Filomat 39:10 (2025), 3315–3320
https://doi.org/10.2298/FIL2510315D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

New (α, β)-order pre-Grüss fractional integral inequalities and applications for CRV

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Abstract. In this paper, we employ Riemann-Liouville fractional integrals to derive two primary integral findings concerning the $\alpha-$ pre-Grüss inequality and the $(\alpha, \beta)-$ pre-Grüss inequality. Our results extend existing integral inequalities reported in the literature. Additionally, we discuss some applications of our results for continuous random variables (CRV, for short) with bounded probability density functions. Some new estimates are provided in this context, along with classical results obtained as special cases from our results.

1. Introduction

We begin the present introduction by recalling the following well-known integral inequality of Grüss [7]:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right|\leq\frac{(M-m)(P-p)}{4} \tag{1}$$

where f and g are two integrable functions on $[a,b]$ that satisfy the conditions:

$$m\leq f(x)\leq M,\ \ p\leq g(x)\leq P;\quad m,M,p,P\in\mathbb{R}, x\in[a,b]. \tag{2}$$

In the case where the integrals exist and f satisfies (2), we have the 1-order pre-Grüss inequality:

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right| \qquad (3)\\ \leq\ &\frac{M-m}{2}\left[\frac{1}{b-a}\int_a^b g^2(x)\,dx-\left(\frac{1}{b-a}\left(\int_a^b g(x)\,dx\right)\right)^2\right]^{\frac{1}{2}}.\end{aligned}$$

2020 *Mathematics Subject Classification*. Primary 26D10; Secondary 26A33.
Keywords. integral inequalities, Riemann-Liouville integral, pre Gruss inequality.
Received: 12 June 2024; Revised: 03 February 2025; Accepted: 06 February 2025
Communicated by Ljubiša D. R. Kočinac
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The inequalities (1) and (3) have garnered considerable interest within the research community, prompting a wealth of explorations into their generalizations, variations, and extensions. A plethora of scholarly works, including those by [5, 8–12], have delved into this topic, offering diverse perspectives and insights into their applications across different domains, especially in probability and statistics [1, 2].
The authors in [4] used an alternative integral approach to prove the following theorem, which generalizes the Grüss result mentioned above:

Theorem 1.1. *Let f and g be two integrable functions on $[0,\infty[$ satisfying the condition (2) on $[0,\infty.[$ Then for all $t>0, \alpha>0$, we have:*

$$\Big|\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}fg(t)-J^{\alpha}f(t)J^{\alpha}g(t)\Big|\leq\Big(\frac{t^{\alpha}}{2\Gamma(\alpha+1)}\Big)^{2}(M-m)(P-p). \tag{4}$$

It is to note that if we apply Theorem 1.1 for $\alpha=1$, we obtain (1) on $[0,t]$.
The main aim of this paper is to derive two primary integral results concerning the $\alpha-$ pre-Gruss inequality and the $(\alpha,\beta)-$ pre-Grüss inequality by using the fractional integrals of Riemann-Liouville. Our results extend existing integral inequalities reported in the literature. Additionally, we discuss some applications of our results for continuous random variables (CRV, for short) with bounded probability density functions. Some new estimates are provided in this context.

2. Preliminaries on fractional calculus

Definition 2.1. *The Riemann-Liouville fractional integral operator of order $\alpha\geq 0$, for an L^1 function f defined over $[a,b]$ is given by:*

$$\begin{aligned} J^{\alpha}f(t) &= \tfrac{1}{\Gamma(\alpha)}\int_a^t(t-\tau)^{\alpha-1}f(\tau)d\tau;\quad \alpha>0,\\ J^{0}f(t) &= f(t). \end{aligned} \tag{5}$$

We recall the property:

$$J^{\alpha}J^{\beta}f(t)=J^{\alpha+\beta}f(t), \alpha\geq 0, \beta\geq 0. \tag{6}$$

For more details, the reader can consult [6].

3. Main results

3.1. *An α-order pre-Gruss fractional integral inequality*

Theorem 3.1. *Let f and g be two integrable functions on $[a,b]$, such that f satisfies the condition (2). Then for all $\alpha>0$, we have:*

$$\begin{aligned} &\Big|\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}fg(t)-J^{\alpha}f(t)J^{\alpha}g(t)\Big| \qquad (7)\\ \leq\ &\Big(\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}(M-m)\Big)\Big[\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}g^{2}(t)-\Big(J^{\alpha}g(t)\Big)^{2}\Big]^{\frac{1}{2}},\ t\in[a,b]. \end{aligned}$$

Based on Lemma 3.2 of [4], we can prove the following lemma over $[a,b]$:

Lemma 3.2. *Let u be an integrable function on $[a,b]$ satisfying the condition of f given in (2). Then for any $\alpha>0$, and $t\in[a,b]$, we have:*

$$\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}u^{2}(t)-\Big(J^{\alpha}u(t)\Big)^{2} \tag{8}$$

$$= \Big(M\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} - J^\alpha u(t)\Big)\Big(J^\alpha u(t) - m\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}\Big)$$
$$-\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J^\alpha (M-u(t))(u(t)-m).$$

Proof. [Proof of Theorem 3.1] Let us consider the quantity defined by the expression:

$$H(\tau,\rho) := (f(\tau)-f(\rho))(g(\tau)-g(\rho));\ \tau,\rho\in(a,t). \tag{9}$$

Multiplying (9) by $\frac{(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}}{\Gamma^2(\alpha)}$; $\tau,\rho\in(a,t)$, then integrating with respect to τ and ρ over $(a,t)^2$, we can observe that

$$\frac{1}{\Gamma^2(\alpha)}\int_a^t\int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}H(\tau,\rho)d\tau d\rho \tag{10}$$
$$= 2\frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha fg(t) - 2J^\alpha f(t)J^\alpha g(t).$$

Thanks to Cauchy Schwarz inequality, it yields that

$$\Big(\frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha fg(t) - J^\alpha f(t)J^\alpha g(t)\Big)^2 \tag{11}$$
$$\leq \Big(\frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha f^2(t) - (J^\alpha f(t))^2\Big)\Big(\frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha g^2(t) - (J^\alpha g(t))^2\Big).$$

Using the fact that $(M-f(x))(f(x)-m)\geq 0$, we deduce by Riemann Liouville integration that the integral inequality

$$\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J^\alpha (M-f(t))(f(t)-m)\geq 0 \tag{12}$$

is valid for any $t\in[a,b]$.
Consequently, we can write

$$\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J^\alpha f^2(t) - \Big(J^\alpha f(t)\Big)^2 \tag{13}$$
$$\leq \Big(M\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} - J^\alpha f(t)\Big)\Big(J^\alpha f(t) - m\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}\Big).$$

Thanks to Lemma 3.2 and taking into account both (11) and (13), we can state that

$$\Big(\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J^\alpha fg(t) - 2J^\alpha f(t)J^\alpha g(t)\Big)^2 \tag{14}$$
$$\leq \Big(M\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} - J^\alpha f(t)\Big)\Big(J^\alpha f(t)$$
$$-m\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}\Big)\Big(\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J^\alpha g^2(t) - \Big(J^\alpha g(t)\Big)^2\Big).$$

On the other hand, since $4pq\leq(p+q)^2; p,q\in\mathbb{R}$, we can write

$$4\Big(M\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} - J^\alpha f(t)\Big)\Big(J^\alpha f(t) - m\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}\Big) \leq \Big(\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}(M-m)\Big)^2. \tag{15}$$

By (14) and (15), we get (7). □

Remark 3.3. *Applying Theorem 3.1 for $\alpha=1, t=b$, we obtain (3).*

3.2. An (α,β)-pre-Gruss fractional integral inequality

Our next result is the following theorem which is a well-known property of the probability density function of any continuous random variable (CRV).

Theorem 3.4. *Let f and g be two integrable functions on $[a,b]$ such that f satisfies the condition (2). Then for all $\alpha>0,\beta>0,t\in[a,b]$ we have:*

$$\begin{aligned}&\Big(\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}fg(t)+\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J^{\alpha}fg(t)-J^{\alpha}f(t)J^{\beta}g(t)-J^{\beta}f(t)J^{\alpha}g(t)\Big)^{2}\\ \leq\ &\Big[\Big(M\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}-J^{\alpha}f(t)\Big)\Big(J^{\beta}f(t)-m\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}\Big)\\ &+\Big(J^{\alpha}f(t)-m\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}\Big)\Big(M\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}-J^{\beta}f(t)\Big)\Big]\\ &\times\Big(\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}g^{2}(t)+\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J^{\alpha}g^{2}(t)-2J^{\alpha}g(t)J^{\beta}g(t)\Big).\end{aligned}\tag{16}$$

The proof of Theorem 3.4 requires the following lemmas, which are extensions of Lemma 3.4 and Lemma 3.5 of the paper [4], valid over any arbitrary interval $[a,b]$.

Lemma 3.5. *Let f and g be two integrable functions on $[a,b]$. Then for any $\alpha>0,\beta>0,t\in[a,b]$, we have:*

$$\begin{aligned}&\Big(\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}fg(t)+\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J^{\alpha}fg(t)-J^{\alpha}f(t)J^{\beta}g(t)-J^{\beta}f(t)J^{\alpha}g(t)\Big)^{2}\\ \leq\ &\Big(\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}f^{2}(t)+\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J^{\alpha}f^{2}(t)-2J^{\alpha}f(t)J^{\beta}f(t)\Big)\\ &\times\Big(\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}g^{2}(t)+\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J^{\alpha}g^{2}(t)-2J^{\alpha}g(t)J^{\beta}g(t)\Big).\end{aligned}\tag{17}$$

Lemma 3.6. *Let u be an integrable function on $[a,b]$ satisfying the condition (2) on $[a,b]$. Then for any $\alpha>0,\beta>0,t\in[a,b]$, we have:*

$$\begin{aligned}&\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}u^{2}(t)+\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J^{\alpha}u^{2}(t)-2J^{\alpha}u(t)J^{\beta}u(t)\\ =\ &\Big(M\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}-J^{\alpha}u(t)\Big)\Big(J^{\beta}u(t)-m\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}\Big)\\ &+\Big(M\frac{(t-a)^{\beta}}{\Gamma(\alpha+1)}-J^{\beta}u(t)\Big)\Big(J^{\alpha}u(t)-m\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}\Big)\\ &-\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}(M-u(t))(u(t)-m)-\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J^{\alpha}(M-u(t))(u(t)-m).\end{aligned}\tag{18}$$

Proof. [Proof of Theorem 3.4] Since $(M-f(t))(f(t)-m)\geq 0$, for any $t\in[a,b]$, then can write

$$-\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}(M-f(t))(f(t)-m)-\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J^{\alpha}(M-f(t))(f(t)-m)\leq 0.\tag{19}$$

Applying Lemma 3.6 to f, then using Lemma 3.5 and by (19), we obtain (16). □

Remark 3.7. (i) *Applying Theorem 3.4 for $\alpha=\beta$, we obtain Theorem 3.1.*
(ii) *Applying Theorem 3.4 for $\alpha=\beta=1,t=b$, we obtain (3).*

4. Applications

We begin this section by recalling the following properties [3, 4]:

$$J^{\alpha}1=\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}, J^{\alpha}[b]=\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{a(b-a)^{\alpha}}{\Gamma(\alpha+1)}$$

and

$$J^{\alpha}\left[b^{2}\right]=\frac{2(b-a)^{\alpha+2}}{\Gamma(\alpha+3)}+2aJ^{\alpha}[b]-\frac{a^{2}(b-a)^{\alpha}}{\Gamma(\alpha+1)}.$$

Let us also recall the following fractional integral expectation [13]:

$$E_{\alpha}(X):=\frac{1}{N_{\alpha}\Gamma(\alpha)}\int_{a}^{b}(b-\tau)^{\alpha-1}\tau f(\tau)d\tau,$$

where $N_{\alpha}:=J^{\alpha}[f(b)]$.
Remarque that when $\alpha=1$, we have $N_1=1$ which is a well known property of the probability density function of any CRV.

Theorem 4.1. *Let X be a CRV having the probability density function f defined over $[a,b]$, such that f satisfies the condition (2). Then for all $\alpha>0$, we have:*

$$\Big|\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}E_{\alpha}(X)-J^{\alpha}b\Big|\leq N_{\alpha}^{-1}\Big(\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}(M-m)\Big)\Big[\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}b^{2}-\left(J^{\alpha}b\right)^{2}\Big]^{\frac{1}{2}}.$$

Proof. [Proof of Theorem 4.1] We take $g(t)=t$ in Theorem 3.1, we can write

$$\Big|\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}tf(t)-J^{\alpha}f(t)J^{\alpha}t\Big|\leq\Big(\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}(M-m)\Big)\Big[\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}t^{2}-\left(J^{\alpha}t\right)^{2}\Big]^{\frac{1}{2}}.$$

Taking $t=b$, yields the following estimate

$$\Big|\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}bf(b)-J^{\alpha}f(b)J^{\alpha}b\Big|\leq\Big(\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}(M-m)\Big)\Big[\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}b^{2}-\left(J^{\alpha}b\right)^{2}\Big]^{\frac{1}{2}}$$

that is

$$N_{\alpha}\Big|\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}E_{\alpha}(X)-J^{\alpha}b\Big|\leq\Big(\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}(M-m)\Big)\Big[\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}b^{2}-\left(J^{\alpha}b\right)^{2}\Big]^{\frac{1}{2}}.$$

□

Remark 4.2. *If we take $\alpha=1$, then $N_1=1$, so we obtain Theorem 4 of [2].*

Theorem 4.3. *Let X be a CRV having the probability density function f defined over $[a,b]$, such that f satisfies the condition (2). Then for any $\alpha>0,\beta>0,t\in[a,b]$, we have:*

$$\begin{aligned}&\Big(N_{\beta}\frac{(b-a)^{\alpha}}{\Gamma(\beta+1)}E_{\beta}(X)+N_{\alpha}\frac{(b-a)^{\beta}}{\Gamma(\alpha+1)}E_{\alpha}(X)-N_{\alpha}J^{\beta}b-N_{\beta}J^{\alpha}b\Big)^{2}\\ \leq\ &\Big[\Big(M\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}-N_{\alpha}\Big)\Big(N_{\beta}-m\frac{(b-a)^{\beta}}{\Gamma(\beta+1)}\Big)\\ &+\Big(N_{\alpha}-m\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\Big)\Big(M\frac{(b-a)^{\beta}}{\Gamma(\beta+1)}-N_{\beta}\Big)\Big]\\ &\times\Big(\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}b^{2}+\frac{(b-a)^{\beta}}{\Gamma(\beta+1)}J^{\alpha}b^{2}-2J^{\alpha}bJ^{\beta}b\Big).\end{aligned}\tag{20}$$

Remark 4.4. *(i) Applying Theorem 4.3 for $\alpha=\beta$, we obtain Theorem 4.1.*
(ii) Applying Theorem 4.3 for $\alpha=\beta=1$, we obtain the inequality (3) .

5. Conclusion

In summary, our study has demonstrated the effectiveness of Riemann-Liouville fractional integrals in deriving significant integral results concerning the α-order pre-Grüss fractional integral inequality and the (α, β)-order pre-Grüss fractional integral inequality. The above main results have extended existing integral inequalities reported in the literature, thereby opening new avenues in this research domain. Additionally, we have explored new applications of our results for continuous random variables with bounded probability density functions, thereby making valuable contributions to probability theory and mathematical analysis. Furthermore, our study has yielded new estimates along with the rediscovery of classical results as special cases of our main theorems. These contributions enrich the existing body of knowledge and provide useful tools for eventually solving practical problems in various application domains.

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