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The φ-mixed volumes

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Abstract. In the paper, our main aim is to introduce a new φ-mixed volume $\overline{V}_{\varphi}(K_1,\ldots,K_n,L_n)$ of $(n+1)$ convex bodies, which obeys classical properties. The new affine geometric quantity in special case yields the classical mixed volume $V(K_1,\ldots,K_n)$, p-mixed quermassintegral $W_{p,i}(K,L)$ and the newly established L_p-multiple mixed volume $V_{\varphi_p}(K_1,\ldots,K_n,L_n)$, respectively. As an application, we establish an Orlicz Alesandrov-Fenchel inequality for the φ-mixed volumes, which follows the classical Alesandrov-Fenchel inequality, L_p-Minkowski inequality for p-mixed quermassintegrals and L_p-Alesandrov-Fenchel inequality, respectively.

1. Introduction

If K is a nonempty closed (not necessarily bounded) convex set in $\mathbb{R}^n$, then (see e.g. [2])

$$h(K,x)=\max\{x\cdot y : y\in K\}, \tag{1.1}$$

for $x\in\mathbb{R}^n$, defines the support function $h(K,x)$ of K, where $x\cdot y$ denotes the usual inner product of x and y in $\mathbb{R}^n$. A nonempty closed convex set is uniquely determined by its support function.

Associated with convex bodies (compact convex subsets with nonempty interiors) $K_1,\ldots,K_n$ is a Borel measure, $S(K_1,\ldots,K_{n-1};\cdot)$, on S^{n-1}, called the mixed surface area measure of convex bodies $K_1,\ldots,K_{n-1}$, which has the property that for each compact convex subset K_n (see e.g [11]),

$$V(K_1,\ldots,K_n)=\frac{1}{n}\int_{S^{n-1}}h(K_n,u)dS(K_1,\ldots,K_{n-1};u). \tag{1.2}$$

In fact, the measure $S(K_1,\ldots,K_{n-1};\cdot)$, can be defined by the property that (1.2) holds for all K_n, and $V(K_1,\ldots,K_n)$ denotes the mixed volume of convex bodies $K_1,\ldots,K_n$. When $K_1=\cdots=K_{n-i-1}=K$ and $K_{n-i}=\cdots=K_{n-1}=B$, $S(K_1,\ldots,K_{n-1};\cdot)$ becomes the i-th mixed surface area measure $S_i(K;u)$.

In the paper, our main aim is to introduce a new concept called it φ-mixed volume $\overline{V}_{\varphi}(K_1,\ldots,K_n,L_n)$ of $(n+1)$ convex body, which obeys classical properties, including continuity, boundedness and affine invariance. The φ-mixed volume $\overline{V}_{\varphi}(K_1,\ldots,K_n,L_n)$ in special case yields the classical mixed volume

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$V(K_1,\dots,K_n)$, p-mixed quermassintegral $W_{p,i}(K,L)$ and the newly established L_p-multiple mixed volume $V_{\varphi_p}(K_1,\dots,K_n,L_n)$, respectively. We establish an Orlicz Alesandrov-Fenchel inequality for the φ-mixed volumes, which follows the classical Alesandrov-Fenchel inequality, L_p-Minkowski inequality for p-mixed quermassintegrals and L_p-Alesandrov-Fenchel inequality, respectively. As applications, some Orlicz Brunn-Minkowski type inequalities are also derived.

We consider a convex and strictly increasing function $\varphi:[0,\infty)\to[0,\infty)$ with $\varphi(0)=0$. Let Φ be the class of convex and strictly increasing functions $\varphi:[0,\infty)\to[0,\infty)$ such that $\varphi(0)=0$. The *φ-mixed volume* $\overline{V}_\varphi(K_1,\dots,K_n,L_n)$ of $(n+1)$ convex bodies $K_1,\dots,K_n,L_n$ is defined by (see Section 3 for the definition)

$$\overline{V}_\varphi(K_1,\dots,K_n,L_n)=\inf\left\{\lambda>0:\int_{S^{n-1}}\varphi\left(\frac{h(K_n,u)}{\lambda h(L_n,u)}\right)dV(K_1,\dots,K_{n-1},L_n;u)\le 1\right\}, \tag{1.3}$$

where $dV(K_1,\dots,K_{n-1},L_n;u)$ denotes mixed volume probability measure of $K_1,\dots,K_{n-1},L_n$, and (see [14])

$$dV(K_1,\dots,K_{n-1},L_n;u)=\frac{1}{nV(K_1,\dots,K_{n-1},L_n)}h(L_n,u)dS(K_1,\dots,K_{n-1};u). \tag{1.4}$$

Remark 1.1 With $\varphi=\varphi_1(t)=t$, (1.3) becomes

$$\overline{V}_{\varphi_1}(K_1,\dots,K_n,L_n)=\frac{V(K_1,\dots,K_n)}{V(K_1,\dots,K_{n-1},L_n)}. \tag{1.5}$$

With $\varphi=\varphi_p(t)=t^p$, and $p\ge 1$, (1.3) yields that

$$\overline{V}_{\varphi_p}(K_1,\dots,K_n,L_n)^p=\frac{V_{\varphi_p}(K_1,\dots,K_n,L_n)}{V(K_1,\dots,K_{n-1},L_n)}. \tag{1.6}$$

where $V_{\varphi_p}(K_1,\dots,K_n,L_n)$ is the L_p-multiple mixed volume of $(n+1)$ convex bodies $K_1,\dots,K_{n-1},L_n$, and (see [14])

$$V_{\varphi_p}(K_1,\dots,K_n,L_n)=\frac{1}{n}\int_{S^{n-1}}\left(\frac{h(K_n,u)}{h(L_n.u)}\right)^p h(L_n,u)dS(K_1,\dots,K_{n-1};u). \tag{1.7}$$

Remark 1.2 Putting $K_1=\cdots=K_{n-i-1}=K$, $K_{n-i}=\cdots=K_{n-1}=B$, $K_n=L$ and $L_n=K$ in (1.3), and let $\varphi=\varphi_p(t)=t^p$, and $p\ge1$, then

$$\overline{V}_{\varphi_p}(K,\dots,K,\underbrace{B,\dots,B}_{i},L,K)=\left(\frac{W_{p,i}(K,L)}{W_i(K)}\right)^{1/p}, \tag{1.8}$$

where $W_i(K)$ is the classical quermassintegral of convex body K, and $W_{p,i}(K,L)$ is the well-known p-mixed quermassintegral of convex bodies K and L, and (see [6])

$$W_{p,i}(K,L)=\frac{1}{n}\int_{S^{n-1}}h(L,u)^p h(K,u)^{1-p}dS_i(K;u).$$

Obviously, the L_p-mixed volume $V_p(K,L)$ of convex bodies K and L is a special case of $W_{p,i}(K,L)$. When $i=0$, (1.8) becomes

$$\overline{V}_{\varphi_p}(\underbrace{K,\dots,K}_{n-1},L,K)=\left(\frac{V_p(K,L)}{V(K)}\right)^{1/p}. \tag{1.9}$$

In Section 4, we establish the following Orlicz Alesandrov-Fenchel inequality for the new φ-mixed volumes of $(n+1)$ convex bodies $K_1,\dots,K_n,L_n$.

Orlicz Alesandrov-Fenchel inequality for φ-mixed volume *If $K_1, \ldots, K_n, L_n$ are convex bodies containing the origin in their interiors, $1 \leq r < n$, $\varphi \in \Phi$ and $\varphi(c_\varphi) = 1$, then*

$$\overline{V}_\varphi(K_1, \ldots, K_n, L_n) \geq \frac{1}{c_\varphi V(K_1, \ldots, K_{n-1}, L_n)} \cdot \prod_{i=1}^{r} V(K_i, \ldots, K_i, K_{r+1}, \ldots, K_n)^{1/r}. \tag{1.10}$$

Remark 1.3 When $\varphi(t) = t$, (1.10) becomes the following classical Alesandrov-Fenchel inequality for mixed volumes of n convex bodies $K_1, \ldots, K_n$ (see e.g. [5]).

The Alesandrov-Fenchel inequality for mixed volumes If $K_1, \ldots, K_n$ are convex bodies containing the origin in their interiors and $1 \leq r < n$, then

$$V(K_1, \ldots, K_n) \geq \prod_{j=1}^{r} V(K_j, \ldots, K_j, K_{r+1}, \ldots, K_n)^{1/r}. \tag{1.11}$$

Unfortunately, the equality conditions of this inequality are, in general, unknown (see the discussion in Schneider [12]).

Remark 1.4 When $\varphi(t) = t^p$ and $p \geq 1$, (1.10) becomes the following L_p-Alesandrov-Fenchel inequality for L_p-multiple mixed volumes of $(n+1)$ convex bodies $K_1, \ldots, K_n, L_n$, which was recently established by Zhao [14].

The L_p-Aleksandrov-Fenchel nequality for L_p-multiple mixed volumes If $K_1, \cdots, K_n, L_n$ are convex bodies containing the origin in its interiors, $1 \leq r \leq n$ and $p \geq 1$, then

$$V_{\varphi_p}(K_1, \cdots, K_n, L_n) \geq \frac{\prod_{i=1}^{r} V(K_i \ldots, K_i, K_{r+1}, \ldots, K_n)^{p/r}}{V(K_1, \cdots, K_{n-1}, L_n)^{p-1}}. \tag{1.12}$$

Remark 1.5 When $r = n - i - 1$, $K_1 = \cdots = K_{n-i-1} = K$, $K_{n-i} = \cdots = K_{n-1} = L$, $K_n = L$ and $L_n = K$, $\varphi(t) = t^p$ and $p \geq 1$, and in view of (2.8), (1.10) becomes the following well-known L_p-Minkowski inequality for p-mixed quermassintegral.

L_p-Minkowski inequality for p-mixed quermassintegral If K and L are convex bodies containing the origin in their interiors, $p > 1$ and $0 \leq i < n - 1$, then

$$W_{p,i}(K, L)^{n-i} \geq W_i(K)^{n-i-p} W_i(L)^p, \tag{1.13}$$

with equality if and only if K and L are homothetic.

2. Notations and preliminaries

The setting for this paper is the n-dimensional Euclidean space $\mathbb{R}^n$. We write $\mathcal{K}^n$ for the set of convex bodies (compact convex subsets with nonempty interiors) of $\mathbb{R}^n$. We write $\mathcal{K}_o^n$ for the set of convex bodies that contain the origin in their interiors. We reserve the letter $u \in S^{n-1}$ for unit vectors, and the letter B for the unit ball centered at the origin. For a compact set K, we write $V(K)$ for the (n-dimensional) Lebesgue measure of K and call this the volume of K. Support function is homogeneous of degree 1, that is,

$$h(K, rx) = rh(K, x), \tag{2.1}$$

for all $x \in \mathbb{R}^n$ and $r \geq 0$.

2.1 Basics regarding convex bodies

For $\phi \in \mathrm{GL}(n)$ write ϕ^t for the transpose of ϕ and ϕ^{-t} for the inverse of the transpose of ϕ. Write $|\phi|$ for the absolute value of the determinant of ϕ. Observe that from the definition of the support function it follows immediately that for $\phi \in \mathrm{GL}(n)$ the support function of the image $\phi K = \{\phi y : y \in K\}$ is given by (see [7])

$$h(\phi K, x) = h(K, \phi^t x), \tag{2.2}$$

Let d denote the Hausdorff metric on $\mathcal{K}^n$, i.e., for $K, L \in \mathcal{K}^n$,

$$d(K, L) = |h(K, u) - h(L, u)|_\infty,$$

where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$.

Let Φ be the class of convex and strictly increasing functions $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(0) = 0$. We say that the sequence $\{\varphi_i\}$, where the $\varphi_i \in \Phi$, is such that $\varphi_i \to \varphi_0 \in \Phi$ provided

$$|\varphi_i - \varphi_0|_I := \max_{t \in I} |\varphi_i(t) - \varphi_0(t)| \to 0,$$

for every compact interval $I \subset \mathbb{R}$.

For $K \in \mathcal{K}_o^n$, r_K and R_K are defined by

$$r_K = \min_{u \in S^{n-1}} h(K, u), \quad R_K = \max_{u \in S^{n-1}} h(K, u). \tag{2.3}$$

2.2 *Mixed volumes*

If $K_i \in \mathcal{K}^n$ $(i = 1, 2, \ldots, r)$ and λ_i $(i = 1, 2, \ldots, r)$ are nonnegative real numbers, then of fundamental importance is the fact that the volume of $\sum_{i=1}^r \lambda_i K_i$ is a homogeneous polynomial in λ_i given by (see e.g. [8])

$$V(\lambda_1 K_1 + \cdots + \lambda_n K_n) = \sum_{i_1, \ldots, i_n} \lambda_{i_1} \ldots \lambda_{i_n} V_{i_1 \ldots i_n}, \tag{2.4}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding r. The coefficient $V_{i_1 \ldots i_n}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$ and is uniquely determined by (2.4), it is called the mixed volume of $K_{i_1}, \ldots, K_{i_n}$, and is written as $V(K_1, \ldots, K_n)$. The mixed volume $V(K_1, \ldots, K_n)$ has recently been given the following representation (see [14]):

$$V(K_1, \cdots, K_n) = \lim_{\varepsilon \to 0^+} \frac{V(K_1, \cdots, K_{n-1}, K_n + \varepsilon \cdot K_n) - V(K_1, \cdots, K_n)}{\varepsilon}. \tag{2.5}$$

This is very interesting that the mixed volume is such a limiting form.

Let $K_1 = \ldots = K_{n-i} = K$ and $K_{n-i+1} = \ldots = K_n = L$, then the mixed volume $V(K_1, \ldots, K_n)$ is written as $V_i(K, L)$. When $i = 1$, $V_i(K, L)$ becomes the classical mixed volume $V_1(K, L)$ of K and L, and

$$V_1(K, L) = \frac{1}{n} \lim_{\varepsilon \to 0^+} \frac{V(K + \varepsilon L) - V(K)}{\varepsilon} = \frac{1}{n} \int_{S^{n-1}} h(L, u) dS(K, u). \tag{2.6}$$

A fundamental inequality for mixed volume $V_1(K, L)$ is the following Minkowski inequality: for $K, L \in \mathcal{K}^n$,

$$V_1(K, L)^n \geq V(K)^{n-1} V(L), \tag{2.7}$$

with equality if and only if K and L are homothetic.

Let $K_1 = \ldots = K_{n-i} = K$ and $K_{n-i+1} = \ldots = K_n = L$, then the mixed volume $V(K_1, \ldots, K_n)$ is written as $V_i(K, L)$. If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = B$, the mixed volumes $V_i(K, B)$ is written as $W_i(K)$ and called as quermassintegrals (or ith mixed quermassintegrals) of K. We write $W_i(K, L)$ for the mixed volume $V(K, \ldots, K, \underbrace{B, \ldots, B}_{i}, L)$ and call as mixed quermassintegrals. Aleksandrov [1] and Fenchel and Jessen [4] (also see Busemann [3] and Schneider [13] have shown that for $K \in \mathcal{K}_o^n$, and $i = 0, 1, \ldots, n-1$, there exists a regular Borel measure $S_i(K,\cdot)$ on S^{n-1}, such that the mixed quermassintegrals $W_i(K,L)$ has the following representation:

$$W_i(K,L)=\frac{1}{n}\int_{S^{n-1}}h(L,u)dS_i(K,u).$$

A fundamental inequality for mixed quermassintegrals stats that: For $K,L\in\mathcal{K}_o^n$ and $0\leq i<n-1$,

$$W_i(K,L)^{n-i}\geq W_i(K)^{n-i-1}W_i(L), \tag{2.8}$$

with equality if and only if K and L are homothetic.

2.3 Mixed p-quermassintegrals

Mixed quermassintegrals are, of course, the first variation of the ordinary quermassintegrals, with respect to Minkowski addition. The mixed quermassintegrals $W_{p,0}(K,L),W_{p,1}(K,L),\ldots,W_{p,n-1}(K,L)$, as the first variation of the ordinary quermassintegrals, with respect to Firey addition: For $K,L\in\mathcal{K}_o^n$, and real $p\geq 1$, defined by (see [6])

$$W_{p,i}(K,L)=\frac{p}{n-i}\lim_{\varepsilon\to 0^+}\frac{W_i(K+_p\varepsilon\cdot L)-W_i(K)}{\varepsilon}, \tag{2.9}$$

where $+_p$ is the p-addition. The mixed p-quermassintegrals $W_{p,i}(K,L)$, for all $K,L\in\mathcal{K}_{oo}^n$, has the following integral representation:

$$W_{p,i}(K,L)=\frac{1}{n}\int_{S^{n-1}}h(L,u)^p dS_{p,i}(K,u), \tag{2.10}$$

where $S_{p,i}(K,\cdot)$ denotes a Borel measure on S^{n-1}. The measure $S_{p,i}(K,\cdot)$ is absolutely continuous with respect to $S_i(K,\cdot)$, and has Radon-Nikodym derivative (see [9])

$$\frac{dS_{p,i}(K,\cdot)}{dS_i(K,\cdot)}=h(K,\cdot)^{1-p}. \tag{2.11}$$

A fundamental inequality for mixed p-quermassintegrals states that: For $K,L\in\mathcal{K}_o^n, p>1$ and $0\leq i<n-1$,

$$W_{p,i}(K,L)^{n-i}\geq W_i(K)^{n-i-p}W_i(L)^p, \tag{2.12}$$

with equality if and only if K and L are homothetic. Obviously, putting $i=0$ in (2.6), the mixed p-quermassintegrals $W_{p,i}(K,L)$ become the well-known L_p-mixed volume $V_p(K,L)$, defined by (see e.g. [10])

$$V_p(K,L)=\frac{1}{n}\int_{S^{n-1}}h(L,u)^p dS_p(K,u). \tag{2.13}$$

2.4 Orlicz multiple mixed volumes

Let us introduce Orlicz multiple mixed volume $(n+1)$ convex bodies $K_1,\cdots,K_n,L_n$.

Definition 2.1 (see [14]) For $\varphi\in\Phi$, we define Orlicz multiple mixed volume of $(n+1)$ convex bodies $K_1,\cdots,K_n,L_n$, denoted by $V_\varphi(K_1,\cdots,K_n,L_n)$, as

$$V_\varphi(K_1,\cdots,K_n,L_n)=:\frac{1}{n}\int_{S^{n-1}}\varphi\left(\frac{h(K_n,u)}{h(L_n,u)}\right)h(L_n,u)dS(K_1,\ldots,K_{n-1};u), \tag{2.14}$$

for all $K_1,\ldots,K_n,L_n\in\mathcal{K}_o^n$.

Apparently, when $\varphi(t)=t^p$ and $p\geq 1$, $V_\varphi(K_1,\cdots,K_n,L_n)$ becomes the L_p multiple mixed volume $V_{\varphi_p}(K_1,\cdots,K_n,L_n)$ stated in the introduction.

A fundamental inequality for Orlicz multiple mixed volume states that:

Orlicz-Aleksandrov-Fenchel inequality (see [14]) *If* $K_1,\cdots,K_n,L_n\in\mathcal{K}_o^n$, $1\leq r\leq n$ *and* $\varphi\in\Phi$, *then*

$$V_\varphi(K_1,\cdots,K_n,L_n)\geq V(K_1,\cdots,K_{n-1},L_n)\cdot\varphi\left(\frac{\prod_{i=1}^r V(K_i\ldots,K_i,K_{r+1},\ldots,K_n)^{\frac{1}{r}}}{V(K_1,\cdots,K_{n-1},L_n)}\right). \tag{2.15}$$

Putting $\varphi(t)=t^p$ and $p\geq 1$ in (2.15), (2.15) becomes the L_p-Aleksandrov-Fenchel inequality (1.12) stated in the introduction.

3. The φ-mixed volume

We first give the definition of φ-mixed volume of $(n+1)$ convex bodies $K_1,\ldots,K_n,L_n$.

Definition 3.1 Let $K_1,\ldots,K_n,L_n\in\mathcal{K}^n$ and $\varphi\in\Phi$, the φ-mixed volume of $(n+1)$ convex bodies $K_1,\ldots,K_n,L_n$, is denoted by $\overline{V}_\varphi(K_1,\ldots,K_n,L_n)$, is defined by

$$\overline{V}_\varphi(K_1,\ldots,K_n,L_n)=\inf\left\{\lambda>0:\int_{S^{n-1}}\varphi\left(\frac{h(K_n,u)}{\lambda h(L_n,u)}\right)dV(K_1,\ldots,K_{n-1},L_n;u)\leq 1\right\}. \tag{3.1}$$

Since $\varphi\in\Phi$, it follows that the function:

$$\lambda\to\int_{S^{n-1}}\varphi\left(\frac{h(K_n,u)}{\lambda h(L_n,u)}\right)dV(K_1,\ldots,K_{n-1},L_n;u)$$

is also strictly decreasing in $(0,\infty)$. This yields that

Lemma 3.2 *If* $K_1,\ldots,K_n,L_n\in\mathcal{K}_o^n$ *and* $\varphi\in\Phi$, *then*

$$\int_{S^{n-1}}\varphi\left(\frac{h(K_n,u)}{\lambda_o h(L_n,u)}\right)dV(K_1,\ldots,K_{n-1},L_n;u)=1$$

if and only if

$$\overline{V}_\varphi(K_1,\ldots,K_n,L_n)=\lambda_o.$$

When $\lambda_0=1$, the φ-mixed volume becomes the-well known Orlicz-multiple mixed volume. This is very interesting.

Lemma 3.3 *If* $K_1,\ldots,K_n,L_n,K_n'\in\mathcal{K}_o^n$, *and* $\varphi\in\Phi$, *then*

(i) For $\gamma>0$,

$$\overline{V}_\varphi(K_1,\ldots,\gamma K_n,L_n)=\gamma\overline{V}_\varphi(K_1,\ldots,K_n,L_n).$$

(i) For $\gamma>0$,

$$\overline{V}_\varphi(K_1,\ldots,K_n,\gamma L_n)=\frac{1}{\gamma}\overline{V}_\varphi(K_1,\ldots,K_n,L_n).$$

(iii)

$$\overline{V}_\varphi(K_1,\ldots,K_{n-1},K_n+K_n',L_n)\leq\overline{V}_\varphi(K_1,\ldots,K_n,L_n)+\overline{V}_\varphi(K_1,\ldots,K_{n-1},K_n',L_n).$$

Proof First, for any $\gamma>0$, we obtain

$$\begin{aligned}\overline{V}_\varphi(K_1,\ldots,K_{n-1},\gamma K_n,L_n) &= \inf\left\{\lambda>0:\int_{S^{n-1}}\varphi\left(\frac{h(\gamma K_n,u)}{\lambda h(L_n,u)}\right)dV(K_1,\ldots,K_{n-1},L_n;u)\leq 1\right\}\\ &= \gamma\inf\left\{\mu>0:\int_{S^{n-1}}\varphi\left(\frac{h(K_n,u)}{\mu h(L_n,u)}\right)dV(K_1,\ldots,K_{n-1}.L_n;u)\leq 1\right\}\\ &= \gamma\overline{V}_\varphi(K_1,\ldots,K_n,L_n),\end{aligned}$$

where $\mu=\dfrac{\lambda}{\gamma}$.

Second, for any $\gamma > 0$, we obtain

$$\begin{aligned}\overline{V}_\varphi(K_1,\ldots,K_{n-1},K_n,\gamma L_n) &= \inf\left\{\lambda>0:\int_{S^{n-1}}\varphi\left(\frac{h(K_n,u)}{\lambda\gamma h(L_n,u)}\right)dV(K_1,\ldots,K_{n-1},L_n;u)\le 1\right\}\\ &= \frac{1}{\gamma}\inf\left\{\delta>0:\int_{S^{n-1}}\varphi\left(\frac{h(K_n,u)}{\delta h(L_n,u)}\right)dV(K_1,\ldots,K_{n-1}.L_n;u)\le 1\right\}\\ &= \frac{1}{\gamma}\overline{V}_\varphi(K_1,\ldots,K_n,L_n),\end{aligned}$$

where $\delta = \lambda\gamma$.

Let $\overline{V}_\varphi(K_1,\ldots,K_n,L_n)=\lambda_1$ and $\overline{V}_\varphi(K_1,\ldots,K_{n-1},K_n',L_n)=\lambda_2$, then

$$\int_{S^{n-1}}\varphi\left(\frac{h(K_n,u)}{\lambda_1 h(L_n,u)}\right)dV(K_1,\ldots,K_{n-1},L_n;u)=1,$$

and

$$\int_{S^{n-1}}\varphi\left(\frac{h(K_n',u)}{\lambda_2 h(L_n,u)}\right)dV(K_1,\ldots,K_{n-1};u)=1.$$

Combining the convexity of the function $s\to\varphi(s/h(L_n,u))$, we obtain

$$\begin{aligned}1 &= \frac{\lambda_1}{\lambda_1+\lambda_2}\int_{S^{n-1}}\varphi\left(\frac{h(K_n,u)}{\lambda_1 h(L_n,u)}\right)dV(K_1,\ldots,K_{n-1};u)\\ &\quad+\frac{\lambda_2}{\lambda_1+\lambda_2}\int_{S^{n-1}}\varphi\left(\frac{h(K_n',u)}{\lambda_2 h(L_n,u)}\right)dV(K_1,\ldots,K_{n-1};u)\\ &\ge \int_{S^{n-1}}\varphi\left(\frac{h(K_n,u)+h(K_n',u)}{(\lambda_1+\lambda_2)h(L_n,u)}\right)dV(K_1,\ldots,K_{n-1};u)\\ &= \int_{S^{n-1}}\varphi\left(\frac{h(K_n+K_n',u)}{(\lambda_1+\lambda_2)h(L_n,u)}\right)dV(K_1,\ldots,K_{n-1};u)\end{aligned}$$

Hence

$$\begin{aligned}\overline{V}_\varphi(K_1,\ldots,K_{n-1},K_n+K_n',L_n) &\le \lambda_1+\lambda_2\\ &= \overline{V}_\varphi(K_1,\ldots,K_n,L_n)+\overline{V}_\varphi(K_1,\ldots,K_{n-1},K_n',L_n).\end{aligned}$$

This completes the proof. □

In the following, we prove that the φ-mixed volume functional $\overline{V}_\varphi(K_1,\ldots,K_n,L_n)$ is continuous.

Lemma 3.4 *If $K_1,\ldots,K_n,L_n\in\mathcal{K}_o^n$, and $\varphi\in\Phi$, then φ-mixed volume functional $\overline{V}_\varphi(K_1,\ldots,K_n,L_n)$: $\underbrace{\mathcal{K}_o^n\times\cdots\times\mathcal{K}_o^n}_{n+1}\to[0,\infty)$ is continuous with respect to the Hausdorff metric.*

Proof To see this, indeed, let $K_{ij}\in\mathcal{S}^n$, $i\in\mathbb{N}\cup\{0\}$, $j=1,\ldots,n$, be such that $K_{ij}\to K_{0j}$ as $i\to\infty$ and $L_{in}\to L_{0n}$ as $i\to\infty$. Noting that

$$\begin{aligned}&\overline{V}_\varphi(K_{i1},\ldots,K_{in},L_{in})\\ &=\inf\left\{\lambda>0:\int_{S^{n-1}}\varphi\left(\frac{h(K_{in},u)}{\lambda h(L_{in},u)}\right)dV(K_{i1},\ldots,K_{i(n-1)},L_{in};u)\le 1\right\}\\ &=\inf\left\{\lambda>0:\frac{1}{nV(K_{i1},\ldots,K_{i(n-1)},L_{in})}\int_{S^{n-1}}\varphi\left(\frac{h(K_{in},u)}{\lambda h(L_{in},u)}\right)h(L_{in},u)dS(K_{i1},\ldots,K_{i(n-1)};u)\le 1\right\}\end{aligned}$$

Since the mixed area measures is weakly continuous, i.e.

$$dS(K_{i1},\ldots,K_{i(n-1)};u)\to dS(K_{01},\ldots,K_{0(n-1)};u)\ \text{ weakly on }\ S^{n-1}.$$

Since $h(K_{in},u)\to h(K_{0n},u)$ and $h(L_{in},u)\to h(L_{0n},u)$, uniformly on S^{n-1}, and φ is continuous, implies that for any $\lambda>0$

$$\varphi\left(\frac{h(K_{in},u)}{\lambda h(L_{in},u)}\right)\to\varphi\left(\frac{h(K_{0n},u)}{\lambda h(L_{0n},u)}\right).$$

Further

$$\int_{S^{n-1}}\varphi\left(\frac{h(K_{in},u)}{\lambda h(L_{in},u)}\right)dV(K_{i1},\ldots,K_{i(n-1)},L_{in};u)\to\int_{S^{n-1}}\varphi\left(\frac{h(K_{0n},u)}{\lambda h(L_{0n},u)}\right)dV(K_{01},\ldots,K_{0(n-1)},L_{0n};u).$$

Hence

$$\begin{aligned}\lim_{i\to\infty}\overline{V}_\varphi(K_{i1},\ldots,K_{in},L_{in}) &= \inf\left\{\lambda>0:\int_{S^{n-1}}\varphi\left(\frac{h(K_{0n},u)}{\lambda h(L_{0n},u)}\right)dV(K_{01},\ldots,K_{0(n-1)},L_{0n};u)\le 1\right\}\\ &= \overline{V}_\varphi(K_{01},\ldots,K_{0n},L_{0n}).\end{aligned}$$

This shows that the φ-mixed volume $\overline{V}_\varphi(K_1,\ldots,K_n,L_n)$ is continuous. □

Lemma 3.5 *If $K_1,\ldots,K_n,L_n\in\mathcal{K}_o^n$, and $\varphi_i\in\Phi$, $i\in\mathbb{N}$, then*

$$\varphi_i\to\varphi\in\Phi\Rightarrow\overline{V}_{\varphi_i}(K_1,\ldots,K_{n-1},K_n,L_n)\to\overline{V}_\varphi(K_1,\ldots,K_n,L_n).$$

Proof We note that $\varphi_i\to\varphi\in\Phi$, implies that

$$\varphi_i\left(\frac{h(K_n,u)}{\lambda h(L_n,u)}\right)\to\varphi\left(\frac{h(K_n,u)}{\lambda h(L_n,u)}\right)\in\Phi.$$

Further

$$\int_{S^{n-1}}\varphi_i\left(\frac{h(K_n,u)}{\lambda h(L_n,u)}\right)dV(K_1,\ldots,K_{n-1},L_n;u)\to\int_{S^{n-1}}\varphi\left(\frac{h(K_n,u)}{\lambda h(L_n,u)}\right)dV(K_1,\ldots,K_{n-1},L_n;u).$$

Hence

$$\begin{aligned}\lim_{i\to\infty}\overline{V}_{\varphi_i}(K_1,\ldots,K_n,L_n) &= \inf\left\{\lambda>0:\int_{S^{n-1}}\varphi\left(\frac{h(K_n,u)}{\lambda h(L_n,u)}\right)dV(K_1,\ldots,K_{n-1},L_n;u)\le 1\right\}\\ &= \overline{V}_\varphi(K_1,\ldots,K_n,L_n).\end{aligned}$$

□

Lemma 3.6 *If $K_1,\ldots,K_n,L_n\in\mathcal{K}_o^n$, and $\varphi\in\Phi$, then Orlicz mixed volume $\overline{V}_\varphi(K_1,\ldots,K_n,L_n):\underbrace{\mathcal{K}_o^n\times\cdots\times\mathcal{K}_o^n}_{n+1}\to[0,\infty)$ is bounded.*

Proof For $\varphi\in\Phi$, there must be a real number $0<c_\varphi<\infty$ such that $\varphi(c_\varphi)=1$, and let

$$\overline{V}_\varphi(K_1,\ldots,K_n,L_n)=\lambda_0.$$

Hence

$$\begin{aligned} 1 &= \varphi(c_\varphi) \\ &= \int_{S^{n-1}} \varphi\left(\frac{h(K_n,u)}{\lambda_0 h(L_n,u)}\right) dV(K_1,\ldots,K_{n-1},L_n;u) \\ &\geq \varphi\left(\int_{S^{n-1}} \frac{h(L_n,u)}{\lambda_0 h(K_n,u)} dV(K_1,\ldots,K_{n-1},L_n;u)\right) \\ &\geq \varphi\left(\int_{S^{n-1}} \frac{r_{L_n}}{\lambda_0 R_{K_n}} dV(K_1,\ldots,K_{n-1},L_n;u)\right) \\ &= \varphi\left(\frac{r_{L_n}}{\lambda_0 R_{K_n}}\right). \end{aligned}$$

Since φ is monotone increasing on $[0,\infty)$, from this we obtain the lower bound,

$$\lambda_0 \geq \frac{r_{L_n}}{c_\varphi R_{K_n}}.$$

In a similar approach, we can obtain upper bound for $h(\mathbf{\Pi}_\varphi(K_1,\ldots,K_n,u)$,

$$\lambda_0 \leq \frac{R_{L_n}}{c_\varphi r_{K_n}}.$$

This completes the proof. □

We easily find that the φ-mixed volume $\overline{V}_\varphi(K_1,\ldots,K_n,L_n)$ is invariant under simultaneous unimodular centro-affine transformation.

Lemma 3.7 *If $K_1,\ldots,K_n,L_n \in \mathcal{K}_o^n$, $\phi \in \mathrm{SL}(n)$, and $\varphi \in \Phi$, then*

$$\overline{V}_\varphi(\phi K_1,\ldots,\phi K_n,\phi L_n) = \overline{V}_\varphi(K_1,\ldots,K_n,L_n). \tag{3.7}$$

Proof From (2.2) and (3.1), we obtain

$$\begin{aligned} \overline{V}_\varphi(\phi K_1,\ldots,\phi K_{n-1},K_n,\phi L_n) &= \inf\left\{\lambda>0: \frac{1}{V(\phi K_1,\ldots,\phi K_{n-1},\phi L_n)}\int_{S^{n-1}}\varphi\left(\frac{h(K_n,u)}{\lambda h(\phi L_n,u)}\right)\right. \\ &\times \left. h(\phi L_n,u)dS(\phi K_1,\ldots,\phi K_{n-1};u)\leq 1\right\} \\ &= \inf\left\{\lambda>0: \frac{1}{V(K_1,\ldots,K_{n-1},L_n)}\int_{S^{n-1}}\varphi\left(\frac{h(K_n,u)}{\lambda h(L_n,\phi^t u)}\right)\right. \\ &\times \left. h(L_n,\phi^t u)dS(K_1,\ldots,K_{n-1};\phi^t u)\leq 1\right\} \\ &= \inf\left\{\lambda>0: \frac{1}{V(K_1,\ldots,K_{n-1},L_n)}\int_{S^{n-1}}\varphi\left(\frac{h(K_n,\phi^{-t}u)}{\lambda h(L_n,u)}\right)\right. \\ &\times \left. h(L_n,u)dS(K_1,\ldots,K_{n-1};u)\leq 1\right\} \\ &= \inf\left\{\lambda>0: \frac{1}{V(K_1,\ldots,K_{n-1},L_n)}\int_{S^{n-1}}\varphi\left(\frac{h(\phi^{-1}K_n,u)}{\lambda h(L_n,u)}\right)\right. \\ &\times \left. h(L_n,u)dS(K_1,\ldots,K_{n-1};u)\leq 1\right\} \\ &= \overline{V}_\varphi(K_1,\ldots,K_{n-1},\phi^{-1}K_n,L_n). \end{aligned}$$

Hence

$$\overline{V}_{\varphi}(\phi K_1,\ldots,\phi K_n,\phi L_n)=\overline{V}_{\varphi}(K_1,\ldots,K_n,L_n).$$

This completes the proof. □

4. Orlicz Alesandrov-Fenchel inequality for φ-mixed volumes

Theorem 4.1 (Orlicz Alesandrov-Fenchel inequality for φ-mixed volume) *If $K_1,\ldots,K_n,L_n\in\mathcal{K}_o^n$, $1\leq r<n$, $\varphi\in\Phi$ and $\varphi(c_\varphi)=1$, then*

$$\overline{V}_{\varphi}(K_1,\ldots,K_n,L_n)\geq\frac{1}{c_\varphi V(K_1,\ldots,K_{n-1},L_n)}\cdot\prod_{i=1}^{r}V(K_i,\ldots,K_i,K_{r+1},\ldots,K_n)^{1/r}. \tag{4.1}$$

Proof For $\varphi\in\Phi$, let

$$\overline{V}_{\varphi}(K_1,\ldots,K_n,L_n)=\lambda. \tag{4.2}$$

Then

$$\frac{1}{nV(K_1,\ldots,K_{n-1},L_n)}\int_{S^{n-1}}\varphi\left(\frac{h(K_n,u)}{\lambda h(L_n,u)}\right)h(L_n,u)dS(K_1,\ldots,K_{n-1};u)=1.$$

Hence

$$\frac{1}{nV(K_1,\ldots,K_{n-1},L_n)\overline{V}_{\varphi}(K_1,\ldots,K_n,L_n)}\int_{S^{n-1}}\varphi\left(\frac{h(K_n,u)}{h(\lambda L_n,u)}\right)h(\lambda L_n,u)dS(K_1,\ldots,K_{n-1};u)=1. \tag{4.3}$$

From (3.1) and (4.3), we have

$$\overline{V}_{\varphi}(K_1,\ldots,K_n,L_n)=\frac{V_{\varphi}(K_1,\ldots,K_n,\lambda L_n)}{V(K_1,\ldots,K_{n-1},L_n)}. \tag{4.4}$$

From (4.4) and by using the Orlicz-Aleksandrov-Fenchel inequality (2.15), we obtain

$$\overline{V}_{\varphi}(K_1,\ldots,K_n,L_n)\geq\frac{V(K_1,\ldots,K_{n-1},\lambda L_n)}{V(K_1,\ldots,K_{n-1},L_n)}\cdot\varphi\left(\frac{\prod_{i=1}^{r}V(K_i,\ldots,K_i,K_{r+1},\ldots,K_{n-1},K_n)^{1/r}}{V(K_1,\ldots,K_{n-1},\lambda L_n)}\right).$$

For $\varphi\in\Phi$, there must be a real number $0<c_\varphi<\infty$ such that $\varphi(c_\varphi)=1$, further

$$1=\varphi(c_\varphi)\geq\varphi\left(\frac{\prod_{i=1}^{r}V(K_i,\ldots,K_i,K_{r+1},\ldots,K_n)^{1/r}}{V(K_1,\ldots,K_{n-1},\lambda L_n)}\right).$$

In view of the monotonicity of the function φ, we have

$$\overline{V}_{\varphi}(K_1,\ldots,K_n,L_n)\geq\frac{1}{c_\varphi V(K_1,\ldots,K_{n-1},L_n)}\cdot\prod_{i=1}^{r}V(K_i,\ldots,K_i,K_{r+1},\ldots,K_n)^{1/r}.$$

This completes the proof. □

As an application, we get the following Orlicz Brunn-Minkowski type inequality for φ-mixed volumes.

Theorem 4.2 (Orlicz Brunn-Minkowski inequality for φ-mixed volumes) *If $K_1,\ldots,K_n,L_n,L_{n+1}\in\mathcal{K}_o^n$, $1\leq r<n$, $\varphi\in\Phi$ and $\varphi(c_\varphi)=1$, then*

$$\overline{V}_{\varphi}(K_1,\ldots,K_n,L_{n+1})+\overline{V}_{\varphi}(K_1,\ldots,K_{n-1},L_n,L_{n+1})$$
$$\leq\frac{1}{c_\varphi V(K_1,\ldots,K_{n-1},L_{n+1})}\cdot\prod_{i=1}^{r}V(K_i,\ldots,K_i,K_{r+1},\ldots,K_{n-1},K_n+L_n)^{1/r}. \tag{4.5}$$

Proof This follows immediately from Lemma 3.3 and Theorem 4.1 □

Corollary 4.3 (L_p-Brunn-Minkowski inequality for φ-mixed volumes) *If $K_1,\ldots,K_n,L_n,L_{n+1}\in\mathcal{K}_o^n$, $p\geq 1$, $1\leq r<n$, then*

$$\overline{V}_{\varphi_p}(K_1,\ldots,K_n,L_{n+1})+\overline{V}_{\varphi_p}(K_1,\ldots,K_{n-1},L_n,L_{n+1})$$
$$\leq\frac{1}{V(K_1,\ldots,K_{n-1},L_{n+1})}\cdot\prod_{i=1}^{r}V(K_i,\ldots,K_i,K_{r+1},\ldots,K_{n-1},K_n+L_n)^{1/r}. \tag{4.6}$$

Proof This follows immediately from Theorem 4.2 with $\varphi=\varphi_p(t)=t^p$ and $p\geq 1$. □

Corollary 4.4 (Brunn-Minkowski type inequality) *If $K_1,\ldots,K_n,L_n,L_{n+1}\in\mathcal{K}_o^n$, $1\leq r<n$, then*

$$V(K_1,\ldots,K_n)+V(K_1,\ldots,K_{n-1},L_n)\leq\prod_{i=1}^{r}V(K_i,\ldots,K_i,K_{r+1},\ldots,K_{n-1},K_n+L_n)^{1/r}. \tag{4.7}$$

Proof This follows immediately from Theorem 4.2 and (1.5). □

Apparently, in view of (2.7), (4.7) becomes the following classical Brunn-Minkowski inequality: If $K,L\in\mathcal{K}_o^n$, then

$$V(K+L)^{1/n}\geq V(K)^{1/n}+V(L)^{1/n},$$

with equality if and only if K and L are homothetic.

Corollary 4.5 (L_p-Brunn-Minkowski inequality for L_p-multiple mixed volumes) *If $K_1,\ldots,K_n,L_n,L_{n+1}\in\mathcal{K}_o^n$, $p\geq 1$, $1\leq r<n$, then*

$$V_{\varphi_p}(K_1,\ldots,K_n,L_{n+1})^{1/p}+V_{\varphi_p}(K_1,\ldots,K_{n-1},L_n,L_{n+1})^{1/p}$$
$$\leq\frac{1}{V(K_1,\ldots,K_{n-1},L_{n+1})^{(p-1)/p}}\cdot\prod_{i=1}^{r}V(K_i,\ldots,K_i,K_{r+1},\ldots,K_{n-1},K_n+L_n)^{1/r}. \tag{4.8}$$

Proof This follows immediately from Corollary 4.3 and (1.6). □

Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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