Filomat 39:10 (2025), 3297–3313
https://doi.org/10.2298/FIL2510297D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Bipolar fuzzy soft filter and its application to multi-criteria group decision-making

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**Abstract.** The convergence theory is not only a basic theory of topology but also has wide applications in other fields including information technology, economics and computer science. The convergence of filters is also one of the most important tools used in topology to characterize certain concepts such as the closure of a set, continuous mapping, Hausdorff space and so on. Besides, multi-criteria group decision making (for short MCGDM) aims to make unanimous decision based on different criterions to find the most accurate solution of real world problems and so that the MCGDM plays a very important role in our daily life problems. In this paper, taking into account all of these, we firstly introduce the notion of a bipolar fuzzy soft filter (for short BFS-filter) by using bipolar fuzzy soft sets (for short BFS-sets). Also, we define the idea of an ultra BFS-filter and establish some of its properties. Moreover, we investigate the convergence of BFS-filters in a bipolar fuzzy soft topological space (BFS-topological space) with related results. After introducing the concepts of a bipolar fuzzy soft continuity (BFS-continuity) and a bipolar fuzzy soft Hausdorfness (BFS-Hausdorffness), with the aid of the convergence of BFS-filters, we discuss the characterizations of these concepts. Next, we develop a multi-criteria group decision-making method based on the BFS-filters to deal with uncertainties in our daily life. Finally, we present a numerical example to make a decision for selection of best alternative.

## 1. Introduction

Classical methods are inadequate due to the existence of various uncertainties in solving the complex problems in the fields of economics, engineering and environment. In order to overcome with this uncertainty, many theories have been presented. The most known theories are fuzzy set theory introduced by Zadeh [33] in 1965 and rough set theory which was introduced in 1982 by Pawlak [22]. Both of these theories are useful tools to deal with uncertainties. However, as pointed by Molodtsov [20], these theories have their own difficulties and inadequacies because of parameterization tool not being enough. So, Molodtsov [20] invented a new notion named soft set, which handles ambiguities and imprecisions in the parametric manners. Then, a lot of researchers have utilized this theory as a powerful tool to define uncertainties. For

2020 *Mathematics Subject Classification*. Primary 54A20; Secondary 06D72, 54A40.
*Keywords*. Bipolar fuzzy soft set, BFS-filter, ultra BFS-filter, convergence, BFS-continuity, BFS-Hausdorffness, multi-criteria group decision making.
Received: 03 June 2024; Revised: 22 January 2025; Accepted: 28 January 2025
Communicated by Ljubiša D. R. Kočinac
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example, Maji et al. [19] investigated the terms such as subset, union, intersection and complement for the soft sets. Moreover, Maji et al. [18] introduced a more general concept, which is a combination of fuzzy set and soft set; the fuzzy soft set. Ali et al. [3] proposed new operations of the algebraic nature on soft sets and studied their properties. Shabir and Naz [28] investigated soft topological spaces. Later, Al-shami [5] applied soft compactness on ordered settings to expect the missing values on the information systems. Moreover, Alcantud [2] introduced the formal model consisting of convex soft geometries and studied how he can associate a convex geometry with each convex soft geometry, and conversely. By using a combination of soft sets and grey numbers, Voskoglou [32] introduced a new parametric decision-making method. Next, Kharal and Ahmad [15] introduced a mapping on the classes of fuzzy soft sets and also studied the properties of fuzzy soft images. Then, Varol and Aygün [31] established the fuzzy soft topological spaces. Later, Peng and Garg [23] solved the comparison problem by new score function in intuitionistic fuzzy soft (IFS) environment and explored some novel properties of IFS matrix.

Fuzzy sets are unable to represent the satisfaction degree to counter-property although they are able to represent uncertainties in membership degree assignments. In order to get over this problem, Lee [17] introduced the concept of a bipolar valued fuzzy set which the membership degree range is [-1,1], making the coexistence of negativity and positivity. In a bipolar valued fuzzy set, the membership value 0 of an element shows that the element is irrelevant to the corresponding property, the membership degree (0, 1] of an element means that the element somewhat satisfies the property, and the membership degree [-1, 0) of an element shows that the element somewhat satisfies the implicit counter-property. This concept is significant in human thought because human decision making is based on positive and negative thinkings. Afterwards, Abdullah et al. [1] and Naz and Shabir [21] defined independently bipolar fuzzy soft sets, combining both the bipolar fuzzy sets and the soft sets. Riaz and Tehrim [26] initiated the idea of a bipolar fuzzy soft topology and discussed certain aspects of BFS-topology. Moreover, Riaz and Tehrim [24] indicated the concept of mappings between BFS-sets and applied this concept to the problem of medical diagnosis. Later, Gwak et al. [13] presented the concept of a bipolar complex intuitionistic fuzzy soft set and explained its basic operations including complement, union, and intersection with some appropriate examples. In recent years, there has been a considerable literature on the BFS-sets and their applications [4, 9, 11, 14, 25, 27, 30, 34].

In topology, a subfield of mathematics, filters are used to study the basic topological concepts such as convergent, continuity, compactness, and more. Also, the filters play an important role in investigating different domains of mathematics like analysis and algebra. Therefore, the problem of extensions of filters have been tackled by many authors. By Vicente and Aranguren [10], the notion of a fuzzy filter appeared for the first time. Afterwards, Kim et al. [16] proposed a new definition of fuzzy filter. Şahin and Küçük [29] defined the soft filters and studied some of their properties. By using fuzzy soft sets, Çetkin and Aygün [6] introduced fuzzy soft filters on the base of definition suggested by Kim et al. [16]. Demir et al. [8] established the convergence theory of fuzzy soft filters. Also, with the use of Q-neighborhoods, Gao and Wu [12] redefined the concept of fuzzy soft filter convergence. Dalkılıç and Demirtaş [7] presented the notion of a bipolar soft filter and investigated some of its basic features.

Inspired by these works we introduce the concept of a BFS-filter and study its convergence properties. Then, we obtain the basic results of filters under bipolar fuzzy soft environment and provide suitable examples to illustrate the effectiveness of the proposed results. Also, we discuss the notion of a BFS-continuous mapping between the BFS-topological spaces and analyze its connection with the BFS-filters. In addition, we prove that a BFS-filter converges to unique bipolar fuzzy soft point in a BFS-Hausdorff space. Afterwards, we construct an algorithm based on the BFS-filters. Finally, we apply it to a real-world problem to demonstrate the applicability of the obtained algorithm.

## 2. Preliminaries

In this section, we recall some basic notions regarding the BFS-sets which will be used in the sequel.

Throughout this paper, $U$ be a universe of alternatives (objects) and $E$ be a set of specified parameters (criteria or attributes) unless otherwise explicit.

**Definition 2.1.** ([17]) *Consider a universal set $U$. A set having form*

$$\eta = \{(u, \delta^+_\eta(u), \delta^-_\eta(u)) : u \in U\}$$

*denotes a bipolar fuzzy set on $U$, where $\delta^+_\eta(u)$ denotes the positive memberships ranges over $[0, 1]$ and $\delta^-_\eta(u)$ denotes the negative memberships ranges over $[-1, 0]$.*

**Definition 2.2.** ([17]) *Let $\eta_1$ and $\eta_2$ be two bipolar fuzzy sets on $U$. Then, their intersection and union are defined as follows:*

(*i*) $\eta_1 \wedge \eta_2 = \Big\{\Big(u, \min\{\delta^+_{\eta_1}(u), \delta^+_{\eta_2}(u)\}, \max\{\delta^-_{\eta_1}(u), \delta^-_{\eta_2}(u)\}\Big) : u \in U\Big\}$.

(*ii*) $\eta_1 \vee \eta_2 = \Big\{\Big(u, \max\{\delta^+_{\eta_1}(u), \delta^+_{\eta_2}(u)\}, \min\{\delta^-_{\eta_1}(u), \delta^-_{\eta_2}(u)\}\Big) : u \in U\Big\}$.

**Definition 2.3.** ([1, 21]) *Consider a universal set $U$ and a set of parameters $E$. Let $A \subseteq E$ and define a mapping $\Omega : E \to BF^U$, where $BF^U$ represents the family of all bipolar fuzzy subsets of $U$. Then, $\Omega_A$ is called a BFS-set on $U$, where*

$$\Omega_A = \{\langle e, \Omega(e)\rangle : e \in E\}$$

*such that $\delta^+_{\Omega(e)}(u) = \delta^-_{\Omega(e)}(u) = 0$ for all $e \notin A$ and all $u \in U$.*

Note that the set of all bipolar fuzzy soft sets on $U$ with the attributes from $E$ is denoted by $(BF^U)^E$.

In order to better understand the above definition, consider the following illustrative example.

**Example 2.4.** *Suppose that Mrs. X wants to buy a model graphics card and let $E = \{e_1 =$ CUDA cores, $e_2 =$ base clock rate, $e_3 =$ GPU boost rate, $e_4 =$ memory capacity$\}$ be the set of decision variables. Afterwards, consider the set of three types of model graphics cards $U = \{u_1, u_2, u_3\}$ by keeping in view the requirements of Mrs. X. After a research, we show that a website has assigned the numerical values for each decision variable to three model graphics cards, taking into account the positive and negative feedbacks based on the customers. The tabular representation of these numerical values is as follows:*

**Table 1**
Tabular reprentation of positive feedbacks

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|---|---|---|---|---|
| $u_1$ | 0.2 | 0.5 | 0.38 | 0.6 |
| $u_2$ | 0.3 | 0.73 | 0.84 | 0.25 |
| $u_3$ | 0.46 | 0.26 | 0.25 | 0.45 |

**Table 2**
Tabular reprentation of negative feedbacks

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|---|---|---|---|---|
| $u_1$ | −0.43 | −0.65 | −0.45 | −0.45 |
| $u_2$ | −0.44 | −0.55 | −0.42 | −0.55 |
| $u_3$ | −0.15 | −0.35 | −0.85 | −0.55 |

*Therefore, the following bipolar fuzzy soft set on U with the set E of decision variables reporting the positive-negative informations is obtained:*

$$\Omega_A = \left\{ \begin{array}{l} \langle e_1, \Omega(e_1) = \{(u_1, 0.2, -0.43), (u_2, 0.3, -0.44), (u_3, 0.46, -0.15)\}\rangle, \\ \langle e_2, \Omega(e_2) = \{(u_1, 0.5, -0.65), (u_2, 0.73, -0.55), (u_3, 0.26, -0.35)\}\rangle, \\ \langle e_3, \Omega(e_3) = \{(u_1, 0.38, -0.45), (u_2, 0.84, -0.42), (u_3, 0.25, -0.85)\}\rangle, \\ \langle e_4, \Omega(e_4) = \{(u_1, 0.6, -0.45), (u_2, 0.25, -0.55), (u_3, 0.45, -0.55)\}\rangle \end{array} \right\}.$$

**Definition 2.5.** ([34])

(*i*) *A BFS-set* $\Omega_E \in (BF^U)^E$ *is called an absolute BFS-set, denoted by* $U_E$, *if* $\delta^+_{\Omega(e)}(u) = 1$ *and* $\delta^-_{\Omega(e)}(u) = -1$ *for all* $u \in U$ *and all* $e \in E$.

(*ii*) *A BFS-set* $\Omega_A \in (BF^U)^E$ *is called a null BFS-set, denoted by* $\phi_A$, *if* $\delta^+_{\Omega(e)}(u) = \delta^-_{\Omega(e)}(u) = 0$ *for all* $u \in U$ *and all* $e \in A$.

**Definition 2.6.** ([1, 21]) *Let* $\Omega^1_{A_1}, \Omega^2_{A_2} \in (BF^U)^E$. *Then,*

(*i*) *The union of* $\Omega^1_{A_1}$ *and* $\Omega^2_{A_2}$ *is a bipolar fuzzy soft set* $\Omega^3_{A_3}$ *over U such that for all* $e \in E$, $\Omega^3(e) = \Omega^1(e) \vee \Omega^2(e)$ *and denoted by* $\Omega^3_{A_3} = \Omega^1_{A_1} \tilde{\cup}\, \Omega^2_{A_2}$.

(*ii*) *The intersection of* $\Omega^1_{A_1}$ *and* $\Omega^2_{A_2}$ *is a bipolar fuzzy soft set* $\Omega^3_{A_3}$ *over U such that for all* $e \in E$, $\Omega^3(e) = \Omega^1(e) \wedge \Omega^2(e)$ *and denoted by* $\Omega^3_{A_3} = \Omega^1_{A_1} \tilde{\cap}\, \Omega^2_{A_2}$.

**Definition 2.7.** ([1, 21]) *The complement of a BFS-set* $\Omega_A \in (BF^U)^E$ *is shown by* $(\Omega_A)^c = \Omega^c_{A_1}$ *where* $\Omega^c : E \to BF^U$ *is a mapping defined by* $\delta^+_{\Omega^c(e)}(u) = 1 - \delta^+_{\Omega(e)}(u)$ *and* $\delta^-_{\Omega^c(e)}(u) = -1 - \delta^-_{\Omega(e)}(u)$ *for all* $e \in E$ *and* $u \in U$.

**Definition 2.8.** ([34]) *Let* $\Omega^1_{A_1}, \Omega^2_{A_2} \in (BF^U)^E$. *Then,* $\Omega^1_{A_1}$ *is a BFS-subset of* $\Omega^2_{A_2}$ *if* $\delta^+_{\Omega^1(e)}(u) \leq \delta^+_{\Omega^2(e)}(u)$, $\delta^-_{\Omega^1(e)}(u) \geq \delta^-_{\Omega^2(e)}(u)$, *which is shown by* $\Omega^1_{A_1} \tilde{\subseteq} \Omega^2_{A_2}$.

**Theorem 2.9.** ([21]) *Let* $\Omega^1_{A_1}, \Omega^2_{A_2}$ *be two BFS-sets over U. Then, the following statements hold:*

(*i*) $((\Omega^1_{A_1})^c)^c = \Omega^1_{A_1}$.

(*ii*) *If* $\Omega^1_{A_1} \tilde{\subseteq} \Omega^2_{A_2}$, *then* $(\Omega^2_{A_2})^c \tilde{\subseteq} (\Omega^1_{A_1})^c$.

(*iii*) $(\Omega^1_{A_1} \tilde{\cap}\, \Omega^2_{A_2})^c = (\Omega^1_{A_1})^c \tilde{\cup}\, (\Omega^2_{A_2})^c$.

(*iv*) $(\Omega^1_{A_1} \tilde{\cup}\, \Omega^2_{A_2})^c = (\Omega^1_{A_1})^c \tilde{\cap}\, (\Omega^2_{A_2})^c$.

**Definition 2.10.** ([9]) *Let* $\Omega_A \in (BF^U)^E$ *with* $A = \{e\}$. *If there is a* $u \in U$ *such that* $\delta^+_{\Omega(e)}(u) \neq 0$ *or* $\delta^-_{\Omega(e)}(u) \neq 0$ *and* $\delta^+_{\Omega(e)}(u') = \delta^-_{\Omega(e)}(u') = 0$ *for all* $u' \in U \backslash \{u\}$, *then* $\Omega_A$ *is called a BFS-point in U. It is denoted by* $e^{(p,n)}_u$.

Let $\mathcal{P}(U, E)$ be the family of all BFS-points on $U$.

**Definition 2.11.** ([9]) *The BFS-point* $e^{(p,n)}_u$ *is said to belongs to a BFS-set* $\Omega_A$, *denoted by* $e^{(p,n)}_u \tilde{\in} \Omega_A$, *if* $p \leq \delta^+_{\Omega(e)}(u)$ *and* $n \geq \delta^-_{\Omega(e)}(u)$.

**Proposition 2.12.** ([9]) *Let* $\{\Omega^i_{A_i} : i \in J\}$ *be a family of BFS-sets over U and* $e^{(p,n)}_u$ *be a BFS-point. Then, the following statements hold:*

(*i*) $e^{(p,n)}_u \tilde{\in} \widetilde{\bigcap}_{i\in J} \Omega^i_{A_i}$ *if and only if* $e^{(p,n)}_u \tilde{\in}\, \Omega^i_{A_i}$ *for all* $i \in J$.

(ii) $e_u^{(p,n)} \tilde{\in} \widetilde{\bigcup}_{i \in J} \Omega^i_{A_i}$ *if there exists an* $i_0 \in J$ *such that* $e_u^{(p,n)} \tilde{\in} \Omega^{i_0}_{A_{i_0}}$.

**Definition 2.13.** ([24]) *Let* $(BF^U)^E$ *and* $(BF^V)^D$ *be two the families of all bipolar fuzzy soft sets on U and V with the parameters from E and D, respectively. Assume that* $\mathfrak{u} : U \to V$ *and* $\mathfrak{g} : E \to D$ *be two mappings. Then, the mapping* $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : (BF^U)^E \to (BF^V)^D$ *is called a BFS-mapping from U to V, defined as the following :*

*(i) Let* $\Omega_A \in (BF^U)^E$. *Then,* $\mathfrak{f}(\Omega_A) = (\mathfrak{f}(\Omega))_{A_1}$ *is the BFS-set over V with the parameters from D given by* $\mathfrak{f}(\Omega_A) = \{\langle d, \mathfrak{f}(\Omega)(d)\rangle : d \in D\}$ *such that* $\mathfrak{f}(\Omega)(d) = \{(v, \delta^+_{\mathfrak{f}(\Omega)(d)}(v), \delta^-_{\mathfrak{f}(\Omega)(d)}(v)) : v \in V\}$, *where*

$$\delta^+_{\mathfrak{f}(\Omega)(d)}(v) = \begin{cases} \sup\{\delta^+_{\Omega(e)}(u) : u \in \mathfrak{u}^{-1}(v), e \in \mathfrak{g}^{-1}(d) \cap A\}, & \text{if } \mathfrak{u}^{-1}(v) \neq \emptyset, \mathfrak{g}^{-1}(d) \cap A \neq \emptyset, \\ 0, & \text{if otherwise,} \end{cases}$$

$$\delta^-_{\mathfrak{f}(\Omega)(d)}(v) = \begin{cases} \inf\{\delta^-_{\Omega(e)}(u) : u \in \mathfrak{u}^{-1}(v), e \in \mathfrak{g}^{-1}(d) \cap A\}, & \text{if } \mathfrak{u}^{-1}(v) \neq \emptyset, \mathfrak{g}^{-1}(d) \cap A \neq \emptyset, \\ 0, & \text{if otherwise.} \end{cases}$$

*Then,* $\mathfrak{f}(\Omega_A)$ *is called a BFS-image of BFS-set* $\Omega_A$ *under* $\mathfrak{f}$.

*(ii) Let* $\Omega^1_{A_1} \in (BF^V)^D$. *Then,* $\mathfrak{f}^{-1}(\Omega^1_{A_1}) = (\mathfrak{f}^{-1}(\Omega^1))_A$ *is the BFS-set over U with the parameters from E given by* $\mathfrak{f}^{-1}(\Omega^1_{A_1}) = \{\langle e, \mathfrak{f}^{-1}(\Omega^1)(e)\rangle : e \in E\}$ *such that* $\mathfrak{f}^{-1}(\Omega^1)(e) = \{(u, \delta^+_{\mathfrak{f}^{-1}(\Omega^1)(e)}(u), \delta^-_{\mathfrak{f}^{-1}(\Omega^1)(e)}(u)) : u \in U\}$, *where*

$$\delta^+_{\mathfrak{f}^{-1}(\Omega^1)(e)}(u) = \begin{cases} \delta^+_{\Omega^1(\mathfrak{g}(e))}(\mathfrak{u}(u)), & \text{if } \mathfrak{g}(e) \in A_1, \\ 0, & \text{if otherwise,} \end{cases}$$

$$\delta^-_{\mathfrak{f}^{-1}(\Omega^1)(e)}(u) = \begin{cases} \delta^-_{\Omega^1(\mathfrak{g}(e))}(\mathfrak{u}(u)), & \text{if } \mathfrak{g}(e) \in A_1, \\ 0, & \text{if otherwise.} \end{cases}$$

*Then,* $\mathfrak{f}^{-1}(\Omega^1_{A_1})$ *is called a BFS-inverse image of BFS-set* $\Omega^1_{A_1}$.

**Theorem 2.14.** ([24]) *Let* $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : (BF^U)^E \to (BF^V)^D$ *be a BFS-mapping. Then, for* $\Omega^1_{A_1}$, $\Omega^2_{A_2} \in (BF^U)^E$ *and* $\Gamma^1_{B_1}$, $\Gamma^2_{B_2} \in (BF^V)^D$, *the following properties are satisfied:*

(i) $\mathfrak{f}(\phi_A) = \phi_A$, $\mathfrak{f}^{-1}(\phi_A) = \phi_A$.

(ii) $\mathfrak{f}(\Omega^1_{A_1} \tilde{\cup} \Omega^2_{A_2}) = \mathfrak{f}(\Omega^1_{A_1}) \tilde{\cup} \mathfrak{f}(\Omega^2_{A_2})$.

(iii) $\mathfrak{f}^{-1}(\Gamma^1_{B_1} \tilde{\cup} \Gamma^2_{B_2}) = \mathfrak{f}^{-1}(\Gamma^1_{B_1}) \tilde{\cup} \mathfrak{f}^{-1}(\Gamma^2_{B_2})$.

(iv) $\mathfrak{f}(\Omega^1_{A_1} \tilde{\cap} \Omega^2_{A_2}) \tilde{\subseteq} \mathfrak{f}(\Omega^1_{A_1}) \tilde{\cap} \mathfrak{f}(\Omega^2_{A_2})$.

(v) $\mathfrak{f}^{-1}(\Gamma^1_{B_1} \tilde{\cap} \Gamma^2_{B_2}) = \mathfrak{f}^{-1}(\Gamma^1_{B_1}) \tilde{\cap} \mathfrak{f}^{-1}(\Gamma^2_{B_2})$.

(vi) $\Omega^1_{A_1} \tilde{\subseteq} \mathfrak{f}^{-1}(\mathfrak{f}(\Omega^1_{A_1}))$, $\mathfrak{f}(\mathfrak{f}^{-1}(\Gamma^1_{B_1})) \tilde{\subseteq} \Gamma^1_{B_1}$.

(vii) *If* $\Omega^1_{A_1} \tilde{\subseteq} \Omega^2_{A_2}$, *then* $\mathfrak{f}(\Omega^1_{A_1}) \tilde{\subseteq} \mathfrak{f}(\Omega^2_{A_2})$.

(viii) *If* $\Gamma^1_{B_1} \tilde{\subseteq} \Gamma^2_{B_2}$, *then* $\mathfrak{f}^{-1}(\Gamma^1_{B_1}) \tilde{\subseteq} \mathfrak{f}^{-1}(\Gamma^2_{B_2})$.

**Definition 2.15.** ([26]) *A family* $\tau$ *of BFS-sets over U is said to be a BFS-topology on U if it satisfies the following properties:*
*(BFST1)* $U_E$ *and* $\phi_A$ *are members of* $\tau$,
*(BFST2) If* $\Omega^i_{A_i} \in \tau$ *for all* $i \in J$, *an index set, then* $\widetilde{\bigcup}_{i \in J} \Omega^i_{A_i} \in \tau$,
*(BFTS3) If* $\Omega^1_{A_1}$, $\Omega^2_{A_2} \in \tau$, *then* $\Omega^1_{A_1} \tilde{\cap} \Omega^2_{A_2} \in \tau$.

*We say* $(U, \tau, E)$ *is a BFS-topological space. A member in* $\tau$ *is called a BFS-open set and its complement is called a BFS-closed set.*

**Definition 2.16.** ([9]) *Let $(U, \tau, E)$ be a BFS-topological space. A BFS-set $\Omega_A$ is called a BFS-neighborhood of a BFS-point $e_u^{(p,n)}$ if there is a BFS-open set $\Omega^1_{A_1}$ such that $e_u^{(p,n)} \tilde{\in} \Omega^1_{A_1} \tilde{\subseteq} \Omega_A$.*

*The collection of all BFS-neighborhoods of $e_u^{(p,n)}$ is called a BFS-neighborhood system of $e_u^{(p,n)}$ and denoted by $\mathcal{N}(e_u^{(p,n)})$.*

**Theorem 2.17.** ([9])*] Let $(U, \tau, E)$ be a BFS-topological space and $\mathcal{N}(e_u^{(p,n)})$ be a BFS-neighborhood system of a BFS-point $e_u^{(p,n)}$. Then, we get the followings:*

*(BFSN1) $\mathcal{N}(e_u^{(p,n)}) \neq \phi$.*

*(BFSN2) If $\Omega_A \in \mathcal{N}(e_u^{(p,n)})$, then $e_u^{(p,n)} \tilde{\in} \Omega_A$.*

*(BFSN3) If $\Omega_A \in \mathcal{N}(e_u^{(p,n)})$ and $\Omega_A \tilde{\subseteq} \Omega^1_{A_1}$, then $\Omega^1_{A_1} \in \mathcal{N}(e_u^{(p,n)})$.*

*(BFSN4) If $\Omega^1_{A_1}, \Omega^2_{A_2} \in \mathcal{N}(e_u^{(p,n)})$, then $\Omega^1_{A_1} \tilde{\cap} \Omega^2_{A_2} \in \mathcal{N}(e_u^{(p,n)})$.*

*(BFSN5) If $\Omega_A \in \mathcal{N}(e_u^{(p,n)})$, then there exists an $\Gamma_B \in \mathcal{N}(e_u^{(p,n)})$ with $\Gamma_B \tilde{\subseteq} \Omega_A$ and $\Omega_A \in \mathcal{N}(d_v^{(p',n')})$ for all $d_v^{(p',n')} \tilde{\in} \Gamma_B$.*

**Theorem 2.18.** ([9]) *Let each BFS-point $e_u^{(p,n)} \in \mathcal{P}(U, E)$ be satisfy the condition*

$$if\ e_u^{(p,n)} \tilde{\in} \widetilde{\bigcup}_{i \in J} \Omega^i_{A_i},\ then\ there\ exists\ an\ i_0 \in J\ such\ that\ e_u^{(p,n)} \tilde{\in} \Omega^{i_0}_{A_{i_0}}.$$

*If for each BFS-point $e_u^{(p,n)}$, there is a collection $\mathcal{N}(e_u^{(p,n)})$ of subsets of $(BF^U)^E$ such that the conditions (BFSN1)-(BFSN5) are satisfied, then there is a BFS-topology $\tau$ on $U$ such that, for each $e_u^{(p,n)} \in \mathcal{P}(U, E)$, $\mathcal{N}(e_u^{(p,n)})$ is the $\tau$-BFS neighborhood system of $e_u^{(p,n)}$.*

**Theorem 2.19.** ([9]) *Let $(U, \tau, E)$ be a BFS-topological space and $\Omega_A \in (BF^U)^E$. Then, $\Omega_A$ is a BFS-neighborhood of each of its BFS-points if and only if it is a BFS-open set.*

## 3. Bipolar fuzzy soft filter

In this section, we bring out the notion of a BFS-filter by using bipolar fuzzy soft sets and study some fundamental properties of it. Also, we present the concepts of a BFS-filter base and an ultra BFS-filter and obtain their related properties. Next, we show how a BFS-topology is derived from a BFS-filter.

**Definition 3.1.** A BFS-filter $\mathcal{F}$ on $U$ is a nonempty collection of subsets of $(BF^U)^E$ if it satisfies the following conditions:

$(BFSF1)$ $\phi_A \notin \mathcal{F}$,

$(BFSF2)$ If $\Omega^1_{A_1}, \Omega^2_{A_2} \in \mathcal{F}$, then $\Omega^1_{A_1} \tilde{\cap} \Omega^2_{A_2} \in \mathcal{F}$,

$(BFSF3)$ If $\Omega^1_{A_1} \in \mathcal{F}$ and $\Omega^1_{A_1} \tilde{\subseteq} \Omega^2_{A_2}$ then $\Omega^2_{A_2} \in \mathcal{F}$.

**Example 3.2.** *For each $\alpha \in (0, 1]$,*

$$\mathcal{F}_\alpha = \{\Omega_E \in (BF^U)^E : \delta^+_{\Omega(e)}(u) \geq \alpha,\ \delta^-_{\Omega(e)}(u) \leq -\alpha\ for\ all\ e \in E\ and\ u \in U\}$$

*is a BFS-filter on $U$.*

**Definition 3.3.** Let $\Omega_A \in (BF^U)^E$.

($i$) If there are at most finitely many $e \in A$ such that $\Omega(e) \neq \overline{0}$, then $\Omega_A$ is called a finite BFS-set.

($ii$) If there are at most countably many $e \in A$ such that $\Omega(e) \neq \overline{0}$, then $\Omega_A$ is called a countable BFS-set.

**Example 3.4.** *Let $U = \{u_1, u_2\}$, $E = \{e_1, e_2, e_3, ...\}$. Then,*

$$\Omega_E = \left\{ \begin{array}{l} \langle e_1, \Omega(e_1) = \{(u_1, 0.2, 0), (u_2, 0.3, -0.44)\}\rangle, \\ \langle e_2, \Omega(e_2) = \{(u_1, 0.5, -0.65), (u_2, 0, -0.55)\}\rangle, \\ \langle e_3, \Omega(e_3) = \{(u_1, 0, 0), (u_2, 0, 0)\}\rangle, \\ \langle e_4, \Omega(e_4) = \{(u_1, 0, 0), (u_2, 0, 0)\}\rangle, \\ \vdots \end{array} \right\}$$

*(that is, $\Omega(e) = \bar{0}$ for all $e \in E \backslash \{e_1, e_2\}$) is a finite BFS-set.*

**Example 3.5.** *Let $U = \{u_1, u_2\}$, $E = \{e_i : i \in \mathbb{R}\}$ and $A = \{e_i : i \in \mathbb{N}\}$. Consider a BFS-set $\Omega_A = \{\langle e, \Omega(e)\rangle : e \in E\}$. Then, $\Omega_A$ is a countable BFS-set.*

**Remark 3.6.** *It is clear that the union of a finite family of finite BFS-sets is a finite BFS-set and also the union of a countably infinite family of countable BFS-sets is a countable BFS-set.*

**Example 3.7.** *(i) Let U be any set and E be an infinite set. Then,*

$$\mathcal{F} = \{\Omega_A \in (BF^U)^E : (\Omega_A)^c \ is\ finite\ BFS-set\}$$

*is a BFS-filter on U.*

*(ii) Let U be any set and E be an uncountable set. Then,*

$$\mathcal{F} = \{\Omega_A \in (BF^U)^E : (\Omega_A)^c \ is\ countable\ BFS-set\}$$

*is a BFS-filter on U.*

**Definition 3.8.** Let $\mathcal{B}$ be a subcollection of a BFS-filter $\mathcal{F}$ such that for every $\Omega_A \in \mathcal{F}$, there is an $\Omega^1_{A_1} \in \mathcal{B}$ with $\Omega^1_{A_1} \tilde{\subseteq} \Omega_A$. We call such an $\mathcal{B}$ a BFS-base of $\mathcal{F}$.

From definition above we can see that $\mathcal{B}$ satisfies:
$(B_1)$ $\phi_A \notin \mathcal{B}$,
$(B_2)$ for every $\Omega^1_{A_1}, \Omega^2_{A_2} \in \mathcal{B}$, there exists a $\Omega^3_{A_3} \in \mathcal{B}$ with $\Omega^3_{A_3} \tilde{\subseteq} \Omega^1_{A_1} \widetilde{\cap} \Omega^2_{A_2}$.
Conversely, given a collection $\mathcal{B}$ satisfying $(B_1)$ and $(B_2)$, then putting

$$\mathcal{F} = \{\Omega_A : \Omega^1_{A_1} \tilde{\subseteq} \Omega_A \ for\ some\ \Omega^1_{A_1} \in \mathcal{B}\}$$

we get a BFS-filter $\mathcal{F}$ which contains $\mathcal{B}$ as a BFS-base. $\mathcal{B}$ is said to generate $\mathcal{F}$ .

**Example 3.9.** *Let $\Omega_A$ be a non-null BFS-set. Thus, $\mathcal{B} = \{\Omega_A\}$ is a BFS-base for a BFS-filter on U.*

**Theorem 3.10.** *Let $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : (BF^U)^E \to (BF^V)^D$ be a BFS-mapping and let $\mathcal{F}$ be a BFS-filter on U. Then, $\mathcal{B}^* = \{\mathfrak{f}(\Omega_A) : \Omega_A \in \mathcal{F}\}$ is a BFS-base for a BFS-filter $\mathfrak{f}(\mathcal{F})$ on V.*

*Proof.* $(B_1)$ is obvious.

$(B_2)$ Suppose that $\mathfrak{f}(\Omega^1_{A_1})$, $\mathfrak{f}(\Omega^2_{A_2}) \in \mathcal{B}^*$. Then, we have to show that $\mathfrak{f}(\Omega^3_{A_3}) \in \mathcal{B}^*$ such that $\mathfrak{f}(\Omega^3_{A_3}) \subseteq \mathfrak{f}(\Omega^1_{A_1}) \tilde{\cap} \mathfrak{f}(\Omega^2_{A_2})$. Since $\Omega^1_{A_1} \tilde{\cap} \Omega^2_{A_2} \in \mathcal{F}$, we obtain $\mathfrak{f}(\Omega^1_{A_1} \tilde{\cap} \Omega^2_{A_2}) \in \mathcal{B}^*$. Then, by Theorem 2.14, this implies $\mathfrak{f}(\Omega^1_{A_1} \tilde{\cap} \Omega^2_{A_2}) \tilde{\subseteq} \mathfrak{f}(\Omega^1_{A_1}) \tilde{\cap} \mathfrak{f}(\Omega^2_{A_2})$. Thus, the proof ends. □

**Definition 3.11.** Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two BFS-filters on $U$. Then, we say that $\mathcal{F}_2$ is finer than $\mathcal{F}_1$ (or $\mathcal{F}_1$ is coarser than $\mathcal{F}_2$) if $\mathcal{F}_2 \supseteq \mathcal{F}_1$ .

**Definition 3.12.** If a BFS-filter $\mathcal{F}$ on $U$ has the property that there is no BFS-filter on $U$ which is finer than $\mathcal{F}$, then $\mathcal{F}$ is called an ultra BFS-filter on $U$.

**Theorem 3.13.** *Let $\mathcal{F}$ be a BFS-filter on U. Then, there exists an ultra BFS-filter $\mathcal{F}^*$ on U such that $\mathcal{F} \subseteq \mathcal{F}^*$.*

*Proof.* Let $\mathbb{F}$ be a collection that contains all the BFS-filters finer than $\mathcal{F}$ on $U$ such that it is partially ordered by the relation "$\supseteq$" given in Definition 3.11. Consider an chain $\{\mathcal{F}_i : i \in J\} \subseteq \mathbb{F}$. Then $\bigcup_{i\in J} \mathcal{F}_i$ is a BFS-filter on $U$ and also it is an upper bound of $\{\mathcal{F}_i : i \in J\}$. Using Zorn's Lemma, we see that $\mathbb{F}$ has a maximal element $\mathcal{F}^*$. Hence, $\mathcal{F}^*$ is a ultra BFS-filter containing $\mathcal{F}$. □

**Lemma 3.14.** *Let $\mathcal{A}$ be a family of BFS-sets satisfying the finite intersection property. Then, there exists a BFS-filter $\mathcal{F}$ on U with $\mathcal{A} \subseteq \mathcal{F}$.*

*Proof.* Consider that $\mathcal{F}$ is the family consisting of all $\Omega_A \in (BF^U)^E$ such that there exists a finite set $\{\Omega^1_{A_1}, \Omega^2_{A_2}, \Omega^3_{A_3}, ..., \Omega^n_{A_n}\} \subseteq \mathcal{A}$ satisfying $\Omega^1_{A_1} \tilde{\cap}\, \Omega^2_{A_2} \tilde{\cap}\, \Omega^3_{A_3} \tilde{\cap} ... \tilde{\cap}\, \Omega^n_{A_n} \tilde{\subseteq} \Omega_A$. So, $\mathcal{F}$ is a BFS-filter including $\mathcal{A}$. □

**Theorem 3.15.** *Let $\mathcal{F}$ be a BFS-filter on U. Then, the following results hold:*

(*i*) *$\mathcal{F}$ is an ultra BFS-filter on U if and only if all $\Gamma_B \in (BF^U)^E$ satisfying $\Gamma_B \tilde{\cap}\, \Omega_A \neq \phi_A$ for every $\Omega_A \in \mathcal{F}$ is contained in $\mathcal{F}$.*

(*ii*) *If $\mathcal{F}$ is an ultra BFS-filter on U and $\Omega^1_{A_1} \tilde{\cup}\, \Omega^2_{A_2} \in \mathcal{F}$, then we have $\Omega^1_{A_1} \in \mathcal{F}$ or $\Omega^2_{A_2} \in \mathcal{F}$.*

(*iii*) *If $\mathcal{F}$ is an ultra BFS-filter on U, then for all $\Omega_A \in (BF^U)^E$, we get $\Omega_A \in \mathcal{F}$ or $(\Omega_A)^c \in \mathcal{F}$.*

*Proof.* (i) Let $\mathcal{F}$ be an ultra BFS-filter on $U$. Consider that $\Gamma_B \in (BF^U)^E$ satisfying $\Omega_A \tilde{\cap}\, \Gamma_B \neq \phi_A$ for all $\Omega_A \in \mathcal{F}$. Take $\mathcal{M} = \mathcal{F} \cup \{\Gamma_B\}$. Because $\mathcal{M}$ has the finite intersection property, from Lemma 3.14, there is a BFS-filter $\mathcal{L}$ on $U$ with $\mathcal{M} \subseteq \mathcal{L}$. So, from Definition 3.12, it is easily seen that $\mathcal{F} = \mathcal{L}$. Thus, we obtain $\Gamma_B \in \mathcal{F}$.
Conversely, assume that $\mathcal{F}$ includes all $\Gamma_B \in (BF^U)^E$ such that $\Omega_A \tilde{\cap}\, \Gamma_B \neq \phi_A$ for all $\Omega_A \in \mathcal{F}$. Let us create a BFS-filter $\mathcal{M}$ satisfying $\mathcal{F} \subseteq \mathcal{M}$. Therefore, we have $\Lambda_C \in (BF^U)^E$ such that $\Lambda_C \in \mathcal{M}$ and $\Lambda_C \notin \mathcal{F}$. Now, choose any $\Omega_A \in \mathcal{F}$. Therefore, by the property of BFS-filter, we have $\Omega_A \tilde{\cap}\, \Lambda_C \neq \phi_A$. Thus, from hypothesis, it follows that $\Lambda_C \in \mathcal{F}$, which leads to a contradiction.
(ii) Let $\Omega^1_{A_1}, \Omega^2_{A_2} \notin \mathcal{F}$ and assume that $\Omega_A = \Omega^1_{A_1} \tilde{\cup}\, \Omega^2_{A_2} \in \mathcal{F}$. From (i) there are $\Gamma^1_{B_1}, \Gamma^2_{B_2} \in \mathcal{F}$ such that $\Omega^1_{A_1} \tilde{\cap}\, \Gamma^1_{B_1} = \Omega^2_{A_2} \tilde{\cap}\, \Gamma^2_{B_2} = \phi_A$. If $\Gamma_B = \Gamma^1_{B_1} \tilde{\cap}\, \Gamma^2_{B_2}$, then we possess $\Gamma_B \in \mathcal{F}$ and $\Gamma_B \tilde{\cap}\, \Omega_A = \phi_A$. It can be understood from these expressions that $\Omega_A \notin \mathcal{F}$, a contradiction.
(iii) Let $\mathcal{F}$ be an ultra BFS-filter on $U$. Suppose that $\Omega_A, (\Omega_A)^c \notin \mathcal{F}$. From (ii) we know that $\Omega_A \tilde{\cup}\, (\Omega_A)^c = \Omega^1_{A_1} \notin \mathcal{F}$. Then, by (i), there exists an $\Omega^2_{A_2} \in \mathcal{F}$ satisfying $\Omega^2_{A_2} \tilde{\cap}\, \Omega^1_{A_1} = \phi_A$. From $\Omega^2_{A_2} \neq \phi_A$, we have an $e \in A_2$ and a $u \in U$ such that $\delta^+_{\Omega^2(e)}(u) \neq 0$ or $\delta^-_{\Omega^2(e)}(u) \neq 0$. Choose that $\delta^+_{\Omega^2(e)}(u) \neq 0$. Also, we know that $\delta^+_{\Omega^1(e)}(u) \neq 0$ for these $e \in E$ and $u \in U$. As a result, we obtain $min\{\delta^+_{\Omega^2(e)}(u), \delta^+_{\Omega^1(e)}(u)\} \neq 0$, which is a contradiction. □

**Theorem 3.16.** *Let $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : (BF^U)^E \to (BF^V)^D$ be a BFS-mapping and let $\mathcal{B}$ be a BFS-base for an ultra BFS-filter on U. Then, $\mathcal{B}^* = \{\mathfrak{f}(\Omega_A) : \Omega_A \in \mathcal{B}\}$ is a BFS-base for an ultra BFS-filter on V.*

*Proof.* Firstly, we need to verify the conditions $(B_1)$ and $(B_2)$.
$(B_1)$ is clear.
$(B_2)$ Let $\mathfrak{f}(\Omega^1_{A_1}), \mathfrak{f}(\Omega^2_{A_2}) \in \mathcal{B}^*$. Since $\mathcal{B}$ is a BFS-base, there is an $\Omega^3_{A_3} \in \mathcal{B}$ such that $\Omega^3_{A_3} \tilde{\subseteq} \Omega^1_{A_1} \tilde{\cap}\, \Omega^2_{A_2}$. Accordingly, by Theorem 2.14, we obtain $\mathfrak{f}(\Omega^3_{A_3}) \tilde{\subseteq} \mathfrak{f}(\Omega^1_{A_1}) \tilde{\cap}\, \mathfrak{f}(\Omega^2_{A_2})$. Hence $\mathcal{B}^*$ is a BFS-base for a BFS-filter on $V$.

Let $\mathcal{F}^*$ be the BFS-filter on $V$ generated by $\mathcal{B}^*$. Let us show that $\mathcal{F}^*$ is an ultra BFS-filter on $V$. Assume that $\mathcal{G}$ is a BFS-filter such that $\mathcal{F}^* \neq \mathcal{G}$ and $\mathcal{F}^* \subseteq \mathcal{G}$. Then, there exists a $\Gamma_B \in (BF^V)^D$ with $\Gamma_B \in \mathcal{G}, \Gamma_B \notin \mathcal{F}^*$. Consider $\mathcal{F}$ is an ultra BFS-filter on $U$ generated by $\mathcal{B}$ and take a $\Lambda_C \in \mathcal{F}$. So, there is a $\Lambda^1_{C_1} \in \mathcal{B}$ with $\Lambda^1_{C_1} \tilde{\subseteq} \Lambda_C$. Owing to $\mathfrak{f}(\Lambda^1_{C_1}), \Gamma_B \in \mathcal{G}$, we get $\Gamma^1_{B_1} = \mathfrak{f}(\Lambda^1_{C_1}) \tilde{\cap}\, \Gamma_B \neq \phi_A$. For this reason, there are a $d \in B_1$ and a

$v \in V$ such that $\delta^+_{\Gamma^1(d)}(v) \neq 0$ or $\delta^-_{\Gamma^1(d)}(v) \neq 0$. Let us choose $\delta^+_{\Gamma^1(d)}(v) \neq 0$. Then, we find a positive real number $\alpha < \delta^+_{\Gamma^1(d)}(v)$. From the definition of $\mathfrak{f}(\Lambda^1_{C_1})$, there exist an $e \in E$ and a $u \in U$ with $\mathfrak{g}(e) = d$, $\mathfrak{u}(u) = v$ and $\alpha < \delta^+_{\Lambda^1(e)}(u)$. Hence,

$$\begin{aligned}\alpha &< min\{\delta^+_{\Lambda^1(e)}(u), \delta^+_{\Gamma^1(\mathfrak{g}(e))}(\mathfrak{u}(u))\}\\ &= min\{\delta^+_{\Lambda^1(e)}(u), \delta^+_{\mathfrak{f}^{-1}(\Gamma^1)(e)}(u)\}.\end{aligned}$$

This implies that $\Lambda^1_{C_1} \tilde{\cap} \mathfrak{f}^{-1}(\Gamma^1_{B_1}) \neq \phi_A$. Therefore, by $\Lambda^1_{C_1} \tilde{\subseteq} \Lambda_C$ and $\Gamma^1_{B_1} \tilde{\subseteq} \Gamma_B$, we get $\Lambda_C \tilde{\cap} \mathfrak{f}^{-1}(\Gamma_B) \neq \phi_A$. From Theorem 3.15 (i), we obtain $\mathfrak{f}^{-1}(\Gamma_B) \in \mathcal{F}$ and so that there exists a $\Gamma^2_{B_2} \in \mathcal{B}$ such that $\Gamma^2_{B_2} \tilde{\subseteq} \mathfrak{f}^{-1}(\Gamma_B)$. Since $\mathfrak{f}(\Gamma^2_{B_2}) \tilde{\subseteq} \Gamma_B$ and $\mathfrak{f}(\Gamma^2_{B_2}) \in \mathfrak{B}^*$, we have $\Gamma_B \in \mathcal{F}^*$. But this contradicts the fact that $\Gamma_B \notin \mathcal{F}^*$. Thus, $\mathcal{F}^*$ is an ultra BFS-filter on $V$. □

**Definition 3.17.** Let $\mathcal{F}$ be a BFS-filter on $U$. If $\tilde{\bigcap}\{\Omega_A : \Omega_A \in \mathcal{F}\} = \phi_A$, then $\mathcal{F}$ is said to be a BFS-filter free.

**Theorem 3.18.** *Every ultra BFS-filter $\mathcal{F}$ is a BFS-filter free.*

*Proof.* Suppose that $\tilde{\bigcap}\{\Omega_A : \Omega_A \in \mathcal{F}\} \neq \phi_A$. In this case, we have a BFS-point $e_u^{(p,n)}$such that for every $\Omega_A \in \mathcal{F}$, $e_u^{(p,n)} \tilde{\in} \Omega_A$. Let $p \neq 0$ and $n = 0$. Take BFS-point $e_u^{(p_1,0)}$ with $p_1 < p$. Therefore, for every $\Omega_A \in \mathcal{F}$, we obtain $e_u^{(p_1,0)} \tilde{\cap}\, \Omega_A \neq \phi_A$. From Theorem 3.15 (i) it follows that $e_u^{(p_1,0)} \in \mathcal{F}$. Thus, we get $e_u^{(p,0)} \tilde{\in}\, e_u^{(p_1,0)}$, which yields a contradiction. The other cases are similar to this one and so we skip the details. □

**Theorem 3.19.** *Let each BFS-point $e_u^{(p,n)} \in \mathcal{P}(U,E)$ satisfy the condition:*

$$\text{if } e_u^{(p,n)} \tilde{\in} \widetilde{\bigcup}_{i\in J} \Omega^i_{A_i}, \text{ then there exists an } i_0 \in J \text{ such that } e_u^{(p,n)} \tilde{\in}\, \Omega^{i_0}_{A_{i_0}}.$$

*If for each BFS-point $e_u^{(p,n)} \in \mathcal{P}(U,E)$, there is a BFS-filter $\mathcal{F}(e_u^{(p,n)})$ satisfying the below properties, then there is a BFS-topology $\tau$ on $U$ such that for all $e_u^{(p,n)} \in \mathcal{P}(U,E)$, $\mathcal{F}(e_u^{(p,n)})$ is the $\tau$-BFS neighborhood system of $e_u^{(p,n)}$:*

(*i*) *If $\Omega_A \in \mathcal{F}(e_u^{(p,n)})$, then $e_u^{(p,n)} \tilde{\in}\, \Omega_A$.*

(*ii*) *If $\Omega_A \in \mathcal{F}(e_u^{(p,n)})$, then there is an $\Omega^1_{A_1} \in \mathcal{F}(e_u^{(p,n)})$ with $\Omega^1_{A_1} \tilde{\subseteq} \Omega_A$ and $\Omega_A \in \mathcal{N}(d_v^{(p',n')})$ for all $d_v^{(p',n')} \tilde{\in}\, \Omega^1_{A_1}$.*

*Proof.* Since $\mathcal{F}(e_u^{(p,n)})$ satisfies properties (BFSN1)-(BFSN5), we can easily prove it from Theorem 2.18. □

## 4. Convergence of BFS-filters

In this section, we introduce and study the notion of convergence for BFS-filters in the BFS-topological spaces by means of the concept of a BFS-neighborhood of a BFS-point given by Demir and Saldamlı [9]. This enable us to give some results about BFS-Hausdorff spaces. Moreover, we give the idea of a BFS-continuous mapping and characterize it in the light of the convergence of BFS-filters. Note that these ideas seem to be extremely beneficial for the development of BFS-topology. At the same time, these concepts will be of great use to provide theoretical foundation to design decision-making problems in next section.

**Definition 4.1.** Let $(U, \tau, E)$ be a BFS-topological space, $\mathcal{F}$ be a BFS-filter on $U$, and $e_u^{(p,n)} \in \mathcal{P}(U,E)$. Then:

(*i*) The BFS-filter $\mathcal{F}$ is said to converge to $e_u^{(p,n)}$ if $\mathcal{N}(e_u^{(p,n)}) \subseteq \mathcal{F}$, and it is denoted by $\mathcal{F} \to e_u^{(p,n)}$.

(*ii*) The BFS-point $e_u^{(p,n)}$ is called a BFS-cluster point of $\mathcal{F}$ if every BFS-set of a family of BFS-neighborhoods of $e_u^{(p,n)}$ meets every BFS-set of $\mathcal{F}$, and we write $\mathcal{F} \infty\, e_u^{(p,n)}$.

It is easily seen that if $\mathcal{F} \to e_u^{(p,n)}$, then $\mathcal{F} \infty\, e_u^{(p,n)}$. On the other hand, the converse may not be true as explained in the following example.

**Example 4.2.** *Let $U = \{u_1, u_2\}$ and $E = \{e_1, e_2\}$. Take $\Omega_E^1, \Omega_E^2 \in (BF^U)^E$, where*

$$\Omega_E^1 = \left\{ \begin{array}{l} \langle e_1, \Omega(e_1) = \{(u_1, 0.4, -0.2), (u_2, 0.7, -0.4)\}\rangle, \\ \langle e_2, \Omega(e_2) = \{(u_1, 0.5, -0.4), (u_2, 0.3, -0.8)\}\rangle \end{array} \right\},$$

$$\Omega_E^2 = \left\{ \begin{array}{l} \langle e_1, \Omega(e_1) = \{(u_1, 0.3, -0.8), (u_2, 0.6, -0.4)\}\rangle, \\ \langle e_2, \Omega(e_2) = \{(u_1, 0.2, -0.4), (u_2, 0.3, -0.8)\}\rangle \end{array} \right\}.$$

*Then, $\tau = \{\phi_A, U_E, \Omega_E^1, \Omega_E^2, \Omega_E^1 \tilde{\cup}\, \Omega_E^2, \Omega_E^1 \tilde{\cap}\, \Omega_E^2\}$ is a BFS-topology over $U$. Moreover, $\mathcal{F} = \{\Omega_E \in (BF^U)^E : \Omega_E^1 \tilde{\subseteq} \Omega_E\}$ is a BFS-filter on $U$. One can easily check that $\mathcal{F}$ has a BFS-cluster point $(e_1)_{u_1}^{(0.3,-0.8)}$ but do not converge to this BFS-point.*

**Theorem 4.3.** *Let $(U, \tau, E)$ be a BFS-topological space. Consider a BFS-filter $\mathcal{F}$ on $U$. Then, $\mathcal{F} \infty\, e_u^{(p,n)}$ if and only if there is a BFS-filter $\mathcal{G}$ satisfying $\mathcal{F} \subseteq \mathcal{G}$ and $\mathcal{G} \to e_u^{(p,n)}$.*

*Proof.* Let $\mathcal{F} \infty\, e_u^{(p,n)}$. One can readily verify that the family

$$\mathcal{B} = \{\Omega_{A_1}^1 \tilde{\cap}\, \Omega_{A_2}^2 : \Omega_{A_1}^1 \in \mathcal{N}(e_u^{(p,n)}),\ \Omega_{A_2}^2 \in \mathcal{F}\}$$

is a BFS-base for a BFS-filter $\mathcal{G}$. Also, it is finer than $\mathcal{F}$ and converges to $e_u^{(p,n)}$.
Conversely, let $\mathcal{F} \subseteq \mathcal{G}$ and $\mathcal{G} \to e_u^{(p,n)}$. In this case, $\mathcal{G}$ contains all the BFS-neighborhoods of $e_u^{(p,n)}$ and all the BFS sets of $\mathcal{F}$. Thus, from the fact that $\mathcal{G}$ is a BFS-filter it follows that $\mathcal{F} \infty\, e_u^{(p,n)}$. □

**Definition 4.4.** Let $(U, \tau, E)$ be a BFS-topological space and $e_u^{(p,n)} \in \mathcal{P}(U, E)$. Then, $e_u^{(p,n)}$ is called in the adherence of a BFS-set $\Omega_A$ over $U$ provided that for every $\Omega_{A_1}^1 \in \mathcal{N}((e_u^{(p,n)})^c)$, we have $\Omega_{A_1}^1 \tilde{\nsubseteq} (\Omega_A)^c$, where $(e_u^{(p,n)})^c = e_u^{(1-p,-1-n)}$.

**Example 4.5.** *Let $U = \{u_1, u_2\}$ and $E = \{e_1, e_2\}$. Let $\Omega_E^1, \Omega_E^2 \in (BF^U)^E$ be defined by*

$$\Omega_E^1 = \left\{ \begin{array}{l} \langle e_1, \Omega(e_1) = \{(u_1, 0.6, -0.8), (u_2, 0.4, -0.6)\}\rangle, \\ \langle e_2, \Omega(e_2) = \{(u_1, 0.5, -0.7), (u_2, 0, -0.55)\}\rangle \end{array} \right\},$$

$$\Omega_E^2 = \left\{ \begin{array}{l} \langle e_1, \Omega(e_1) = \{(u_1, 0.4, -0.2), (u_2, 0.6, -0.3)\}\rangle, \\ \langle e_2, \Omega(e_2) = \{(u_1, 0.5, -0.3), (u_2, 0.8, -0.45)\}\rangle \end{array} \right\}.$$

*Then, $\tau = \{\phi_A, U_E, \Omega_E^1, \Omega_E^2, \Omega_E^1 \tilde{\cup}\, \Omega_E^2, \Omega_E^1 \tilde{\cap}\, \Omega_E^2\}$ is a BFS-topology over $U$. Moreover, since $\Omega_E^3 \tilde{\nsubseteq} (\Omega_E^1)^c$ for all $\Omega_E^3 \in \mathcal{N}\Big(((e_1)_{u_1}^{(0.5,-0.9)})^c\Big)$, we verify that $(e_1)_{u_1}^{(0.5,-0.9)}$ is in the adherence of $\Omega_E^1$.*

**Theorem 4.6.** *Let $(U, \tau, E)$ be a BFS-topological space, $\Omega_A \in (BF^U)^E$, and $e_u^{(p,n)} \in \mathcal{P}(U, E)$. Then, the following statements are satisfied:*

*(i)* *$\Omega_A \in \tau$ if and only if whenever $\mathcal{F}$ is a BFS-filter on $U$ such that $\mathcal{F} \to e_u^{(p,n)}$ and $e_u^{(p,n)} \tilde{\in} \Omega_A$, then $\Omega_A \in \mathcal{F}$.*

*(ii)* *$e_u^{(p,n)}$ is in the adherence of $\Omega_A$ if and only if there is a BFS-filter $\mathcal{F}$ on $U$ with $(\Omega_A)^c \notin \mathcal{F}$ and $\mathcal{F} \to (e_u^{(p,n)})^c$.*

*Proof.* (i) Necessity of the condition follows from the definition of BFS-open set.
Now, consider $e_u^{(p,n)} \tilde{\in} \Omega_A$ and put $\mathcal{N}(e_u^{(p,n)}) = \mathcal{F}$. By hypothesis, we have $\Omega_A \in \mathcal{F}$. So, from Theorem 2.19, the proof is concluded.

(ii) Let $(e_u^{(p,n)})$ be in the adherence of $\Omega_A$. Then, for every $\Omega^1_{A_1} \in \mathcal{N}((e_u^{(p,n)})^c)$, we have $\Omega^1_{A_1} \tilde{\nsubseteq} (\Omega_A)^c$. If $\mathcal{F} = \mathcal{N}((e_u^{(p,n)})^c)$, we obtain $\mathcal{F} \to (e_u^{(p,n)})^c$ and $(\Omega_A)^c \notin \mathcal{F}$.
Conversely, let $\mathcal{F}$ be a BFS-filter on $U$ satisfying $\mathcal{F} \to (e_u^{(p,n)})^c$ and $(\Omega_A)^c \notin \mathcal{F}$. In this case, for all $\Omega^1_{A_1} \in \mathcal{N}((e_u^{(p,n)})^c)$, we get $\Omega^1_{A_1} \tilde{\nsubseteq} (\Omega_A)^c$. Indeed, suppose that there exists an $\Omega^1_{A_1} \in \mathcal{N}((e_u^{(p,n)})^c)$ such that $\Omega^1_{A_1} \tilde{\subseteq} (\Omega_A)^c$. Therefore, we obtain $(\Omega_A)^c \in \mathcal{F}$, which is a contradiction. Thus, $e_u^{(p,n)}$ is in the adherence of $\Omega_A$. □

**Definition 4.7.** Let $(U, \tau_1, E)$, $(V, \tau_2, D)$ be two BFS-topological spaces and $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : (U, \tau_1, E) \to (V, \tau_2, D)$ be a BFS-mapping. The BFS-mapping $\mathfrak{f}$ is said to be BFS-continuous at $e_u^{(p,n)} \tilde{\in} U_E$ provided that for each BFS-neighborhood $\Gamma_B$ of $\mathfrak{f}(e_u^{(p,n)})$ there exists a BFS-neighborhood $\Omega_A$ of $e_u^{(p,n)}$ such that $\mathfrak{f}(\Omega_A) \tilde{\subseteq} \Gamma_B$.

**Theorem 4.8.** *Let $(U, \tau_1, E)$, $(V, \tau_2, D)$ be two BFS-topological spaces and $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : (U, \tau_1, E) \to (V, \tau_2, D)$ be a BFS-mapping. Then, the following statements are equivalent:*

(*i*) $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : (U, \tau_1, E) \to (V, \tau_2, D)$ *is a BFS-continuous mapping at* $e_u^{(p,n)} \tilde{\in} U_E$.

(*ii*) *For all* $\Gamma_B \in \mathcal{N}(\mathfrak{f}(e_u^{(p,n)}))$, *there exists an* $\Omega_A \in \mathcal{N}(e_u^{(p,n)})$ *such that* $\Omega_A \tilde{\subseteq} \mathfrak{f}^{-1}(\Gamma_B)$.

(*iii*) *For all* $\Gamma_B \in \mathcal{U}(\mathfrak{f}(e_u^{(p,n)}))$, *we have* $\mathfrak{f}^{-1}(\Gamma_B) \in \mathcal{N}(e_u^{(p,n)})$, *where* $\mathcal{U}(\mathfrak{f}(e_u^{(p,n)}))$ *consists of all bipolar fuzzy soft open sets containing* $\mathfrak{f}(e_u^{(p,n)})$.

(*iv*) *For all* $\Gamma_B \in \mathcal{N}(\mathfrak{f}(e_u^{(p,n)}))$, *we have* $\mathfrak{f}^{-1}(\Gamma_B) \in \mathcal{N}(e_u^{(p,n)})$.

*Proof.* We shall prove that (i) ⇒ (ii). Since $\mathfrak{f}$ is BFS-continuous at $e_u^{(p,n)} \tilde{\in} U_E$, there is an $\Omega_A \in \mathcal{N}(e_u^{(p,n)})$ such that $\mathfrak{f}(\Omega_A) \tilde{\subseteq} \Gamma_B$. Then, from Theorem 2.14, we obtain $\Omega_A \tilde{\subseteq} \mathfrak{f}^{-1}(\mathfrak{f}(\Omega_A)) \tilde{\subseteq} \mathfrak{f}^{-1}(\Gamma_B)$.
The implications (ii) ⇒ (iii) and (iii) ⇒ (iv) are obvious by the property (BFSN3) and the definition of BFS-neighborhood.
To prove that (iv) ⇒ (i) let $\Gamma_B \in \mathcal{N}(\mathfrak{f}(e_u^{(p,n)}))$. By (iv), we obtain $\mathfrak{f}^{-1}(\Gamma_B) \in \mathcal{N}(e_u^{(p,n)})$. Take $\mathfrak{f}^{-1}(\Gamma_B) = \Omega_A$. From Theorem 2.14, we get $\mathfrak{f}(\Omega_A) = \mathfrak{f}(\mathfrak{f}^{-1}(\Gamma_B) \tilde{\subseteq} \Gamma_B$, completing the proof. □

**Example 4.9.** *Let $U = \{u_1, u_2, u_3\}$, $E = \{e_1, e_2\}$ and $D = \{d_1, d_2\}$. Let $\Omega^1_E$ and $\Omega^2_E$ be two BFS-sets in $(BF^U)^E$ with*

$$\Omega^1_E = \left\{ \begin{array}{l} \langle e_1, \Omega^1(e_1) = \{(u_1, 0.51, -0.33), (u_2, 0.32, -0.21), (u_3, 0.22, -0.31)\}\rangle, \\ \langle e_2, \Omega^1(e_2) = \{(u_1, 0.32, -0.43), (u_2, 0.51, -0.33), (u_3, 0.23, -0.51)\}\rangle \end{array} \right\},$$

$$\Omega^2_E = \left\{ \begin{array}{l} \langle e_1, \Omega^2(e_1) = \{(u_1, 0.31, -0.52), (u_2, 0.21, -0.32), (u_3, 0.33, -0.34)\}\rangle, \\ \langle e_2, \Omega^2(e_2) = \{(u_1, 0.52, -0.32), (u_2, 0.51, -0.53), (u_3, 0.55, -0.36)\}\rangle \end{array} \right\}.$$

*Then, $\tau_1 = \{\phi_A, U_E, \Omega^1_E, \Omega^2_E, \Omega^1_E \tilde{\cup} \Omega^2_E, \Omega^1_E \tilde{\cap} \Omega^2_E\}$ is a BFS-topology over U with the set E of parameters. Let us consider two BFS-sets $\Omega^3_D$ and $\Omega^4_D$ in $(BF^U)^D$ satisfying*

$$\Omega^3_D = \left\{ \begin{array}{l} \langle d_1, \Omega^3(d_1) = \{(u_1, 0.65, -0.45), (u_2, 0.65, -0.33), (u_3, 0.25, -0.65)\}\rangle, \\ \langle d_2, \Omega^3(d_2) = \{(u_1, 0.35, -0.55), (u_2, 0.45, -0.9), (u_3, 0, -0.21)\}\rangle \end{array} \right\},$$

$$\Omega^4_D = \left\{ \begin{array}{l} \langle d_1, \Omega^4(d_1) = \{(u_1, 0.28, -0.15), (u_2, 0.18, -0.75), (u_3, 0.5, -0.4)\}\rangle, \\ \langle d_2, \Omega^4(d_2) = \{(u_1, 0.33, -0.43), (u_2, 0.53, -0.67), (u_3, 0.7, -0.8)\}\rangle \end{array} \right\}.$$

*Therefore,* $\tau_2 = \{\phi_A, U_D, \Omega^3_D, \Omega^4_D, \Omega^3_D \tilde{\cup} \Omega^4_D, \Omega^3_D \tilde{\cap} \Omega^4_D\}$ *is a BFS-topology over* $U$ *with the set* $D$ *of parameters. Now, take a BFS-mapping* $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : (U, \tau_1, E) \to (U, \tau_2, D)$ *such that*

$$\mathfrak{u}(u_1) = u_1, \quad \mathfrak{u}(u_2) = u_2, \quad \mathfrak{u}(u_3) = u_3,$$
$$\mathfrak{g}(e_1) = d_1, \quad \mathfrak{g}(e_2) = d_1$$

*and choose a BFS-point* $(e_1)^{(0,51,-0.33)}_{u_1}$. *Then, we get* $\mathfrak{f}((e_1)^{(0,51,-0.33)}_{u_1}) = (d_1)^{(0,51,-0.33)}_{u_1}$ *and it follows that* $\mathcal{U}((d_1)^{(0,51,-0.33)}_{u_1}) = \{U_D, \Omega^3_D, \Omega^3_D \tilde{\cup} \Omega^4_D\}$. *Thus, the BFS-mapping* $\mathfrak{f}$ *is BFS-continuous at* $(e_1)^{(0,51,-0.33)}_{u_1}$ *because the BFS-sets* $\mathfrak{f}^{-1}(\Omega^3_D)$ *and* $\mathfrak{f}^{-1}(\Omega^3_D \tilde{\cup} \Omega^4_D)$ *are BFS-neighborhoods of* $(e_1)^{(0,51,-0.33)}_{u_1}$, *where*

$$\mathfrak{f}^{-1}(\Omega^3_D) = \left\{ \begin{array}{l} \langle e_1, \mathfrak{f}^{-1}(\Omega^3)(e_1) = \{(u_1, 0.65, -0.45), (u_2, 0.65, -0.33), (u_3, 0.25, -0.65)\}\rangle, \\ \langle e_2, \mathfrak{f}^{-1}(\Omega^3)(e_2) = \{(u_1, 0.65, -0.45), (u_2, 0.65, -0.33), (u_3, 0.25, -0.65)\}\rangle \end{array} \right\},$$

$$\mathfrak{f}^{-1}(\Omega^3_D \tilde{\cup} \Omega^4_D) = \mathfrak{f}^{-1}(\Omega^5_D) = \left\{ \begin{array}{l} \langle e_1, \mathfrak{f}^{-1}(\Omega^5)(e_1) = \{(u_1, 0.65, -0.45), (u_2, 0.65, -0.75), (u_3, 0.5, -0.65)\}\rangle, \\ \langle e_2, \mathfrak{f}^{-1}(\Omega^5)(e_2) = \{(u_1, 0.65, -0.45), (u_2, 0.65, -0.75), (u_3, 0.5, -0.65)\}\rangle \end{array} \right\}.$$

**Theorem 4.10.** *Let* $(U, \tau_1, E), (V, \tau_1, D)$ *be two BFS-topological spaces and* $e^{(p,n)}_u \in \mathcal{P}(U, E)$. *A BFS-mapping* $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : (U, \tau_1, E) \to (V, \tau_2, D)$ *is BFS-continuous if and only if for every BFS-filter* $\mathcal{F}$ *on* $U$ *which converges to* $e^{(p,n)}_u$, *the BFS-filter* $\mathfrak{f}(\mathcal{F})$ *converges to* $\mathfrak{f}(e^{(p,n)}_u)$.

*Proof.* Consider $\mathcal{F} \to e^{(p,n)}_u$ and let $\Gamma_B$ be a BFS-neighborhood of $\mathfrak{f}(e^{(p,n)}_u)$ in $\mathcal{P}(V, D)$. By BFS-continuity, there exists an $\Omega_A \in \mathcal{N}(e^{(p,n)}_u)$ such that $\mathfrak{f}(\Omega_A) \tilde{\subseteq} \Gamma_B$. Therefore, we obtain $\Omega_A \in \mathcal{F}$. Thus, in view of Theorem 3.10, we get $\Gamma_B \in \mathfrak{f}(\mathcal{F})$.
For the converse, let $\Gamma_B \in \mathcal{N}(\mathfrak{f}(e^{(p,n)}_u))$. If considering $\mathcal{F} = \mathcal{N}(e^{(p,n)}_u)$, then $\mathcal{F} \to e^{(p,n)}_u$. From hypothesis, it follows that $\mathfrak{f}(\mathcal{F}) \to \mathfrak{f}(e^{(p,n)}_u)$. So, $\Gamma_B \in \mathfrak{f}(\mathcal{F})$ and thus, by Theorem 3.10, there exists a BFS-neighborhood $\Omega_A$ of $e^{(p,n)}_u$ such that $\mathfrak{f}(\Omega_A) \tilde{\subseteq} \Gamma_B$. This completes the proof. □

**Definition 4.11.** Let $(e_1)^{(p_1,n_1)}_{u_1}, (e_2)^{(p_2,n_2)}_{u_2} \in \mathcal{P}(U, E)$. These two BFS-points are called equal if $e_1 = e_2$, $u_1 = u_2$ and $(p_1, n_1) = (p_2, n_2)$. Moreover, $(e_1)^{(p_1,n_1)}_{u_1} \neq (e_2)^{(p_2,n_2)}_{u_2} \Leftrightarrow u_1 \neq u_2$ or $e_1 \neq e_2$ or $(p_1, n_1) \neq (p_2, n_2)$.

**Definition 4.12.** A BFS-topological space $(U, \tau, E)$ is said to be a BFS-Hausdorff space if for each pair $(e_1)^{(p_1,n_1)}_{u_1}$, $(e_2)^{(p_2,n_2)}_{u_2}$ of distinct BFS-points in $\mathcal{P}(U, E)$, there are an $\Omega^1_{A_1} \in \mathcal{N}((e_1)^{(p_1,n_1)}_{u_1})$ and an $\Omega^2_{A_2} \in \mathcal{N}((e_2)^{(p_2,n_2)}_{u_2})$ satisfying $\Omega^1_{A_1} \tilde{\cap} \Omega^2_{A_2} = \phi_A$.

**Theorem 4.13.** *A BFS-topological space* $(U, \tau, E)$ *is a BFS-Hausdorff space if and only if each BFS-filter on* $U$ *converges to a unique BFS-point.*

*Proof.* Consider a BFS-Hausdorff space $(U, \tau, E)$ and take a BFS-filter $\mathcal{F}$ on $U$. Assume that $\mathcal{F}$ converges to $(e_1)^{(p_1,n_1)}_{u_1}$ and $(e_2)^{(p_2,n_2)}_{u_2}$ with $(e_1)^{(p_1,n_1)}_{u_1} \neq (e_2)^{(p_2,n_2)}_{u_2}$. Therefore, there exist an $\Omega^1_{A_1} \in \mathcal{N}((e_1)^{(p_1,n_1)}_{u_1})$ and an $\Omega^2_{A_2} \in \mathcal{N}((e_2)^{(p_2,n_2)}_{u_2})$ such that $\Omega^1_{A_1} \tilde{\cap} \Omega^2_{A_2} = \phi_A$. From our assumption it follows that $\Omega^1_{A_1}, \Omega^2_{A_2} \in \mathcal{F}$. Hence, by the definition of BFS-filter, we obtain $\Omega^1_{A_1} \tilde{\cap} \Omega^2_{A_2} \neq \phi_A$, which yields a contradiction.
For sufficiency, suppose that each BFS-filter on $U$ converges to a unique BFS-point. On the other hand, assume that $(U, \tau, E)$ is not a BFS-Hausdorff space. So, there exist distinct BFS-points $(e_1)^{(p_1,n_1)}_{u_1}$ and $(e_2)^{(p_2,n_2)}_{u_2}$ in $\mathcal{P}(U, E)$ such that for any $\Omega^1_{A_1} \in \mathcal{N}((e_1)^{(p_1,n_1)}_{u_1})$ and $\Omega^2_{A_2} \in \mathcal{N}((e_2)^{(p_2,n_2)}_{u_2})$, we have $\Omega^1_{A_1} \tilde{\cap} \Omega^2_{A_2} \neq \phi_A$. Then,

$$\mathcal{F} = \left\{\Omega^1_{A_1} \tilde{\cap} \Omega^2_{A_2} : \Omega^1_{A_1} \in \mathcal{N}((e_1)^{(p_1,n_1)}_{u_1}),\ \Omega^2_{A_2} \in \mathcal{N}((e_2)^{(p_2,n_2)}_{u_2})\right\}$$

is a BFS-filter on $U$. One can be easily checked that $\mathcal{F}$ is finer than $\mathcal{N}((e_1)_{u_1}^{(p_1,n_1)})$ and $\mathcal{N}((e_2)_{u_2}^{(p_2,n_2)})$. Thus, we obtain $\mathcal{F} \to (e_1)_{u_1}^{(p_1,n_1)}$ and $\mathcal{F} \to (e_2)_{u_2}^{(p_2,n_2)}$, which contradicts our hypothesis. □

The property of being BFS-Hausdorff in Theorem 4.13 is not superfluous. To see this, consider the following example.

**Example 4.14.** *Let us take Example 4.5 as a BFS-topological space. Then, this space is not a BFS-Hausdorff space. Moreover, it is easily seen that* $\mathcal{N}((e_1)_{u_1}^{(0.3,-0.1)}) = \mathcal{N}((e_2)_{u_1}^{(0.4,-0.2)})$. *Now, consider* $\mathcal{F} = \mathcal{N}((e_1)_{u_1}^{(0.3,-0.1)}) = \mathcal{N}((e_2)_{u_1}^{(0.4,-0.2)})$. *Thus, we have* $\mathcal{F} \to (e_1)_{u_1}^{(0.3,-0.1)}$ *and* $\mathcal{F} \to (e_2)_{u_1}^{(0.4,-0.2)}$, *that is, the BFS-filter* $\mathcal{F}$ *converges to more than one BFS-point.*

## 5. Application of BFS-filter to multi-criteria group decision-making

Multi-criteria group decision-making (MCGDM) is utilized to deal with the information when many attributes of alternatives are evaluated by a group of decision-makers. In this regard, we construct a MCGDM method with the aid of BFS-topology and BFS-filter. This is the first kind of such methods in this direction as we employ the convergence of BFS-filters.

Now, we establish an algorithm for problems that are characterized by BFS-filter and BFS-topology as follows. We also apply it to real situations in order to prove its importance and adaptability.

**Algorithm 1** Selection of an alternative

**Input:**

1: $U = \{u_1, u_2, ..., u_n\}$, a universe of objects, and $E = \{e_1, e_2, ..., e_m\}$, a set of attributes.
2: Insert three BFS-sets $\Omega^1_{A_1}$, $\Omega^2_{A_2}$ and $\Gamma_B$ according to the opinions of different persons.

**Calculation:**

3: Construct a BFS-topology $\tau$ such that $\Omega^1_{A_1}$ and $\Omega^2_{A_2}$ are BFS-open sets in $\tau$.
4: Construct a BFS-filter $\mathcal{F}$ on $U$ with $\mathcal{B} = \{\Gamma_B\}$ as a BFS-base.
5: Find the BFS-points $(e_i)_{u_j}^{(p_{ij},n_{ij})}$ that $\mathcal{F}$ converges in $(U, \tau, E)$ such that

$$p_{ij} \in (I^+_{ij})^\ell \subseteq [0,1] \text{ and } n_{ij} \in (I^-_{ij})^\ell \subseteq [-1,0]$$

where $\ell \in L_{ij} = \{1, 2, ..., k\}$.
6: Compute $\delta_{ij} = \frac{1}{|L_{ij}|}\sum_{\ell=1}^{n} \inf(I^+_{ij})^\ell + \inf(I^-_{ij})^\ell$ and $\Delta_{ij} = \frac{1}{|L_{ij}|}\sum_{\ell=1}^{n} \sup(I^+_{ij})^\ell + \sup(I^-_{ij})^\ell$ for each $u_j \in U$ and each $e_i \in E$.
7: Compute $\alpha^*_j = \frac{1}{|E|}\sum_{i=1}^{m} \delta_{ij}$ and $\beta^*_j = \frac{1}{|E|}\sum_{i=1}^{m} \Delta_{ij}$ for each $u_j \in U$.
8: Compute $s_j = \alpha^*_j + \beta^*_j$ for each $u_j \in U$.
9: Determine $s_k = \max_{j=1,2,...,n} s_j$.

**Output:**

10: $s_k$ will be the decision. If there are more than one values of $k$, then any one of them could be selected as best option.

### 5.1. Numerical Example

Printers are an indispensable office equipment in many workplaces because they can be used for scanning, copying, printing out high quality photos and much more. So, with the developing technology, it is very important to choose the most suitable printer depending on determined goals. Therefore, using Algorithm 1, we solve a printer problem on BFS-filter and BFS-topology.

Suppose that Mrs. X and Mr. Y, who work in the same office in a university, want to purchase a printer to use together. So, they determine some criteria for purchasing an all-in-one printer that can print, scan, copy and fax documents as follows: $E = \{e_1, e_2, e_3, e_4\}$ where

$e_1$ = *Color resolution* (*max*)
$e_2$ = *Color print speed* (*max*)
$e_3$ = *Dimension*
$e_4$ = *Scan resolution* (*max*).

After searching websites related to the "all-in-one printers", four printer alternatives are identified as meeting the defined criteria. Let $U = \{u_1, u_2, u_3, u_4\}$ be the set of printers which are according to requirements of Mrs. X and Mr. Y. Having moderate knowledge of the printers, they construct the input data $\Omega^1_{A_1}$ and $\Omega^2_{A_2}$ given below in the context of bipolar fuzzy soft sets, where the positive membership and negative membership degrees describe their opinions regarding a certain criteria of a printer and their opinions regarding opposite criteria of this printer, respectively:

$$\Omega^1_{A_1} = \left\{ \begin{array}{l} \langle e_1, \Omega^1(e_1) = \{(u_1, 0.6, -0.3), (u_2, 0.5, -0.2), (u_3, 0.6, -0.3), (u_4, 0.4, -0.1)\}\rangle, \\ \langle e_2, \Omega^1(e_2) = \{(u_1, 0.4, -0.1), (u_2, 0.4, -0.2), (u_3, 0.5, -0.1), (u_4, 0.5, -0.2)\}\rangle, \\ \langle e_3, \Omega^1(e_3) = \{(u_1, 0.7, -0.4), (u_2, 0.6, -0.3), (u_3, 0.8, -0.3), (u_4, 0.7, -0.2)\}\rangle, \\ \langle e_4, \Omega^1(e_4) = \{(u_1, 0.5, -0.3), (u_2, 0.6, -0.2), (u_3, 0.4, -0.1), (u_4, 0.4, -0.3)\}\rangle \end{array} \right\},$$

$$\Omega^2_{A_2} = \left\{ \begin{array}{l} \langle e_1, \Omega^2(e_1) = \{(u_1, 0.5, -0.4), (u_2, 0.5, -0.3), (u_3, 0.7, -0.2), (u_4, 0.6, -0.3)\}\rangle, \\ \langle e_2, \Omega^2(e_2) = \{(u_1, 0.5, -0.3), (u_2, 0.7, -0.4), (u_3, 0.5, -0.3), (u_4, 0.5, -0.2)\}\rangle, \\ \langle e_3, \Omega^2(e_3) = \{(u_1, 0.6, -0.2), (u_2, 0.4, -0.1), (u_3, 0.6, -0.3), (u_4, 0.7, -0.3)\}\rangle, \\ \langle e_4, \Omega^2(e_4) = \{(u_1, 0.6, -0.4), (u_2, 0.5, -0.1), (u_3, 0.5, -0.2), (u_4, 0.4, -0.2)\}\rangle \end{array} \right\}.$$

Now, we construct a bipolar fuzzy soft topology given by

$$\tau = \{U_E, \phi_A, \Omega^1_{A_1}, \Omega^2_{A_2}, \Omega^1_{A_1} \tilde{\cap}\, \Omega^2_{A_2}, \Omega^1_{A_1} \tilde{\cup}\, \Omega^2_{A_2}\}.$$

Also, they want to get an expert opinion in order to make the best selection and so they apply to the expert who has worked in the field of printer technology. Based on feedback of the customers, the expert construct a BFS-set $\Gamma_B$ over $U$ as follows:

$$\Gamma_B = \left\{ \begin{array}{l} \langle e_1, \Gamma(e_1) = \{(u_1, 0.5, -0.1), (u_2, 0.6, -0.2), (u_3, 0.4, -0.2), (u_4, 0.5, -0.2)\}\rangle, \\ \langle e_2, \Gamma(e_2) = \{(u_1, 0.4, -0.2), (u_2, 0.6, -0.3), (u_3, 0.5, -0.3), (u_4, 0.6, -0.3)\}\rangle, \\ \langle e_3, \Gamma(e_3) = \{(u_1, 0.7, -0.3), (u_2, 0.7, -0.3), (u_3, 0.6, -0.2), (u_4, 0.7, -0.3)\}\rangle, \\ \langle e_4, \Gamma(e_4) = \{(u_1, 0.5, -0.2), (u_2, 0.5, -0.2), (u_3, 0.4, -0.1), (u_4, 0.3, -0.1)\}\rangle \end{array} \right\}.$$

Let us consider a BFS-filter $\mathcal{F}$ on $U$ whose BFS-base is $\mathcal{B} = \{\Gamma_B\}$. Now, we investigate the BFS-points satisfying $\mathcal{F} \to (e_i)^{(p_{ij}, n_{ij})}_{u_j}$. Tables 3 and 4 represent the required intervals of $(I^+_{ij})^\ell$ and $(I^-_{ij})^\ell$, respectively. Next, using Step 6 of Algorithm 1, we compute $\delta_{ij}$ and $\Delta_{ij}$ for each $u_j \in U$ and each $e_i \in E$, as displayed in Tables 5 and 6. Moreover, for each $u_j \in U$, we obtain $\alpha^*_j$ and $\beta^*_j$ whose the tabular representations are in Table 7. Attending at its last column, it is readily seen that $s_3 = \max\{s_j\}$. Thus, the best printer for Mrs. X and Mr. Y is $u_3$.

**Table 3**
Tabular representation of $(I_{ij}^{+})^{\ell}$

| $\ell$ | 1 | 2 | 3 |
|---|---|---|---|
| $(I_{11}^{+})^{\ell}$ | $[0, 0.6]$ | $(0.6, 1]$ | $(0.6, 1]$ |
| $(I_{21}^{+})^{\ell}$ | $[0, 0.5]$ | $(0.5, 1]$ | $(0.5, 1]$ |
| $(I_{31}^{+})^{\ell}$ | $[0, 0.7]$ | $(0.7, 1]$ | $(0.7, 1]$ |
| $(I_{41}^{+})^{\ell}$ | $[0, 0.6]$ | $(0.6, 1]$ | $(0.6, 1]$ |
| $(I_{12}^{+})^{\ell}$ | $[0, 0.5]$ | $(0.5, 1]$ | $(0.5, 1]$ |
| $(I_{22}^{+})^{\ell}$ | $[0, 0.7]$ | $(0.7, 1]$ | $(0.7, 1]$ |
| $(I_{32}^{+})^{\ell}$ | $[0, 0.6]$ | $(0.6, 1]$ | $(0.6, 1]$ |
| $(I_{42}^{+})^{\ell}$ | $[0, 0.6]$ | $(0.6, 1]$ | $(0.6, 1]$ |
| $(I_{13}^{+})^{\ell}$ | $[0, 0.7]$ | $(0.7, 1]$ | $(0.7, 1]$ |
| $(I_{23}^{+})^{\ell}$ | $[0, 0.5]$ | $(0.5, 1]$ | $(0.5, 1]$ |
| $(I_{33}^{+})^{\ell}$ | $[0, 0.8]$ | $(0.8, 1]$ | $(0.8, 1]$ |
| $(I_{43}^{+})^{\ell}$ | $[0, 0.5]$ | $(0.5, 1]$ | $(0.5, 1]$ |
| $(I_{14}^{+})^{\ell}$ | $[0, 0.6]$ | $(0.6, 1]$ | $(0.6, 1]$ |
| $(I_{24}^{+})^{\ell}$ | $[0, 0.5]$ | $(0.5, 1]$ | $(0.5, 1]$ |
| $(I_{34}^{+})^{\ell}$ | $[0, 0.7]$ | $(0.7, 1]$ | $(0.7, 1]$ |
| $(I_{44}^{+})^{\ell}$ | $[0, 0.4]$ | $(0.4, 1]$ | $(0.4, 1]$ |

**Table 4**
Tabular representation of $(I_{ij}^{-})^{\ell}$

| $\ell$ | 1 | 2 | 3 |
|---|---|---|---|
| $(I_{11}^{-})^{\ell}$ | $[-1, -0.4)$ | $[-0.4, 0]$ | $[-1, -0.4)$ |
| $(I_{21}^{-})^{\ell}$ | $[-1, -0.3)$ | $[-0.3, 0]$ | $[-1, -0.3)$ |
| $(I_{31}^{-})^{\ell}$ | $[-1, -0.4)$ | $[-0.4, 0]$ | $[-1, -0.4)$ |
| $(I_{41}^{-})^{\ell}$ | $[-1, -0.4)$ | $[-0.4, 0]$ | $[-1, -0.4)$ |
| $(I_{12}^{-})^{\ell}$ | $[-1, -0.3)$ | $[-0.3, 0]$ | $[-1, -0.3)$ |
| $(I_{22}^{-})^{\ell}$ | $[-1, -0.4)$ | $[-0.4, 0]$ | $[-1, -0.4)$ |
| $(I_{32}^{-})^{\ell}$ | $[-1, -0.3)$ | $[-0.3, 0]$ | $[-1, -0.3)$ |
| $(I_{42}^{-})^{\ell}$ | $[-1, -0.2)$ | $[-0.2, 0]$ | $[-1, -0.2)$ |
| $(I_{13}^{-})^{\ell}$ | $[-1, -0.3)$ | $[-0.3, 0]$ | $[-1, -0.3)$ |
| $(I_{23}^{-})^{\ell}$ | $[-1, -0.3)$ | $[-0.3, 0]$ | $[-1, -0.3)$ |
| $(I_{33}^{-})^{\ell}$ | $[-1, -0.3)$ | $[-0.3, 0]$ | $[-1, -0.3)$ |
| $(I_{43}^{-})^{\ell}$ | $[-1, -0.2)$ | $[-0.2, 0]$ | $[-1, -0.2)$ |
| $(I_{14}^{-})^{\ell}$ | $[-1, -0.3)$ | $[-0.3, 0]$ | $[-1, -0.3)$ |
| $(I_{24}^{-})^{\ell}$ | $[-1, -0.2)$ | $[-0.2, 0]$ | $[-1, -0.2)$ |
| $(I_{34}^{-})^{\ell}$ | $[-1, -0.3)$ | $[-0.3, 0]$ | $[-1, -0.3)$ |
| $(I_{44}^{-})^{\ell}$ | $[-1, -0.3)$ | $[-0.3, 0]$ | $[-1, -0.3)$ |

**Table 5**
Tabular representation of $\delta_{ij}$

| $\delta_{ij}$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ |
|---|---|---|---|---|
| $e_1$ | −0.4 | −0.43 | −0.3 | −0.37 |
| $e_2$ | −0.43 | −0.33 | −0.43 | −0.4 |
| $e_3$ | −0.33 | −0.37 | −0.23 | −0.3 |
| $e_4$ | −0.4 | −0.33 | −0.4 | −0.5 |

**Table 6**
Tabular representation of $\Delta_{ij}$

| $\Delta_{ij}$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ |
|---|---|---|---|---|
| $e_1$ | 0.6 | 0.63 | 0.7 | 0.67 |
| $e_2$ | 0.63 | 0.63 | 0.63 | 0.7 |
| $e_3$ | 0.63 | 0.67 | 0.73 | 0.7 |
| $e_4$ | 0.6 | 0.73 | 0.7 | 0.6 |

**Table 7**
Final score table

| $\alpha_j^* = \frac{1}{\lvert E\rvert}\sum_{i=1}^{4}\delta_{ij}$ | $\beta_j^* = \frac{1}{\lvert E\rvert}\sum_{i=1}^{4}\Delta_{ij}$ | $s_j = \alpha_j^* + \beta_j^*$ |
|---|---|---|
| $\alpha_1^* = -0.39$ | $\beta_1^* = 0.62$ | $s_1 = 0.23$ |
| $\alpha_2^* = -0.37$ | $\beta_2^* = 0.67$ | $s_2 = 0.3$ |
| $\alpha_3^* = -0.34$ | $\beta_3^* = 0.69$ | $s_3 = 0.35$ |
| $\alpha_4^* = -0.39$ | $\beta_4^* = 0.67$ | $s_4 = 0.28$ |

## 6. Comparison analysis and advantages

The idea of a BFS-filter is the generalization of prevalent filters such as fuzzy filters [10], soft filters [29], bipolar soft filters [7] and fuzzy soft filters [6] as stated below:

(1) If we use just one parameter and ignore the the negative membership degree, then the BFS-filter will coincide with the fuzzy filter.

(2) If we ignore the negative membership degree and the fuzzy value set of each parameter becomes a crisp set, then the BFS-filter will coincide with the soft filter.

(3) If the fuzzy value set of each parameter becomes a crisp set, then the BFS-filter will coincide with the bipolar soft filter.

(4) If we ignore the negative membership degree, then the BFS-filter will coincide with the fuzzy soft filter.

Decision-making problems have not been studied in the previous filter structures mentioned above, and so that our work is the first to explore filter-based decision making problems. Therefore, as compared to prevailing ideas, the BFS-filter structure has an impressive and distinguished advantage. Another remarkable advantage of this proposed structure is that it builds upon a rigorous theoretical foundation. In addition, in order to describe a novel filter structure, we combine three concepts, namely parameterization, fuzziness, and bipolarity, which make the BFS-filter structure more accurate, outstanding and exclusive when compared with existing methods, as discussed above.

## 7. Discussion and conclusion

Bipolar fuzzy set is one of the best approaches to expand on human thinkings because human decisions are based on positive and negative thoughts. Negative thinking indicates what is restricted or impossible while positive thinking demonstrates what is taken into account or possible. Besides, soft set theory is the most developed tool to describe human decision analysis by displaying uncertain and not clearly defined objects in a parametric manner. Therefore, we opt bipolar fuzzy soft set theory which is an extension of fuzzy set theory together with soft set and bipolarity.

Multi-attribute group decision-making problem (MCGDM) is the study of identifying and choosing the preferences of multiple decision makers to make the objective and scientific evaluation of each alternative under multiple attributes or indicators. Therefore, MCGDM is an important part of decision science and also has wide applications in other fields including information technology, economics and computer science. On the other hand, the study of filters is a very natural way to describe convergence in a topological space and commonly employed in many fields. Also, the notion of filter convergence structures is an important tool for interpreting topology since it can be utilized to introduce basic properties of topological spaces such as separation axioms, continuity, compactness and connectedness.

Joining all these ideas and discussions prompt us to make an attempt to uncover this research. Firstly, we define the concepts of a BFS-filter and an ultra BFS-filter. Then, we establish some of their basic results and illustrate them with corresponding examples. Also, we extend the convergence structures to bipolar fuzzy soft setting and give the relations between bipolar fuzzy soft convergence structures and BFS-topology. Moreover, we prove that a BFS-filter converges to at most one BFS-point in the BFS-Hausdorff spaces. In addition, we characterize the continuity of BFS-mappings between BFS-topological spaces by means of the convergence of BFS-filters. Next, we propose an algorithm with the help of BFS-filter and BFS-topology for modeling uncertainties in the multi-criteria group decision-making (MCGDM) problems. Moreover, we demonstrate the utility and applicability of the proposed algorithm by applying it to a realistic example. Thus, this paper contains the first rigorous analysis of filter structure in bipolar fuzzy soft environment and gives an idea for the beginning of new study.

It is expected that these theoretical studies will pave the way to further investigation of novel approaches for BFS-filter and BFS-topology. Moreover, the developed algorithm can be applied in solving other MCGDM problems having uncertainties. In the future, this pioneering analysis can promote the study of relationships with other types of BFS-topological structures like BFS-compact spaces, BFS-connected spaces or others. Also, one can extend our work to other bipolar fuzzy soft models like bipolar fuzzy N-soft sets, bipolar fuzzy soft expert sets, bipolar complex fuzzy soft sets, pythagorean bipolar fuzzy soft sets and rough fuzzy bipolar soft sets to study filter structures and MCGDM methods on these models.

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