. Thus $g \in [B_1, \phi_1]^w[f_1]$, which shows that $p = g(y) \in (f_1(y) - \phi_1(y), f_1(y) + \phi_1(y))$. □

Theorem 3.4. *Let (X, d) be a metric space and $\mathcal{B}$ be a bornology on X with a closed base such that $\mathcal{K} \subseteq \mathcal{B}$ and $\mathcal{B}$ has a countable base. Then $(C(X), \tau_\beta^{w})$ is pseudo-complete.*

Proof. Let $\mathcal{B}_0 = \{B_n : n \in \mathbb{N}\}$ be a countable base for $\mathcal{B}$. We can assume that $\mathcal{B}_0$ is an increasing base for $\mathcal{B}$ i.e. $B_n \subseteq B_{n+1} \forall\, n$. Consider a collection $\mathfrak{B}_j = \{[B_n, \phi]^w[f] : f \in C(X), \phi \in C_j^+(X), n \in \mathbb{N}\}$ for every $j \in \mathbb{N}$, where $C_j^+(X) = \{\phi \in C^+(X) : \phi(x) < \frac{1}{2^j}\ \forall x \in X\}$. Clearly, each $\mathfrak{B}_j$ is a π-base for $(C(X), \tau_\beta^{w})$. Now, let $U_j \in \mathfrak{B}_j$ for every j such that $\overline{U_{j+1}} \subseteq U_j$. Without loss of generality, U_j can be taken as $[B_{n_j}, \phi_j]^w[f_j]$ with $n_j < n_{j+1}\ \forall j$, where $\phi_j \in C_j^+(X)$, $f_j \in C(X)$ and $B_{n_j} \in \mathcal{B}_0$, for all j. By Lemma 3.3, for every j, we have $[f_{j+1}(x) - \phi_{j+1}(x), f_{j+1}(x) + \phi_{j+1}(x)] \subseteq (f_j(x) - \phi_j(x), f_j(x) + \phi_j(x))\ \forall x \in B_{n_j}$. Since for every j, $\phi_j(x) < \frac{1}{2^j}\ \forall x \in X$ and $(B_{n_j})_{j\in\mathbb{N}}$ is increasing, by Cantor's intersection theorem, for every m, we have

$\bigcap_{j=m}^{\infty}(f_j(x)-\phi_j(x), f_j(x)+\phi_j(x)) = \{f(x)\}\ \forall x \in B_{n_m}$ for some $f(x) \in \mathbb{R}$. This defines a function f such that $f \in \bigcap_{m=1}^{\infty} U_m$. Also note that for every m, $|f(x)-f_j(x)| < \frac{1}{2^j}$, $\forall x \in B_{n_m}$, $j \geq m$. It shows that (f_n) converges uniformly to f on B_{n_m}, for every m. Since every nonempty compact set in X belongs to $\mathcal{B}$, (f_n) converges uniformly to f on each compact subset of X, which gives $f \in C(X)$. □

Recall that a bornology $\mathcal{B}$ with a closed base on a metric space (X, d), is shielded from closed sets if and only if $\tau_{\beta}^{sw} = \tau_{\beta}^{w}$ on $C(X)$. (See [8]). We have the following corollary to Theorem 3.4.

Corollary 3.5. *Let (X, d) be a metric space and $\mathcal{B}$ be a bornology on X with a closed base such that $\mathcal{K} \subseteq \mathcal{B}$, $\mathcal{B}$ is shielded from closed sets and $\mathcal{B}$ has a countable base. Then $(C(X), \tau_{\beta}^{sw})$ is pseudo-complete.*

Remark 3.6. *Note that the conditions taken in Theorem 3.4 and Corollary 3.5 for pseudo-completeness of $(C(X), \tau_{\beta}^{w})$ and $(C(X), \tau_{\beta}^{sw})$ are equivalent to the complete metrizability of τ_{β} and τ_{β}^{s}, respectively. But the converse need not be true in any case. Let X be an uncountable discrete metric space and $\mathcal{B} = \mathcal{K}$, then $(C(X), \tau_{\beta}^{sw}) = (C(X), \tau_{\beta}^{w}) = C_k(X)$ is pseudo-complete, by Corollary 4.5 in [20]. But $\mathcal{B}$ has not countable base.*

Now we give a characterization for complete metrizability of $(C(X), \tau_{\beta}^{sw})$.

Theorem 3.7. *Let (X, d) be a metric space and $\mathcal{B}$ be a bornology on X with a closed base. Then the following are equivalent.*

(*i*) *For every complete metric space (Y, ρ), $(C(X, Y), \tau_{\beta}^{sw})$ is completely metrizable.*

(*ii*) *$(C(X), \tau_{\beta}^{sw})$ is completely metrizable.*

(*iii*) *Every nonempty compact set in X belongs to $\mathcal{B}$ and $\mathcal{B}$ has a countable base consisting of compact sets.*

(*iv*) *$\mathcal{B} = \mathcal{K}$ and X is hemicompact.*

(*v*) *$\mathcal{B} = \mathcal{K}$ and X is locally compact and separable.*

Proof. (*i*) ⇒ (*ii*) It is immediate.

(*ii*) ⇒ (*iii*) If $(C(X), \tau_{\beta}^{sw})$ is completely metrizable, then by Lemma 3.1, every nonempty compact set in X belongs to $\mathcal{B}$ and from Theorem 3.2 in [9], $\mathcal{B}$ has a countable base consisting of compact sets.

(*iii*) ⇒ (*i*) If $\mathcal{B}$ has a compact base, then $\tau_{\beta} = \tau_{\beta}^{sw}$ and the complete metrizability of τ_{β} follows from Theorem 3.1 in [15].

(*iii*) ⇔ (*iv*) Observe that $\mathcal{B} = \mathcal{K}$ if and only if every nonempty compact set in X belongs to $\mathcal{B}$ and $\mathcal{B}$ has a compact base. Also, $\mathcal{K}$ has a countable base consisting of compact sets if and only if X is hemicompact.

(*iv*) ⇔ (*v*) It follows from the fact that a metric space is hemicompact if and only if it is locally compact and separable. □

Similarly, we can prove the following result for the topology τ_{β}^{w}.

Theorem 3.8. *Let (X, d) be a metric space and $\mathcal{B}$ be a bornology on X with a closed base. Then the following are equivalent.*

(*i*) *For every complete metric space (Y, ρ), $(C(X, Y), \tau_{\beta}^{w})$ is completely metrizable.*

(*ii*) *$(C(X), \tau_{\beta}^{w})$ is completely metrizable.*

(*iii*) *Every nonempty compact set in X belongs to $\mathcal{B}$ and $\mathcal{B}$ has a countable base consisting of compact sets.*

(*iv*) *$\mathcal{B} = \mathcal{K}$ and X is hemicompact.*

(*v*) *$\mathcal{B} = \mathcal{K}$ and X is locally compact and separable.*

Remark 3.9. *It is known that $(C(X), \tau^w)$ is always pseudo-complete, (see [11], [19]). Then for $X = \mathbb{R}$ with the usual metric and $\mathcal{B} = \mathcal{P}_0(X)$, $(C(X), \tau^{sw}_\beta) = (C(X), \tau^w_\beta) = (C(X), \tau^w)$ is pseudo-complete but $\mathcal{B} \nsubseteq \mathcal{K}$. So by Theorem 3.2 in [9], $(C(X), \tau^{sw}_\beta)$ is not metrizable.*

We give an example that shows the metrizability of $(C(X), \tau^{sw}_\beta)$ is not equivalent to the complete metrizability of $(C(X), \tau^{sw}_\beta)$ which discards Corollary 2 in [6]. Moreover, the pseudocompactness of X and $\mathcal{B}$ having a countable base need not imply the complete metrizability of $(C(X), \tau^{sw}_\beta)$.

Example 3.10. *Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \bigcup \{0\}$ with the usual metric and $\mathcal{B}$ be the bornology $\mathcal{F}$ of all finite subsets of X with a countable base $\{\{0, 1, \frac{1}{2}, ..., \frac{1}{n}\} : n \in \mathbb{N}\}$. Since X is a countable non-discrete compact metric space, $(C(X), \tau^{sw}_\beta) = C_p(X)$ is metrizable but not completely metrizable. Also note that $X \notin \mathcal{B}$, then by Lemma 3.1, $C_p(X)$ is not Baire.*

The next example shows that the complete metrizability of $(C(X), \tau^{sw}_\beta)$ need not imply the pseudocompactness of X.

Example 3.11. *Let $X = \mathbb{R}$ with the usual metric and $\mathcal{B}$ be the bornology on X with base $\{[-n, n] : n \in \mathbb{N}\}$. Now $\mathcal{B} = \mathcal{K}$ and X is hemicompact but X is not pseudocompact.*

Note that the complete metrizability of $(C(X), \tau^{sw}_\beta)$ implies the complete metrizability of $(C(X), \tau_\beta)$ and $(C(X), \tau^s_\beta)$. But the following example shows that the converse need not hold.

Example 3.12. *Let $X = \mathbb{R}$ with the usual metric and $\mathcal{B}$ be the bornology on $\mathbb{R}$ with base $\{[n, \infty) : n \in \mathbb{Z}\}$. Observe that every nonempty compact set in X belongs to $\mathcal{B}$. Also, $\mathcal{B}$ is shielded from closed sets as each $[n, \infty)$ is shield of itself. Then both τ_β and τ^s_β are completely metrizable but $\mathcal{B} \nsubseteq \mathcal{K}$, hence $(C(X), \tau^{sw}_\beta)$ is not completely metrizable.*

Recall that, a space X is *pointwise countable type* if each point of X is contained in a compact subset of X having countable character. A subset B of a space X is said to have *countable character* if there exists a countable family $\{W_n : n \in \mathbb{N}\}$ of open subsets of X such that each W_n contains B and for any open set W containing B, there exists some W_n contained in W.

Before showing equivalence of complete metrizability of $(C(X), \tau^{sw}_\beta)$ to other completeness properties, we state the followings.

Lemma 3.13. *Let (X, d) be a metric space, D a dense subset of X and B a compact subset of D. Then B has a countable character in D if and only if B has a countable character in X.*

Proof. Let B has a countable character $\{W_n : n \in \mathbb{N}\}$ in D, then for each n, choose U_n, open subsets of X such that $W_n = U_n \cap D$. Then it is easy to see that B has countable character $\{U_n : n \in \mathbb{N}\}$ in X. □

Theorem 3.14. *The space $(C(X), \tau^{sw}_\beta)$ is a space of point countable type if and only if $(C(X), \tau^{sw}_\beta)$ contains a dense subspace of point countable type.*

Proof. It follows from Lemma 3.13 and homogeneity of $(C(X), \tau^{sw}_\beta)$. □

Theorem 3.15. *Let (X, d) be a metric space and $\mathcal{B}$ be a bornology on X with a closed base. Then the following are equivalent.*

(*i*) *$(C(X), \tau^{sw}_\beta)$ is Čech-complete.*

(*ii*) *$(C(X), \tau^{sw}_\beta)$ is sieve-complete.*

(*iii*) $(C(X), \tau_\beta^{sw})$ *is partition-complete.*

(*iv*) $(C(X), \tau_\beta^{sw})$ *is almost Čech-complete.*

(*v*) $(C(X), \tau_\beta^{sw})$ *is pseudo-complete q-space.*

(*vi*) $(C(X), \tau_\beta^{sw})$ *is Baire q-space.*

(*vii*) *Every nonempty compact set in X belongs to $\mathcal{B}$ and $\mathcal{B}$ has a countable base consisting of compact sets.*

(*viii*) $(C(X), \tau_\beta^{sw})$ *is completely metrizable.*

Proof. (*i*) ⇒ (*ii*) ⇒ (*iii*) ⇒ (*iv*) follows from Proposition 4.4 in [23].

(*iv*) ⇒ (*v*) Since $(C(X), \tau_\beta^{sw})$ has a dense subspace of point countable type, by Theorem 3.14 $(C(X), \tau_\beta^{sw})$ is of point countable type, hence q-space. Now the implication follows from the fact that any almost Čech-complete space is pseudo-complete, (see [1]).

(*v*) ⇒ (*vi*) It is immediate.

(*vi*) ⇒ (*vii*) It can be proved in the same manner as the proof of (*ii*) ⇒ (*iii*) in Theorem 3.7.

(*vii*) ⇒ (*viii*) It directly follows from Theorem 3.7.

(*viii*) ⇒ (*i*) It is obvious. □

Since Theorem 3.14 holds for $(C(X), \tau_\beta^{w})$ also, we have the following result whose proof is similar to the proof of Theorem 3.15.

Theorem 3.16. *Let (X, d) be a metric space and $\mathcal{B}$ be a bornology on X with a closed base. Then the following are equivalent.*

(*i*) $(C(X), \tau_\beta^{w})$ *is Čech-complete.*

(*ii*) $(C(X), \tau_\beta^{w})$ *is sieve-complete.*

(*iii*) $(C(X), \tau_\beta^{w})$ *is partition-complete.*

(*iv*) $(C(X), \tau_\beta^{w})$ *is almost Čech-complete.*

(*v*) $(C(X), \tau_\beta^{w})$ *is pseudo-complete q-space.*

(*vi*) $(C(X), \tau_\beta^{w})$ *is Baire q-space.*

(*vii*) *Every nonempty compact set in X belongs to $\mathcal{B}$ and $\mathcal{B}$ has a countable base consisting of compact sets.*

(*viii*) $(C(X), \tau_\beta^{w})$ *is completely metrizable.*

4. Polishness of τ_β^{sw}

A space X is called Polish if it is completely metrizable and separable.

We show the equivalence of Polishness and complete metrizability of $(C(X), \tau_\beta^{sw})$. Moreover, the complete metrizability of $(C(X), \tau_\beta^{sw})$ agrees with the Poilshness of $(C(X), \tau_\beta^{s})$.

Theorem 4.1. *Let (X, d) be a metric space and $\mathcal{B}$ be a bornology on X with a closed base. Then the following are equivalent.*

(*i*) $(C(X), \tau_\beta^{sw})$ *is completely metrizable.*

(*ii*) $(C(X), \tau_\beta^{sw})$ *is Polish.*

Proof. It is sufficient to prove $(i) \Rightarrow (ii)$. If $(C(X), \tau_\beta^{sw})$ is completely metrizable, then $\mathcal{B} = \mathcal{K}$ and X is hemicompact. So, $\tau_\beta^{sw} = \tau_k$. And it is well known that if X is hemicompact metric space, then $(C(X), \tau_k)$ is separable. □

Similarly, we can prove that the Polishness of $(C(X), \tau_\beta^{w})$ is equivalent to the complete metrizability.

Theorem 4.2. *Let (X, d) be a metric space and $\mathcal{B}$ be a bornology on X with a closed base. Then the following are equivalent.*

(*i*) *$(C(X), \tau_\beta^{w})$ is completely metrizable.*

(*ii*) *$(C(X), \tau_\beta^{w})$ is Polish.*

Polishness of τ_β and τ_β^s not only gives the coincidence of these topologies with the compact-open topology but also the bornology $\mathcal{B}$ is $\mathcal{K}$, (see [15]).

Theorem 4.3. *Let (X, d) be a metric space and $\mathcal{B}$ be a bornology on X with a closed base. Then the following are equivalent.*

(*i*) *$(C(X), \tau_\beta^{s})$ is Polish.*

(*ii*) *$\tau_\beta^{s} = \tau_k$ on $C(X)$ and X is hemicompact.*

(*iii*) *$(C(X), \tau_\beta)$ is Polish.*

(*iv*) *$\tau_\beta = \tau_k$ on $C(X)$ and X is hemicompact.*

(*v*) *$\mathcal{B} = \mathcal{K}$ and X is hemicompact.*

Proof. $(i) \Leftrightarrow (ii)$ By Theorem 4.1 in [15].

$(iii) \Leftrightarrow (iv)$ By Theorem 4.2 in [15].

$(ii) \Leftrightarrow (v) \Leftrightarrow (iv)$ From Theorem 4.8 in [8], it is easy to see that in both the cases $\tau_\beta^{s} = \tau_k$ and $\tau_\beta = \tau_k$, we have $\mathcal{B} = \mathcal{K}$. □

From Theorems 3.7, 3.8, 3.15, 3.16, 4.1, 4.2 and 4.3, we have the following.

Corollary 4.4. *Let (X, d) be a metric space and $\mathcal{B}$ be a bornology on X with a closed base. Then the following are equivalent.*

(*i*) *For every complete metric space (Y, ρ), $(C(X, Y), \tau_\beta^{sw})$ is completely metrizable.*

(*ii*) *For every complete metric space (Y, ρ), $(C(X, Y), \tau_\beta^{w})$ is completely metrizable.*

(*iii*) *$(C(X), \tau_\beta^{sw})$ is completely metrizable.*

(*iv*) *$(C(X), \tau_\beta^{w})$ is completely metrizable.*

(*v*) *$(C(X), \tau_\beta^{sw})$ is Čech-complete.*

(*vi*) *$(C(X), \tau_\beta^{w})$ is Čech-complete.*

(*vii*) *$(C(X), \tau_\beta^{sw})$ is sieve-complete.*

(*viii*) *$(C(X), \tau_\beta^{w})$ is sieve-complete.*

(*ix*) *$(C(X), \tau_\beta^{sw})$ is partition-complete.*

(x) $(C(X), \tau^{w}_{\beta})$ *is partition-complete.*

(xi) $(C(X), \tau^{sw}_{\beta})$ *is almost Čech-complete.*

(xii) $(C(X), \tau^{w}_{\beta})$ *is almost Čech-complete.*

(xiii) $(C(X), \tau^{sw}_{\beta})$ *is pseudo-complete q-space.*

(xiv) $(C(X), \tau^{w}_{\beta})$ *is pseudo-complete q-space.*

(xv) $(C(X), \tau^{sw}_{\beta})$ *is Baire q-space.*

(xvi) $(C(X), \tau^{w}_{\beta})$ *is Baire q-space.*

(xvii) $(C(X), \tau^{sw}_{\beta})$ *is Polish.*

(xviii) $(C(X), \tau^{s}_{\beta})$ *is Polish.*

(xix) $(C(X), \tau^{w}_{\beta})$ *is Polish.*

(xx) $(C(X), \tau_{\beta})$ *is Polish.*

(xxi) *Every nonempty compact set in X belongs to* $\mathcal{B}$ *and* $\mathcal{B}$ *has a countable base consisting of compact sets.*

(xxii) $\mathcal{B} = \mathcal{K}$ *and X is hemicompact.*

(xxiii) $\mathcal{B} = \mathcal{K}$ *and X is locally compact and separable.*

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