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Some further construction methods for uninorms on bounded lattices via uninorms

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Abstract. In this paper, we further investigate new construction methods for uninorms on bounded lattices via given uninorms. More specifically, we first construct new uninorms on arbitrary bounded lattices by extending a given uninorm on a subinterval of the lattices under necessary and sufficient conditions on the given uninorm. Moreover, based on the resulting uninorms, we can obtain another sufficient and necessary condition under which S_1^* is a t-conorm (T_1^* is a t-norm) in [32]. Furthermore, using closure operators (interior operators), we also provide new construction methods for uninorms by extending the given uninorm on a subinterval of a bounded lattice under some additional constraints and simultaneously investigate the additional constraints carefully and systematically. Meanwhile, some illustrative examples for the above construction methods of uninorms on bounded lattices are provided.

1. Introduction

Yager and Rybalor [37] introduced the notions of uninorms with a neutral element in the interior of the unit interval $[0, 1]$ which are generalizations of t-norms and t-conorms. These operators also have been proved to play an important role in other fields, such as neural networks, decision-making, expert systems and so on (see, e.g., [16, 20–22, 26, 27, 31, 33, 38, 39]).

Since the bounded lattice L is more general than $[0, 1]$, the studies of uninorms on $[0, 1]$ have been extended to L . Uninorms on L, were first proposed in [29] as a unification of t-norms and t-conorms on L. Since then, a lot of researchers have used many tools to construct uninorms on the bounded lattices, such as t-norms (t-conorms) (see, e.g., [1, 2, 4–6, 8–12, 17, 18, 29, 34]), closure operators (interior operators) (see, e.g., [13, 15, 23, 30, 40]), t-subnorms (t-subconorms) (see, e.g., [25, 28, 36, 41]), additive generators [24] and uninorms (see, e.g., [14, 35]).

As we see, in fact, in [14] and [35], the methods to construct uninorms both start from a given uninorm on a subinterval of a bounded lattice L and then extend it to L. The more important point is that these methods generalize some known construction methods for uninorms in the literature. So, in this paper, we still study the construction methods for uninorms via uninorms defined on the subinterval $[0, a]$ (or $[b, 1]$) of L. In section 3, we can construct new uninorms just by extending a given uninorm U^* on a subinterval of

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L under necessary and sufficient conditions on U^*. The resulting uninorms can provide another sufficient and necessary condition under which S_1^* is a t-conorm (T_1^* is a t-norm) in [32]. In section 4, based on closure operators (interior operators), we can obtain new uninorms by extending a given uninorm U^* on a subinterval of L. Moreover, if we take $e = b$, $e = a$, $e = 1$ and $e = 0$, respectively, then all uninorms, constructed in sections 3 and 4, are the existing results in the literature.

2. Preliminaries

In this section, we recall some conceptions and results, which will be used in this manuscript.

Definition 2.1. ([3]) A lattice $(L, \leq)$ is bounded if it has top element 1 and bottom element 0, that is, there exist $1, 0 \in L$ such that $0 \leq x \leq 1$ for all $x \in L$.

Throughout this article, unless stated otherwise, we denote L as a bounded lattice in the above definition.

Definition 2.2. ([3]) Let $a, b \in L$ with $a \leq b$. A subinterval $[a, b]$ of L is defined as

$$[a, b] = \{x \in L : a \leq x \leq b\}.$$

Similarly, we can define $[a, b) = \{x \in L : a \leq x < b\}, (a, b] = \{x \in L : a < x \leq b\}$ and $(a, b) = \{x \in L : a < x < b\}$. If a and b are incomparable, then we use the notation $a \parallel b$. If a and b are comparable, then we use the notation $a \nparallel b$.

Moreover, I_a denotes the set of all incomparable elements with a, that is, $I_a = \{x \in L \mid x \parallel a\}$. I_a^b denotes the set of elements that are incomparable with a but comparable with b, that is, $I_a^b = \{x \in L \mid x \parallel a \text{ and } x \nparallel b\}$. $I_{a,b}$ denotes the set of elements that are incomparable with both a and b, that is, $I_{a,b} = \{x \in L \mid x \parallel a \text{ and } x \parallel b\}$. Obviously, $I_a^a = \emptyset$ and $I_{a,a} = I_a$.

Definition 2.3. ([32]) If an operation $T : L^2 \to L$ is associative, commutative and increasing with respect to both variables, and has the neutral element $1 \in L$, that is, $T(1, x) = x$ for all $x \in L$, then it is called a t-norm on L.

Definition 2.4. ([5]) If an operation $S : L^2 \to L$ is associative, commutative and increasing with respect to both variables, and has the neutral element $0 \in L$, that is, $S(0, x) = x$ for all $x \in L$, then it is called a t-conorm on L.

Definition 2.5. ([29]) If an operation $U : L^2 \to L$ is associative, commutative and increasing with respect to both variables, and has the neutral element $e \in L$, that is, $U(e, x) = x$ for all $x \in L$, then it is called a uninorm on L.

Proposition 2.6. ([29]) *Let U be a uninorm on L with the neutral element $e \in L \setminus \{0, 1\}$. Then the following results hold obviously.*

(1) $T_e = U \mid [0, e]^2 \to [0, e]$ *is a t-norm on* $[0, e]$.

(2) $S_e = U \mid [e, 1]^2 \to [e, 1]$ *is a t-conorm on* $[e, 1]$.

Definition 2.7. ([5]) Let U be a uninorm on L with the neutral element $e \in L \setminus \{0, 1\}$.

(1) If an element $x \in L$ satisfies $U(x, x) = x$, then it is called an idempotent element of U.

(2) If a uninorm U satisfies $U(x, x) = x$ for all $x \in L$, then it is called an idempotent uninorm on L.

Definition 2.8. ([5]) Let U be a uninorm on L with the neutral element $e \in L \setminus \{0, 1\}$.

(1) If $U(0, 1) = 0$, then U is a conjunctive uninorm.

(2) If $U(0,1) = 1$, then U is a disjunctive uninorm.

Definition 2.9. ([19]) A mapping $cl : L \to L$ is called a closure operator on L if, for all $x, y \in L$, it satisfies the following conditions:

(1) $x \leq cl(x)$;

(2) $cl(x \vee y) = cl(x) \vee cl(y)$;

(3) $cl(cl(x)) = cl(x)$.

Definition 2.10. ([30]) A mapping $int : L \to L$ is called an interior operator on L if, for all $x, y \in L$, it satisfies the following conditions:

(1) $int(x) \leq x$;

(2) $int(x \wedge y) = int(x) \wedge int(y)$;

(3) $int(int(x)) = int(x)$.

Definition 2.11. ([41]) Let L be a bounded lattice and $e \in L \setminus \{0,1\}$.

We denote by $\mathcal{U}^*_{min}$ the class of all uninorms U on L with neutral element e satisfying the following condition:

$U(x,y) = y$ for all $(x,y) \in (e,1] \times [0,e)$.

Similarly, we denote by $\mathcal{U}^*_{max}$ the class of all uninorms U on L with neutral element e satisfying the following condition:

$U(x,y) = y$ for all $(x,y) \in [0,e) \times (e,1]$.

Proposition 2.12. ([28]) *Let S be a nonempty set and $A_1, A_2, \ldots, A_n$ be subsets of S. Let H be a commutative binary operation on S. Then H is associative on $A_1 \cup A_2 \cup \ldots \cup A_n$ if and only if all of the following statements hold:*

(*i*) *for every combination $\{i, j, k\}$ of size 3 chosen from $\{1, 2, \ldots, n\}$, $H(x, H(y,z)) = H(H(x,y),z) = H(y, H(x,z))$ for all $x \in A_i, y \in A_j, z \in A_k$;*

(*ii*) *for every combination $\{i, j\}$ of size 2 chosen from $\{1, 2, \ldots, n\}$, $H(x, H(y,z)) = H(H(x,y),z)$ for all $x \in A_i, y \in A_i, z \in A_j$;*

(*iii*) *for every combination $\{i, j\}$ of size 2 chosen from $\{1, 2, \ldots, n\}$, $H(x, H(y,z)) = H(H(x,y),z)$ for all $x \in A_i, y \in A_j, z \in A_j$;*

(*iv*) *for every $i \in \{1, 2, \ldots, n\}$, $H(x, H(y,z)) = H(H(x,y),z)$ for all $x, y, z \in A_i$.*

In Theorem 4.8 of [32], if we take $b = 1$, then we can obtain the following Proposition 2.13(1). Meanwhile, its dual result is given by Proposition 2.13(2).

Proposition 2.13. ([32]) *Let $b, a \in L \setminus \{0,1\}$.*

(1) *For a t-norm $V : [b,1]^2 \to [b,1]$, an ordinal sum extension T^*_1 of V to L defined by*

$$T^*_1(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [b,1]^2, \\ x \wedge y & \text{otherwise,} \end{cases}$$

is a t-norm if and only if $x \parallel y$ for all $x \in I_b$ and $y \in [b,1)$.

(2) *For a t-conorm $W : [0,a]^2 \to [0,a]$, an ordinal sum extension S^*_1 of W to L defined by*

$$S^*_1(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in [0,a]^2, \\ x \vee y & \text{otherwise,} \end{cases}$$

is a t-conorm if and only if $x \parallel y$ for all $x \in I_a$ and $y \in (0,a]$.

3. New construction methods for uninorms via uninorms on bounded lattices

In this section, we focus on the construction methods for uninorms on a bounded lattice L by extending a given uninorm U^* on a subinterval $[0,a]$ (or $[b,1]$) of L under sufficient and necessary conditions on U^*. Based on the new uninorms, we can obtain another sufficient and necessary condition under which S_1^* is a t-conorm (T_1^* is a t-norm) in [32].

Theorem 3.1. *Let U^* be a uninorm on $[0,a]$ with a neutral element $e \in L$ for $a \in L \setminus \{0,1\}$. Then $U_1 : L^2 \to L$ defined by*

$$U_1(x,y) = \begin{cases} U^*(x,y) & \text{if } (x,y) \in [0,a]^2, \\ x & \text{if } (x,y) \in I_{e,a} \times [0,e], \\ y & \text{if } (x,y) \in [0,e] \times I_{e,a}, \\ x \vee y & \text{otherwise}, \end{cases}$$

is a uninorm on L with the neutral element e if and only if U^* satisfies the following conditions:

(1) *for $z \in I_{e,a}$, if $(x,y) \in ((I_e^a \cup (e,a]) \cap \{l \in L \mid l \nparallel z\})^2$, then $U^*(x,y) \nparallel z$;*

(2) *for $z \in I_{e,a}$, if $(x,y) \in [0,a] \times ((I_e^a \cup (e,a]) \cap \{l \in L \mid l \parallel z\}) \cup ((I_e^a \cup (e,a]) \cap \{l \in L \mid l \parallel z\}) \times [0,a]$, then $U^*(x,y) \parallel z$;*

(3) *for $z \in I_a^e$, if $(x,y) \in ((I_e^a \cup (e,a]) \cap \{l \in L \mid l \nparallel z\})^2$, then $U^*(x,y) \nparallel z$;*

(4) *for $z \in I_a^e$, if $(x,y) \in [0,a] \times ((I_e^a \cup (e,a]) \cap \{l \in L \mid l \parallel z\}) \cup ((I_e^a \cup (e,a]) \cap \{l \in L \mid l \parallel z\}) \times [0,a]$, then $U^*(x,y) \parallel z$.*

Proof. Necessity. Suppose that $U_1(x,y)$ is a uninorm with the neutral element e. Next we need to show that U^* satisfies the conditions (1), (2), (3) and (4).

(1). For $z \in I_{e,a}$, if $(x,y) \in ((I_e^a \cup (e,a]) \cap \{l \in L \mid l \nparallel z\})^2$, then $U^*(x,y) \nparallel z$.

Assume that for $z \in I_{e,a}$, there exists $(x,y) \in ((I_e^a \cup (e,a]) \cap \{l \in L \mid l \nparallel z\})^2$ such that $U^*(x,y) \parallel z$. Then $U_1(x,z) = z$ and $U_1(x,y) = U^*(x,y)$. Since $U^*(x,y) \parallel z$, this contradicts the increasingness property of $U_1(x,y)$. Thus, for $z \in I_{e,a}$, if $(x,y) \in ((I_e^a \cup (e,a]) \cap \{l \in L \mid l \nparallel z\})^2$, then $U^*(x,y) \nparallel z$.

(2). For $z \in I_{e,a}$, if $(x,y) \in [0,a] \times ((I_e^a \cup (e,a]) \cap \{l \in L \mid l \parallel z\}) \cup ((I_e^a \cup (e,a]) \cap \{l \in L \mid l \parallel z\}) \times [0,a]$, then $U^*(x,y) \parallel z$.

Now we give the proof of that for $z \in I_{e,a}$, if $(x,y) \in [0,a] \times ((I_e^a \cup (e,a]) \cap \{l \in L \mid l \parallel z\})$, then $U^*(x,y) \parallel z$, and the other case is obvious by the commutativity of U^*. Assume that for $z \in I_{e,a}$, there exists $(x,y) \in [0,a] \times ((I_e^a \cup (e,a]) \cap \{l \in L \mid l \parallel z\})$ such that $U^*(x,y) \nparallel z$. Then $U_1(x, U_1(y,z)) = U_1(x, z \vee a) = z \vee a$ and $U_1(U_1(x,y), z) = U_1(U^*(x,y), z) = z$. Since $z \vee a \neq z$, this contradicts the associativity of $U_1(x,y)$. Thus, for $z \in I_{e,a}$, if $(x,y) \in [0,a] \times ((I_e^a \cup (e,a]) \cap \{l \in L \mid l \parallel z\}) \cup ((I_e^a \cup (e,a]) \cap \{l \in L \mid l \parallel z\}) \times [0,a]$, then $U^*(x,y) \parallel z$.

(3). For $z \in I_a^e$, if $(x,y) \in ((I_e^a \cup (e,a]) \cap \{l \in L \mid l \nparallel z\})^2$, then $U^*(x,y) \nparallel z$.

Assume that for $z \in I_a^e$, there exists $(x,y) \in ((I_e^a \cup (e,a]) \cap \{l \in L \mid l \nparallel z\})^2$ such that $U^*(x,y) \parallel z$. Then $U_1(x, U_1(y,z)) = U_1(x,z) = z$ and $U_1(U_1(x,y), z) = U_1(U^*(x,y), z) = z \vee a$. Since $z \vee a \neq z$, this contradicts the associativity of U_1. Thus, for $z \in I_a^e$, if $(x,y) \in ((I_e^a \cup (e,a]) \cap \{l \in L \mid l \nparallel z\})^2$, then $U^*(x,y) \nparallel z$.

(4). For $z \in I_a^e$, if $(x,y) \in [0,a] \times ((I_e^a \cup (e,a]) \cap \{l \in L \mid l \parallel a\}) \cup ((I_e^a \cup (e,a]) \cap \{l \in L \mid l \parallel z\}) \times [0,a]$, then $U^*(x,y) \parallel z$.

Now we prove that for $z \in I_a^e$, if $(x,y) \in [0,a] \times ((I_e^a \cup (e,a]) \cap \{l \in L \mid l \parallel z\})$, then $U^*(x,y) \parallel z$, and the other case is obvious by the commutativity of U^*. Assume that for $z \in I_a^e$, there exists $(x,y) \in [0,a] \times ((I_e^a \cup (e,a]) \cap \{l \in L \mid l \parallel z\})$ such that $U^*(x,y) \nparallel z$. Then $U_1(x, U_1(y,z)) = U_1(x, z \vee a) = z \vee a$ and $U_1(U_1(x,y), z) = U_1(U^*(x,y), z) = z$. Since $z \vee a \neq z$, this contradicts the associativity of U_1. Thus, for $z \in I_a^e$, if $(x,y) \in [0,a] \times ((I_e^a \cup (e,a]) \cap \{l \in L \mid l \parallel a\}) \cup ((I_e^a \cup (e,a]) \cap \{l \in L \mid l \parallel z\}) \times [0,a]$, then $U^*(x,y) \parallel z$.

Sufficiency. First, we can see that U_1 is commutative and e is the neutral element of U_1. Hence, we only need to prove the increasingness and the associativity of U_1.

I. Increasingness: Next, we prove that if $x \leq y$, then $U_1(x,z) \leq U_1(y,z)$ for all $z \in L$. It is easy to verify that $U_1(x,z) \leq U_1(y,z)$ if both x and y belong to one of the intervals $[0,e], I_e^a, (e,a], I_a^e, I_{e,a}$ or $(a,1]$ for all $z \in L$. The residual proof can be split into all possible cases.

1. $x \in [0,e]$

1.1. $y \in I_e^a \cup (e, a]$

1.1.1. $z \in [0, e] \cup I_e^a \cup (e, a]$

$U_1(x, z) = U^*(x, z) \le U^*(y, z) = U_1(y, z)$

1.1.2. $z \in I_a^e \cup I_{e,a} \cup (a, 1]$

$U_1(x, z) = z \le y \vee z = U_1(y, z)$

1.2. $y \in I_a^e \cup I_{e,a} \cup (a, 1]$

1.2.1. $z \in [0, e]$

$U_1(x, z) = U^*(x, z) \le x < y = U_1(y, z)$

1.2.2. $z \in I_e^a \cup (e, a]$

$U_1(x, z) = U^*(x, z) \le z < y \vee z = U_1(y, z)$

1.2.3. $z \in I_a^e \cup I_{e,a} \cup (a, 1]$

$U_1(x, z) = z \le y \vee z = U_1(y, z)$

2. $x \in I_e^a$

2.1. $y \in (e, a]$

2.1.1. $z \in [0, e] \cup I_e^a \cup (e, a]$

$U_1(x, z) = U^*(x, z) \le U^*(y, z) = U_1(y, z)$

2.1.2. $z \in I_a^e \cup I_{e,a} \cup (a, 1]$

$U_1(x, z) = x \vee z \le y \vee z = U_1(y, z)$

2.2. $y \in I_a^e$

2.2.1. $z \in [0, e]$

$U_1(x, z) = U^*(x, z) \le x < y = U_1(y, z)$

2.2.2. $z \in I_e^a \cup (e, a]$

If $z \parallel y$, then $U_1(x, z) = U^*(x, z) \le a < y \vee a = y \vee z = U_1(y, z)$.

If $z \nparallel y$, then $U_1(x, z) = U^*(x, z) < y = y \vee z = U_1(y, z)$.

2.2.3. $z \in I_a^e \cup I_{e,a} \cup (a, 1]$

$U_1(x, z) = x \vee z \le y \vee z = U_1(y, z)$

2.3. $y \in I_{e,a}$

2.3.1. $z \in [0, e]$

$U_1(x, z) = U^*(x, z) \le x < y = U_1(y, z)$

2.3.2. $z \in I_e^a$

If $z \parallel y$, then $U_1(x, z) = U^*(x, z) \le a < y \vee a = y \vee z = U_1(y, z)$.

If $z \nparallel y$, then $U_1(x, z) = U^*(x, z) < y = y \vee z = U_1(y, z)$.

2.3.3. $z \in (e, a]$

$U_1(x, z) = U^*(x, z) \le a < y \vee a = y \vee z = U_1(y, z)$

2.3.4. $z \in I_a^e \cup I_{e,a} \cup (a, 1]$

$U_1(x, z) = x \vee z \le y \vee z = U_1(y, z)$

2.4. $y \in (a, 1]$

2.4.1. $z \in [0, e] \cup I_e^a \cup (e, a]$

$U_1(x, z) = U^*(x, z) \le a < y = U_1(y, z)$

2.4.2. $z \in I_a^e \cup I_{e,a} \cup (a, 1]$

$U_1(x, z) = x \vee z \le y \vee z = U_1(y, z)$

3. $x \in (e, a]$

3.1. $y \in I_a^e$

3.1.1. $z \in [0, e]$

$U_1(x, z) = U^*(x, z) \le x < y = U_1(y, z)$

3.1.2. $z \in I_e^a \cup (e, a]$

If $z \parallel y$, then $U_1(x, z) = U^*(x, z) \le a < y \vee a = y \vee z = U_1(y, z)$.

If $z \nparallel y$, then $U_1(x, z) = U^*(x, z) < y = y \vee z = U_1(y, z)$.

3.1.3. $z \in I_a^e \cup I_{e,a} \cup (a, 1]$

$U_1(x, z) = x \vee z \le y \vee z = U_1(y, z)$

3.2. $y \in (a, 1]$

3.2.1. $z \in [0, e] \cup I_e^a \cup (e, a]$

$U_1(x,z) = U^*(x,z) \le a < y = U_1(y,z)$

3.2.2. $z \in I_a^e \cup I_{e,a} \cup (a,1]$

$U_1(x,z) = x \vee z \le y \vee z = U_1(y,z)$

4. $x \in I_a^e, y \in (a,1], z \in L$

$U_1(x,z) = x \vee z \le y \vee z = U_1(y,z)$

5. $x \in I_{e,a}, y \in I_a^e \cup (a,1]$

5.1. $z \in [0,e]$

$U_1(x,z) = x \le y = U_1(y,z)$

5.2. $z \in I_e^a \cup (e,a] \cup I_a^e \cup I_{e,a} \cup (a,1]$

$U_1(x,z) = x \vee z \le y \vee z = U_1(y,z)$

II. Associativity: It can be shown that $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z)$ for all $x,y,z \in L$. By Proposition 2.12, we just verify the following cases.

1. If $x,y,z \in [0,e] \cup I_e^a \cup (e,a]$, then $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y,U_1(x,z))$ for U^* is associative.

2. If $x,y,z \in I_a^e \cup I_{e,a} \cup (a,1]$, then $U_1(x,U_1(y,z)) = U_1(x,y \vee z) = x \vee y \vee z = U_1(x \vee y,z) = U_1(U_1(x,y),z)$ and $U_1(y,U_1(x,z)) = U_1(y,x \vee z) = x \vee y \vee z$. Thus $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y,U_1(x,z))$.

3. If $x,y \in [0,e]$ and $z \in I_a^e \cup I_{e,a} \cup (a,1]$, then $U_1(x,U_1(y,z)) = U_1(x,z) = z = U_1(U^*(x,y),z) = U_1(U_1(x,y),z)$.

4. If $x,y \in I_e^a, z \in I_a^e \cup I_{e,a}, x \nparallel z$ and $y \nparallel z$, then $U_1(x,U_1(y,z)) = U_1(x,z) = z = U_1(U^*(x,y),z) = U_1(U_1(x,y),z)$.

If $x,y \in I_e^a, z \in I_a^e \cup I_{e,a}$ and at least one of x,y is incomparable with z, then $U_1(x,U_1(y,z)) = U_1(x,y \vee z) = x \vee y \vee z = z \vee a = U^*(x,y) \vee z = U_1(U^*(x,y),z) = U_1(U_1(x,y),z)$.

5. If $x,y \in I_e^a$ and $z \in (a,1]$, then $U_1(x,U_1(y,z)) = U_1(x,z) = z = U_1(U^*(x,y),z) = U_1(U_1(x,y),z)$.

6. If $x,y \in (e,a], z \in I_a^e, x \nparallel z$ and $y \nparallel z$, then $U_1(x,U_1(y,z)) = U_1(x,z) = z = U_1(U^*(x,y),z) = U_1(U_1(x,y),z)$.

If $x,y \in (e,a], z \in I_a^e$ and at least one of x and y is incomparable with z, then $U_1(x,U_1(y,z)) = U_1(x,y \vee z) = x \vee y \vee z = z \vee a = U^*(x,y) \vee z = U_1(U^*(x,y),z) = U_1(U_1(x,y),z)$.

7. If $x,y \in (e,a]$ and $z \in I_{e,a}$, then $U_1(x,U_1(y,z)) = U_1(x,y \vee z) = x \vee y \vee z = z \vee a = U^*(x,y) \vee z = U_1(U^*(x,y),z) = U_1(U_1(x,y),z)$.

8. If $x,y \in (e,a]$ and $z \in (a,1]$, then $U_1(x,U_1(y,z)) = U_1(x,z) = z = U_1(U^*(x,y),z) = U_1(U_1(x,y),z)$.

9. If $x \in [0,e]$ and $y,z \in I_a^e \cup I_{e,a} \cup (a,1]$, then $U_1(x,U_1(y,z)) = U_1(x,y \vee z) = y \vee z = U_1(y,z) = U_1(U_1(x,y),z)$ and $U_1(y,U_1(x,z)) = U_1(y,z) = y \vee z$. Thus $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y,U_1(x,z))$.

10. If $x \in I_e^a \cup (e,a]$, $y,z \in I_a^e \cup I_{e,a} \cup (a,1]$, then $U_1(x,U_1(y,z)) = U_1(x,y \vee z) = x \vee y \vee z = U_1(x \vee y,z) = U_1(U_1(x,y),z)$ and $U_1(y,U_1(x,z)) = U_1(y,x \vee z) = x \vee y \vee z$. Thus $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y,U_1(x,z))$.

11. If $x \in [0,e], y \in I_e^a, z \in I_a^e \cup I_{e,a}$ and $y \nparallel z$, then $U_1(x,U_1(y,z)) = U_1(x,z) = z = U_1(U^*(x,y),z) = U_1(U_1(x,y),z)$ and $U_1(y,U_1(x,z)) = U_1(y,z) = z$. Thus $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y,U_1(x,z))$.

If $x \in [0,e], y \in I_e^a, z \in I_a^e \cup I_{e,a}$ and $y \parallel z$, then $U_1(x,U_1(y,z)) = U_1(x,y \vee z) = x \vee y \vee z = z \vee a = U^*(x,y) \vee z = U_1(U^*(x,y),z) = U_1(U_1(x,y),z)$ and $U_1(y,U_1(x,z)) = U_1(y,z) = y \vee z = z \vee a$. Thus $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y,U_1(x,z))$.

12. If $x \in [0,e], y \in I_e^a$ and $z \in (a,1]$, then $U_1(x,U_1(y,z)) = U_1(x,z) = z = U_1(U^*(x,y),z) = U_1(U_1(x,y),z)$ and $U_1(y,U_1(x,z)) = U_1(y,z) = z$. Thus $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y,U_1(x,z))$.

13. If $x \in [0,e], y \in (e,a], z \in I_a^e$ and $y \nparallel z$, then $U_1(x,U_1(y,z)) = U_1(x,z) = z = U_1(U^*(x,y),z) = U_1(U_1(x,y),z)$ and $U_1(y,U_1(x,z)) = U_1(y,z) = z$. Thus $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y,U_1(x,z))$.

If $x \in [0,e], y \in (e,a], z \in I_a^e$ and $y \parallel z$, then $U_1(x,U_1(y,z)) = U_1(x,y \vee z) = x \vee y \vee z = z \vee a = U^*(x,y) \vee z = U_1(U^*(x,y),z) = U_1(U_1(x,y),z)$ and $U_1(y,U_1(x,z)) = U_1(y,z) = y \vee z = z \vee a$. Thus $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y,U_1(x,z))$.

14. If $x \in [0,e], y \in (e,a]$ and $z \in I_{e,a}$, then $U_1(x,U_1(y,z)) = U_1(x,y \vee z) = x \vee y \vee z = z \vee a = U^*(x,y) \vee z = U_1(U^*(x,y),z) = U_1(U_1(x,y),z)$ and $U_1(y,U_1(x,z)) = U_1(y,z) = y \vee z = z \vee a$. Thus $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y,U_1(x,z))$.

15. If $x \in [0,e], y \in (e,a]$ and $z \in (a,1]$, then $U_1(x,U_1(y,z)) = U_1(x,z) = z = U_1(U^*(x,y),z) = U_1(U_1(x,y),z)$ and $U_1(y,U_1(x,z)) = U_1(y,z) = z$. Thus $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y,U_1(x,z))$.

16. If $x \in I_e^a, y \in (e,a], z \in I_a^e, x \nparallel z$ and $y \nparallel z$, then $U_1(x,U_1(y,z)) = U_1(x,z) = z = U_1(U^*(x,y),z) = U_1(U_1(x,y),z)$ and $U_1(y,U_1(x,z)) = U_1(y,z) = z$. Thus $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y,U_1(x,z))$.

If $x \in I_e^a, y \in (e,a], z \in I_a^e$ and at least one of x, y is incomparable with z, then $U_1(x, U_1(y,z)) = U_1(x, y \vee z) = x \vee y \vee z = z \vee a = U^*(x,y) \vee z = U_1(U^*(x,y),z) = U_1(U_1(x,y),z)$ and $U_1(y, U_1(x,z)) = U_1(y, x \vee z) = y \vee x \vee z = z \vee a$. Thus $U_1(x, U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y, U_1(x,z))$.

17. If $x \in I_e^a, y \in (e,a]$ and $z \in I_{e,a}$, then $U_1(x, U_1(y,z)) = U_1(x, y \vee z) = x \vee y \vee z = z \vee a = U^*(x,y) \vee z = U_1(U^*(x,y),z) = U_1(U_1(x,y),z)$ and $U_1(y, U_1(x,z)) = U_1(y, x \vee z) = x \vee y \vee z = z \vee a$. Thus $U_1(x, U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y, U_1(x,z))$.

18. If $x \in I_e^a, y \in (e,a]$ and $z \in (a,1]$, then $U_1(x, U_1(y,z)) = U_1(x,z) = z = U_1(U^*(x,y),z) = U_1(U_1(x,y),z)$ and $U_1(y, U_1(x,z)) = U_1(y,z) = z$. Thus $U_1(x, U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y, U_1\ (x,z))$. □

Remark 3.2. Theorem 3.1 seem to be restrained for there are some conditions on the given uninorm U^*. However, these conditions are necessary for our construction methods. On one hand, these additional conditions are necessary and sufficient; on the other hand, in case of $e = a$ or $e = 0$, these conditions naturally hold and then Theorem 3.1 is the existing result in the literature as follows. These show the rationality of these conditions and our uninorm in some degree.

If we take $e = a$ in Theorem 3.1, then we can obtain the existing result in the literature.

Remark 3.3. In Theorem 3.1, if taking $e = a$, then $[0,a] = [0,e]$, $I_{e,a} = I_e$, $I_e^a \cup I_a^e \cup (e,a] = \emptyset$ and U^* is a t-norm on $[0,a]$. Moreover, the conditions (1), (2), (3) and (4) in Theorem 3.1 naturally hold.

By the above fact, if taking $e = a$ in Theorem 3.1, then we retrieve the uninorm $U_t : L^2 \to L$ constructed by Çaylı, Karaçal, and Mesiar ([5], Theorem 1) as follow:

$$U_t(x,y) = \begin{cases} T_e(x,y) & \text{if } (x,y) \in [0,e]^2, \\ x & \text{if } (x,y) \in I_e \times [0,e], \\ y & \text{if } (x,y) \in [0,e] \times I_e, \\ x \vee y & \textit{otherwise}. \end{cases}$$

Lemma 3.4. *In Theorem 3.1, if $e = 0$, then the condition* (4) *holds.*

Proof. If $e = 0$, then we can rewrite the condition (4) as follow: for $z \in I_a$, if $(x,y) \in [0,a] \times ((0,a] \cap \{l \in L \mid l \parallel z\}) \cup ((0,a] \cap \{l \in L \mid l \parallel z\}) \times [0,a]$, then $U^*(x,y) \parallel z$. Next we just prove that for $z \in I_a$, if $(x,y) \in [0,a] \times ((0,a] \cap \{l \in L \mid l \parallel z\})$, then $U^*(x,y) \parallel z$. The other case in the above is obvious by the commutativity of U^*. Obviously, U^* is a t-conorm on $[0,a]$. Assume that for $z \in I_a$, there exists $(x,y) \in [0,a] \times ((0,a] \cap \{l \in L \mid l \parallel z\})$ such that $U^*(x,y) \nparallel z$, that is, $U^*(x,y) < z$. For $(x,y) \in [0,a] \times ((0,a] \cap \{l \in L \mid l \parallel z\})$, we can obtain that $U^*(x,y) \in [x \vee y, a]$. If $U^*(x,y) < z$, then $y \leq x \vee y \leq U^*(x,y) < z$. This contradicts with the fact $y \in \{l \in L \mid l \parallel z\}$. Hence, $U^*(x,y) \parallel z$. □

In Theorem 3.1, if taking $e = 0$, then U^* is a t-conorm on $[0,a]$ and $I_{e,a} \cup I_e^a = \emptyset$. Thus, $I_e^a \cup (e,a] = (e,a] = (0,a]$ and the conditions (1), (2) and (4) hold by Lemma 3.4. In this case, we can obtain the following proposition.

Proposition 3.5. *Let S be a t-conorm on $[0,a]$ for $a \in L \setminus \{0,1\}$. Then the function $S_1 : L^2 \to L$ defined by*

$$S_1(x,y) = \begin{cases} S(x,y) & \textit{if } (x,y) \in [0,a]^2, \\ x \vee y & \textit{otherwise}, \end{cases}$$

is a t-conorm if and only if for $z \in I_a$, if $(x,y) \in ((0,a] \cap \{l \in L \mid l \nparallel z\})^2$, then $S(x,y) \nparallel z$.

Remark 3.6. In Proposition 3.5, we give a sufficient and necessary constraint condition under which S_1 is a t-conorm. Obviously, this condition differs from that in Proposition 2.13(2). More precisely, our condition is based on the viewpoint of t-conorms; the condition in Proposition 2.13(2) is based on the viewpoint of L.

The next example illustrates the construction method of uninorms on bounded lattices in Theorem 3.1.

Example 3.7. Given a bounded lattice L_1 drawn in Fig.1 and a uninorm $U^* : [0,a]^2 \to [0,a]$ shown in Table 1. It is clear that U^* satisfies the conditions in Theorem 3.1 on L_1. Based on Theorem 3.1, a uninorm U_1 on L_1, shown in Table 2, can be obtained.

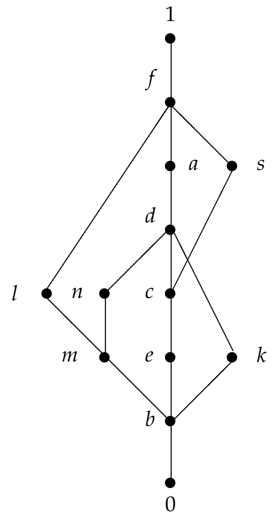

Fig.1. The lattice L_1

Table 1: U^* on $[0, a]$.

U^*	0	b	e	c	m	n	k	d	a
0	0	0	0	c	m	n	k	d	a
b	0	b	b	c	m	n	k	d	a
e	0	b	e	c	m	n	k	d	a
c	c	c	c	c	d	d	d	d	a
m	m	m	m	d	m	n	d	d	a
n	n	n	n	d	n	n	d	d	a
k	k	k	k	d	d	d	k	d	a
d	d	d	d	d	d	d	d	d	a
a	a	a	a	a	a	a	a	a	a

Table 2: U_1 on L_1.

U_1	0	b	e	c	m	n	k	d	a	l	s	f	1
0	0	0	0	c	m	n	k	d	a	l	s	f	1
b	0	b	b	c	m	n	k	d	a	l	s	f	1
e	0	b	e	c	m	n	k	d	a	l	s	f	1
c	c	c	c	c	d	d	d	d	a	f	s	f	1
m	m	m	m	d	m	n	d	d	a	l	f	f	1
n	n	n	n	d	n	n	d	d	a	f	f	f	1
k	k	k	k	d	d	d	k	d	a	f	f	f	1
d	d	d	d	d	d	d	d	d	a	f	f	f	1
a	a	a	a	a	a	a	a	a	a	f	f	f	1
l	l	l	l	f	l	f	f	f	f	l	f	f	1
s	s	s	s	s	f	f	f	f	f	f	s	f	1
f	f	f	f	f	f	f	f	f	f	f	f	f	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1

Remark 3.8. Let U_1 be a uninorm in Theorem 3.1.

(1) U_1 is disjunctive, i.e., $U_1(0,1) = 1$.

(2) If $a = 1$, then $U_1 = U^*$.

(3) U_1 is idempotent if and only if U^* is idempotent.

(4) $U_1 \in \mathcal{U}^*_{max}$ if and only if $U^* \in \mathcal{U}^*_{max}$.

Remark 3.9. By Remark 3.8(4), we can easily construct the uninorms, which need not belong to the class of $\mathcal{U}^*_{max}$. In Theorem 3.1, if $U^* \notin \mathcal{U}^*_{max}$, then the uninorm U_1 does not belong to $\mathcal{U}^*_{max}$. Next, we give a example for the uninorm U^* on $[0,a]$ of L_2, shown in Table 3, satisfying $U^* \notin \mathcal{U}^*_{max}$ and the conditions in Theorem 3.1. Therefore, we can easily construct a uninorm U_1 by U^* such that $U_1 \notin \mathcal{U}^*_{max}$.

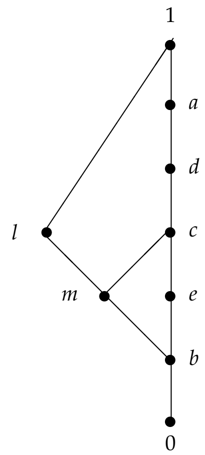

Fig.2. The lattice L_2

Table 3: U^* on $[0,a]$.

U^*	0	b	e	m	c	d	a
0	0	0	0	m	c	c	a
b	0	b	b	m	c	c	a
e	0	b	e	m	c	d	a
m	m	m	m	m	a	a	a
c	c	c	c	a	a	a	a
d	c	c	d	a	a	a	a
a	a	a	a	a	a	a	a

Next, we give the dual result of Theorem 3.1.

Theorem 3.10. *Let U^* be a uninorm on $[b,1]$ with a neutral element e for $b \in L\backslash\{0,1\}$. Then the function $U_2 : L^2 \to L$ defined by*

$$U_2(x,y) = \begin{cases} U^*(x,y) & \text{if } (x,y) \in [b,1]^2, \\ x & \text{if } (x,y) \in I_{e,b} \times [e,1], \\ y & \text{if } (x,y) \in [e,1] \times I_{e,b}, \\ x \wedge y & \text{otherwise}, \end{cases}$$

is a uninorm on L with the neutral element $e \in L$ if and only if U^* satisfies the following conditions:

(1) *for $z \in I_{e,b}$, if $(x,y) \in ((I^b_e \cup [b,e)) \cap \{l \in L \mid l \nparallel z\})^2$, then $U^*(x,y) \nparallel z$;*

(2) *for* $z \in I_{e,b}$, *if* $(x, y) \in [b, 1] \times ((I_e^b \cup [b, e)) \cap \{l \in L \mid l \parallel z\}) \cup ((I_e^b \cup [b, e)) \cap \{l \in L \mid l \parallel z\}) \times [b, 1]$, *then* $U^*(x, y) \parallel z$;

(3) *for* $z \in I_b^e$, *if* $(x, y) \in ((I_e^b \cup [b, e)) \cap \{l \in L \mid l \nparallel z\})^2$, *then* $U^*(x, y) \nparallel z$;

(4) *for* $z \in I_b^e$, *if* $(x, y) \in [b, 1] \times ((I_e^b \cup [b, e)) \cap \{l \in L \mid l \parallel z\}) \cup ((I_e^b \cup [b, e)) \cap \{l \in L \mid l \parallel z\}) \times [b, 1]$, *then* $U^*(x, y) \parallel z$.

Proof. It can be proved immediately by the proof similar to Theorem 3.1. □

If we take $e = b$ in Theorem 3.10, then we can obtain the existing result in the literature.

Remark 3.11. If taking $e = b$ in Theorem 3.10, then $[b, 1] = [e, 1]$, $I_{e,b} = I_e$, $I_e^b \cup I_b^e \cup [b, e) = \emptyset$ and U^* is a t-conorm on $[b, 1]$. Moreover, the conditions (1), (2), (3) and (4) in Theorem 3.10 naturally hold.

By the above fact, if taking $e = b$ in Theorem 3.1, then we retrieve the uninorm $U_s : L^2 \to L$ constructed by Çaylı, Karaçal, and Mesiar ([5], Theorem 1) as follow:

$$U_s(x, y) = \begin{cases} S_e(x, y) & \text{if } (x, y) \in [e, 1]^2, \\ x & \text{if } (x, y) \in I_e \times [e, 1], \\ y & \text{if } (x, y) \in [e, 1] \times I_e, \\ x \wedge y & \textit{otherwise}. \end{cases}$$

Remark 3.12. In Theorem 3.10, if taking $e = 1$, then U^* is a t-norm on $[b, 1]$ and $I_{e,b} \cup I_e^b = \emptyset$. Thus, the conditions (1) and (2) hold and $I_e^b \cup [b, e) = [b, e) = [b, 1)$. In this case, the condition (4) in Theorem 3.10 holds.

By Remark 3.12, if $e = 1$ in Theorem 3.10, then the following proposition holds.

Proposition 3.13. *Let T be a t-norm on $[b, 1]$ for $b \in L \setminus \{0, 1\}$. Then the function $T_1 : L^2 \to L$ defined by*

$$T_1(x, y) = \begin{cases} T(x, y) & \textit{if } (x, y) \in [b, 1]^2, \\ x \wedge y & \textit{otherwise}, \end{cases}$$

is a t-norm on L if and only if for $z \in I_b$, if $(x, y) \in ([b, 1) \cap \{l \in L | \, l \nparallel z\})^2$, then $T(x, y) \nparallel z$.

Remark 3.14. In Proposition 3.13, we give a sufficient and necessary condition under which T_1 is a t-norm. Obviously, this condition differs from that in Proposition 2.13(1). More precisely, our constraint condition is based on the viewpoint of t-norms; the constraint condition in Proposition 2.13(1) is based on the viewpoint of L.

Remark 3.15. Let U_2 be a uninorm in Theorem 3.10.

(1) U_2 is conjunctive, i.e., $U_2(0, 1) = 0$.

(2) If $b = 0$, then $U_2 = U^*$.

(3) U_2 is idempotent if and only if U^* is idempotent.

(4) $U_2 \in \mathcal{U}^*_{min}$ if and only if $U^* \in \mathcal{U}^*_{min}$.

4. Constructing uninorms via given uninorms based on closure and interior operators

In this section, we mainly construct new uninorms by extending given uninorms based on interior operators and closure operators.

For convenience, $\mathcal{U}^*_\perp$ denotes the class of all uninorms U on L with neutral element e satisfying $U(x, y) \in [0, e]$ implies $(x, y) \in [0, e]^2$. Similarly, $\mathcal{U}^*_\top$ denotes the class of all uninorms U on L with neutral element e satisfying $U(x, y) \in [e, 1]$ implies $(x, y) \in [e, 1]^2$.

Theorem 4.1. *Let U^* be a uninorm on $[0, a]$ with a neutral element e for $a \in L \setminus \{0, 1\}$ and cl be a closure operator. Let $U_3 : L^2 \to L$ be a function defined as follow:*

$$U_3(x,y) = \begin{cases} U^*(x,y) & \text{if } (x,y) \in [0,a]^2, \\ x & \text{if } (x,y) \in (L \setminus [0,a]) \times [0,e], \\ y & \text{if } (x,y) \in [0,e] \times (L \setminus [0,a]), \\ cl(x) \vee cl(y) & \text{if } (x,y) \in I_{e,a} \times I_{e,a}, \\ 1 & \text{otherwise.} \end{cases}$$

(1) *Suppose that $cl(x) \vee cl(y) \in I_{e,a}$ for all $x, y \in I_{e,a}$.*

(i) *Let us assume that $U^* \in \mathcal{U}^*_\perp$. Then U_3 is a uninorm with the neutral element $e \in L$ if and only if $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I_e^a$.*

(ii) *Let us assume that $I_{e,a} \cup I_a^e \cup (a, 1) \neq \emptyset$. Then U_3 is a uninorm with the neutral element $e \in L$ if and only if $U^* \in \mathcal{U}^*_\perp$ and $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I_e^a$.*

(2) *Suppose that $cl(x) \vee cl(y) \in I_a^e \cup (a, 1]$ for all $x, y \in I_{e,a}$.*

(i) *Let us assume that $x \parallel y$ for all $x \in I_{e,a}$, $y \in I_e^a$ and $U^* \in \mathcal{U}^*_\perp$. Then U_3 is a uninorm with the neutral element $e \in L$.*

(ii) *Let us assume that $cl(x) \vee cl(y) < 1$ for all $x, y \in I_{e,a}$ and $I_{e,a} \cup I_a^e \cup (a, 1) \neq \emptyset$. Then U_3 is a uninorm with the neutral element $e \in L$ if and only if $x \parallel y$ for all $x \in I_{e,a}$, $y \in I_e^a$ and $U^* \in \mathcal{U}^*_\perp$.*

Proof. (1)(i) Necessity. Let $U_3(x, y)$ be a uninorm with a neutral element e and $cl(x) \vee cl(y) \in I_{e,a}$ for all $x, y \in I_{e,a}$. We prove that $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I_e^a$.

Assume that there exist $x \in I_{e,a}$ and $y \in I_e^a$ such that $x \nparallel y$, i.e., $y < x$. Then $U_3(x, y) = 1$ and $U_3(x, x) = cl(x) \vee cl(x) = cl(x)$. Since $cl(x) < 1$, the increasingness property of U_3 is violated. Thus $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I_e^a$.

Sufficiency. By the definition of U_3, U_3 is commutative and e is the neutral element of U_3. Thus, we only need to prove the increasingness and the associativity of U_3.

I. Increasingness: We prove that if $x \leq y$, then $U_3(x, z) \leq U_3(y, z)$ for all $z \in L$. It is easy to verify that $U_3(x, z) \leq U_3(y, z)$ if both x and y belong to one of the intervals $[0, e], I_e^a, (e, a], I_a^e, I_{e,a}$ or $(a, 1]$ for all $z \in L$. The proof is split into all possible cases.

1. $x \in [0, e]$

1.1. $y \in I_e^a \cup (e, a]$

1.1.1. $z \in [0, e] \cup I_e^a \cup (e, a]$

$U_3(x, z) = U^*(x, z) \leq U^*(y, z) = U_3(y, z)$

1.1.2. $z \in I_a^e \cup I_{e,a} \cup (a, 1]$

$U_3(x, z) = z \leq 1 = U_3(y, z)$

1.2. $y \in I_a^e \cup (a, 1]$

1.2.1. $z \in [0, e]$

$U_3(x, z) = U^*(x, z) \leq x < y = U_3(y, z)$

1.2.2. $z \in I_e^a \cup (e, a]$

$U_3(x, z) = U^*(x, z) \leq a < 1 = U_3(y, z)$

1.2.3. $z \in I_a^e \cup I_{e,a} \cup (a, 1]$

$U_3(x, z) = z \leq 1 = U_3(y, z)$

1.3. $y \in I_{e,a}$

1.3.1. $z \in [0, e]$

$U_3(x, z) = U^*(x, z) \leq x < y = U_3(y, z)$

1.3.2. $z \in I_e^a \cup (e, a]$

$U_3(x, z) = U^*(x, z) \leq a < 1 = U_3(y, z)$

1.3.3. $z \in I_a^e \cup (a, 1]$

$U_3(x, z) = z \leq 1 = U_3(y, z)$

1.3.4. $z \in I_{e,a}$
$U_3(x,z) = z \leq cl(y) \vee cl(z) = U_3(y,z)$

2. $x \in I_e^a$

2.1. $y \in (e,a]$

2.1.1. $z \in [0,e] \cup I_e^a \cup (e,a]$
$U_3(x,z) = U^*(x,z) \leq U^*(y,z) = U_3(y,z)$

2.1.2. $z \in I_a^e \cup I_{e,a} \cup (a,1]$
$U_3(x,z) = 1 = U_3(y,z)$

2.2. $y \in I_a^e \cup (a,1]$

2.2.1. $z \in [0,e]$
$U_3(x,z) = U^*(x,z) \leq x < y = U_3(y,z)$

2.2.2. $z \in I_e^a \cup (e,a]$
$U_3(x,z) = U^*(x,z) \leq a < 1 = U_3(y,z)$

2.2.3. $z \in I_a^e \cup I_{e,a} \cup (a,1]$
$U_3(x,z) = 1 = U_3(y,z)$

3. $x \in (e,a], y \in I_a^e \cup (a,1]$

3.1. $z \in [0,e]$
$U_3(x,z) = U^*(x,z) \leq x < y = U_3(y,z)$

3.2. $z \in I_e^a \cup (e,a]$
$U_3(x,z) = U^*(x,z) \leq a < 1 = U_3(y,z)$

3.3. $z \in I_a^e \cup I_{e,a} \cup (a,1]$
$U_3(x,z) = 1 = U_3(y,z)$

4. $x \in I_a^e, y \in (a,1]$

4.1. $z \in [0,e]$
$U_3(x,z) = x \leq y = U_3(y,z)$

4.2. $z \in I_e^a \cup (e,a] \cup I_a^e \cup I_{e,a} \cup (a,1]$
$U_3(x,z) = 1 = U_3(y,z)$

5. $x \in I_{e,a}, y \in I_a^e \cup (a,1]$

5.1. $z \in [0,e]$
$U_3(x,z) = x < y = U_3(y,z)$

5.2. $z \in I_e^a \cup (e,a] \cup I_a^e \cup (a,1]$
$U_3(x,z) = 1 = U_3(y,z)$

5.3. $z \in I_{e,a}$
$U_3(x,z) = cl(x) \vee cl(z) < 1 = U_3(y,z)$

II. Associativity: We demonstrate that $U_3(x, U_3(y,z)) = U_3(U_3(x,y),z)$ for all $x,y,z \in L$. By Proposition 2.12, we just consider the following cases.

1. If $x,y,z \in [0,e] \cup I_e^a \cup (e,a]$, then $U_3(x,U_3(y,z)) = U_3(U_3(x,y),z) = U_3(y,U_3(x,z))$ for U^* is associative.

2. If $x,y,z \in I_a^e \cup (a,1]$, then $U_3(x,U_3(y,z)) = U_3(x,1) = 1 = U_3(1,z) = U_3(U_3(x,y),z)$.

3. If $x,y,z \in I_{e,a}$, then $U_3(x,U_3(y,z)) = U_3(x,cl(y) \vee cl(z)) = cl(x) \vee cl(y) \vee cl(z) = U_3(cl(x) \vee cl(y),z) = U_3(U_3(x,y),z)$.

4. If $x,y \in [0,e]$ and $z \in I_a^e \cup I_{e,a} \cup (a,1]$, then $U_3(x,U_3(y,z)) = U_3(x,z) = z = U_3(U^*(x,y),z) = U_3(U_3(x,y),z)$.

5. If $x,y \in I_e^a \cup (e,a]$ and $z \in I_a^e \cup I_{e,a} \cup (a,1]$, then $U_3(x,U_3(y,z)) = U_3(x,1) = 1 = U_3(U^*(x,y),z) = U_3(U_3(x,y),z)$ and $U_3(y,U_3(x,z)) = U_3(y,1) = 1$. Thus $U_3(x,U_3(y,z)) = U_3(U_3(x,y),z) = U_3(y,U_3(x,z))$.

6. If $x,y \in I_a^e$ and $z \in I_{e,a}$, then $U_3(x,U_3(y,z)) = U_3(x,1) = 1 = U_3(1,z) = U_3(U_3(x,y),z)$.

7. If $x,y \in I_{e,a}$ and $z \in (a,1]$, then $U_3(x,U_3(y,z)) = U_3(x,1) = 1 = U_3(cl(x) \vee cl(y),z) = U_3(U_3(x,y),z)$.

8. If $x \in [0,e]$ and $y,z \in I_a^e \cup (a,1]$, then $U_3(x,U_3(y,z)) = U_3(x,1) = 1 = U_3(y,z) = U_3(U_3(x,y),z)$ and $U_3(y,U_3(x,z)) = U_3(y,z) = 1$. Thus $U_3(x,U_3(y,z)) = U_3(U_3(x,y),z) = U_3(y,U_3(x,z))$.

9. If $x \in [0,e]$ and $y,z \in I_{e,a}$, then $U_3(x,U_3(y,z)) = U_3(x,cl(y) \vee cl(z)) = cl(y) \vee cl(z) = U_3(y,z) = U_3(U_3(x,y),z)$.

10. If $x \in I_e^a \cup (e,a]$ and $y,z \in I_a^e \cup (a,1]$, then $U_3(x,U_3(y,z)) = U_3(x,1) = 1 = U_3(1,z) = U_3(U_3(x,y),z)$ and $U_3(y,U_3(x,z)) = U_3(y,1) = 1$. Thus $U_3(x,U_3(y,z)) = U_3(U_3(x,y),z) = U_3(y,U_3(x,z))$.

11. If $x \in I_e^a \cup (e,a]$ and $y,z \in I_{e,a}$, then $U_3(x,U_3(y,z)) = U_3(x,cl(y) \vee cl(z)) = 1 = U_3(1,z) = U_3(U_3(x,y),z)$.

12. If $x \in I_a^e$ and $y, z \in I_{e,a}$, then $U_3(x, U_3(y,z)) = U_3(x, cl(y) \vee cl(z)) = 1 = U_3(1,z) = U_3(U_3(x,y),z)$.

13. If $x \in I_{e,a}$ and $y, z \in (a,1]$, then $U_3(x, U_3(y,z)) = U_3(x,1) = 1 = U_3(1,z) = U_3(U_3(x,y),z)$. Thus $U_3(x, U_3(y,z)) = U_3(U_3(x,y),z)$.

14. If $x \in [0,e], y \in I_e^a \cup (e,a]$ and $z \in I_a^e \cup I_{e,a} \cup (a,1]$, then $U_3(x, U_3(y,z)) = U_3(x,1) = 1 = U_3(U^*(x,y),z) = U_3(U_3(x,y),z)$ and $U_3(y, U_3(x,z)) = U_3(y,z) = 1$. Thus $U_3(x,U_3(y,z)) = U_3(U_3(x,y),z) = U_3(y,U_3(x,z))$.

15. If $x \in [0,e], y \in I_a^e$ and $z \in I_{e,a}$, then $U_3(x,U_3(y,z)) = U_3(x,1) = 1 = U_3(y,z) = U_3(U_3(x,y),z)$ and $U_3(y,U_3(x,z)) = U_3(y,z) = 1$. Thus $U_3(x,U_3(y,z)) = U_3(U_3(x,y),z) = U_3(y,U_3(x,z))$.

16. If $x \in [0,e], y \in I_{e,a}$ and $z \in (a,1]$, then $U_3(x,U_3(y,z)) = U_3(x,1) = 1 = U_3(y,z) = U_3(U_3(x,y),z)$ and $U_3(y,U_3(x,z)) = U_3(y,z) = 1$. Thus $U_3(x,U_3(y,z)) = U_3(U_3(x,y),z) = U_3(y,U_3(x,z))$.

17. If $x \in I_e^a \cup (e,a], y \in I_a^e$ and $z \in I_{e,a}$, then $U_3(x,U_3(y,z)) = U_3(x,1) = 1 = U_3(1,z) = U_3(U_3(x,y),z)$ and $U_3(y,U_3(x,z)) = U_3(y,1) = 1$. Thus $U_3(x,U_3(y,z)) = U_3(U_3(x,y),z) = U_3(y,U_3(x,z))$.

18. If $x \in I_e^a \cup (e,a], y \in I_{e,a}$ and $z \in (a,1]$, then $U_3(x,U_3(y,z)) = U_3(x,1) = 1 = U_3(1,z) = U_3(U_3(x,y),z)$ and $U_3(y,U_3(x,z)) = U_3(y,1) = 1$. Thus $U_3(x,U_3(y,z)) = U_3(U_3(x,y),z) = U_3(y,U_3(x,z))$.

19. If $x \in I_a^e, y \in I_{e,a}$ and $z \in (a,1]$, then $U_3(x,U_3(y,z)) = U_3(x,1) = 1 = U_3(1,z) = U_3(U_3(x,y),z)$ and $U_3(y,U_3(x,z)) = U_3(y,1) = 1$. Thus $U_3(x,U_3(y,z)) = U_3(U_3(x,y),z) = U_3(y,U_3(x,z))$.

(1)(ii) We just prove that If $I_{e,a} \cup I_a^e \cup (a,1) \neq \emptyset$, then the condition $U^* \in \mathcal{U}_\perp^*$ is necessary. The proof can be split into all possible cases.

a. $U^*(x,y) \notin [0,e]$ for all $(x,y) \in [0,e] \times (I_e^a \cup (e,a]) \cup (I_e^a \cup (e,a]) \times [0,e] \cup I_e^a \times I_e^a$.

Now we give the proof of $U^*(x,y) \notin [0,e]$ for all $(x,y) \in [0,e] \times (I_e^a \cup (e,a]) \cup I_e^a \times I_e^a$, and the other case is obvious by the commutativity of U^*. Assume that there exists $(x,y) \in [0,e] \times (I_e^a \cup (e,a]) \cup I_e^a \times I_e^a$ such that $U^*(x,y) \in [0,e]$. If $z \in I_{e,a} \cup I_a^e \cup (a,1)$, then $U_3(x,U_3(y,z)) = U_3(x,1) = 1$ and $U_3(U_3(x,y),z) = U_3(U^*(x,y),z) = z$. Since $z < 1$, the associativity of U_3 is violated. Thus $U^*(x,y) \notin [0,e]$ for all $(x,y) \in [0,e) \times (I_e^a \cup (e,a]) \cup (I_e^a \cup (e,a]) \times [0,e) \cup I_e^a \times I_e^a$.

b. $U^*(x,y) \notin [0,e]$ for all $(x,y) \in (e,a]^2 \cup (e,a] \times I_e^a \cup I_e^a \times (e,a]$.

Now we just prove that $U^*(x,y) \notin [0,e]$ for all $(x,y) \in (e,a]^2 \cup (e,a] \times I_e^a$, and the other case is obvious by the commutativity of U^*. By the increasingness of U^*, we can obtain that $y = U^*(e,y) \leq U^*(x,y)$. Since $y \in I_e^a \cup (e,a]$, $U^*(x,y) \notin [0,e]$. Thus $U^*(x,y) \notin [0,e]$ for all $(x,y) \in (e,a]^2 \cup (e,a] \times I_e^a \cup I_e^a \times (e,a]$.

(2)(i) By the definition of U_3, U_3 is commutative and e is the neutral element of U_3. Thus, we only need to show the increasingness and the associativity of U_3.

I. Increasingness: We prove that if $x \leq y$, then $U_3(x,z) \leq U_3(y,z)$ for all $z \in L$. Next, it is enough to check the cases that are different from those in the proof of Theorem 4.1(1)(i).

1. $x \in [0,e], y \in I_{e,a}$
 1.1. $z \in I_{e,a}$
 $U_3(x,z) = z < cl(y) \vee cl(z) = U_3(y,z)$
2. $x \in I_{e,a}$
 2.1. $y \in I_{e,a}$
 2.1.1. $z \in I_{e,a}$
 $U_3(x,z) = cl(x) \vee cl(z) \leq cl(y) \vee cl(z) = U_3(y,z)$
 2.2. $y \in I_a^e \cup (a,1]$
 2.2.1. $z \in I_{e,a}$
 $U_3(x,z) = cl(x) \vee cl(z) \leq 1 = U_3(y,z)$

II. Associativity: It can be shown that $U_3(x,U_3(y,z)) = U_3(U_3(x,y),z)$ for all $x,y,z \in L$. By Proposition 2.12 and the proof of Theorem 4.1(1)(i), we just check the cases that are different from the cases in the proof of Theorem 4.1(1)(i).

1. If $x,y,z \in I_{e,a}$, then $U_3(x,U_3(y,z)) = U_3(x,cl(y) \vee cl(z)) = 1 = U_3(cl(x) \vee cl(y),z) = U_3(U_3(x,y),z)$.

2. If $x \in [0,e]$ and $y,z \in I_{e,a}$, then $U_3(x,U_3(y,z)) = U_3(x,cl(y) \vee cl(z)) = cl(y) \vee cl(z) = U_3(y,z) = U_3(U_3(x,y),z)$.

3. If $x \in I_e^a \cup (e,a] \cup I_a^e$ and $y,z \in I_{e,a}$, then $U_3(x,U_3(y,z)) = U_3(x,cl(y) \vee cl(z)) = 1 = U_3(1,z) = U_3(U_3(x,y),z)$.

4. If $x,y \in I_{e,a}$ and $z \in (a,1]$, then $U_3(x,U_3(y,z)) = U_3(x,1) = 1 = U_3(cl(x) \vee cl(y),z) = U_3(U_3(x,y),z)$.

(2)(ii) In the following, we only prove that if $cl(x) \vee cl(y) < 1$ for all $x,y \in I_{e,a}$ and $I_{e,a} \cup I_a^e \cup (a,1) \neq \emptyset$, then $x \parallel y$ for all $x \in I_{e,a}, y \in I_e^a$ and $U^* \in \mathcal{U}_\perp^*$ are necessary. First, assume that there exist $x \in I_{e,a}$ and $y \in I_e^a$ such that $x \nparallel y$, i.e., $y < x$. Then $U_3(x,x) = cl(x) \vee cl(x)$ and $U_3(x,y) = 1$. Since $cl(x) \vee cl(x) < 1$, the increasingness of U_3 is violated. Thus $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I_e^a$. Then we can obtain that $U^* \in \mathcal{U}_\perp^*$ is necessary by the proof similar to Theorem 4.1(1)(ii). □

If we take $e = 0$ and $e = a$ in Theorem 4.1, respectively, then we can obtain some existing results in the literature.

Remark 4.2. If taking $e = 0$ in Theorem 4.1, then $[e, a] = [0, a]$, $I_a^e = I_a$, $I_{e,a} \cup I_e^a \cup [0, e) = \emptyset$ and U^* is a t-conorm on $[0, a]$. Moreover, based on the above case, the conditions in Theorem 4.1 naturally hold.

By the above fact, if taking $e = 0$ in Theorem 4.1, then we retrieve the t-conorm $S_2^* : L^2 \to L$ constructed by Çaylı ([7], Theorem 1) as follow:

$$S_2^*(x,y) = \begin{cases} S_e(x,y) & \text{if } (x,y) \in [0,a]^2, \\ x & \text{if } (x,y) \in (I_a \cup (a,1]) \times \{0\}, \\ y & \text{if } (x,y) \in \{0\} \times (I_a \cup (a,1]), \\ 1 & \textit{otherwise}. \end{cases}$$

Remark 4.3. If taking $e = a$ in Theorem 4.1, then $[0, a] = [0, e]$, $I_{e,a} = I_e$, $I_a^e \cup I_e^a \cup (e, a] = \emptyset$ and U^* is a t-norm on $[0, a]$. Moreover, in this case, $U^* \in \mathcal{U}_\perp^*$ and $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I_e^a$ naturally hold.

By the above fact, if taking $e = a$ in Theorem 4.1, then we retrieve the uninorm $U_{cl,1}^1 : L^2 \to L$ under the conditions $cl(x) \vee cl(y) \in I_e$ for all $x, y \in I_e$ or $cl(x) \vee cl(y) \in (e, 1]$ for all $x, y \in I_e$, constructed by Zhao and Wu ([40], Proposition 3.1) as follow:

$$U_{cl,1}^1(x,y) = \begin{cases} T_e(x,y) & \text{if } (x,y) \in [0,e]^2, \\ x & \text{if } (x,y) \in I_e \times [0,e] \cup (e,1] \times [0,e], \\ y & \text{if } (x,y) \in [0,e] \times I_e \cup [0,e] \times (e,1], \\ cl(x) \vee cl(y) & \text{if } (x,y) \in I_e \times I_e, \\ 1 & \textit{otherwise}. \end{cases}$$

Moreover, if taking $cl(x) = x \vee a$ for $x \in L$ in $U_{cl,1}^1$, then we retrieve the uninorm $U_e^T : L^2 \to L$ constructed by Çaylı ([8], Theorem 2.23) as follow:

$$U_e^T(x,y) = \begin{cases} T_e(x,y) & \text{if } (x,y) \in [0,e]^2, \\ x & \text{if } (x,y) \in (L \setminus [0,e]) \times [0,e], \\ y & \text{if } (x,y) \in [0,e] \times (L \setminus [0,e]), \\ x \vee y \vee e & \text{if } (x,y) \in I_e \times I_e, \\ 1 & \textit{otherwise}. \end{cases}$$

The next example illustrates the construction method of uninorms on bounded lattices in Theorem 4.1.

Example 4.4. Given a bounded lattice L_3 depicted in Fig.3., a uninorm $U^* : [0, a]^2 \to [0, a]$ shown in Table 4 and a closure operator $cl : L_3 \to L_3$ defined by $cl(x) = x$ for all $x \in L_3$. It is easy to see that L_3, U^* and cl satisfy the conditions in Theorem 4.1(1)(i) on L_3. By Theorem 4.1, we can obtain a uninorm U_3 on L_4 with the neutral element e, shown in Table 5.

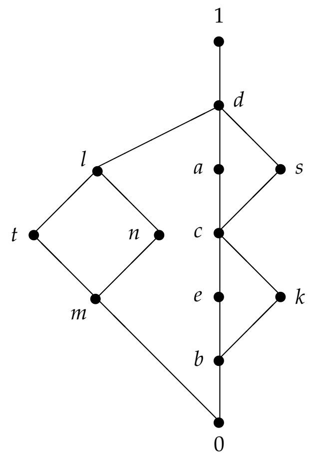

Fig.3. The lattice L_3.

Table 4: U^* on $[0, a]$.

U^*	0	b	e	k	c	a
0	0	0	0	k	c	a
b	0	b	b	k	c	a
e	0	b	e	k	c	a
k	k	k	k	k	c	a
c	c	c	c	c	c	a
a	a	a	a	a	a	a

Table 5: U_3 on L_3.

U_3	0	b	e	k	c	a	m	t	n	l	s	d	1
0	0	0	0	k	c	a	m	t	n	l	s	d	1
b	0	b	b	k	c	a	m	t	n	l	s	d	1
e	0	b	e	k	c	a	m	t	n	l	s	d	1
k	k	k	k	k	c	a	1	1	1	1	1	1	1
c	c	c	c	c	c	a	1	1	1	1	1	1	1
a	a	a	a	a	a	a	1	1	1	1	1	1	1
m	m	m	m	1	1	1	m	t	n	l	1	1	1
t	t	t	t	1	1	1	t	t	l	l	1	1	1
n	n	n	n	1	1	1	n	l	n	l	1	1	1
l	l	l	l	1	1	1	l	l	l	l	1	1	1
s	s	s	s	1	1	1	1	1	1	1	1	1	1
d	d	d	d	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1

Remark 4.5. (1) In Theorem 4.1(1), the condition $cl(x) \vee cl(y) \in I_{e,a}$ for all $x, y \in I_{e,a}$ can not be omitted, in general.

(2) Similarly, in Theorem 4.1(2), the condition $cl(x) \vee cl(y) \in I_a^e \cup (a, 1]$ for all $x, y \in I_{e,a}$ can not be omitted.

The next example illustrates the facts in Remark 4.5. That is, if the conditions in Theorem 4.1 do not hold, then the associativity of U_3 can be violated.

Example 4.6. Given a bounded lattice L_4 depicted in Fig.4., a uninorm $U^* : [0,a]^2 \to [0,a]$ shown in Table 6 and a closure operator $cl : L_4 \to L_4$ defined by $cl(m) = b$ for $m \in L_4$ and $cl(x) = x$ for all $x \in L_4 \setminus \{m\}$. It is easy to see that $U^* \in \mathcal{U}^*_\perp$ and cl does not satisfy the conditions in Remark 4.5, i.e., $cl(m) \vee cl(k) = b \vee k = b \in I^e_a \cup (a,1]$ and $cl(k) \vee cl(k) = k \vee k = k \in I_{e,a}$ for $m, k \in I_{e,a}$. By Theorem 4.1, we can obtain a function U_3 on L_4, shown in Table 7. Since $U_3(k, U_3(k,m)) = U_3(k,b) = 1$ and $U_3(U_3(k,k),m) = U_3(k,m) = b$ for $k, m \in L_4$, the function U_3 does not satisfy associativity. Thus, the function U_3 is not a uninorm on L_4.

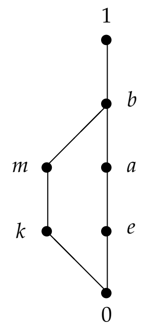

Fig.4. The lattice L_4.

Table 6: U^* on $[0,a]$.

U^*	0	e	a
0	0	0	a
e	0	e	a
a	a	a	a

Table 7: U_3 on L_4.

U_3	0	e	a	k	m	b	1
0	0	0	a	k	m	b	1
e	0	e	a	k	m	b	1
a	a	a	a	1	1	1	1
k	k	k	1	k	b	1	1
m	m	m	1	b	b	1	1
b	b	b	1	1	1	1	1
1	1	1	1	1	1	1	1

Remark 4.7. Let U_3 be a uninorm in Theorem 4.1.

(1) U_3 is disjunctive, i.e., $U_3(0,1) = 1$.

(2) If $a = 1$, then $U_3 = U^*$.

(3) U_3 is not idempotent, in general. In fact, if $(a,1) \neq \emptyset$, then there exists x such that $x \in (a,1)$ and then $U_3(x,x) = 1 \neq x$. This is a contradiction with that U_3 be idempotent.

(4) $U_3 \in \mathcal{U}^*_{max}$ if and only if $U^* \in \mathcal{U}^*_{max}$.

Also, we give the dual result of Theorem 4.1.

Theorem 4.8. *Let U^* be a uninorm on $[b,1]$ with a neutral element e for $b \in L \setminus \{0,1\}$ and int be an interior operator. Let $U_4 : L^2 \to L$ be a function defined as follow:*

$$U_4(x,y)=\begin{cases}U^*(x,y) & \text{if } (x,y)\in[b,1]^2,\\ x & \text{if } (x,y)\in(L\setminus[b,1])\times[e,1],\\ y & \text{if } (x,y)\in[e,1]\times(L\setminus[b,1]),\\ int(x)\wedge int(y) & \text{if } (x,y)\in I_{e,b}\times I_{e,b},\\ 0 & \text{otherwise.}\end{cases}$$

(1) *Suppose that $int(x)\wedge int(y)\in I_{e,b}$ for all $x,y\in I_{e,b}$.*

(i) *Let us assume that $U^*\in\mathcal{U}_{\top}^*$. Then U_4 is a uninorm with the neutral element $e\in L$ if and only if $x\parallel y$ for all $x\in I_{e,b}$ and $y\in I_e^b$.*

(ii) *Let us assume that $I_{e,b}\cup I_b^e\cup(0,b)\neq\emptyset$. Then U_4 is a uninorm with the neutral element $e\in L$ if and only if $U^*\in\mathcal{U}_{\top}^*$ and $x\parallel y$ for all $x\in I_{e,b}$ and $y\in I_e^b$.*

(2) *Suppose that $int(x)\wedge int(y)\in I_b^e\cup[0,b)$ for all $x,y\in I_{e,b}$.*

(i) *Let us assume that $x\parallel y$ for all $x\in I_{e,b}$ and $y\in I_e^b$ and $U^*\in\mathcal{U}_{\top}^*$. Then U_4 is a uninorm with the neutral element $e\in L$.*

(ii) *Let us assume that $I_{e,b}\cup I_b^e\cup(0,b)\neq\emptyset$ and $0<int(x)\wedge int(y)$ for all $x,y\in I_{e,b}$. Then U_4 is a uninorm on L with the neutral element $e\in L$ if and only if $x\parallel y$ for all $x\in I_{e,b}$, $y\in I_e^b$ and $U^*\in\mathcal{U}_{\top}^*$.*

Proof. It can be proved immediately by the proof similar to Theorem 4.1. □

If we take $e=1$ and $e=b$ in Theorem 4.8, respectively, then we can obtain some existing results in the literature.

Remark 4.9. If taking $e=1$ in Theorem 4.8, then $[b,e]=[b,1]$, $I_b^e=I_b$, $I_{e,b}\cup I_e^b\cup(e,1]=\emptyset$ and U^* is a t-norm on $[b,1]$. Moreover, in this case, the conditions in Theorem 4.8 naturally hold.

By the above fact, if taking $e=1$ in Theorem 4.8, then we retrieve the following t-norm $T_2^*:L^2\to L$ constructed by Çaylı ([7], Theorem 1) as follow:

$$T_2^*(x,y)=\begin{cases}T_e(x,y) & \text{if } (x,y)\in[b,1]^2,\\ x & \text{if } (x,y)\in(I_b\cup[0,b))\times\{1\},\\ y & \text{if } (x,y)\in\{1\}\times(I_b\cup[0,b)),\\ 0 & \text{otherwise.}\end{cases}$$

Remark 4.10. If taking $e=b$ in Theorem 4.8, then $[b,1]=[e,1]$, $I_{e,b}=I_e$, $I_b^e\cup I_e^b\cup[b,e)=\emptyset$ and U^* is a t-conorm on $[b,1]$. Moreover, in this case, $U^*\in\mathcal{U}_{\top}^*$ and $x\parallel y$ for all $x\in I_{e,b}$ and $y\in I_e^b$ naturally hold.

By the above fact, if taking $e=b$ in Theorem 4.8, then we retrieve the uninorm $U_{int,1}^0:L^2\to L$ under the conditions $int(x)\wedge int(y)\in I_e$ for all $x,y\in I_e$ or $int(x)\wedge int(y)\in[0,e)$ for all $x,y\in I_e$, constructed by Zhao and Wu ([40], Corollary 4.1) as follow:

$$U_{int,1}^0(x,y)=\begin{cases}S_e(x,y) & \text{if } (x,y)\in[e,1]^2,\\ x & \text{if } (x,y)\in I_e\times[e,1]\cup[e,1]\times[0,e),\\ y & \text{if } (x,y)\in[e,1]\times I_e\cup[0,e)\times[e,1],\\ int(x)\wedge int(y) & \text{if } (x,y)\in I_e\times I_e,\\ 0 & \text{otherwise.}\end{cases}$$

Moreover, if taking $int(x)=x\wedge b$ for $x\in L$ in $U_{int,1}^0$, then we retrieve the following uninorm $U_e^S:L^2\to L$ constructed by Çaylı ([8], Theorem 2.23) as follow:

$$U_e^S(x,y)=\begin{cases}S_e(x,y) & \text{if } (x,y)\in[e,1]^2,\\ x & \text{if } (x,y)\in(L\setminus[e,1])\times[e,1],\\ y & \text{if } (x,y)\in[e,1]\times(L\setminus[e,1]),\\ x\wedge y\wedge e & \text{if } (x,y)\in I_e\times I_e,\\ 0 & \text{otherwise.}\end{cases}$$

Remark 4.11. (1) In Theorem 4.8(1), we observe that the condition $int(x) \wedge int(y) \in I_{e,b}$ for all $x, y \in I_{e,b}$ can not be omitted, in general.

(2) Similarly, in Theorem 4.8(2), the condition $int(x) \wedge int(y) \in I_b^e \cup [0, b)$ for all $x, y \in I_{e,b}$ can not be omitted, in general.

Remark 4.12. Let U_4 be a uninorm in Theorem 4.8.

(1) U_4 is conjunctive, i.e., $U_4(0, 1) = 0$.

(2) If $b = 0$, then $U_4 = U^*$.

(3) U_4 is not idempotent, in general. In fact, if $(0, b) \neq \emptyset$, then there exists x such that $x \in (0, b)$ and then $U_4(x, x) = 0 \neq x$. This is a contradiction with that U_4 be idempotent.

(4) $U_4 \in \mathcal{U}^*_{min}$ if and only if $U^* \in \mathcal{U}^*_{min}$.

5. Conclusions

The new construction methods for uninorms on L using a uninorm defined on a subinterval of L were introduced in [14] and [35]. In this paper, we continue investigating the construction methods based on different tools under some additional constraints.

We give some remarks about the results in this paper as follows.

(1) These methods generalize some construction methods in the literature, such as Theorem 1 in [5], Theorem 1 in [7], Proposition 3.1 and Corollary 4.1 in [40], Theorem 3.1 and Theorem 3.5 in [6], Theorem 2.23 in [8], and also bring some interesting results, such as Propositions 3.5 and 3.13.

(2) Although the additional constraint conditions on the given uninorms are always needed for the construction methods, we try to investigate the additional constraints carefully and systematically and show that some additional constraints are sufficient and necessary.

(3) In [41], Zhang et al. introduced the classes of uninorms U^*_{min} and $\mathcal{U}^*_{max}$. In this paper, we show that whether U belongs to $\mathcal{U}^*_{min}$ ($\mathcal{U}^*_{max}$) depends on whether U^* belongs to $\mathcal{U}^*_{min}$ ($\mathcal{U}^*_{max}$), where U is constructed based on U^*.

The new methods for uninorms on L in this paper provide a novel perspective to study the constructions of uninorms and we believe that they can work well to investigate other operators in the future research.

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