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On some basic character of differentiation in neutrosophic normed spaces

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Abstract. In this presented article, we have exemplified and studied the Fréchet differentiation of nonlinear operators between neutrosophic normed spaces as a generalization of notions given [34] and introduced nonlinear theory of neutrosophic bounded operators by introducing chain rule and some algebraic properties of Fréchet differentiation of operators between neutrosophic normed spaces.

1. Introduction

The concept behind fuzzy logic was preliminary defined by L. A. Zadeh [35] in the year 1965, wherein specific elements had a degree of membership. In real requisition, although, the information of an object corresponding to a fuzzy concept may be insufficient, i.e., the addition of the membership degree and the non-membership degree of an element in a universe corresponding to a vague scheme may be less than one. But in the fuzzy set postulate, there is no means to incorporate the lack of knowledge with the membership degrees. The expected solution is to use the intuitionistic fuzzy set (in short, IFS) which was put forward by Atanassov [2] has pertained extensively in several fields of mathematics, engineering, economics, and science. Mursaleen et al. [25, 26] explored the statistical and ideal convergence in intuitionistic fuzzy topological space. These bring magnificent motivation to use IF-sets and IF-operators in an application. The concept of the IF-set is probably most useful in conclusion-making problems [3, 4, 31]. As an extension of fuzzy set in the year 2004, Park [27] explained the concept of intuitionistic fuzzy metric space. Furthermore, Park along with Saadati procured this concept in the norm and introduced intuitionistic fuzzy normed space-(IFNS) and IF-bounded linear operators [28]. We explored simple approaches of nonlinear functional analysis of operators equations along with Frechet derivative. Although, many questions in mathematics cannot be solved by the classical approach. For example, the top most fascinating usage of fuzzy topology in quantum particle physics occurs in the string and ϵ^{∞}-theory of El-Naschie [7–11]. Mursaleen and

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Mohiuddin [24] proposed Frechet differentiation of nonlinear operators in between IF-normed spaces as a generalization of notions given for the present and the fuzzy topology [1, 12–14, 22] which proved as best recognized theories on discontinuity. The neutrosophic set hypothesis was introduced by Smarandache [30]. Furthermore, Smarandache [30] generalized Atanassov's intuitionistic fuzzy sets. This set is an expansion of IFS no matter if the sum of neutrosophic components is 1^-, or 1^+, or $= 1$. For the specific case, when the sum of $T + M + F = 1$ (as in IFS), on applying the neutrosophic set operators (NSO), one can get different results by applying the intuitionistic fuzzy operators (IFO), since the IFS neglect the indeterminacy, while NSO take into consideration the indeterminacy at the same level as in the generated neutrosophic set, the components T(truth), M(indeterminacy) and F(falsity) were respectively, and having the value of in between $]0^-, 1^+[$. We can understand the neutrosophic sets in a more flexible and workable way by addressing the intuitionistic fuzziness and uncertainty in the traditional IFS. Bera and Mahapatra [5, 6] aligned the neutrosophic soft normed linear space and revealed convexity, metric and Cauchy sequence on it. Khan et al. [15, 18, 19] examined the bounded and continuous linear operators in neutrosophic norm spaces using a literature survey. A vector norm cannot be obtained under a number of conditions, thus the idea of a neutrosophic norm. Yilmaz [32–34] defined some basic properties of differentiation in intuitionistic fuzzy normed spaces, which may produce a helpful functional tool to explain the operator equations assuming such operators as a generalization of notions. In this paper, we want to put forward a nonlinear theory of neutrosophic bounded operators by suggesting a chain rule and some algebraic properties of Fréchet differentiation of operators in between NNS. So here, we have tried a proper approach to nonlinear functional analysis of operator equations by applying Fréchet differentiation of operators in between NNS.

2. Preliminaries

Definition 2.1. ([20, 29]) A continuous t-norm is a binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ with the following conditions:

(i) $\diamond$ is associative and commutative;

(ii) $\diamond$ is continuous;

(iii) $\eta_1 \diamond 1 = \eta_1, \ \forall \ \eta_1 \in [0,1]$;

(iv) $\eta_1 \diamond \eta_2 \leq \eta_3 \diamond \eta_4$ *whenever* $\eta_1 \leq \eta_3$ *and* $\eta_2 \leq \eta_4$, *for each* $\eta_1, \eta_2, \eta_3, \eta_4 \in [0,1]$.

Definition 2.2. ([16, 17, 29]) A continuous t-conorm is a binary operation $\star : [0,1] \times [0,1] \to [0,1]$ with the following conditions

(i) $\star$ is associative and commutative;

(ii) $\star$ is continuous;

(iii) $\eta_1 \star 0 = \eta_1, \forall \ \eta_1 \in [0,1]$;

(iv) $\eta_1 \star \eta_2 \leq \eta_3 \star \eta_4$ *whenever* $\eta_1 \leq \eta_3$ *and* $\eta_2 \leq \eta_4$, *for each* $\eta_1, \eta_2, \eta_3, \eta_4 \in [0,1]$.

Definition 2.3. ([18, 23]) Let U be a linear space and $\mathcal{G} = \{< \zeta, \mathsf{T}(\zeta), \mathsf{M}(\zeta), \mathsf{F}(\zeta) >: \zeta \in U\}$ be a normed space in such a way that $\mathcal{G} : U \times R^+ \to [0,1]$. Let $\diamond$ and $\star$ are continuous $t-$ norm and continuous $t-$ conorm respectively. Then the four-tuple $(U, \mathcal{G}, \diamond, \star)$ is called neutrosophic normed space (NNS), if it satisfy the following axioms, $\forall \ \zeta, y, z \in U$ and $d, t > 0$;

(i) $0 \leq \mathsf{T}(\zeta, t), \quad \mathsf{M}(\zeta, t), \quad \mathsf{F}(\zeta, t) \leq 1,$

(ii) $0 \leq \mathsf{T}(\zeta, d) + \mathsf{M}(\zeta, d) + \mathsf{F}(\zeta, d) \leq 3,$

(iii) $\mathsf{T}(\zeta, d) = 0$ for $d \leq 0,$

(vi) $\mathsf{T}(\zeta, d) = 1$ for $d > 0$ iff $\zeta = 0$

(v) $\mathbf{T}(\gamma\zeta, d) = \mathbf{T}(\zeta, \frac{d}{|\gamma|}) \quad \forall \quad \gamma \neq 0, \;\; d > 0,$

(vi) $\mathsf{T}(\zeta, d) \diamond \mathsf{T}(y, t) \leq \mathsf{T}(\zeta + y, d + t)$

(vii) $\mathsf{T}(\zeta, .)$ is continuous non-decreasing function, $\lim_{d\to\infty} \mathsf{T}(\zeta, d) = 1$

(viii) $\mathsf{M}(\zeta, d) = 1$ for $d \leq 0$,

(ix) $\mathsf{M}(\zeta, d) = 0$ for $d > 0$ iff $\zeta = 0$

(x) $\mathsf{M}(\gamma\zeta, d) = \mathsf{M}(\zeta, \frac{d}{|\gamma|}), \forall \gamma \neq 0, d > 0,$

(xi) $\mathsf{M}(\zeta, d) \star \mathsf{M}(y, t) \geq \mathsf{M}(\zeta + y, d + t),$

(xii) $\mathsf{M}(\zeta, .)$ is continuous non-increasing function, $\lim_{d\to\infty} \mathsf{M}(\zeta, d) = 0,$

(xiii) $\mathsf{F}(\zeta, d) = 1$ for $d \leq 0$,

(xiv) $\mathsf{F}(\zeta, d) = 0$ for $d \;\; > \;\; 0$ iff $\zeta = 0$,

(xv) $\mathsf{F}(\gamma\zeta, d) = \mathsf{F}(\zeta, \frac{d}{|\gamma|}) \; \forall \quad \gamma \neq 0, d > 0,$

(xvi) $\mathsf{F}(\zeta, d) \star \mathsf{F}(y, t) \geq \mathsf{F}(\zeta + y, d + t),$

(xvii) $\mathsf{F}(\zeta, .)$ is continuous non-increasing function, $\lim_{d\to\infty} \mathsf{F}(\zeta, d) = 0.$

In this case $\mathcal{G} = \{\mathsf{T}, \mathsf{M}, \mathsf{F}\}$ is called neutrosophic norm (NN).

Compared to norm space, neutrosophic normed space is more general. A suitable example illustrates this.
Let $(U, \mathcal{G}, \diamond, \star)$ is NNS. Assume $\zeta \diamond y = \zeta y$ and $\zeta \star y = \zeta + y - \zeta y$ for all $\zeta, y \in U$ and $d > 0$ with the condition

$$\mathsf{T}(\zeta, d) > 0 \;\; and \;\; \mathsf{M}(\zeta, d) < 1, \; \mathsf{F}(\zeta, d) < 1 \Rightarrow \zeta = 0 \;\; for\, all \;\; d > 0.$$

Let $\|\zeta\|_\gamma = \inf\Big\{d > 0 : \mathsf{T}(\zeta, d) \geq \gamma \;\; and \;\; \mathsf{M}(\zeta, d) \leq 1 - \gamma, \; \mathsf{F}(\zeta, d) \leq 1 - \gamma\Big\}, \forall\, \gamma \in (0, 1)$. Then $\{\|.\|_\gamma : \gamma \in (0, 1)\}$ is an ascending family of norms on U. These norms are said to be γ-norms on U compatible to neutrosophic norm $(\mathsf{T}, \mathsf{M}, \mathsf{F})$.

3. Neutrosophic continuity

Proposition 3.1. *Each* $\mathring{B}\Big(0, \frac{1}{v}, \frac{1}{v}\Big)$ *is balanced, absorbing and convex neighborhood of* 0.

Proof. To see $\mathring{B}\Big(0, \frac{1}{v}, \frac{1}{v}\Big)$ is balanced, we should prove that
$\eta\mathring{B}\Big(0, \frac{1}{v}, \frac{1}{v}\Big) \subseteq \mathring{B}\Big(0, \frac{1}{v}, \frac{1}{v}\Big)$ for $|\eta| \leq 1$. Let $\zeta \in \eta\mathring{B}\Big(0, \frac{1}{v}, \frac{1}{v}\Big)$, then $\exists$ a $w \in \mathring{B}\Big(0, \frac{1}{v}, \frac{1}{v}\Big)$ such that $\zeta = \eta w$. Thus

$$\mathsf{T}\Big(\zeta, \frac{1}{v}\Big) = \mathsf{T}\Big(\eta w, \frac{1}{v}\Big) = \mathsf{T}\Big(w, \frac{1}{|\eta| v}\Big) \geq \mathsf{T}\Big(w, \frac{1}{v}\Big) > 1 - \frac{1}{v},$$

$$\mathsf{M}\Big(\zeta, \frac{1}{v}\Big) = \mathsf{M}\Big(\eta w, \frac{1}{v}\Big) = \mathsf{M}\Big(w, \frac{1}{|\eta| v}\Big) \leq \mathsf{M}\Big(w, \frac{1}{v}\Big) < \frac{1}{v},$$

$$\mathsf{F}\Big(\zeta, \frac{1}{v}\Big) = \mathsf{F}\Big(\eta w, \frac{1}{v}\Big) = \mathsf{F}\Big(w, \frac{1}{|\eta| v}\Big) \leq \mathsf{F}\Big(w, \frac{1}{v}\Big) < \frac{1}{v}$$

since $|\eta| \leq 1$. This means, $\zeta \in \mathring{B}\big(0, \frac{1}{v}, \frac{1}{v}\big)$.
To see $\mathring{B}\big(0, \frac{1}{v}, \frac{1}{v}\big)$ is absorbing, let $\zeta \in U$ be arbitrary. Then we should obtain an $\lambda = \lambda(\zeta) > 0$ such that $\eta\zeta \in \mathring{B}\big(0, \frac{1}{v}, \frac{1}{v}\big)$ for every $|\eta| \leq \lambda$.
Remember that $\eta\zeta \in \mathring{B}\big(0, \frac{1}{v}, \frac{1}{v}\big)$ iff

$$\mathsf{T}\Big(w, \frac{1}{|\eta| v}\Big) > 1 - \frac{1}{v},\ \mathsf{M}\Big(w, \frac{1}{|\eta| v}\Big) < \frac{1}{v} \text{ and } \mathsf{F}\Big(w, \frac{1}{|\eta| v}\Big) < \frac{1}{v}.$$

By the fact that $\mathsf{T}\big(\zeta, \diamond\big) : (0, \infty) \to [0, 1]$ is continuous and $\lim_{d\to\infty} \mathsf{T}\big(\zeta, d\big) = 1$ we write $\mathsf{T}\big(\zeta, \frac{1}{|\eta| v}\big) \to 1$ as $|\eta| \to 0$.
Hence, we can choosen an $\lambda_1 > 0$ such that $\mathsf{T}\big(\zeta, \frac{1}{|\eta| v}\big) > 1 - \frac{1}{v}$ for $|\eta| \leq \lambda_1$.
By the similar conditions on M, we can find another $\lambda_2 > 0$ such that $\mathsf{M}\big(\zeta, \frac{1}{|\eta| v}\big) < \frac{1}{v}$ for $|\eta| \leq \lambda_2$.
Similar conditions on F, we can find another $\lambda_3 > 0$ such that $\mathsf{F}\big(\zeta, \frac{1}{|\eta| v}\big) < \frac{1}{v}$ for $|\eta| \leq \lambda_3$. Take $\lambda = \min\{\lambda_1, \lambda_2, \lambda_3\}$.
Finally, suppose $\zeta, w \in \mathring{B}\big(0, \frac{1}{v}, \frac{1}{v}\big)$ and $0 \leq \gamma \leq 1$. Then

$$\begin{aligned}
\mathsf{T}\Big(\gamma\zeta + (1-\gamma)w, \frac{1}{v}\Big) &= \mathsf{T}\Big(\gamma\Big(\zeta + \frac{1-\gamma}{\gamma} w\Big), \frac{1}{v}\Big) = \mathsf{T}\Big(\zeta + \frac{1-\gamma}{\gamma} w\Big), \frac{1}{\gamma v}\Big) \\
&= \mathsf{T}\Big(\zeta + \frac{1-\gamma}{\gamma} w, \frac{1}{v} + \Big(\frac{1-\gamma}{\gamma}\Big)\frac{1}{v}\Big) \\
&\geq \min\Big\{\mathsf{T}\Big(\zeta, \frac{1}{v}\Big), \mathsf{T}\Big(\frac{1-\gamma}{\gamma} w, \Big(\frac{1-\gamma}{\gamma}\Big)\frac{1}{v}\Big)\Big\} \\
&= \Big\{\mathsf{T}\Big(\zeta, \frac{1}{v}\Big), \mathsf{T}\Big(w, \frac{1}{v}\Big)\Big\} > 1 - \frac{1}{v}
\end{aligned}$$

and it isn't hard to see in a similar way that

$$\mathsf{M}\Big(\gamma\zeta + (1-\gamma)w, \frac{1}{v}\Big) \leq \max\Big\{\mathsf{M}\Big(\zeta, \frac{1}{v}\Big), \mathsf{M}\Big(w, \frac{1}{v}\Big)\Big\} < \frac{1}{v},$$

$$\mathsf{F}\Big(\gamma\zeta + (1-\gamma)w, \frac{1}{v}\Big) \leq \max\Big\{\mathsf{F}\Big(\zeta, \frac{1}{v}\Big), \mathsf{F}\Big(w, \frac{1}{v}\Big)\Big\} < \frac{1}{v},$$

that is $\gamma\zeta + (1-\gamma)w \in \mathring{B}\big(0, \frac{1}{v}, \frac{1}{v}\big)$. □

Now, we create the strong and weak neutrosophic continuity of mappings between NNS.

Definition 3.2. ([21]) Suppose $(U, \mathcal{G}, \diamond, \star)$ be a NNS and $Y \subset U$. Then Y is called neutrosophic open subset U if for every $\zeta \in Y$ there exists some $d > 0$ and $\gamma \in (0,1)$ such that $C(\zeta, \gamma, d) \subseteq Y$, where

$$C(\zeta, \gamma, d) := \Big\{ y : \mathsf{T}(\zeta - y, d) > 1 - \gamma \ \text{ and } \ \mathsf{M}(\zeta - y, d) < \gamma, \ \mathsf{F}(\zeta - y, d) < \gamma \Big\}.$$

Definition 3.3. Suppose $(U, \mathcal{G}_1, \diamond, \star)$ and $(V, \mathcal{G}_2, \diamond, \star)$ be two NNS. A mapping $h : U \to V$ ba a linear operator.

(i) The operator h is called weakly neutrosophic continuous at $\zeta_0 \in U$ if for given $\lambda > 0$ and $\gamma \in (0,1)$ there exists some $\vartheta = \vartheta(\lambda, \gamma) > 0$ such that for all $\zeta \in U$
if $\mathsf{T}_1\big(\zeta - \zeta_0, \vartheta\big) \geq \gamma$, and $\mathsf{M}_1\big(\zeta - \zeta_0, \vartheta\big) \leq 1 - \gamma$, $\mathsf{F}_1\big(\zeta - \zeta_0, \vartheta\big) \leq 1 - \gamma$, then
$\mathsf{T}_2\big(h(\zeta) - h(\zeta_0), \lambda\big) \geq \gamma$ and $\mathsf{M}_2\big(h(\zeta) - h(\zeta_0), \lambda\big) \leq 1 - \gamma$, $\mathsf{F}_2\big(h(\zeta) - h(\zeta_0), \lambda\big) \leq 1 - \gamma$,
if h is weakly neutrosophic continuous at each point of U, then h is said to be weakly neutrosophic continuous on X.

(ii) The operator h is called strongly neutrosophic continuous at $\zeta_0 \in U$ if for given $\lambda > 0$ and $\gamma \in (0,1)$ there exists some $\vartheta = \vartheta(\lambda, \gamma) > 0$ such that for all $\zeta \in U$
$\mathsf{T}_2\big(h(\zeta) - h(\zeta_0), \lambda\big) \geq \mathsf{T}_1\big(\zeta - \zeta_0, \vartheta\big)$ and $\mathsf{M}_2\big(h(\zeta) - h(\zeta_0), \lambda\big) \leq \mathsf{M}_1\big(\zeta - \zeta_0, \vartheta\big)$,
$\mathsf{F}_2\big(h(\zeta) - h(\zeta_0), \lambda\big) \leq \mathsf{F}_1\big(\zeta - \zeta_0, \lambda\big)$, for all $\zeta \in U$.

(iii) The operator h is called weakly neutrosophic bounded (for short, WNB) on U if for given $\gamma \in (0,1)$ $\exists$ some, $p_\gamma > 0$ such that
$\mathsf{T}_1(\zeta, \frac{d}{p_\gamma}) \geq \gamma \Rightarrow \mathsf{T}_2\big(h(\zeta), d\big) \geq \gamma$ and $\mathsf{M}_1(\zeta, \frac{d}{p_\gamma}) \leq 1 - \gamma \Rightarrow \mathsf{M}_2\big(h(\zeta), d\big) \leq 1 - \gamma$,
$\mathsf{F}_1(\zeta, \frac{d}{p_\gamma}) \leq 1 - \gamma \Rightarrow \mathsf{F}_2\big(h(\zeta), d\big) \leq 1 - \gamma$,
for all $\zeta \in U$ and $d > 0$. Let $E'(U, V)$ indicate the set of all WNB linear operators.

(iv) The operator h is called strongly neutrosophic bounded (for short, SN-B) on U if for given $\gamma \in (0,1)$, $\exists$ some, $K > 0$ such that

$$\mathsf{T}_2\big(h(\zeta), d\big) \geq \mathsf{T}_1\big(\zeta, \frac{d}{K}\big) \ and \ \mathsf{M}_2\big(h(\zeta), d\big) \leq \mathsf{M}_1\big(\zeta, \frac{d}{K}\big), \mathsf{F}_2\big(h(\zeta), d\big) \leq \mathsf{F}_1\big(\zeta, \frac{d}{K}\big),$$

for all $\zeta \in U$ and $d > 0$. Let $E(U, V)$ denote the set of all SN-B linear operators.

(i) A linear operator $h : (U, \mathcal{G}_1, \diamond, \star)$ onto $(V, \mathcal{G}_2, \diamond, \star)$ is called strongly neutrosophic bounded (briefly, SN-bounded) if there exists a constant $k > 0$ such that $\mathsf{T}_2(h(\zeta), d) \geq \mathsf{T}_1(k\zeta, d)$ and $\mathsf{M}_2(h(\zeta), d) \leq \mathsf{M}_1(k\zeta, d)$, $\mathsf{F}_2(h(\zeta), d) \leq \mathsf{F}_1(k\zeta, d)$ for all $\zeta \in U$ and $d > 0$.

(ii) A linear operator $h : (U, \mathcal{G}_1, \diamond, \star)$ onto $(V, \mathcal{G}_2, \diamond, \star)$ is called weakly neutrosophic bounded (briefly, WN-bounded) if for any $\gamma \in (0,1)$, there exists a constant $k_\gamma > 0$ such that, for every $\zeta \in U$ and $d > 0$,
$\mathsf{T}_1(k_\gamma \zeta, d) \geq \gamma$ and $\mathsf{M}_1(k_\gamma \zeta, d) \leq 1 - \gamma$, $\mathsf{F}_1(k_\gamma \zeta, d) \leq 1 - \gamma \Rightarrow \mathsf{T}_2(h(\zeta), d) \geq \gamma$ and $\mathsf{M}_2(h(\zeta), d) \leq 1 - \gamma$, $\mathsf{F}_2(h(\zeta), d) \leq 1 - \gamma$.

It is clear that every *SN*-bounded operator is *WN*-bounded, but not conversely. In fact, the definition of *SN*-bounded operator is first proposed in [10, Definition 6.1.] as neutrosophic bounded linear operator without any particular limits on $\diamond$ and $\star$. They also present that every neutrosophic bounded linear operator is continuous. The converse of this result is evident. This gives the neutrosophic analogue of the general conjecture: a linear operator is bounded if and only if is continuous in simple normed spaces. We are going to propose *SN*(*WN*)-continuity of functions in the next section.

4. Limits and compactness of mappings in neutrosophic-settings

Definition 4.1. Consider $(U, \mathcal{G}_1, \diamond, \star)$ onto $(V, \mathcal{G}_2, \diamond, \star)$ be two NNS and $h : U \to V$ be a linear mapping. Then,

(i) $\mathcal{L}$ is called strong neutrosophic limit of linear operator h at some $\zeta_0 \in U$ iff for each $\lambda > 0, \exists$ some $\vartheta = \vartheta(\lambda) > 0$ such that

$$\mathsf{T}_2\Big(h(\zeta) - \mathcal{L}, \lambda\Big) \geq \mathsf{T}_1\Big(\zeta - \zeta_0, \vartheta\Big) \text{ and } \mathsf{M}_2\Big(h(\zeta) - \mathcal{L}, \lambda\Big) \leq \mathsf{M}_1\Big(\zeta - \zeta_0, \vartheta\Big),$$
$$\mathsf{F}_2\Big(h(\zeta) - \mathcal{L}, \vartheta\Big) \leq \mathsf{F}_1\Big(\zeta - \zeta_0, \vartheta\Big).$$

In this case, we write **(Strong Neutrosophic-SN)**-$\lim_{\zeta \to \zeta_0} h(\zeta) = \mathcal{L}$, which also means that

$$\lim_{\mathsf{T}_1(\zeta-\zeta_0,\vartheta)\to 1} \mathsf{T}_2\Big(h(\zeta) - \mathcal{L}, \lambda\Big) = \mathcal{L}, (SN) \textit{ and } \lim_{\mathsf{M}_1(\zeta-\zeta_0,\vartheta)\to 0} \mathsf{M}_2\Big(h(\zeta) - \mathcal{L}, \lambda\Big) = \mathcal{L}(SN),$$
$$\lim_{\mathsf{F}_1(\zeta-\zeta_0,\vartheta)\to 0} \mathsf{F}_2\Big(h(\zeta) - \mathcal{L}, \lambda\Big) = \mathcal{L}(SN) \quad \textit{or}$$

$$\Big\{\mathsf{T}_2\Big(h(\zeta) - \mathcal{L}, \lambda\Big) = \mathcal{L}, (SN) \textit{ as } \mathsf{T}_1(\zeta - \zeta_0, \vartheta) \to 1, \textit{ and } \mathsf{M}_2\Big(h(\zeta) - \mathcal{L}, \lambda\Big) = \mathcal{L}, (SN) \textit{ as}$$
$$\mathsf{M}_1(\zeta - \zeta_0, \vartheta) \to 0, \ \ \mathsf{F}_2\Big(h(\zeta) - \mathcal{L}, \lambda\Big) = \mathcal{L}, (SN) \textit{ as } \mathsf{F}_1(\zeta - \zeta_0, \vartheta) \to 0, \ \forall d > 0\Big\}.$$

(ii) $\mathcal{L}$ is called weak neutrosophic limit of h at some $\zeta_0 \in U$ iff for given $\lambda > 0$ and $\gamma \in (0, 1)$, $\exists$ some $\vartheta = \vartheta(\lambda, \gamma) > 0$ s.t

$\mathsf{T}_1\Big(\zeta - \zeta_0, \vartheta\Big) \geq \gamma \Rightarrow \mathsf{T}_2\Big(h(\zeta) - \mathcal{L}, \lambda\Big) \geq \gamma$ and $\mathsf{M}_1\Big(\zeta - \zeta_0, \vartheta\Big) \leq 1 - \gamma \Rightarrow \mathsf{M}_2\Big(h(\zeta) - \mathcal{L}, \lambda\Big) \leq 1 - \gamma$, $\mathsf{F}_1\Big(\zeta - \zeta_0, \vartheta\Big) \leq 1 - \gamma \Rightarrow \mathsf{F}_2\Big(h(\zeta) - \mathcal{L}, \lambda\Big) \leq 1 - \gamma$.

In this case, we write **(Weak Neutrosophic-WN)**-$\lim_{\zeta \to \zeta_0} h(\zeta) = \mathcal{L}$, which also means that

$$\lim_{\mathsf{T}_1(\zeta-\zeta_0,\vartheta)\to 1} \mathsf{T}_2\Big(h(\zeta) - \mathcal{L}, \lambda\Big) = \mathcal{L}, (WN) \textit{ and } \lim_{\mathsf{M}_1(\zeta-\zeta_0,\vartheta)\to 0} \mathsf{M}_2\Big(h(\zeta) - \zeta, \lambda\Big) = \mathcal{L}(SN),$$
$$\lim_{\mathsf{F}_1(\zeta-\zeta_0,\vartheta)\to 0} \mathsf{F}_2\Big(h(\zeta) - \mathcal{L}, \lambda\Big) = \mathcal{L}(SN), \quad \text{or}$$

$$\Big\{\mathsf{T}_2\Big(h(\zeta) - \mathcal{L}, \lambda\Big) = \mathcal{L}, (WN) \textit{ as } \mathsf{T}_1(\zeta - \zeta_0, \vartheta) \to 1, \textit{ and } \mathsf{M}_2\Big(h(\zeta) - \mathcal{L}, \lambda\Big) = \mathcal{L},$$
$$(WN) \textit{ as } \mathsf{M}_1(\zeta - \zeta_0, \vartheta) \to 0, \mathsf{F}_2\Big(h(\zeta) - \mathcal{L}, \lambda\Big) = \mathcal{L}, (WN) \textit{ as } \mathsf{F}_1(\zeta - \zeta_0, \vartheta) \to 0\Big\}$$

for all $d > 0$.

Proposition 4.2. **SN** − lim *implies* **WN** − lim *but not conversely, in general. Further,* **SN** − lim = **WN** − lim *whenever* **SN** − lim *exists.*

Proof. The implications easily can be seen from the definition. Now let, **(Strong Neutrosophic-SN)**− $\lim_{\zeta \to \zeta_0} h(\zeta) = \mathcal{L}$, and **(Weak Neutrosophic-WN)**− $\lim_{\zeta \to \zeta_0} h(\zeta) = \mathcal{L}_1$. Then

$$\mathsf{T}_2(\mathcal{L}_1 - \mathcal{L}, 2d) = \mathsf{T}_2(\mathcal{L}_1 - h(\zeta) + h(\zeta) - \mathcal{L}, 2d) \geq \min\{\mathsf{T}_2(h(\zeta) - \mathcal{L}_1, d), \mathsf{T}_2(h(\zeta) - \mathcal{L}, d)\}$$

for each $d > 0$. Hence we get $\mathsf{T}_2(\mathcal{L}_1 - \mathcal{L}, 2d) = 1$, for all $d > 0$, since $\mathsf{T}_2(h(\zeta) - \mathcal{L}_1, d) \to 1$ and $\mathsf{T}_2(h(\zeta) - \mathcal{L}, d) \to 1$ as $\mathsf{T}_1(\zeta - \zeta_0, d) \to 1$. So $\mathcal{L} = \mathcal{L}_1$ from the definition (It doesn't need to use inference on M_1, M_2 and F_1 , F_2 here.). □

The next example shows why the converse implication may not be true.

Example 4.3. Consider U be a neutrosophic norm space and let

$$\mathsf{T}_1(\zeta, d) = \begin{cases} \frac{d}{d+\|\zeta\|} & \text{if } d \ > \ \| \zeta \|, \\ 0 & \text{otherwise;} \end{cases} \quad , \mathsf{T}_2(\zeta, d) = \begin{cases} 1, & \text{if } d > \| \zeta \|, \\ 0, & \text{if } d \leq \| \zeta \|; \end{cases}$$

and

$$\mathsf{M}_1(\zeta, d) = \begin{cases} \frac{\|\zeta\|}{d+\|\zeta\|} & \text{if } d \ > \ \| \zeta \|, \\ 1 & \text{otherwise;} \end{cases} \quad , \quad \mathsf{M}_2(\zeta, d) = \begin{cases} 1, & \text{if } d \leq \| \zeta \|, \\ 0, & \text{if } d > \| \eta \|; \end{cases}$$

$$\mathsf{F}_1(\zeta, d) = \begin{cases} \frac{\|\zeta\|}{d} & \text{if } d \ > \ \| \eta \|, \\ 1 & \text{otherwise;} \end{cases} \quad , \quad \mathsf{F}_2(\zeta, d) = \begin{cases} 1, & \text{if } d \leq \| \zeta \|, \\ 0, & \text{if } d > \| \zeta \|. \end{cases}$$

It is simple to perform that $\mathcal{G}_1 = (\mathsf{T}_1, \mathsf{M}_1, \mathsf{F}_1)$ and $\mathcal{G}_2 = (\mathsf{T}_2, \mathsf{M}_2, \mathsf{F}_2)$ are neutrosophic norms on U. Consider the identity function $h(\zeta) = \zeta$ from $(U, \mathcal{G}_1, \diamond)$ onto $(U, \mathcal{G}_2, \star)$. Then, $\mathbf{WN} - \lim_{\zeta \to \mathbf{0}} \mathbf{h}(\zeta) = \mathbf{0}$.
Let us show this. Take some $\lambda > 0$ and $\gamma \in (0, 1)$. Then

$$\mathsf{T}_2(h(\zeta) - \mathcal{L}, \lambda) = \mathsf{T}_2(\zeta, \lambda) \geq 1 - \gamma \text{ and } \mathsf{M}_2(h(\zeta) - \mathcal{L}, \lambda) = \mathsf{M}_2(\zeta, \lambda) \leq \gamma,$$
$$\mathsf{F}_2(h(\zeta) - \mathcal{L}, \lambda) = \mathsf{F}_2(\zeta, \lambda) \leq \gamma$$

$\Rightarrow \lambda > \|\zeta\|$. So taking $\vartheta = \frac{(1-\gamma)\lambda}{\gamma} > 0$ we get

$$\begin{aligned} \mathsf{T}_1(\zeta - \zeta_0, \vartheta) = \mathsf{T}_1(\zeta, \vartheta) &= \frac{(1-\gamma)\lambda}{\gamma} . \frac{\gamma}{(1-\gamma)\lambda + \gamma\|\zeta\|} \\ &\geq \frac{\lambda(1-\gamma)}{(1-\gamma)\lambda + \gamma\lambda} = 1 - \gamma, \end{aligned}$$

$\mathsf{T}_1(\zeta - \zeta_0, \vartheta) \geq 1 - \gamma \Rightarrow \mathsf{T}_2(h(\zeta) - \mathcal{L}, \lambda) \geq 1 - \gamma$ and, similar way
$\mathsf{M}_1(\zeta - \zeta_0, \vartheta) \leq \gamma \Rightarrow \mathsf{M}_2(h(\zeta) - \mathcal{L}, \lambda) \leq \gamma$,
$\mathsf{F}_1(\zeta - \zeta_0, \vartheta) \leq \gamma \Rightarrow \mathsf{F}_2(h(\zeta) - \mathcal{L}, \lambda) \leq \gamma$.
However, $SN - \lim_{\zeta \to 0} h(\zeta) = 0$ doesn't exist. Because, for $\|\zeta\| = \lambda$, there is no $\vartheta > 0$ satisfying the condition

$$\mathsf{T}_1(\zeta, \lambda) = 0 \geq \mathsf{T}_1(\zeta, \vartheta) = \frac{\vartheta}{\|\zeta\| + \vartheta} = \frac{\vartheta}{\lambda + \vartheta}.$$

Definition 4.4. Let $(U, \mathcal{G}_1, \diamond, \star) \to (V, \mathcal{G}_2, \diamond, \star)$ be two NNS and $h : U \to V$ be a mapping. Then,
(i) h is called strong neutrosophic continuous (SN-continuous, for short) at some $\zeta_0 \in U$ if and only if

$$SN - \lim_{\zeta \to 0} h(\zeta) = h(\zeta_0).$$

(ii) h is called weak neutrosophic continuous (WN-continuous, for short) at some $\zeta_0 \in U$ if and only if

$$WN - \lim_{\zeta \to 0} h(\zeta) = h(\zeta_0).$$

Remark 4.5. The main difference between $SN - lim$ and $WN - lim$ is in determining the ϑ for given λ and γ. In the strong neutrosophic limit, ϑ is determined entirely by λ and not by γ, but in the weak neutrosophic limit, it is determined by both λ and γ. This is shown better when the neutrosophic normed spaces satisfy the condition (β).

Definition 4.6. [22] Suppose $(\mathcal{U}, \mathcal{G}, \diamond, \star)$ be a NNS. A sequence $\zeta = (\zeta_i)$ is called a Cauchy sequence w.r.t $\mathcal{G}$, if for each $\lambda > 0$ and $d > 0$, $\exists\, d \in \mathbb{N}$ such that $\mathsf{T}(\zeta_i - \zeta_k, d) > 1 - \lambda$, $\mathsf{M}(\zeta_i - \zeta_k, d) < \lambda$ and $\mathsf{F}(\zeta_i - \zeta_k, d) < \lambda$ for all $i, k \geq d$.

Definition 4.7. Suppose $(\mathcal{U}, \mathcal{G}_1, \diamond, \star)$ and $(V, \mathcal{G}_2, \diamond, \star)$ be two NNS. A mapping h from $(\mathcal{U}, \mathcal{G}_1, \diamond, \star) \to (V, \mathcal{G}_2, \diamond, \star)$ is called neutrosophic continuous at $\zeta_0 \in U$ if for any given $\lambda > 0$, $\exists\, d = d(c, \lambda)$, $i = i(c, \lambda) \in (0, 1)$ such that $\forall\, \zeta \in U$ and for all $c \in (0, 1)$,

$$\mathsf{T}_1(\zeta - \zeta_0, d) > 1 - \gamma \Longrightarrow \mathsf{T}_2(h(\zeta) - h(\zeta_0), \lambda) > 1 - c,$$

$$\mathsf{M}_1(\zeta - \zeta_0, d) < \gamma \Longrightarrow \mathsf{M}_2(h(\zeta) - h(\zeta_0), \lambda) < c,$$

$$\mathsf{F}_1(\zeta - \zeta_0, d) < \gamma \Longrightarrow \mathsf{F}_2(h(\zeta) - h(\zeta_0), \lambda) < c.$$

Proposition 4.8. *Let $(\mathcal{U}, \mathcal{G}_1, \diamond, \star) \to (V, \mathcal{G}_2, \diamond, \star)$ be two NNS satisfying the condition (β) and $h : U \to V$ be a mapping. Then,*

(i) $WN - \lim_{\zeta \to 0} h(\zeta) = \mathcal{L} \iff$

for each $\gamma \in (0, 1)$, $\lim_{\|\zeta - \zeta_0\|^1_\gamma \to 0} \|h(\zeta) - \mathcal{L}\|^2_\gamma = 0.$

(ii) $SN - \lim_{\zeta \to 0} h(\zeta) = \mathcal{L} \iff$

$\lim_{\|\zeta - \zeta_0\|^1_\gamma \to 0} \|h(\zeta) - \mathcal{L}\|^2_\gamma = 0$, *uniformly in* γ

where $\|.\|^1_\gamma$ *and* $\|.\|^2_\gamma$ *are the* $\gamma-$ *norms of the neutrosophic norms* $\mathcal{G}_1 = (\mathsf{T}_1, \mathsf{M}_1, \mathsf{F}_1)$ *and* $\mathcal{G}_2 = (\mathsf{T}_2, \mathsf{M}_2, \mathsf{F}_2)$, *respectively.*

Theorem 4.9. *If a sequence (ζ_p) is SN-convergent then it is WN-convergent to the same limit, but not conversely. It is simple that SN-convergence implies WN-convergence. But not converse,*

Proof. Let us just prove the second part because the first one is easy. Suppose $SN - \lim_{\zeta \to 0} h(\zeta) = \mathcal{L}$, i.e, given $\lambda > 0$, there exists some $\vartheta = \vartheta(\lambda) > 0$ such that for every $\zeta \in \mathcal{U}$,

$\mathsf{T}_2(h(\zeta) - \mathcal{L}, \lambda) \geq \mathsf{T}_1(\zeta - \zeta_0, \vartheta)$ *and* $\mathsf{M}_2(h(\zeta) - \mathcal{L}, \lambda) \leq \mathsf{M}_1(\zeta - \zeta_0, \vartheta)$,
$\mathsf{F}_2(h(\zeta) - \mathcal{L}, \lambda) \leq \mathsf{F}_1(\zeta - \zeta_0, \vartheta)$.

Now for every $\gamma \in (0, 1)$, if

$$\|\zeta - \zeta_0\|^1_\gamma = \inf\left\{\vartheta > 0 : \mathsf{T}_1(\zeta - \zeta_0, \vartheta) > \gamma \text{ and } \mathsf{M}_1(\zeta - \zeta_0, \vartheta) < 1 - \gamma, \mathsf{F}_1(\zeta - \zeta_0, \vartheta) < 1 - \gamma\right\} \leq \vartheta,$$

then $\mathsf{T}_1(\zeta - \zeta_0, \vartheta) \geq \gamma$ and $\mathsf{M}_1(\zeta - \zeta_0, \vartheta) < 1 - \gamma$, $\mathsf{F}_1(\zeta - \zeta_0, \vartheta) < 1 - \gamma$ by the hypothesis. This means $\|h(\zeta) - \mathcal{L}\|^2_\gamma \leq \lambda$. Since ϑ doesn't depend on the γ this shows that

$$\lim_{\|\zeta - \zeta_0\|^1_\gamma \to 0} \|h(\zeta) - \mathcal{L}\|^2_\gamma = 0$$

uniformly in γ.

Conversely, let $\lim_{\|\zeta-\zeta_0\|^1_\gamma\to 0}\|h(\zeta)-\mathcal{L}\|^2_\gamma = 0$ uniformly in γ. For given $\lambda > 0$, there exists some $\vartheta = \vartheta(\lambda) > 0$ such that for every $\zeta \in U$

$$\|\zeta-\zeta_0\|^1_\gamma \le \vartheta \Rightarrow \|h(\zeta)-\mathcal{L}\|^2_\gamma \le \lambda, \text{ for all } \gamma \in (0,1)$$

Consider some $\mathsf{T}_1(\zeta-\zeta_0,\vartheta) > \ell$ and $\mathsf{M}_1(\zeta-\zeta_0,\vartheta) < 1-\ell$, $\mathsf{F}_1(\zeta-\zeta_0,\vartheta) < 1-\ell$. Observe that

$$\mathsf{T}_1(\zeta-\zeta_0,\vartheta) = \sup\left\{\gamma : \gamma \in (0,1) : \|\zeta-\zeta_0\|^1_\gamma \le \vartheta\right\} \textit{ and}$$

$$\mathsf{M}_1(\zeta-\zeta_0,\vartheta) = \inf\left\{1-\gamma : \gamma \in (0,1) : \|\zeta-\zeta_0\|^1_\gamma \le \vartheta\right\}$$

$$\mathsf{F}_1(\zeta-\zeta_0,\vartheta) = \inf\left\{1-\gamma : \gamma \in (0,1) : \|\zeta-\zeta_0\|^1_\gamma \le \vartheta\right\}.$$

Hence, there exists some $\gamma_0 \in (0,1)$ such that $\ell < \gamma_0$ and $\|\zeta-\zeta_0\|^1_\gamma \le \vartheta$. Hence $\|h(\zeta)-\mathcal{L}\|^2_{\gamma_0} \le \lambda$, by the hypothesis, and so,

$$\mathsf{T}_2(h(\zeta)-\mathcal{L},\lambda) \ge \gamma_0 > \ell \text{ and } \mathsf{M}_2(h(\zeta)-\mathcal{L},\lambda) \le 1-\gamma_0 < 1-\ell, \mathsf{F}_2(h(\zeta)-\mathcal{L},\lambda) \le 1-\gamma_0 < 1-\ell.$$

Consequently, this follows that

$$\mathsf{T}_2(h(\zeta)-\mathcal{L},\lambda) \ge \mathsf{T}_1(\zeta-\zeta_0,\vartheta) \textit{ and } \mathsf{M}_2(h(\zeta)-\mathcal{L},\lambda) \le \mathsf{M}_1(\zeta-\zeta_0,\vartheta),$$
$$\mathsf{F}_2(h(\zeta)-\mathcal{L},\lambda) \le \mathsf{F}_1(\zeta-\zeta_0,\vartheta).$$

by the definitions of supremum and infimum. □

Definition 4.10. Consider $\{\zeta_p\}$ be a sequence in an NNS $(U, \mathcal{G}, \diamond, \star)$. Then

(i) It is called weakly convergent, briefly, *WN*-convergent, to $\zeta \in U$ and is denoted by $\zeta_p \xrightarrow{WN} \zeta$ iff, for each $\gamma \in (0,1)$ and $\lambda > 0$ and there exists some $p_0 = p_0(\gamma,\lambda)$ such that $p \ge p_0 \Rightarrow \mathsf{T}(\zeta_p-\zeta,\lambda) \ge 1-\gamma$ and $\mathsf{M}(\zeta_p-\zeta,\lambda) \le \gamma$, $\mathsf{F}(\zeta_p-\zeta,\lambda) \le \gamma$.

(ii) It is called strong convergent, briefly, *SN*-convergent, to $\zeta \in U$ and is denoted by $\zeta_p \xrightarrow{SN} \zeta$ iff, for each $\gamma \in (0,1)$, there exists some $p_0 = p_0(\gamma)$ such that $p \ge p_0 \Rightarrow \mathsf{T}(\zeta_p-\zeta,d) \ge 1-\gamma$ and $\mathsf{M}(\zeta_p-\zeta,d) \le \gamma$, $\mathsf{F}(\zeta_p-\zeta,d) \le \gamma$, for all $d > 0$.

Hence we can derive the definitions of the *SN*(*WN*)-Cauchy sequence and *SN*(*WN*)-complete *NNS* from the above definition as is in the classical cases. Proving that each *SN*(*WN*)-convergent sequence is *SN*(*WN*)-Cauchy is an ordinary task.

Proposition 4.11. *Consider* $\{\zeta_p\}$ *be a sequence in an NNS* $(U, \mathcal{G}, \diamond, \star)$ *and, satisfying the condition* (β) *Then*

(i) $\zeta_p \xrightarrow{WN} \zeta \iff \lim_{p\to\infty}\|\zeta_p-\zeta\|_\gamma = 0$, *for each* $\gamma \in (0,1)$.

(ii) $\zeta_p \xrightarrow{SN} \zeta \iff \lim_{p\to\infty}\|\zeta_p-\zeta\|_\gamma = 0$, *uniformly in* $\gamma \in (0,1)$,

where $\|.\|_\gamma$ *are the* γ*-norms of the NNS* $(\mathsf{T},\mathsf{M},\mathsf{F})$.

Every *SN*-convergent sequence is also a *WN*-convergent sequence. However, as the following example shows, the inverse of this assertion may not be true.

Example 4.12. Consider U be a NNS and define

$$\mathsf{T}(\zeta,d)=\begin{cases}\frac{d-\|\zeta\|}{d+\|\zeta\|} & \text{if } d>\|\zeta\|\\ 0 & \text{if } d\leq\|\zeta\|;\end{cases}$$

$$\mathsf{M}(\zeta,d)=\begin{cases}\frac{2\|\zeta\|}{d+\|\zeta\|} & \text{if } d\ <\ \|\ \zeta\ \|\\ 1 & \text{if } d\leq\|\ \zeta\ \|;\end{cases}$$

$$\mathsf{F}(\zeta,d)=\begin{cases}\frac{2\|\zeta\|}{d+\|\zeta\|} & \text{if } d\ <\ \|\ \zeta\ \|\\ 1 & \text{if } d\leq\|\ \zeta\ \|.\end{cases}$$

on U. We can find γ-norms of neutrosophic norm $(\mathsf{T}_1,\mathsf{M}_1,\mathsf{F}_1)$ since satisfies the followings condition (β). Thus

$$\mathsf{T}(\zeta,d)\geq\gamma\iff\frac{d-\|\zeta\|}{d+\|\zeta\|}\geq\gamma\iff\frac{1+\gamma}{1-\gamma}\parallel\zeta\parallel\leq d,$$

$$\mathsf{M}(\zeta,d)\leq 1-\gamma\iff\frac{2\|\zeta\|}{d+\|\zeta\|}\leq 1-\gamma\iff\frac{1+\gamma}{1-\gamma}\parallel\zeta\parallel\leq d,$$

$$\mathsf{F}(\zeta,d)\leq 1-\gamma\iff\frac{2\|\zeta\|}{d+\|\zeta\|}\leq 1-\gamma\iff\frac{1+\gamma}{1-\gamma}\parallel\zeta\parallel\leq d,$$

This shows that

$$\parallel\zeta\parallel_\gamma=\inf\left\{d>0:\mathsf{T}(\zeta,d)\geq\gamma\ \ and\ \ \mathsf{M}(\zeta,d)\leq 1-\gamma,\mathsf{F}(\zeta,d)\leq 1-\gamma\right\}=\frac{1+\gamma}{1-\gamma}\parallel\zeta\parallel.$$

$\mathsf{T}\Big(\zeta,\frac{1+\gamma}{1-\gamma}\parallel\zeta\parallel\Big)=\gamma$ and $\mathsf{M}\Big(\zeta,\frac{1+\gamma}{1-\gamma}\parallel\zeta\parallel\Big)=1-\gamma$, $\mathsf{F}\Big(\zeta,\frac{1+\gamma}{1-\gamma}\parallel\zeta\parallel\Big)=1-\gamma$
whence,

$$\frac{1+\gamma}{1-\gamma}\parallel\zeta\parallel\in\left\{d>0:\mathsf{T}(\zeta,d)\geq\gamma\ \ and\ \ \mathsf{M}(\zeta,d)\leq 1-\gamma,\mathsf{F}(\zeta,d)\leq 1-\gamma\right\}.$$

This means $\parallel\zeta\parallel_\gamma=\frac{1+\gamma}{1-\gamma}\parallel\zeta\parallel$. Let $r\in Y_U=\{\zeta\in U:\|\zeta\|=1\}$ be fixed and define the sequence $\{\zeta_p\}=\left\{\frac{r}{p}\right\}$. We now show that the sequence $\{\zeta_p\}=\left\{\frac{r}{p}\right\}$. WN-convergent to 0. This is easy since, for each $\gamma\in(0,1)$,
$\parallel\zeta-0\parallel_\gamma=\frac{1+\gamma}{1-\gamma}\frac{\|r\|}{p}\to 0$, as $p\to\infty$.
However, this convergence is not uniform in γ. For given $\lambda>0$,

$$\parallel\zeta\parallel_\gamma=\frac{1+\gamma}{1-\gamma}\frac{\parallel r\parallel}{p}<\lambda\iff\frac{1+\gamma}{(1-\gamma)\lambda}<p,$$

whence, we cannot find desired p_0 since $\frac{1+\gamma}{(1-\gamma)\lambda}\to\infty$ as $\gamma\to 1$.

Definition 4.13. A subset C in an NNS $(U,\mathcal{G},\diamond,\star)$ is called $SN(WN)$-compact if each sequence of elements of C has a $SN(WN)$-convergent subsequence.
Obviously, every $SN-$ compact set is $WN-$ compact, but not vice-versa.

Example 4.14. Take $U=\mathbb{C}$ in the Example 4.12. Then the unit sphere
$C_U=\{\zeta\in U:\|\zeta\|=1\}$ is $WN-$ compact in $(U,\mathcal{G},\diamond,\star)$. However, it is not SN-compact. Indeed; the sequence $\left(\frac{1}{p}\right)$ cannot have a SN-convergent subsequence as is shown in the last part of the Example 4.12.

Definition 4.15. The $SN(WN)$-closure of a subset B in an NNS $(U, \mathcal{G}, \diamond, \star)$ is denoted by $C^{-s}(C^{-w})$ and defined by the set of all $\zeta \in U$ such that there exists a sequence $\{\zeta_p\}$ in $X_U = \{\zeta \in U : \|\zeta\| \leq 1\}$ such that $\zeta_p \xrightarrow{SN(WN)} \zeta$. We say that C is $SN(WN)$-closed whenever $C^{-s}(C^{-w}) = C$.

It is easy to see that $C^{-s} \subseteq C^{-w}$. Let us present an example showing that this inclusion may be strict

Example 4.16. Let U be a normed space. Again consider the NNS $(U, \mathcal{G}, \diamond, \star)$ and Example 4.12 and let $X_U = \{\zeta \in U : \|\zeta\| < 1\}$. Then

$$X_U^{-w} = C_U = \{\zeta \in U : \|\zeta\| \leq 1\}$$

Let us show this. For every $\zeta \in C_U$ we must find a sequence $\{\zeta_p\}_{p=1}^{\infty} \subset X_U$ such that $\|\zeta_p - \zeta\|_\gamma \to 0$, as $p \to \infty$ for each $\gamma \in (0,1)$, This is accomplished by taking $\zeta_p = \left(1 - \frac{1}{p}\right)\zeta$ since each $\zeta_p \in X_U$ and

$$\| \zeta_p - \zeta \|_\gamma = \left(\frac{1+\gamma}{1-\gamma}\right) \| \zeta_p - \zeta \|$$

$$\left(\tfrac{1+\gamma}{1-\gamma}\right)\tfrac{1}{p+1} \to 0, \text{ as } p \to \infty, \text{ for each } \gamma \in (0,1).$$

However, $X_U^{-s} = X_U$. Indeed; if $\zeta \in X_U^{-s}$ then there exists $\{\zeta_p\}_{p=1}^{\infty} \subset X_U$ such that $\|\zeta_p - \zeta\|_\gamma \to 0$ uniformly in γ as $p \to \infty$. This means, given $\lambda > 0$, there exists an integer $p_0(\lambda) > 0$ such that for $\lambda \geq \lambda_0$ and for every $\gamma \in (0,1)$,

$$\| \zeta_p - \zeta \|_\gamma \leq \lambda.$$

On the other hand,

$$\| \zeta \| \leq \| \zeta_p - \zeta \| + \| \zeta_p \|$$

$$< \| \zeta_p - \zeta \| + 1$$

$$= \left(\frac{1-\gamma}{1+\gamma}\right) \| \zeta_p - \zeta \|_\gamma + 1$$

$$= \left(\tfrac{1-\gamma}{1+\gamma}\right)\lambda + 1, \text{ for } \lambda \geq \lambda_0, \text{ and for every } \gamma \in (0,1).$$

By letting $\lambda \to 0$ we get $\| \zeta \| < 1$ that is, $\zeta \in X_U$. Note that, there is no danger of $\gamma \to 1$ as $\lambda \to 0$ since changes on λ (*via* p_0) doesn't effect γ. Hence, $X_U^{-s} \subseteq X_U$.

Definition 4.17. Let $(U, \mathcal{G}_1, \diamond, \star) \to (V, \mathcal{G}_2, \diamond, \star)$ be two NNS and $h : U \to V$ be a mapping. Then h is said to be $SN(WN)$-compact if for every neutrosophic bounded $C \subset U$ the subset $h(C)$ is relatively $SN(WN)$-compact that is, the $SN(WN)$-clousre of $h(C)$ is $SN(WN)$-compact.

The following theorem, which has a similar proof to its classical counterpart, is an easy and quick characterization of $SN(WN)$-compact operators.

Theorem 4.18. *Let $(U, \mathcal{G}_1, \diamond, \star) \to (V, \mathcal{G}_2, \diamond, \star)$ be two NNS and $h : U \to V$ be a mapping. Then h is said to be SN(WN)-compact if and only if it maps every neutrosophic bounded sequence $\{\zeta_p\}$ in U onto a sequence $\{h\{\zeta_p\}\}$ in V which has a SN(WN)-convergent subsequence.*

Remark 4.19. Of course, every *SN*-compact operator is also a *WN*-compact operator, but not the other way around. The identity operator on *U* in Example 4.12 is clearly not SN-compact, but it is *WN*-compact.

Theorem 4.20. *Consider $(U, \mathcal{G}_1, \diamond, \star) \to (V, \mathcal{G}_2, \diamond, \star)$ be two NNS satisfying the condition (β) and $h : U \to V$ be a mapping. Suppose that h is WN-compact. Then h is an ordinary compact operator from the normed space $\left(U, \|\,.\,\|_\gamma\right)$ into $\left(V, \|\,.\,\|_\gamma\right)$, for each $\gamma \in (0,1)$*

Proof. Let $\gamma \in (0,1)$ be arbitrary and pick some arbitrary bounded sequence $\{\zeta_p\} \subset \left(U, \|\,.\,\|_\gamma\right)$ and say $\mathcal{K} = \sup_p \| \zeta_p \| < \infty$. So
$\mathsf{T}_1\left(\zeta_p, \mathcal{K}\right) \geq \gamma$ and $\mathsf{M}_1\left(\zeta_p, \mathcal{K}\right) \leq 1-\gamma,\ \ \mathsf{F}_1\left(\zeta_p, \mathcal{K}\right) \leq 1-\gamma$, for each $\ p = 1, 2, 3, \dots$

by the definition of $\left(U, \|\,.\,\|_\gamma\right)$. Thus, $\{\zeta_p\}$ is an neutrosophic bounded sequence in *U*. There exist a *WN*-convergent subsequence $\{h\{\zeta_p\}\}$ of $\{h\{\zeta_{p_k}\}\}$ in *V* by the hypothesis. Hence $\{h\{\zeta_{p_k}\}\}$ is convergent in the normed space $\left(V, \|\,.\,\|_\gamma\right)$ by the Proposition 4.11. □

Theorem 4.21. *Consider $((U, \mathcal{G}_1, \diamond, \star) \to (V, \mathcal{G}_2, \diamond, \star)$ be two NNS satisfying the condition (β) and $h : U \to V$ be a mapping. Then every SN(WN)-compact linear opreator $h : U \to V$ is SN(WN)-continuous.*

Definition 4.22. Let $(U, \mathcal{G}_1, \diamond, \star)$ and $(V, \mathcal{G}_2, \diamond, \star)$ be two *NNS*, $X \subseteq U$ be an neutrosophic open subset and $h : X \to V$ probably non-linear. Then,

(i) *h* is called strong neutrosophic Fréchet differentiable at $\zeta_0 \in X$ if there exists a strong neutrosophic bounded linear operator *L* from $(U, \mathcal{G}_1, \diamond, \star)$ to $(V, \mathcal{G}_2, \diamond, \star)$ such that, given $\lambda > 0$, there exists some $\vartheta = \vartheta(\lambda) > 0$ such that

$$\mathsf{T}_2\left(\frac{h(\zeta_0 + j) - h(\zeta_0) - Lj}{1 - \mathsf{T}_1(j, \vartheta)}, \lambda\right) \geq \mathsf{T}_1(j, \vartheta) = \mathsf{T}_1(\zeta - \zeta_0, \vartheta)$$

and

$$\mathsf{M}_2\left(\frac{h(\zeta_0 + j) - h(\zeta_0) - Lj}{1 - \mathsf{M}_1(j, \vartheta)}, \lambda\right) \leq \mathsf{M}_1(j, \vartheta) = \mathsf{M}_1(\zeta - \zeta_0, \vartheta),$$

$$\mathsf{F}_2\left(\frac{h(\zeta_0 + j) - h(\zeta_0) - Lj}{1 - \mathsf{F}_1(j, \vartheta)}, \lambda\right) \leq \mathsf{F}_1(j, \vartheta) = \mathsf{F}_1(\zeta - \zeta_0, \vartheta).$$

where $j = \zeta - \zeta_0$. In this case, it is written that $L = \mathcal{D}_{SN} h[\zeta_0]$

(ii) *h* is called weak neutrosophic Fréchet differentiable at $\zeta_0 \in X$ if there exists a weak neutrosophic bounded linear operator *L* from $(U, \mathcal{G}_1, \diamond, \star)$ to $(V, \mathcal{G}_2, \diamond, \star)$ such that, given $\lambda > 0$, and $\gamma \in (0,1)$ there exists some $\vartheta = \vartheta(\lambda, \gamma) > 0$ such that

$$\mathsf{T}_1(j, \vartheta) \geq 1 - \gamma \Rightarrow \mathsf{T}_2\left(\frac{h(\zeta_0 + j) - h(\zeta_0) - Lj}{1 - \mathsf{T}_1(j, \vartheta)}, \lambda\right) \geq 1 - \gamma$$

$$\mathsf{M}_1(j,\vartheta) \le \gamma \Rightarrow \mathsf{M}_2\left(\frac{h(\zeta_0 + j) - h(\zeta_0) - Lj}{1 - \mathsf{M}_1(j,\vartheta)}, \lambda\right) \le \gamma$$

$$\mathsf{F}_1(j,\vartheta) \le \gamma \Rightarrow \mathsf{F}_2\left(\frac{h(\zeta_0 + j) - h(\zeta_0) - Lj}{1 - \mathsf{F}_1(j,\vartheta)}, \lambda\right) \le \gamma$$

where $h = \zeta - \zeta_0$. In this case, it is written that $L = \mathcal{D}_{WN}h[\zeta_0]$
h is called SN(WN)-Fréchet differentiable on U if it is SN(WN)-Fréchet differentiable at every point of X.

Proposition 4.23. *A strongly (weakly) neutrosophic bounded linear operator h is SN(WN)-Fréchet differentiable at every point ζ_0 and $\mathcal{D}_{SN(WN)}h[\zeta_0] = L$.*

Proof. This is explicit since

$$\mathsf{T}_2\left(\frac{h(\zeta_0 + j) - h(\zeta_0) - Lj}{1 - \mathsf{T}_1(j,d)}, d\right) = \mathsf{T}_2(0,d) = 1,$$

$$\mathsf{M}_2\left(\frac{h(\zeta_0 + j) - h(\zeta_0) - Lj}{1 - \mathsf{M}_1(j,d)}, d\right) = \mathsf{M}_2(0,d) = 0,$$

$$\mathsf{F}_2\left(\frac{h(\zeta_0 + j) - h(\zeta_0) - Lj}{1 - \mathsf{F}_1(j,d)}, d\right) = \mathsf{F}_2(0,d) = 0, \quad \forall\ d > 0$$

□

Proposition 4.24. *If h is SN(WN)-Fréchet differentiable at $\zeta_0 \in U$ then it is strong (weak) neutrosophic continuous at ζ_0.*

Proof. We take the following inequalities. For given $d > 0$,

$$\begin{aligned}
\mathsf{T}_2(h(\zeta) - h(\zeta_0), d) &= \mathsf{T}_2(h(\zeta) - h(\zeta_0) - Lj + Lj, d + d\mathsf{T}_1(j,d) - d\mathsf{T}_1(j,d)) \\
&\ge \mathsf{T}_2(h(\zeta) - h(\zeta_0) - Lj, d(1 - \mathsf{T}_1(j,d))) \diamond (Lj, d\mathsf{T}_1(j,d)) \\
&= \mathsf{T}_2\left(\frac{h(\zeta) - h(\zeta_0) - Lj}{(1 - \mathsf{T}_1(j,d))}, d\right) \diamond \mathsf{T}_2\left(\frac{Lj}{\mathsf{T}_1(j,d)}, d\right),
\end{aligned}$$

$$\begin{aligned}
\mathsf{M}_2(h(\zeta) - h(\zeta_0), d) &= \mathsf{M}_2(h(\zeta) - h(\zeta_0) - Lj + Lj, d + d\mathsf{M}_1(j,d) - d\mathsf{M}_1(j,d)) \\
&\le \mathsf{M}_2(h(\zeta) - h(\zeta_0) - Lj, d\mathsf{M}_1(j,d)) \star (Lj, d(1 - \mathsf{M}_1(j,d)) \\
&= \mathsf{M}_2\left(\frac{h(\zeta) - h(\zeta_0) - Lj}{(\mathsf{M}_1(j,d))}, d\right) \star \mathsf{M}_2\left(\frac{Lj}{1 - \mathsf{M}_1(j,d)}, d\right),
\end{aligned}$$

$$\begin{aligned}
\mathsf{F}_2(h(\zeta) - h(\zeta_0), d) &= \mathsf{F}_2(h(\zeta) - h(\zeta_0) - Lj + Lj, d + d\mathsf{F}_1(j,d) - d\mathsf{F}_1(j,d)) \\
&\le \mathsf{F}_2(h(\zeta) - h(\zeta_0) - Lj, d\mathsf{F}_1(j,d)) \star (Lj, d(1 - \mathsf{F}_1(j,d))
\end{aligned}$$

$$= \mathsf{F}_2\left(\frac{h(\zeta) - h(\zeta_0) - Lj}{(\mathsf{F}_1(j,d))}, d\right) \star \mathsf{F}_2\left(\frac{Lj}{1 - \mathsf{F}_1(j,d)}, d\right).$$

Since h is SN(WN)-Fréchet differentiable at $\zeta_0 \in H$, it follows that

$$\mathsf{T}_2(h(\zeta) - h(\zeta_0), d) \geq 1 \diamond \mathsf{T}_2\left(\frac{Lj}{\mathsf{T}_1(j,d)}, d\right),$$

and

$$\mathsf{M}_2(h(\zeta) - h(\zeta_0), d) \leq 0 \star \mathsf{M}_2\left(\frac{Lj}{1 - \mathsf{M}_1(j,d)}, d\right),$$

$$\mathsf{F}_2(h(\zeta) - h(\zeta_0), d) \leq 0 \star \mathsf{F}_2\left(\frac{Lj}{1 - \mathsf{F}_1(j,d)}, d\right),$$

where $L = \mathcal{D}_{SN(WN)}h[\zeta_0]$. Therefore h is strong(weak) neutrosophic continuous. □

Theorem 4.25. *Suppose $(U, \mathcal{G}_1, \diamond, \star)$ and $(V, \mathcal{G}_2, \diamond, \star)$ be two NNS, $X \subseteq U$ be an neutrosophic open subset and $h : X \to V$. If h is SN-Fréchet differentiable at some $\zeta_0 \in X$ then it is WN-Fréchet differentiable at some ζ_0 with the same derivative but not conversely. The proof of the above theorm follows directly from the Proposition 4.11.*

Example 4.26. *Consider the linear spaces $U = V = \ell^\infty$, the Banach space of all bounded sequences with the sup norm*

$$\|x\|_\infty = \sup |x_n| \quad \textit{where} \quad x = \{x_n\}_{n=1}^\infty,$$

and define the functions:

$$T(x,t) = \begin{cases} \frac{t^2}{t^2 + 2\|x\|_\infty}, & \textit{if } t > 0 \\ 0, & \textit{if } t \leq 0 \end{cases}$$

$$M(x,t) = \begin{cases} 0, & \textit{if } t > 0 \textit{ and } t^2 > \|x\|_\infty \\ 1, & \textit{if } t \leq 0 \textit{ or } t^2 \leq \|x\|_\infty \end{cases}$$

$$F(x,t) = \begin{cases} 0, & \textit{if } t > 0 \textit{ and } t^2 > \|x\|_\infty \\ 1, & \textit{if } t \leq 0 \textit{ or } t^2 \leq \|x\|_\infty \end{cases}$$

These functions constitute neutrosophic normed spaces. Show that T, M, and F are NF norms on $U = V = \ell_\infty$. Further, consider the shift operator

$$S(x) = S(\{x_1, x_2, \dots\}) = \{0, x_1, x_2, \dots\}$$

on ℓ^∞.

Consider the shift operator $S(x)$ on ℓ_∞:

$$S(x) = \{0, x_1, x_2, x_3, \dots\}.$$

We claim that:

$$S = DwnS[x] \quad \textit{for all } x \in \ell_\infty.$$

The operator $S(x)$ converges weakly to zero, i.e., $S(x) \rightharpoonup 0$, but does not converge strongly since the norm of $S(x)$ is the same as the norm of x. Thus: $S(x)$ converges weakly to 0, that is, $S = DwnS[x]$ for all $x \in \ell_\infty$, but it does not converge strongly to 0, which means $DsnS[x]$ does not exist.

$DsnS[x]$ does not exist.

Theorem 4.27. *Let $(U, \mathcal{G}_1, \diamond, \star)$ and $(V, \mathcal{G}_2, \diamond, \star)$ be two NNS satisfying the condition (β), $X \subseteq U$ be an neutrosophic open subset and $h : X \to V$. Then,*

(i) h is SN-Fréchet differentiable at some $\zeta_0 \in X$ with $L = \mathcal{D}_{SN}h[\zeta_0]$ iff for each $\gamma \in (0, 1)$,

$$\lim_{\|j\|^1_\gamma \to 0} \frac{\|h(\zeta_0 + j) - h(\zeta_0) - Lj\|^2_\gamma}{\|j\|^1_\gamma} = 0.$$

(ii) h is WN-Fréchet differentiable at some $\zeta_0 \in X$ with $L = \mathcal{D}_{WN}h[\zeta_0]$ iff for each $\gamma \in (0, 1)$,

$$\lim_{\|j\|^1_\gamma \to 0} \frac{\|h(\zeta_0 + j) - h(\zeta_0) - Lj\|^2_\gamma}{\|j\|^1_\gamma} = 0.$$

where $\| \,.\, \|^1_\gamma$ and $\| \,.\, \|^2_\gamma$ are the γ-norms of the NNS $(U, \mathcal{G}_1, \diamond, \star)$ and $(V, \mathcal{G}_2, \diamond, \star)$, respectively.

Now let us state a main result including some useful properties of neutrosophic differentiation.

Theorem 4.28. *Let $(U, \mathcal{G}_1, \diamond, \star)$, $(V, \mathcal{G}_1, \diamond, \star)$ and $(W, \mathcal{G}_3, \diamond, \star)$ NNS, (β), $X \subseteq U$ and $Y \subseteq V$ be neutrosophic open subset.*
Let $h, g : X \to V$. be a mappings and $\mathcal{D}_{SN(WN)}h[\zeta_0]$ and $\mathcal{D}_{SN(WN)}g[\zeta_0]$ exists. Then $\mathcal{D}_{SN(WN)}(h + g)[\zeta_0]$ exists and $\mathcal{D}_{SN(WN)}(h + g)[\zeta_0] = \mathcal{D}_{SN(WN)}h[\zeta_0] + \mathcal{D}_{SN(WN)}g[\zeta_0]$.
(ii) Suppose that $h : X \to V$ and $g : Y \to W$. are such that $g \circ h : X \to W$ is defined, $\mathcal{D}_{SN(WN)}h[\zeta_0]$ and $\mathcal{D}_{SN(WN)}g[\zeta_0]$ exists. Then $\mathcal{D}_{SN(WN)}(h + g)[\zeta_0]$ exists and
$\mathcal{D}_{SN(WN)}(g \circ h)[\zeta_0] = \mathcal{D}_{SN(WN)}g[h(\zeta_0)] \circ \mathcal{D}_{SN(WN)}[h(\zeta_0)]$.

Proof. Since proof of the first part is elementary let us prove the second.
We prove the assertion only for SN-Fréchet differential.
Suppose $\mathsf{T}_1(j, d) \neq 1$, $\wedge(\zeta_0, j) = g(h(\zeta_0 + j)) - g(h(\zeta_0))$ and $\mathcal{A} = \mathcal{D}_{SN}g[h(\zeta_0)]$ $\mathcal{B} = \mathcal{D}_{SN}[h(\zeta_0)]$.
Now, let $g(h(\zeta_0)) + g(\zeta_0 + j) - h(\zeta_0) - g(h(\zeta_0) = \wedge(\zeta_0, j)$ and, by the hypothesis, given $\lambda > 0$ there exists some $\vartheta(\lambda) > 0$ such that $\mathsf{T}_1(j, d) \neq 1$,
$\wedge(\zeta_0, j) = g(h(\zeta_0 + j)) - g(h(\zeta_0))$,

$$\mathsf{T}_3\left(\frac{\wedge(\zeta_0, j) - (\mathcal{A}(h(\zeta_0 + j)) - h(\zeta_0))}{1 - \mathsf{T}_2(h(\zeta_0 + j) - h(\zeta_0), \vartheta)}, \lambda\right) \geq \mathsf{T}_2(h(\zeta_0 + j) - h(\zeta_0), \vartheta),$$
$$\mathsf{M}_3\left(\frac{\wedge(\zeta_0, j) - (\mathcal{A}(h(\zeta_0 + j)) - h(\zeta_0))}{1 - \mathsf{M}_2(h(\zeta_0 + j) - h(\zeta_0), \vartheta)}, \lambda\right) \leq \mathsf{M}_2(h(\zeta_0 + j) - h(\zeta_0), \vartheta),$$
$$\mathsf{F}_3\left(\frac{\wedge(\zeta_0, j) - (\mathcal{A}(h(\zeta_0 + j)) - h(\zeta_0))}{1 - \mathsf{F}_2(h(\zeta_0 + j) - h(\zeta_0), \vartheta)}, \lambda\right) \leq \mathsf{F}_2(h(\zeta_0 + j) - h(\zeta_0), \vartheta)$$

where $j = \zeta - \zeta_0$. But, since h is SN-continuous at ζ_0, there exist some $\vartheta_1(\vartheta) > 0$ such that

$$\mathsf{T}_2(h(\zeta_0 + j) - h(\zeta_0), \vartheta) \geq \mathsf{T}_1(j, \vartheta_1) \text{ and } \mathsf{M}_2(h(\zeta_0 + j) - h(\zeta_0), \vartheta) \leq \mathsf{M}_1(j, \vartheta_1)$$

$$\mathsf{F}_2(h(\zeta_0 + j) - h(\zeta_0), \vartheta) \leq \mathsf{F}_1(j, \vartheta_1).$$

Hence

$$\mathsf{T}_3\left(\frac{\wedge(\zeta_0,j)-(\mathcal{A}(h(\zeta_0+j))-h(\zeta_0))}{1-\mathsf{T}_1(j,\vartheta_1)},\lambda\right) \geq \mathsf{T}_3\left(\frac{\wedge(\zeta_0,j)-(\mathcal{A}(h(\zeta_0+j))-h(\zeta_0))}{1-\mathsf{T}_2(h(\zeta_0+j)-h(\zeta_0),\vartheta)},\lambda\right)$$

$$\geq \mathsf{T}_2(h(\zeta_0+j)-h(\zeta_0),\vartheta) \geq \mathsf{T}_1(j,\vartheta_1)$$

and

$$\mathsf{M}_3\left(\frac{\wedge(\zeta_0,j)-(\mathcal{A}(h(\zeta_0+j))-h(\zeta_0))}{1-\mathsf{M}_1(j,\vartheta_1)},\lambda\right) \leq \mathsf{M}_3\left(\frac{\wedge(\zeta_0,j)-(\mathcal{A}(h(\zeta_0+j))-h(\zeta_0))}{1-\mathsf{M}_2(h(\zeta_0+j)-h(\zeta_0),\vartheta)},\lambda\right)$$

$$\leq \mathsf{M}_2(h(\zeta_0+j)-h(\zeta_0),\vartheta) \leq \mathsf{M}_1(j,\vartheta_1)$$

$$\mathsf{F}_3\left(\frac{\wedge(\zeta_0,j)-(\mathcal{A}(h(\zeta_0+j))-h(\zeta_0))}{1-\mathsf{F}_1(j,\vartheta_1)},\lambda\right) \leq \mathsf{F}_3\left(\frac{\wedge(\zeta_0,j)-(\mathcal{A}(h(\zeta_0+j))-h(\zeta_0))}{1-\mathsf{F}_2(h(\zeta_0+j)-h(\zeta_0),\vartheta)},\lambda\right)$$

$$\leq \mathsf{F}_2(h(\zeta_0+j)-h(\zeta_0),\vartheta) \leq \mathsf{F}_1(j,\vartheta_1)$$

Now, consider following equality

$$\frac{\wedge(\zeta_0,j)-\mathcal{A}((Bj)}{1-\mathsf{T}_1(j,d)} = \frac{\wedge(\zeta_0,j)-(\mathcal{A}(h(\zeta_0+j))-h(\zeta_0)-Bj-(h(\zeta_0+j))-h(\zeta_0))}{1-\mathsf{T}_1(j,d)}$$

$$= \frac{\wedge(\zeta_0,j)-(\mathcal{A}(h(\zeta_0+j))-h(\zeta_0))}{1-\mathsf{T}_1(j,d)} - \mathcal{A}\left(\frac{(h(\zeta_0+j))-h(\zeta_0)-Bj}{1-\mathsf{T}_1(j,d)}\right).$$

Since $\mathcal{A}$ is SN-bounded linear operator (at 0) there exists some $\vartheta_2(\lambda) > 0$ such that

$\mathsf{T}_3\left(\mathcal{A}\left(\frac{(h(\zeta_0+j))-h(\zeta_0)-Bj}{1-\mathsf{T}_1(j,d)}\right),\lambda\right) \geq \mathsf{T}_2\left(\frac{(h(\zeta_0+j))-h(\zeta_0)-Bj}{1-\mathsf{T}_1(j,d)},\vartheta_2\right)$ and

$\mathsf{M}_3\left(\mathcal{A}\left(\frac{(h(\zeta_0+j))-h(\zeta_0)-Bj}{1-\mathsf{M}_1(j,d)}\right),\lambda\right) \leq \mathsf{M}_2\left(\frac{(h(\zeta_0+j))-h(\zeta_0)-Bj}{1-\mathsf{M}_1(j,d)},\vartheta_2\right),$

$\mathsf{F}_3\left(\mathcal{A}\left(\frac{(h(\zeta_0+j))-h(\zeta_0)-Bj}{1-\mathsf{F}_1(j,d)}\right),\lambda\right) \leq \mathsf{F}_2\left(\frac{(h(\zeta_0+j))-h(\zeta_0)-Bj}{1-\mathsf{F}_1(j,d)},\vartheta_2\right)$, for every $d > 0$.

Further, since $B = \mathcal{D}_{SN(WN)}h[\zeta_0]$ there exists some $\vartheta_3(\vartheta_2) > 0$ such that

$\mathsf{T}_2\left(\frac{(h(\zeta_0+j))-h(\zeta_0)-Bj}{1-\mathsf{T}_1(j,\vartheta_3)},\vartheta_2\right) \geq \mathsf{T}_1(j,\vartheta_3)$ and $\mathsf{M}_2\left(\frac{(h(\zeta_0+j))-h(\zeta_0)-Bj}{1-\mathsf{M}_1(j,\vartheta_3)},\vartheta_2\right) \leq \mathsf{M}_1(j,\vartheta_3)$

$\mathsf{F}_2\left(\frac{(h(\zeta_0+j))-h(\zeta_0)-Bj}{1-\mathsf{F}_1(j,\vartheta_3)},\vartheta_2\right) \leq \mathsf{F}_1(j,\vartheta_3).$

Let $\vartheta_4 = \max\{\vartheta_1,\vartheta_3\}$ and $\vartheta_4 = \min\{\vartheta_1,\vartheta_3\}$ and say $\mathcal{P} = \frac{\wedge(\zeta_0,j)-(\mathcal{A}(h(\zeta_0+j))-h(\zeta_0))}{1-\mathsf{T}_1(j,\vartheta_1)}$ and $\mathcal{Q} = \mathcal{A}\left(\frac{(h(\zeta_0+j))-h(\zeta_0-Bj))}{1-\mathsf{T}_1(j,\vartheta_3)}\right)$. Then

$$\mathsf{T}_3\left(\frac{\wedge(\zeta_0,j)-\mathcal{A}(Bj))}{1-\mathsf{T}_1(j,\vartheta_4)},2\lambda\right) \geq \mathsf{T}_3\left(\mathcal{P}+\mathcal{Q},2\lambda\right)$$

$$\geq \max\{\mathsf{T}_3(\mathcal{P},\lambda),\mathsf{T}_3(\mathcal{Q},\lambda)\}$$
$$\geq \mathsf{T}_1(j,\vartheta_5)$$

and

$$\mathsf{M}_3\left(\frac{\wedge(\zeta_0,j)-\mathcal{A}(Bj))}{1-\mathsf{M}_1(j,\vartheta_4)},2\lambda\right) \leq \mathsf{M}_3\left(\mathcal{P}+\mathcal{Q},2\lambda\right)$$

$$\leq \min\{\mathsf{M}_3(\mathcal{P},\lambda),\mathsf{M}_3(\mathcal{Q},\lambda)\}$$
$$\leq \mathsf{M}_1(j,\vartheta_5),$$

$$\mathsf{F}_3\left(\frac{\wedge(\zeta_0,j)-\mathcal{A}(Bj))}{1-\mathsf{F}_1(j,\vartheta_4)},2\lambda\right)\leq \mathsf{F}_3\Big(\mathcal{P}+\mathcal{Q},2\lambda\Big)$$

$$\leq \min\{\mathsf{F}_3(\mathcal{P},\lambda),\mathsf{F}_3(\mathcal{Q},\lambda)\}$$
$$\leq \mathsf{F}_1(j,\vartheta_5).$$

This proves that $\mathcal{D}_{SN(WN)}(h+g)[\zeta_0]=\mathcal{D}_{SN(WN)}h[\zeta_0]+\mathcal{D}_{SN(WN)}g[\zeta_0]$. □

5. Conclusion

In this paper, we investigated a few key aspects of Fréchet's differentiation of nonlinear operators and established a link between the various notions of "NNS" and the boundedness of linear operators between neutrosophic normed functions. The current work is an extension and amplification of Yilmaz's work [34] in a "NNS", which is more regular than the "IFNS". We have also initiated the study of neutrosophic compact operators in "NNS" and obtained different important properties of them. Hence, some classical results have been generalized.

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