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On the convergence of high-accuracy difference schemes for solving nonstationary second-order equations

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Abstract. In this article, fourth-order accurate difference schemes were proposed and analyzed for systems of second-order ordinary differential equations within the class of non-smooth solutions. Stability conditions and a priori estimates were obtained and theorems on the accuracy of the constructed difference schemes were proven. Additive difference schemes were proposed and the results were applied to the study of multidimensional hyperbolic second-order partial differential equations. Accuracy estimates for spatial and temporal variables were obtained. An algorithm for implementing the method was developed and the scheme was tested. The results of a computational experiment illustrated the effectiveness of the constructed numerical methods for solving hyperbolic equations with non-smooth solutions.

1. Statement of the problem

Mathematical models of many non-stationary processes result in solving second-order ordinary differential equations. For instance, hyperbolic partial differential equations, multidimensional problems in gas dynamics, internal wave theory, electro-magnetoelasticity of piezoelectric and electrically conductive bodies, and geomechanics problems can be reduced to second-order ordinary differential equations when spatial variables are approximated using the finite difference or finite element methods [8, 12, 14, 19]. Numerous approximate methods, such as the finite difference method and the finite element method, have been developed to solve second-order ordinary differential equations. However, these methods typically exhibit low accuracy for smooth solutions of the differential problem. Consequently, it is practical to construct simple, high-order accuracy difference schemes for equations with non-smooth solutions to improve computational efficiency and precision.

Let us consider the abstract Cauchy problem for a nonstationary second-order equation with constant coefficients

$$D\ddot{u} + Au = f, \quad t_0 < t \le T, \tag{1}$$

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$$u(t_0) = u_0, \quad \dot{u}(t_0) = u_1, \tag{2}$$

where A and D are linear operators from $H \to H$, independent of t and constant over time. Here, $A^* = A > 0$ and $D^* = D > 0$, for all $t \geq 0$, $u = u(t) \in H$, and $f = f(t) \in H$. In this notation, $\ddot{u} = d^2u/dt^2$, $\dot{u} = du/dt$, and H denotes a Hilbert space with scalar product (u, ϑ) and norm $\|u\| = \sqrt{(u, u)}$.

In the theory of difference schemes for approximate solutions to problems (1) and (2), three-layer difference schemes of second-order accuracy (in time) or Crank–Nicholson schemes are typically employed [5, 10, 19]. High-order accuracy schemes were developed in [1, 2, 15, 16] based on the finite element method. In [4], estimates for second-order accuracy of the finite difference method for a fourth-order nonlinear Sobolev-type equation were established in the class of smooth solutions. In [3], stable compact difference schemes with fourth-order accuracy and weighting parameters were analyzed in the class of smooth solutions for multidimensional hyperbolic-parabolic equations with constant coefficients. A priori stability and convergence estimates for the difference solution were derived in strong grid norms. In [7], difference schemes for various hyperbolic equations in classes of generalized solutions were constructed and studied. Various estimates of the stability and convergence of difference schemes were obtained. Their dispersion properties were studied and methods for improving the quality of grid solutions were proposed.

2. Construction and study of a difference scheme

Let H be a Hilbert space with a given scalar product (.) and corresponding norm $\|.\|$. We introduce the following operator [11]

$$T^t u = \frac{1}{\tau^2} \int_{t-\tau}^{t+\tau} (\tau - |t - \theta|)\, u(\theta) d\theta. \tag{3}$$

Using Taylor series, we obtain

$$\begin{gathered} u(\theta) = u(t) + (t-\theta)\,\dot{u}(t) + \frac{(t-\theta)^2}{2}\ddot{u}(t) + \ldots + \frac{(t-\theta)^n}{n\,!}u^{(n)}(t) + R_n(\theta), \\ R_n(\theta) = \frac{(t-\theta)^{n+1}}{(n+1)\,!}u^{(n+1)}[\theta + \varepsilon(\theta - t)],\ 0 < \varepsilon < 1. \end{gathered} \tag{4}$$

Substituting (4) into (3) and calculating the resulting integrals we get the following expression

$$T^t u(t) = u(t) + (\tau^2/12)\ddot{u}(t) + O(\tau^4). \tag{5}$$

Then, applying operator T^t to equation (1), we have

$$DT^t\ddot{u} + AT^t u = T^t f. \tag{6}$$

Hence, considering (5) and the properties of operator T^t [11]

$$T^t\ddot{u} = u_{\bar{t}t}, \quad T^t u = u,$$

we obtain the following difference scheme

$$\overline{D}y_{\bar{t}t} + Ay = \varphi, \tag{7}$$

where $y = y^n = y(t_n)$ approximates $u(t)$, $\overline{D} = D + \frac{\tau^2}{12}A$, $y_{\bar{t}t} = (\hat{y} - 2y + \check{y})/\tau^2$, $\hat{y} = y(t_{n+1})$, $\check{y} = y(t_{n-1})$, $\varphi = T^t f$, $y \in H_h$, $\varphi \in H_h$, $t_n \in \omega_\tau$, $\omega_\tau = \{t_n = n\tau,\ n = 0, 1, 2, \ldots\}$, $\tau > 0$ is the grid step, H_h is the grid space with energy norm $\|\vartheta\|_A = \sqrt{(\vartheta, \vartheta)_A}$, $(\vartheta, \vartheta)_A = (A\vartheta, \vartheta)$.

Let us introduce the approximation error $z = y - u$. Then, the substitution of $y = z + u$ into (7) leads to a scheme for the error

$$\overline{D}z_{\bar{t}t} + Az = \psi, \tag{8}$$

where $\psi = \varphi - \overline{D}u_{\bar{t}t} - Au$. Then, considering (6), we obtain

$$\psi = A\left(T^t u - u - \frac{\tau^2}{12}u_{\bar{t}t}\right)$$

or

$$\psi = A\eta, \quad \eta = T^t u - u - \frac{\tau^2}{12}u_{\bar{t}t}.$$

Let us estimate functional η using the Bramble-Hilbert lemma [11]. To do this, we first write it in the following form:

$$\eta(u) = \frac{1}{\tau^2}\int_{t_n-\tau}^{t_n+\tau}(\tau - |t_n - \theta|)u(\theta)d\theta - u(t_n) - \frac{1}{12}\left[u(t_n+\tau) - 2u(t_n) + u(t_n-\tau)\right].$$

The change of variables formula $\frac{\theta - t_n}{\tau} = \zeta, \quad \theta = t_n + \tau\zeta, \quad d\theta = \tau d\zeta$ gives us

$$\eta(u) = \int_0^1 (1 - |\zeta|)u(t_n + \tau\zeta)d\zeta - u(t_n) - \frac{1}{12}\left[u(t_n+\tau) - 2u(t_n) + u(t_n-\tau)\right]. \tag{9}$$

If we introduce function $\overline{u}(\zeta) = u(t_n + \tau\zeta)$, then from (9), we obtain

$$\eta(\overline{u}) = \int_0^1 (1 - |\zeta|)\overline{u}(\zeta)d\zeta - \overline{u}(0) - \frac{1}{12}\left[\overline{u}(1) - 2\overline{u}(0) + \overline{u}(-1)\right].$$

By direct verification, one can see that this functional vanishes on polynomials up to the fourth power in variable ζ. Therefore, $\eta(\overline{u})$ is restricted to continuous functions $\eta(\overline{u}) \in C[0,1]$. Moreover, it is restricted to $\eta(\overline{u}) \in W_2^4[0,1]$. Considering that

$$\left|\eta(\overline{u})\right| = \left|T^t\overline{u} - \overline{u} - \frac{\tau^2}{12}\overline{u}_{\bar{t}t}\right| \le M\sum_{m=0}^{4}\left[\int_0^1\left(\frac{d^m\overline{u}}{d\zeta^m}\right)^2 d\zeta\right]^{1/2}$$

and using the Bramble–Hilbert lemma, we obtain the following estimate:

$$\left|\eta(\overline{u})\right| \le \overline{M}\left[\int_0^1\left(\frac{d^4\overline{u}}{d\zeta^4}\right)^2 d\zeta\right]^{1/2}.$$

Returning to the former variables, one can find the following inequality

$$\left|\eta(u)\right| \le \overline{M}\tau^{7/2}\left[\int_{t_n}^{t_{n+1}}\left(\frac{d^4 u}{dt^4}\right)^2 dt\right]^{1/2}, \quad \forall t \in [t_n, t_{n+1}]$$

or

$$\left\|\eta(u)\right\|^2 \le \overline{M}^2\tau^7\left[\int\limits_{t_n}^{t_{n+1}}\left(\frac{d^4u}{dt^4}\right)^2 dt\right]^{1/2} \le \sum_{n=0}^{m-1}\int\limits_{t_n}^{t_{n+1}}\overline{M}^2\tau^7\left\|\frac{d^4u}{dt^4}(t)\right\|^2 dt' =$$

$$= \overline{M}^2\tau^7\left\|\frac{d^4u}{dt^4}(t)\right\|^2\cdot\tau = \overline{M}^2\tau^8\left\|\frac{d^4u}{dt^4}(t)\right\|^2, \quad \forall t\in[t_n, t_{n+1}].$$

Hence

$$\left\|\eta(u)\right\| \le \overline{M}\tau^4\left\|\frac{d^4u}{dt^4}(t)\right\|,$$

i.e., approximation error of scheme (7) is $\psi = O(\tau^4)$.

To achieve the order of approximation $O(\tau^4)$ of initial conditions, we replace du/dt by y_t and using the Taylor expansion with equation (1) and formulas (3), (6), we obtain the initial conditions for scheme (7)

$$y(0) = u_0, \quad \dot{y}(0) = \overline{u}_1, \tag{10}$$

$$\overline{u}_1 = 0.5\tau D^{-1}\left[\left(E - \frac{\tau^2}{12}A\right)(Au_0 - f^0) - \frac{\tau}{3}\dot{f}^0 - \frac{\tau^2}{12}\ddot{f}^0\right] + \left(E + \frac{\tau^2}{6}D^{-1}A\right)u_1.$$

Then, initial conditions for (8) have the following form:

$$z(0) = 0, \quad \dot{z}(0) = \psi^1. \tag{11}$$

Approximation error of the second initial condition is

$$\left\|\psi^1\right\| = \|z_t(0)\| = \left\|\overline{u}_1 - u_t(0)\right\| = O(\tau^4).$$

Based on the results for the three-layer difference scheme (7), (10) obtained in [10], the following theorem holds.

Theorem 2.1. *Let the operators satisfy the following conditions: $A^* = A > 0$, and $D^* = D > 0$. Assume the stability condition:*

$$\overline{D} > \frac{1+\varepsilon}{4}\tau^2 A, \tag{12}$$

where $\varepsilon > 0$ is any number independent of τ. Then, for the solution of problem (7), (10)*, there is an a priori estimate given by:*

$$\left\|y^{n+1}\right\|_{\overline{D}} \le \sqrt{\frac{1+\varepsilon}{\varepsilon}}\left(\left\|y(0)\right\|_{\overline{D}} + \left\|\overline{D}y_t(0)\right\|_{A^{-1}} + \sum_{s=1}^{n}\tau\left\|\varphi^s\right\|_{A^{-1}}\right). \tag{13}$$

Consequently, based on (13), approximation error of scheme (8) $\psi = O(\tau^4)$, and initial conditions (11), we obtain the following result.

Theorem 2.2. *Let operators satisfy $A^* = A > 0$ and $D^* = D > 0$. Suppose $u(t) \in W_2^4[0,T]$ and $f(t) \in C^2[0,T]$. In addition, assume that condition* (12) *is satisfied. Then the solution of the difference scheme* (7), (10) *converges to the exact solution of the differential problem* (1), (2) *and the following accuracy estimate holds:*

$$\left\|u(t) - y(t)\right\|_{\overline{D}} \le M\tau^4, \quad M > 0. \tag{14}$$

3. Schemes with weights

Based on scheme (7), (10) one can write the following class of difference schemes with weights:

$$\bar{D}y_{\bar{t}t} + Ay^{(\sigma_1,\sigma_2)} = \varphi, \quad t_n \in \omega_\tau, \tag{15}$$

with initial conditions (10), where

$$\bar{D} = D + \frac{\tau^2}{12}A, \quad y^{(\sigma_1,\sigma_2)} = \sigma_1 \hat{y} + (1-\sigma_1-\sigma_2)y + \sigma_2 \overset{\vee}{y}.$$

To study the stability of scheme (15), (10), one can write it to the canonical form of three-layer difference schemes

$$\mathrm{D}y_{\bar{t}t} + \mathrm{B}y_{\overset{\circ}{t}} + \mathrm{A}y = \Phi, \quad t_n \in \omega_\tau. \tag{16}$$

Substituting the following identity into (15)

$$y^{(\sigma_1,\sigma_2)} = y + \tau(\sigma_1-\sigma_2)y_{\overset{\circ}{t}} + \frac{\tau^2}{2}(\sigma_1+\sigma_2)y_{\bar{t}t}, \tag{17}$$

where $y_{\overset{\circ}{t}} = \left(\hat{y} - \overset{\vee}{y}\right)/(2\tau)$, we obtain

$$\left(D + \frac{\tau^2}{12}A + \frac{\tau^2}{2}(\sigma_1+\sigma_2)A\right)y_{\bar{t}t} + \tau(\sigma_1-\sigma_2)Ay_{\overset{\circ}{t}} + Ay = \Phi,$$

i. e.

$$\mathrm{D} = D + \frac{\tau^2}{12}A + \frac{\tau^2}{2}(\sigma_1+\sigma_2)A, \quad \mathrm{B} = \tau(\sigma_1-\sigma_2)A, \quad \mathrm{A} = A, \quad \Phi = \varphi.$$

For $\sigma_1 = \sigma_2 = 0$, we get an explicit difference scheme (7), (10), and for $\sigma_1 = \sigma_2 = \sigma$, we get a symmetric difference scheme.

Since $A^* = A > 0$, $D^* = D > 0$, then $\mathrm{A} = \mathrm{A}^* > 0$, $\mathrm{D} = \mathrm{D}^* > 0$, $\mathrm{B} = \mathrm{B}^* \geq 0$, if $\sigma_1 - \sigma_2 \geq 0$. Consequently, the stability condition for three-layer difference scheme (16) has the following form:

$$\mathrm{B} \geq 0, \ \mathrm{D} \geq \frac{1+\varepsilon}{4}\tau^2 \mathrm{A},$$

which will be fulfilled under the following conditions

$$\sigma_1 \geq \sigma_2, \ \sigma_1 + \sigma_2 \geq \frac{2+3\varepsilon}{6}. \tag{18}$$

4. On the convergence of a scheme with weights

To study the convergence of scheme (15), (10), we obtain a problem for the approximation error. Considering that $z = y - u$ $(y = u + z)$, we obtain the equation for the error

$$\bar{D}z_{\bar{t}t} + Az^{(\sigma_1,\sigma_2)} = \tilde{\psi}, \quad t_n \in \omega_\tau, \ z(0) = 0, \ \dot{z}(0) = \psi^1,$$

where $\tilde{\psi} = \varphi - \bar{D}u_{\bar{t}t} - Au^{(\sigma_1,\sigma_2)}$. Then, considering (6), we obtain

$$\psi = A\left(T^t u - u^{(\sigma_1,\sigma_2)} - \frac{\tau^2}{12}u_{\bar{t}t}\right)$$

or

$$\psi = A\eta, \quad \eta = T^t u - u^{(\sigma_1,\sigma_2)} - \frac{\tau^2}{12} u_{\bar{t}t}.$$

Let us estimate functional η using the Bramble-Hilbert lemma. To do this we express it in the following form:

$$\eta(u) = \frac{1}{\tau^2} \int_{t_n-\tau}^{t_n+\tau} (\tau - |t_n - \theta|) u(\theta) d\theta - u^{(\sigma_1,\sigma_2)}(t_n) - \frac{1}{12} \left[u(t_n + \tau) - 2u(t_n) + u(t_n - \tau) \right],$$

or using (17):

$$\eta(u) = \frac{1}{\tau^2} \int_{t_n-\tau}^{t_n+\tau} (\tau - |t_n - \theta|) u(\theta) d\theta - \frac{1}{2}(\sigma_1 - \sigma_2) \left[u(t_n + \tau) - u(t_n - \tau) \right] -$$

$$-\frac{1}{2}(\sigma_1 + \sigma_2 + \frac{1}{6}) \left[u(t_n + \tau) - 2u(t_n) + u(t_n - \tau) \right].$$

A change of variables, $\frac{\theta - t_n}{\tau} = \zeta, \quad \theta = t_n + \tau\zeta, \quad d\theta = \tau d\zeta$, gives:

$$\eta(u) = \int_0^1 (1 - |\zeta|) u(t_n + \tau\zeta) d\zeta - u(t_n) - \frac{1}{2}(\sigma_1 - \sigma_2) \left[u(t_n + \tau) - u(t_n - \tau) \right] -$$
$$-\frac{1}{2}\left(\sigma_1 + \sigma_2 + \frac{1}{6}\right) \left[u(t_n + \tau) - 2u(t_n) + u(t_n - \tau) \right]. \tag{19}$$

If we introduce the function $\bar{u}(\zeta) = u(t_n + \tau\zeta)$, then from (19), we obtain:

$$\eta(\bar{u}) = \int_0^1 (1 - |\zeta|) \bar{u}(\zeta) d\zeta - \bar{u}(0) - \frac{1}{2}(\sigma_1 - \sigma_2) \left[\bar{u}(1) - \bar{u}(-1) \right] -$$

$$-\frac{1}{2}\left(\sigma_1 + \sigma_2 + \frac{1}{6}\right) \left[\bar{u}(1) - 2\bar{u}(0) + \bar{u}(-1) \right].$$

Hence, if $\sigma_1 = \sigma_2 = 0$, then we obtain functional (9). If, however, $\sigma_1 = \sigma_2 = \sigma$, then we obtain the following functional

$$\eta(\bar{u}) = \int_0^1 (1 - |\zeta|) \bar{u}(\zeta) d\zeta - \bar{u}(0) - \left(\sigma + \frac{1}{12}\right) \left[\bar{u}(1) - 2\bar{u}(0) + \bar{u}(-1) \right]. \tag{20}$$

Analyzing (20), we obtain the $\|\eta(u)\| = O(\tau^4)$, provided that $u(t) \in W_2^4[0, T]$.
Consequently, the following theorem holds.

Theorem 4.1. *Let the operators satisfy $A^* = A > 0$, and $D^* = D > 0$, and let $\sigma_1 = \sigma_2 = \sigma$. In addition, suppose that the solution to problem* (1), (2) *satisfies $u(t) \in W_2^4[0, T]$ and $f(t) \in C^2[0, T]$. Assume that the stability conditions* (18) *hold, i.e., $\sigma \geq \frac{2+3\varepsilon}{12}$. Then the solution of the difference scheme* (15), (10) *converges to the exact solution of the differential problem* (1), (2), *and the following accuracy estimate holds:*

$$\|u(t) - y(t)\|_D \leq M\tau^4, \quad M > 0.$$

5. Additive schemes

To construct additive difference schemes, we will focus on additive representations of stationary operator A in the following form:

$$A = \sum_{\alpha=1}^{p} A_\alpha, \quad A_\alpha = (A_\alpha)^* > 0, \quad \alpha = \overline{1,p}. \tag{21}$$

The transition from one time layer t_n to another t_{n+1} is related to the solution of problems for individual constant operators A_α in additive expansion (21). Thus, the original problem decomposes into p of simpler sub-problems.

Let $\sigma_1 = \sigma_2 = \sigma$ in (15) (a symmetrical scheme):

$$\bar{D} y_{\bar{t}t} + A y^{(\sigma)} = \varphi, \quad t_n \in \omega_\tau, \tag{22}$$

where $y^{(\sigma)} = \sigma \hat{y} + (1 - 2\sigma) y + \sigma \overset{\vee}{y}$. The initial conditions are given in the form of (10).

Based on scheme (22), we construct an additive scheme. To achieve this, we reduce the difference scheme (22) to the canonical form of a three-layer difference schemes:

$$(\bar{D} + \sigma\tau^2 A) y_{\bar{t}t} + A y = \varphi, \quad t_n \in \omega_\tau. \tag{23}$$

Here, the stability condition $\bar{D} + \sigma\tau^2 A \geq [(1+\varepsilon)/4]\ \tau^2 A$ is satisfied for all $\sigma \geq (2 + 3\varepsilon)/12$. The estimate (13) takes the following form:

$$\left\| y^{n+1} \right\|_{\bar{D}+\sigma\tau^2 A} \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}} \left(\left\| y(0) \right\|_{\bar{D}+\sigma\tau^2 A} + \left\| D y_t(0) \right\|_{A^{-1}} + \sum_{s=1}^{n} \tau \left\| \varphi^s \right\|_{A^{-1}} \right).$$

The additive scheme corresponding to (23) is defined as in [13]:

$$y_{\bar{t}t} + R^{-1} \sum_{\alpha=1}^{p} A_\alpha y = R^{-1} \varphi, \quad t_n \in \omega_\tau, \tag{24}$$

where $R = \sum\limits_{\beta=1}^{p} (\bar{D} + \sigma\tau^2 A_\beta)$.

Thus, based on Theorem 4.1, the following result holds.

Theorem 5.1. *Let the solution to problem* (1), (2) *satisfy* $u(t) \in W_2^4[0,T]$, $f(t) \in C^2[0,T]$, *and let the condition* $\sigma \geq 1/4$ $(\varepsilon = 1/3)$ *be satisfied. Then, the solution to the difference problem* (24) *with operators* $R = R^* > 0$, $A^* = A > 0$ *converges to the solution of the differential problem* (1), (2) *with accuracy* $O(\tau^4)$, *i. e., the following accuracy estimate holds:*

$$\left\| u(t) - y(t) \right\|_R \leq M\tau^4, \quad M > 0 - const.$$

The computational implementation of the scheme is based on solving p of locally one-dimensional problems:

$$\frac{y_{n+1}^{\alpha} - 2y_n + y_{n-1}}{p\tau^2} + R^{-1} A_\alpha y^n = \frac{1}{p} R^{-1} \varphi, \quad \alpha = \overline{1,p}, \quad n = 1, 2, ..., \tag{25}$$

$$y^0 = u_0, \quad R y_t(0) = y_1. \tag{26}$$

We define the approximate solution on the new layer as:

$$y^{n+1} = \frac{1}{p} \sum_{\alpha=1}^{p} y_{n+1}^{(\alpha)}, \quad n = 0, 1, \dots .$$

In this interpretation, we have an additive locally averaged one-dimensional difference scheme (25), (26).

6. Example

Let us consider p-dimensional initial-boundary value problem of hyperbolic type in rectangular domain $\Omega = \{x : x = (x_1, x_2, ..., x_p),\ 0 < x_\alpha < l_\alpha,\ \alpha = \overline{1,p}\}$, $\bar{\Omega} = \Omega \cup \Gamma$. One can seek the solution $w(x,t)$ to the following equation:

$$\frac{\partial^2 w}{\partial t^2} + Lw = f(x,t),\ \ x \in \Omega,\ \ 0 < t \le T \tag{27}$$

with boundary and initial conditions:

$$w(x,t) = 0,\ x \in \Gamma = \partial\Omega,\ \ 0 < t \le T, \tag{28}$$

$$w(x,0) = w_0(x),\ \ \frac{\partial w}{\partial t}(x,0) = u_1(x),\ \ \ x \in \Omega. \tag{29}$$

Here the operator L is defined as $Lw = -\sum\limits_{\alpha=1}^{p} \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x_\alpha)\frac{\partial w}{\partial x_\alpha}\right)$, $0 < c_1 \le k_\alpha(x_\alpha) \le c_2$, c_1, c_2 are positive constants.

Problem (27)–(29) is approximated only in the spatial variables x_α, $\alpha = \overline{1,p}$. To do this, we introduce a uniform rectangular grid with steps h_α, $\alpha = \overline{1,p}$ in the parallelepiped Ω. Define the discrete grid as

$$\bar{\omega}_{h_\alpha} = \left\{x :\ \ x = (x_1, x_2, ..., x_p),\ x_\alpha = i_\alpha h_\alpha,\ \ i_\alpha = \overline{1, N_\alpha - 1},\ \ N_\alpha h_\alpha = l_\alpha\right\},\ \ \alpha = \overline{1,p}\,.$$

On the set of grid functions $y \in H_h$ that vanish for all $x \notin \omega$, we define the difference operator

$$Ay \equiv Ly = -\sum_{\alpha=1}^{p} (a_\alpha y_{\bar{x}_\alpha})_{x_\alpha},\ \ \ x_\alpha \in \omega_{h_\alpha}, \tag{30}$$

where

$$a_1(x_1, x_2, ..., x_p) = k_1(x_1 + 0.5h_1, x_2, ..., x_p),$$

$$a_2(x_1, x_2, ..., x_p) = k_2(x_1, x_2 + 0.5h_2, x_3, ..., x_p),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_p(x_1, x_2, ..., x_p) = k_p(x_1, x_2, ..., x_p + 0.5h_p).$$

Here h_p is a step in direction p.

The grid operator A defined in (30) is known to be self-adjoint and positive. Therefore, after spatial discretization, the continuous problem (27)–(29) reduces to the Cauchy problem (1), (2) with operators $D \equiv E$ and A from (30).

Consider scheme (24) for (1), (2) with operators (30). The approximation error of the scheme is $\psi = O(\tau^4 + |h|^2)$, if $u(x,t) \in W_2^4[0,T;\ C^4(\Omega)]$, $f(x,t) \in C^{2,2}_{t\,x}[\bar{Q}_T]$, $|h|^2 = h_1^2 + h_2^2 + ... + h_p^2$. It is possible to increase the approximation error in x using scheme (24) with operators (30) as in [8]. For example, in the case of $p = 2$ (two-dimensional case), operator A is taken as $\bar{A} = A + \frac{h_1^2+h_2^2}{12}A_1A_2$, which approximates operator L with error $O(|h|^4)$. Then scheme (24) has the following form:

$$\mathfrak{I} y_{\bar{t}t} + \mathfrak{R} y = \tilde{\varphi},\ \ \ t_n \in \omega_\tau, \tag{31}$$

where $\mathfrak{I} = \sum\limits_{\beta=1}^{2} (\tilde{D} + \sigma\tau^2\bar{A}_\beta)$, $\mathfrak{R} = \bar{A}$, $\tilde{D} = E + \frac{\tau^2}{12}A + \frac{h_1^2+h_2^2}{12}A_1A_2$, $\tilde{\varphi} = f^n + \frac{\tau^2}{12}f^n_{tt} + \frac{h_1^2}{12}f^n_{x_1x_1} + \frac{h_2^2}{12}f^n_{x_2x_2}$, $f^n_{tt} = \frac{\partial^2 f}{\partial t^2}$, $f^n_{x_\alpha x_\alpha} = \frac{\partial^2 f}{\partial x_\alpha^2}$, $\alpha = 1, 2$.

To increase the order of approximation of the initial conditions, we introduce a fictitious time layer corresponding to the time point $t = -\tau$, $n = -1$. We expand equation (31) to zero time $t = 0$, $n = 0$:

$$\mathfrak{I}\frac{y^1 - 2y^0 + y^{-1}}{\tau^2} + \mathfrak{R}y^0 = \tilde{\varphi}^0$$

and approximate the second initial condition (2) using the central derivative:

$$\mathfrak{I}y_{\overset{\circ}{t}}(0) \equiv \frac{\mathfrak{I}y^1 - \mathfrak{I}y^{-1}}{2\tau} = \left(\mathfrak{I} - \frac{\tau^2}{6}\mathfrak{R}\right)u_1 + \frac{\tau^2}{6}\frac{\partial f^0}{\partial t}.$$

For smooth solutions, both equations have the fourth order of approximation: the first - for equation (1), the second - for the second initial condition (2). Then, eliminating fictitious value y^{-1}, we obtain the following initial condition:

$$\mathfrak{I}y_t(0) = \left(\mathfrak{I} - \frac{\tau^2}{6}\mathfrak{R}\right)u_1 + \frac{\tau}{2}(\tilde{\varphi}^0 - \mathfrak{R}u_0) + \frac{\tau^2}{6}\frac{\partial f^0}{\partial t}. \tag{32}$$

Next, let us study the constructed fourth-order approximation scheme with corresponding initial conditions $y^0 = u_0$ and (32). Its stability condition is determined as:

$$\mathfrak{I} = \mathfrak{I}^* > 0, \quad \mathfrak{R} = \mathfrak{R}^* > 0, \quad \mathfrak{I} \geq \frac{1+\varepsilon}{4}\tau^2\mathfrak{R}, \tag{33}$$

which will be fulfilled subject to $\sigma \geq (2+3\varepsilon)/12$. Consequently, based on (13), if condition (33) is met, there is an a priori estimate [1] to solve scheme (31):

$$\left\|y_{n+1}\right\|_{\mathfrak{I}} \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}}\left(\left\|y(0)\right\|_{\mathfrak{I}} + \left\|\mathfrak{I}y_t(0)\right\|_{\mathfrak{R}^{-1}} + \sum_{s=1}^{n}\tau\left\|\varphi_s\right\|_{\mathfrak{R}^{-1}}\right).$$

Based on this estimate, the following assertion holds.

Theorem 6.1. *Let the solution to problem* (27)–(29) *be* $u(x,t) \in W_2^4[0,T;\ C^6(\Omega)]$, $f(x,t) \in C_{t\,x}^{2,4}[\bar{Q}_T]$ *and condition* $\sigma \geq (2+3\varepsilon)/12$ *be satisfied. Then scheme* (31) *with initial conditions* $y^0 = u_0$, $\mathfrak{I}y_t(0)$ *and operators* $\mathfrak{I} = \mathfrak{I}^* > 0$, $\mathfrak{R} = \mathfrak{R}^* > 0$ *converges to the solution of the original problem* (27)–(29) *with accuracy* $O(\tau^4 + |h|^4)$, *i.e., there is an accuracy estimate*

$$\left\|u(x,t) - y(x,t)\right\|_{\mathfrak{I}} \leq M(\tau^4 + |h|^4)\,, \quad M > 0 - const.$$

Remark 6.2. In Theorem 6.1, the smoothness condition for $u(x,t)$ is spatially overestimated, $C^6(\Omega)$. This condition can be weakened using the operators of exact difference schemes in space up to $W_2^4(\Omega)$ [7]; this issue requires a separate study.

7. Computational experiment

Let us consider the equation of string vibration, which is the simplest second-order hyperbolic type equation and has all the characteristic features of such equations [9, 17]. Therefore, checking the quality of numerical methods using these equations is a necessary condition for applying them to more complex hyperbolic equations.

Consider the following problem:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \ \ u(0,t) = 1, \ \ u(1,t) = 0, \ \ 0 < t \leq T, \ \ u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0,$$

with exact solution $u(x,t) = H(t-x)$, where is the Heaviside function. This test allows us to compare schemes on non-smooth (discontinuous) solutions.

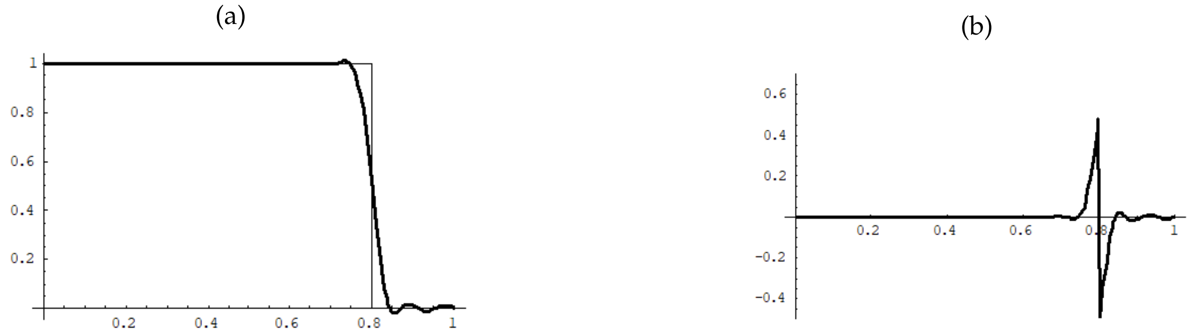

Figure 1: Solution graphs and scheme error: $t = 0.8,\ \tau = 0.025,\ h = 0.03125$

For this problem, consider the fourth-order approximation scheme in time and space (12), (10) with operators $\bar{D} = E + [(\tau^2 + h^2)/12]A,\ A = -y_{\bar{x}x}$. The stability condition depends on parameters $\sigma_1,\ \sigma_2$. If $\sigma_1 = \sigma_2 = \sigma$, then we obtain stability condition $\sigma \geq 1/4$ for $\varepsilon = 1/3$. Figure 1 shows solution graphs: a) $y(x,t)$ solid thick line and $u(x,t)$ solid thin line; b) error $z = y(x,t) - u(x,t)$ of the corresponding test, at time t for the values of the grid parameters h, τ.

To determine the order of convergence in spatial variables and time variables in norm C, the following formulas were used [6]:

$$p_\infty^h = \log_2(\|z(2h,\tau)\|_{L_\infty}/\|z(h,\tau)\|_{L_\infty}),\ p_\infty^\tau = \log_2(\|z(h,2\tau)\|_{L_\infty}/\|z(h,\tau)\|_{L_\infty}).$$

Tables 1 and 2 show the orders of convergence rate in spatial and temporal directions according to an explicit scheme, obtained experimentally. Since the difference solution converges to an exact solution with the fourth order in both variables, to check the rate of convergence, we choose the space step arbitrarily; the time step is determined from the stability condition of the scheme. At that, the deviation of the calculated values from the exact solution in norm C was taken as an error estimate.

Table 1: Convergence rates in spatial direction

Space step	Time step	Error	Order
h=0.01	$\tau = 0.01$	2.69E-08	-
h=0.005	$\tau = 0.01$	6.76E-09	3.992768
h=0.0025	$\tau = 0.01$	1.71E-11	3.9855
h=0.00125	$\tau = 0.01$	4.34E-12	3.974529

Table 2: Convergence rates in temporal direction

Space step	Time step	Error	Order
h=0.01	$\tau = 0.01$	2.69E-08	-
h=0.01	$\tau = 0.005$	6.71E-09	4.003222
h=0.01	$\tau = 0.0025$	1.68E-11	3.997852
h=0.01	$\tau = 0.00125$	4.19E-12	4.003439

8. Conclusions

Difference schemes with weights of fourth-order accuracy were constructed for the abstract Cauchy problem for a second-order equation in the class of generalized solutions. Stability conditions and a priori estimates in energy standards were obtained and, on their basis, accuracy estimates of the constructed schemes were proven. Based on difference schemes with weights, additive difference schemes of the fourth-order accuracy for both variables were proposed. A computational algorithm for locally one-dimensional problems was presented. An example of solving a multidimensional initial-boundary value problem of hyperbolic type in a rectangular domain was given. A theorem was proven on the convergence of the solution of the difference scheme to the solution of the original problem with fourth-order accuracy for all variables. The computational experiments conducted using an explicit scheme illustrated the effectiveness of the constructed numerical methods.

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