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An equivalence relation and groupoid on simplicial morphisms

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Abstract. In this work, we examine the simplicial morphisms homotopies. Firstly, we give the homotopy of free simplicial morphisms and later show that this homotopy relation indicates an equivalence. Then we obtain a groupoid, the simplicial morphisms between two fixed 1-truncated simplicial algebras (with free domain) are the objects, and the homotopies between this simplicial morphisms are the morphisms of this groupoid.

1. Introduction

Let k be a commutative ring. A simplicial (commutative) algebra $\mathcal{A} = (A_n, d_i^n, s_i^n)$ consists of a family of commutative k-algebras $\{A_n\}$ together with face and degeneracy morphisms $d_i^n\colon A_n \to A_{n-1}$, $0 \le i \le n$, $(n \neq 0)$ and $s_i^n\colon A_n \to A_{n+1}$, $0 \le i \le n$ satisfying the simplicial properties which is given in [5, 12]. A simplicial morphism $f\colon \mathcal{A} \to \mathcal{A}'$ is a family of algebra morphisms $f_n\colon A_n \to A'_n$ commuting with the d_i and s_i. Then it can be given as a functor $\mathbf{A}\colon \Delta^{op} \to CommAlg_k$ where Δ is the category of finite ordinal numbers $[n] = \{0 < 1 < \ldots < n\}$ and increasing morphisms. Simplicial (commutative) algebra is involved in algebraic geometry, homological algebra, homotopy theory and algebraic k-theory. In each theory, its internal structures have been relatively little studied. The n-types of simplicial algebras is studied in [7, 8]. Also in [4], Pak S. and Akça İ.İ. worked on the pseudo simplicial groups. Moreover in [1], the higher order Peiffer elements in simplicial Lie algebras are examined. Given a simplicial algebra $\mathcal{A}$, its Moore complex is the chain complex of algebras;

$$M(\mathcal{A}) = \ldots \xrightarrow{d_{(n+1)}} M(A)_n \xrightarrow{d_n} \ldots \xrightarrow{d_3} M(A)_2 \xrightarrow{d_2} M(A)_1 \xrightarrow{d_1} A_0$$

which are defined by $M(A)_n = \bigcap_{i=1}^{n-1} \ker d_i^n \subset A_n$ at level n and the boundary morphisms $d^n\colon M(A)_n \to M(A)_{n-1}$ is the restriction of $d_n^n\colon A_n \to A_{(n-1)}$. If $M(A)_i$ is trivial for $n < i$, it is said that the Moore complex of a simplicial algebra $\mathcal{A}$ has length n. If $n = 2$ then, $M(A)_2 \xrightarrow{d_2} M(A)_1 \xrightarrow{d_1} A_0$ is a 2-crossed module. As a result, this gives an equivalence of categories from the category of simplicial objects with Moore complex of length two and 2-crossed modules of algebras [6, 11, 14].

The homotopy theory of 2-crossed modules of commutative algebras studied in [2]. Then in [3], the concept of a 2-fold homotopy between a pair of 1-fold homotopies connecting 2-crossed module morphisims

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$\mathcal{X} \to \mathcal{X}'$ was defined. Moreover they proved that if the domain 2-crossed module $\mathcal{X}$ is free up to order one, then they obtained a 2-groupoid structure of morphisms of 2-crossed module $\mathcal{X} \to \mathcal{X}'$ and together with their homotopies and 2-fold homotopies. Also, the groupoid structure for modified categories of interest was given in [10].

In [9], it stated that, if $p, q : \mathcal{A} \longrightarrow \mathcal{A}'$ are simplicial morphisms, a homotopy of p onto q is a collection of algebra morphisms $h_i : A_n \to A'_{n+1}$ for each $0 \leq i \leq n$ $(n \geq 0)$, which satisfies the following homotopy identities:

$$\begin{aligned} d_0 h_0 &= p \\ d_{n+1} h_n &= q \\ d_i h_j &= h_{j-1} d_i \quad , j > i \\ d_{j+1} h_{j+1} &= d_{j+1} h_j \\ d_i h_j &= h_j d_{j-1} \quad , j+1 < i \\ s_i h_j &= h_{j+1} s_i \quad , j \geq i \\ s_i h_j &= h_j s_{i-1} \quad , j < i. \end{aligned}$$

Also he briefly mentioned that, the homotopy between free simplicial algebras is an equivalence relation.

In this paper, we clearly show that homotopy relation is an equivalence relation on morphisms between free simplicial algebras. And we prove that this notion of homotopy gives a groupoid with objects that are simplicial morphism between two fixed 1-truncated simplicial algebras (with free domain), the morphisms being the homotopies between these simplicial algebra morphisms.

2. Preliminaries

In this paper, k will be a commutative ring with identity. All of the k-algebras will be commutative.

2.1. Simplicial Algebras

We recommend the reader [5] for additional information on simplicial algebras.

A simplicial algebra is a sequence of algebras $\mathcal{A} = \{A_0, A_1, \ldots, A_n, \ldots\}$ together with morphisms;

$$d_i \colon A_n \to A_{n-1}$$

$$s_i \colon A_n \to A_{n+1}$$

for all $0 \leq i \leq n$. These morphisms are satisfy the following identities:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i, \qquad j > i \\ d_i s_j &= \begin{cases} s_{j-1} d_i, & j > i \\ id, & i = j, j+1 \\ s_j d_{i-1}, & j+1 < i \end{cases} . \\ s_i s_j &= s_j d_{i-1}, \qquad i \leq j \end{aligned}$$

A simplicial morphism $f \colon \mathcal{A} \to \mathcal{A}'$ is a family of morphisms $f_n \colon A_n \to A'_n$ commuting with the morphisms d_i and s_i. Thus we get the category of simplicial algebras, *SimpAlg*.

The Moore complex of a simplicial algebra is the complex:

$$M(\mathcal{A}) = \ldots \xrightarrow{d_{(n+1)}} M(A)_n \xrightarrow{d_n} \ldots \xrightarrow{d_3} M(A)_2 \xrightarrow{d_2} M(A)_1 \xrightarrow{d_1} A_0$$

where

$$\begin{aligned} M(A)_n &= \bigcap_{i \neq n} \ker (d_i : A_n \to A_{n-1}) \\ \partial_n &= d_n \text{ (restricted to } M_n). \end{aligned}$$

The simplicial identities imply that, $\partial_n(M(A)_n)$ is an ideal in $ker(\partial_{n-1})$. The homotopy groups of a simplicial algebra $\mathcal{A}$ are defined as the homology groups of the Moore complex $M(\mathcal{A})$:

$$\Pi_n(A) = ker(\partial_n : M(A)_n \to M(A)_{n-1}) / im(\partial_{n+1} : M(A)_{n+1} \to M(A)_n), \ \ n \geq 0.$$

Moore complex is of length k, if $M(A)_n = 0$ for all $n \geq k+1$. Thus for $r \geq k$, a Moore complex of length k is also of length r.

Let $\mathcal{A}$ be a simplicial algebra. If we forget dimensions of order $> k$, we get a k-truncated simplicial algebra, $tr_k\mathcal{A}$. Thus the category of k-truncated simplicial algebras, $Tr_kSimpAlg$, is obtained.

If $f, g : \mathcal{A} \to \mathcal{A}'$ are two simplicial morphisims, a homotopy $\mathbf{h}$ of f to g is a collection of morphisms:

$$h_i : A_n \to A'_{n+1}\ , \ \ 0 \leq i \leq n$$

for each $n \geq 0$ which satisfies the following properties:

i) $d_0h_0 = f,$
$d_{n+1}h_n = g$

ii) $d_ih_j = h_{j-1}d_i \qquad , j > i$
$d_{j+1}h_{j+1} = d_{j+1}h_j$
$d_ih_j = h_jd_{i-1} \qquad , j+1 < i$

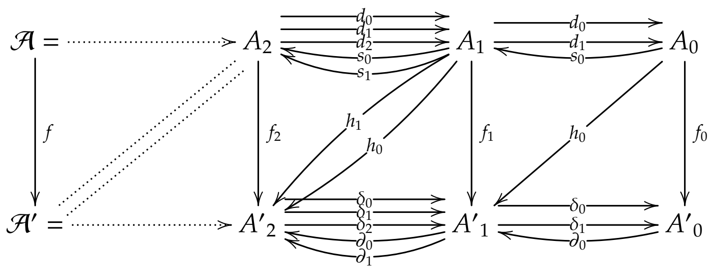
iii) $s_ih_j = h_{j+1}s_i \qquad , j \geq i$
$s_ih_j = h_js_{i-1} \qquad , j < i.$

We denote it by $\mathbf{h} : f \simeq g$. Moreover, we say that the source of $\mathbf{h}$ is f and the target of $\mathbf{h}$ is g, and we denote by $s_i(\mathbf{h}) = f_i$, $t_i(\mathbf{h}) = g_i$.

3. Homotopy in *SimpAlg*

Let $\mathcal{A}$ and $\mathcal{A}'$ be two simplicial algebras, where $\mathcal{A}$ is free and let f, g, u be simplicial morphisms from $\mathcal{A}$ to $\mathcal{A}'$. In this section, we obtain an equivalence relation for homotopies between simplicial morphisms, i.e. we show that $f \simeq f$, if $f \simeq g$ then $g \simeq f$ and if $f \simeq g$ and $g \simeq u$ then $f \simeq u$.

Lemma 3.1. *Let $h^n_i : A_n \to A'_{n+1}$, $(0 \leq i \leq n)$, be the family of morphisms:*



which are given by $h^n_i(x) = \partial^n_i f_n(x)$. Then the collection of h^n_i morphisms $\mathbf{h}$ *is a homotopy of f onto f.*

Proof. *Since for all $0 \leq i \leq n$, ∂^n_i's and f_n's are algbera maps, then $\partial^n_i f_n$'s are all algebra morphisms. For all $a \in A_n$,*

i)

$$\begin{aligned} \delta^{n+1}_0 h^n_0(a) &= \delta^{n+1}_0 \partial^n_0 f_n(a) = f_n(a), \\ \delta^{n+1}_{n+1} h^n_n(a) &= \delta^{n+1}_{n+1} \partial^n_n f_n(a) = f_n(a), \end{aligned}$$

ii) For $j > i$;

$$\begin{aligned}\delta_i^{n+1} h_j^n(a) &= \delta_i^{n+1}\partial_j^n f_n(a)\\ &= \partial_{j-1}^{n-1}\delta_i^n f_n(a)\\ &= \partial_{j-1}^{n-1} f_{n-1} d_i^n(a)\\ &= h_{j-1}^{n-1} d_i^n(a),\end{aligned}$$

iii) For $j+1 < i$;

$$\begin{aligned}\delta_{j+1}^{n+1} h_{j+1}^n(a) &= \delta_{j+1}^{n+1}\partial_{j+1}^n f_n(a)\\ &= \delta_{j+1}^{n+1}\partial_j^n f_n(a)\\ &= \delta_{j+1}^{n+1} h_j^n(a),\end{aligned}$$

$$\begin{aligned}\delta_i^{n+1} h_j^n(a) &= \delta_i^{n+1}\partial_j^n f_n(a)\\ &= \partial_j^{n-1}\delta_{i-1}^n f_n(a)\\ &= \partial_j^{n-1} f_{n-1} d_{i-1}^n(a)\\ &= h_j^{n-1} d_{i-1}^n(a),\end{aligned}$$

iv) For $j \geq i$;

$$\begin{aligned}\partial_i^{n+1} h_j^n(a) &= \partial_i^{n+1}\partial_j^n f_n(a)\\ &= \partial_{j+1}^{n+1}\partial_{i-1}^n f_n(a)\\ &= \partial_{j+1}^{n+1} f_{n+1} s_i^n(a)\\ &= h_{j+1}^{n+1} s_i^n(a),\end{aligned}$$

v) For $j < i$;

$$\begin{aligned}\partial_i^{n+1} h_j^n(a) &= \partial_i^{n+1}\partial_j^n f_n(a)\\ &= \partial_j^{n+1}\partial_{i-1}^n f_n(a)\\ &= \partial_j^{n+1} f_{n+1} s_{i-1}^n(a)\\ &= h_j^{n+1} s_{i-1}^n(a).\end{aligned}$$

That is

$$\mathbf{h} : f \simeq f.$$

Therefore the family of $h_i^n = \partial_i^n f_n$, $(0 \leq i \leq n)$, defines a homotopy of f onto f. □

Let be $\mathbf{h} : f \simeq g$ and $\mathbf{k} : g \simeq u$. We can define the morphisms $h_i^n * k_i^n : A_n \to A'_{n+1}$ as follows,

$$\begin{aligned}h_i^n * k_i^n &= h_i^n + k_i^n - \partial_i^n\left(t_i^n h_i^n\right)\\ &= h_i^n + k_i^n - \partial_i^n g_n\end{aligned}$$

for $0 \leq i \leq n$. This $h_i^n * k_i^n$ morphisms are not algebra morphisms for any simplicial algebras $\mathcal{A}$ and $\mathcal{A}'$.

Definition 3.2. ([13]) Let $\mathcal{A}$ be a simplicial algebra. We can say $\mathcal{A}$ is free, if;

i) A_n is a free algebra with a base set B_n for all $n \geq 0$,

ii) The bases B_n for all $n \geq 0$ are stable under all degeneracy operators. This means that for each $0 \leq i \leq n$, if $a \in A_n$ is a bases element then so is $s_i(a)$.

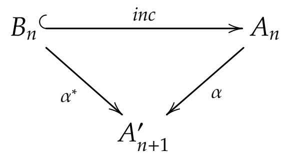
Since A_n is a free algebra for $n \geq 0$, an algebra morphism;

$$\alpha \colon A_n \to A'_{n+1}$$

is identified uniquely by its value on $B_n \subseteq A_n$. Then;

$$\alpha^* \colon B_n \to A'_{n+1}$$

uniquely extends to the algebra morphism α, see the diagram below:

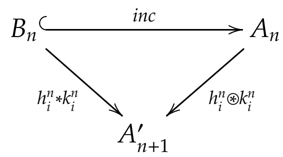


3.1. Concatenation of homotopies

Let $\mathcal{A}$ and $\mathcal{A}'$ be two simplicial algebras, where $\mathcal{A}$ is free simplicial algebra and $f, g, u \colon \mathcal{A} \to \mathcal{A}'$ be simplicial morphisms. Let $\mathbf{h}$ be a simplicial homotopy of f onto g and $\mathbf{k}$ be a simplicial homotopy of g onto u. Then;

$$h_i^n \circledast k_i^n \colon A_n \to A'_{n+1}$$

is an unique algebra morphism with extends the restriction of the $h_i^n * k_i^n$ to B_n:



Lemma 3.3. *For integers $n \geq 0$ and for all $i = 1, 2, \ldots, n$,*

$$w_i^{n(-,-)} \colon A_n \to A'_{n+2}$$

is a k-linear maps such that for each $a, a' \in A_n$,

$$\begin{aligned} w_i^{n(h,k)}(aa') \quad = \quad & \partial_{n+1}^{n+1}(h_i^n(a)k_i^n(a')) - \partial_{n+1}^{n+1}h_i^n(a)\partial_i^{n+1}h_n^n(a') + \partial_{n+1}^{n+1}h_i^n(a)w_i^{n(h,k)}(a') \\ & +\partial_{n+1}^{n+1}(k_i^n(a)h_i^n(a')) - \partial_{n+1}^{n+1}k_i^n(a)\partial_i^{n+1}h_n^n(a') + \partial_{n+1}^{n+1}k_i^n(a)w_i^{n(h,k)}(a') \\ & -\partial_i^{n+1}h_n^n(a)\partial_{n+1}^{n+1}h_i^n(a') - \partial_i^{n+1}h_n^n(a)\partial_{n+1}^{n+1}k_i^n(a') + \partial_i^{n+1}h_n^n(aa') + \partial_i^{n+1}h_n^n(aa') \\ & -\partial_i^{n+1}h_n^n(a)w_i^{n(h,k)}(a') + w_i^{n(h,k)}(a)\partial_{n+1}^{n+1}h_i^n(a') + w_i^{n(h,k)}(a)\partial_{n+1}^{n+1}k_i^n(a') \\ & -w_i^{n(h,k)}(a)\partial_i^{n+1}h_n^n(a') + w_i^{n(h,k)}(a)w_i^{n(h,k)}(a') \end{aligned}$$

and the family of this morphisms satisfy the following identities for $n \geq 0$, $0 \leq i \leq n$;

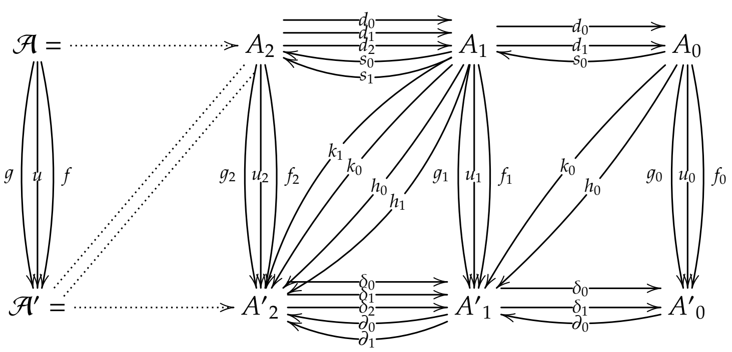
$$
\begin{array}{lll}
i) & \delta_0^{n+2} w_0^{n(h,k)} = 0, & \\
 & \delta_{n+1}^{n+2} w_n^{n(h,k)} = 0, & \\
ii) & \delta_i^{n+2} w_j^{n(h,k)} = w_{j-1}^{(n-1)(h,k)} d_i^n & , j > i \\
 & \delta_{j+1}^{n+2} w_{j+1}^{n(h,k)} = \delta_{j+1}^{n+2} w_j^{n(h,k)}, & \\
 & \delta_i^{n+2} w_j^{n(h,k)} = w_j^{(n-1)(h,k)} d_{i-1}^n & , j+1 < i \\
iii) & \partial_i^{n+2} w_j^{n(h,k)} = w_{j+1}^{(n+1)(h,k)} s_i^n & , j \geq i \\
 & \partial_i^{n+2} w_j^{n(h,k)} = w_j^{(n+1)(h,k)} s_{i-1}^n & , j < i \\
iv) & \begin{array}{l} w_i^{n(\partial f,k)} = 0, \\ w_i^{n(k,\partial g)} = 0. \end{array} & \text{for } k : f \simeq g
\end{array}
$$

Moreover $w_i^{n(h,k)}$ measures the between $h_i^n * k_i^n$ and $h_i^n \circledast k_i^n$ where for $a \in A_n$:

$$
\begin{array}{rcl}
(h_i^n \circledast k_i^n)(a) & = & (h_i^n * k_i^n)(a) + \delta_{n+2}^{n+2} w_i^{n(h,k)}(a) \\
 & = & h_i^n(a) + k_i^n(a) - \partial_i^n g_n(a) + \delta_{n+2}^{n+2} w_i^{n(h,k)}(a)
\end{array}
$$

Remark 3.4. Note that $w_i^{n(h,k)}$ is zero if we choose the element a form the bases $B_n \subseteq A_n$.

Theorem 3.5. *Let f, g and u be simplicial morphisms $\mathcal{A} \to \mathcal{A}'$, $\mathbf{h}$ be simplicial homotopy of f onto g and $\mathbf{k}$ be a simplicial homotopy of g onto u. Then $h \circledast k$ defines a simplicial homotopy of f onto u:*



Proof. Firstly we show that $h_i^n \circledast k_i^n$ is an algebra morphism. Since $h_i^n, k_i^n, \partial_i^n, g_n$ and $w_i^{n(h,k)}$ k-linear morphisms, then $h_i^n \circledast k_i^n$ is a k-linear morphism for all $n \geq 0$ and all $i = 0, 1, \ldots, n$.

For all $a, a' \in A_n$,

$$\begin{array}{rcl}
(h_i^n \circledast k_i^n)(aa') & = & h_i^n(aa') + k_i^n(aa') - \partial_i^n g_n(aa') + \delta_{n+2}^{n+2} w_i^{n(h,k)}(aa') \\
& = & h_i^n(a)h_i^n(a') + k_i^n(a)k_i^n(a') - \partial_i^n g_n(a)\partial_i^n g_n(a') \\
& & +\delta_{n+2}^{n+2}[\partial_{n+1}^{n+1}(h_i^n(a)k_i^n(a')) - \partial_{n+1}^{n+1}h_i^n(a)\partial_{n+1}^{n+1}h_n^n(a') + \partial_{n+1}^{n+1}h_i^n(a)w_i^{n(h,k)}(a') \\
& & +\partial_{n+1}^{n+1}(k_i^n(a)h_i^n(a')) - \partial_{n+1}^{n+1}k_i^n(a)\partial_i^{n+1}h_n^n(a') + \partial_{n+1}^{n+1}k_i^n(a)w_i^{n(h,k)}(a') \\
& & -\partial_i^{n+1}h_n^n(a)\partial_{n+1}^{n+1}h_i^n(a') - \partial_i^{n+1}h_n^n(a)\partial_{n+1}^{n+1}k_i^n(a') + \partial_i^{n+1}h_n^n(aa') + \partial_i^{n+1}h_n^n(aa') \\
& & -\partial_i^{n+1}h_n^n(a)w_i^{n(h,k)}(a') + w_i^{n(h,k)}(a)\partial_{n+1}^{n+1}h_i^n(a') + w_i^{n(h,k)}(a)\partial_{n+1}^{n+1}k_i^n(a') \\
& & -w_i^{n(h,k)}(a)\partial_i^{n+1}h_n^n(a') + w_i^{n(h,k)}(a)w_i^{n(h,k)}(a')] \\
& = & w_i^{n(h,k)}(a)\partial_i^{n+1}h_n^n(a') + w_i^{n(h,k)}(a)w_i^{n(h,k)}(a') \\
& = & [h_i^n(a) + k_i^n(a) - \partial_i^n g_n(a) + \delta_{n+2}^{n+2}w_i^{n(h,k)}(a)][h_i^n(a') + k_i^n(a') - \partial_i^n g_n(a') + \delta_{n+2}^{n+2}w_i^{n(h,k)}(a')] \\
& = & (h_i^n \circledast k_i^n)(a)(h_i^n \circledast k_i^n)(a')
\end{array}$$

Therefore for integer $n \geq 0$ and for all $i = 0, 1, \ldots, n$,

$$h_i^n \circledast k_i^n : A_n \to A'_{n+1}$$

is an algebra morphism.

Now we show that the family of morphisms $h_i^n \circledast k_i^n : A_n \to A'_{n+1}$ satisty the simplicial homotopy identities. For all $a \in A_n$,

i)

$$\begin{array}{rcl}
\delta_0^{n+1}(h_0^n \circledast k_0^n)(a) & = & \delta_0^{n+1}(h_0^n(a) + k_0^n(a) - \partial_0^n g_n(a) + \delta_{n+2}^{n+2}w_0^{n(h,k)}(a)) \\
& = & \delta_0^{n+1}h_0^n(a) + \delta_0^{n+1}k_0^n(a) - \delta_0^{n+1}\partial_0^n g_n(a) + \delta_0^{n+1}\delta_{n+2}^{n+2}w_0^{n(h,k)}(a) \\
& = & f_n(a) + g_n(a) - g_n(a) + \delta_{n+1}^{n+1}\delta_0^{n+2}w_0^{n(h,k)}(a) \\
& = & f_n(a) + \delta_{n+1}^{n+1}(0) \\
& = & f_n(a),
\end{array}$$

$$\begin{array}{rcl}
\delta_{n+1}^{n+1}(h_n^n \circledast k_n^n)(a) & = & \delta_{n+1}^{n+1}(h_n^n(a) + k_n^n(a) - \partial_n^n g_n(a) + \delta_{n+2}^{n+2}w_n^{n(h,k)}(a)) \\
& = & \delta_{n+1}^{n+1}h_n^n(a) + \delta_{n+1}^{n+1}k_n^n(a) - \delta_{n+1}^{n+1}\partial_n^n g_n(a) + \delta_{n+1}^{n+1}\delta_{n+2}^{n+2}w_n^{n(h,k)}(a) \\
& = & g_n(a) + u_n(a) - g_n(a) + \delta_{n+1}^{n+1}\delta_{n+1}^{n+2}w_n^{n(h,k)}(a) \\
& = & u_n(a) - \delta_{n+1}^{n+1}(0) \\
& = & u_n(a),
\end{array}$$

ii) For $j > i$;

$$\begin{array}{rcl}
\delta_i^{n+1}(h_j^n \circledast k_j^n)(a) & = & \delta_i^{n+1}(h_j^n(a) + k_j^n(a) - \partial_j^n g_n(a) + \delta_{n+2}^{n+2}w_j^{n(h,k)}(a)) \\
& = & \delta_i^{n+1}h_j^n(a) + \delta_i^{n+1}k_j^n(a) - \delta_i^{n+1}\partial_j^n g_n(a) + \delta_i^{n+1}\delta_{n+2}^{n+2}w_j^{n(h,k)}(a) \\
& = & h_{j-1}^{n-1}d_i^n(a) + k_{j-1}^{n-1}d_i^n(a) - \partial_{j-1}^{n-1}g_{n-1}d_i^n(a) + \delta_{n+1}^{n+1}\delta_i^{n+2}w_j^{n(h,k)}(a) \\
& = & h_{j-1}^{n-1}d_i^n(a) + k_{j-1}^{n-1}d_i^n(a) - \partial_{j-1}^{n-1}g_{n-1}d_i^n(a) + \delta_{n+1}^{n+1}w_{j-1}^{(n-1)(h,k)}d_i^n(a) \\
& = & [h_{j-1}^{n-1} + k_{j-1}^{n-1} - \partial_{j-1}^{n-1}g_{n-1} + \delta_{n+1}^{n+1}w_{j-1}^{(n-1)(h,k)}]d_i^n(a) \\
& = & (h_{j-1}^{n-1} \circledast k_{j-1}^{n-1})d_i^n(a),
\end{array}$$

$$\begin{array}{rcl}
\delta_{j+1}^{n+1}(h_{j+1}^n \circledast k_{j+1}^n)(a) & = & \delta_{j+1}^{n+1}(h_{j+1}^n(a) + k_{j+1}^n(a) - \partial_{j+1}^n g_n(a) + \delta_{n+2}^{n+2}w_{j+1}^{n(h,k)}(a)) \\
& = & \delta_{j+1}^{n+1}h_{j+1}^n(a) + \delta_{j+1}^{n+1}k_{j+1}^n(a) - \delta_{j+1}^{n+1}\partial_{j+1}^n g_n(a) + \delta_{j+1}^{n+1}\delta_{n+2}^{n+2}w_{j+1}^{n(h,k)}(a) \\
& = & \delta_{j+1}^{n+1}h_j^n(a) + \delta_{j+1}^{n+1}k_j^n(a) - g_n(a) + \delta_{n+1}^{n+1}\delta_{j+1}^{n+2}w_{j+1}^{n(h,k)}(a) \\
& & \delta_{j+1}^{n+1}h_j^n(a) + \delta_{j+1}^{n+1}k_j^n(a) - g_n(a) + \delta_{n+1}^{n+1}\delta_{j+1}^{n+2}w_j^{n(h,k)}(a) \\
& = & \delta_{j+1}^{n+1}[h_j^n(a) + k_j^n(a) - \partial_j^n g_n(a) + \delta_{n+2}^{n+2}w_j^{n(h,k)}(a)] \\
& = & \delta_{j+1}^{n+1}(h_j^n \circledast k_j^n)(a),
\end{array}$$

iii) For $j+1<i$;

$$\begin{aligned}\delta_i^{n+1}(h_j^n \circledast k_j^n)(a) &= \delta_i^{n+1}(h_j^n(a)+k_j^n(a)-\partial_j^n g_n(a)+\delta_{n+2}^{n+2}w_j^{n(h,k)}(a))\\ &= \delta_i^{n+1}h_j^n(a)+\delta_i^{n+1}k_j^n(a)-\delta_i^{n+1}\partial_j^n g_n(a)+\delta_i^{n+1}\delta_{n+2}^{n+2}w_j^{n(h,k)}(a)\\ &= \delta_i^{n+1}h_j^n(a)+\delta_i^{n+1}k_j^n(a)-\partial_j^{n-1}\delta_{i-1}^n g_n(a)+\delta_{n+1}^{n+1}\delta_i^{n+2}w_j^{n(h,k)}(a)\\ &= h_j^{n-1}d_{i-1}^n(a)+k_j^{n-1}d_{i-1}^n(a)-\partial_j^{n-1}g_{n-1}d_{i-1}^n(a)+\delta_{n+1}^{n+1}w_{j-1}^{(n-1)(h,k)}d_{i-1}^n(a)\\ &= [h_j^{n-1}+k_j^{n-1}-\partial_j^{n-1}g_{n-1}+\delta_{n+1}^{n+1}w_{j-1}^{(n-1)(h,k)}]d_{i-1}^n(a)\\ &= (h_j^{n-1}\circledast k_j^{n-1})d_{i-1}^n(a),\end{aligned}$$

iv) For $j \geq i$;

$$\begin{aligned}\partial_i^{n+1}(h_j^n \circledast k_j^n)(a) &= \partial_i^{n+1}(h_j^n(a)+k_j^n(a)-\partial_j^n g_n(a)+\delta_{n+2}^{n+2}w_j^{n(h,k)}(a))\\ &= \partial_i^{n+1}h_j^n(a)+\partial_i^{n+1}k_j^n(a)-\partial_i^{n+1}\partial_j^n g_n(a)+\partial_i^{n+1}\delta_{n+2}^{n+2}w_j^{n(h,k)}(a)\\ &= h_{j+1}^{n+1}s_i^n(a)+k_{j+1}^{n+1}s_i^n(a)-\partial_{j+1}^{n+1}\partial_i^n g_n(a)+\delta_{n+3}^{n+3}\partial_i^{n+2}w_j^{n(h,k)}(a)\\ &= h_{j+1}^{n+1}s_i^n(a)+k_{j+1}^{n+1}s_i^n(a)-\partial_{j+1}^{n-1}g_{n+1}s_i^n(a)+\delta_{n+3}^{n+3}w_j^{(n+1)(h,k)}s_i^n(a)\\ &= [h_{j+1}^{n+1}(a)+k_{j+1}^{n+1}(a)-\partial_{j+1}^{n-1}g_{n+1}(a)+\delta_{n+3}^{n+3}w_j^{(n+1)(h,k)}]s_i^n(a)\\ &= (h_{j+1}^{n+1}\circledast k_{j+1}^{n+1})s_i^n(a),\end{aligned}$$

v) For $j<i$;

$$\begin{aligned}\partial_i^{n+1}(h_j^n \circledast k_j^n)(a) &= \partial_i^{n+1}(h_j^n(a)+k_j^n(a)-\partial_j^n g_n(a)+\delta_{n+2}^{n+2}w_j^{n(h,k)}(a))\\ &= \partial_i^{n+1}h_j^n(a)+\partial_i^{n+1}k_j^n(a)-\partial_i^{n+1}\partial_j^n g_n(a)+\partial_i^{n+1}\delta_{n+2}^{n+2}w_j^{n(h,k)}(a)\\ &= h_j^{n+1}s_{i-1}^n(a)+k_j^{n+1}s_{i-1}^n(a)-\partial_j^{n+1}\partial_{i-1}^n g_n(a)+\delta_{n+3}^{n+3}\partial_i^{n+2}w_j^{n(h,k)}(a)\\ &= h_j^{n+1}s_{i-1}^n(a)+k_j^{n+1}s_{i-1}^n(a)-\partial_j^{n+1}g_{n+1}s_{i-1}^n(a)+\delta_{n+3}^{n+3}w_j^{(n+1)(h,k)}s_{i-1}^n(a)\\ &= [h_j^{n+1}(a)+k_j^{n+1}(a)-\partial_j^{n-1}g_{n+1}(a)+\delta_{n+3}^{n+3}w_j^{(n+1)(h,k)}]s_{i-1}^n(a)\\ &= (h_j^{n+1}\circledast k_j^{n+1})s_{i-1}^n(a).\end{aligned}$$

Therefore $\mathbf{h}\circledast\mathbf{k}$ is a simplicial homotopy of f onto u. That is if $\mathbf{h}: f\simeq g$ and $\mathbf{k}: g\simeq u$ then,

$$\mathbf{h}\circledast\mathbf{k}: f\simeq u.$$

□

In this section, it is shown that $\partial_n^i f_n$ is a homotopy of f onto f for the simplicial morphism $f: \mathcal{A}\to\mathcal{A}'$. Then $\mathbf{h}\circledast\mathbf{k}$ is defined for homotopies $\mathbf{h}: f\simeq g$ and $\mathbf{k}: g\simeq u$. This operation $\mathbf{h}\circledast\mathbf{k}$ is a homotopy of f onto u. Therefore for every $\mathbf{h}$ and $\mathbf{k}$ homotopies such that $t(\mathbf{h})=s(\mathbf{k})$. Now we will show that the simplicial homotopy relation is an equivalence relation with the homotopy $\partial_n^i f_n = f\simeq f$ and the operation $\mathbf{h}\circledast\mathbf{k}$. Then we compose groupoid structure with objects being the simplicial morphisms $f: \mathcal{A}\to\mathcal{A}$, the morphisms being the homotopies between that.

Lemma 3.6. *For $\mathbf{k}: f\simeq g$, $\partial_i^n f_n\circledast k_i^n = k_i^n$ and $k_i^n\circledast\partial_i^n g_n = k_i^n$.*

Proof. Let $\mathbf{k}: f\simeq g$. Then we have,

$$\begin{aligned}(\partial_i^n f_n\circledast k_i^n)(a) &= \partial_i^n f_n(a)+k_i^n(a)-\partial_i^n f_n(a)+\delta_{n+2}^{n+2}w_i^{n(\partial f,k)}(a)\\ &= k_i^n(a) \qquad\qquad \because w_i^{n(\partial f,k)}(a)=0\end{aligned}$$

for all $a\in A_n$. Similarly,

$$\begin{aligned}(k_i^n * \partial_i^n g_n)(a) &= k_i^n(a)+\partial_i^n g_n(a)-\partial_i^n g_n(a)+\delta_{n+2}^{n+2}w_i^{n(k,\partial g)}(a)\\ &= k_i^n(a). \qquad\qquad \because w_i^{n(k,\partial g)}(a)=0\end{aligned}$$

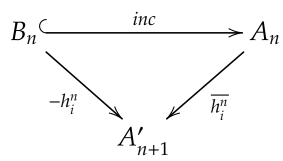
for all $a\in A_n$. □

3.2. Inverse of a simplicial homotopy

Let $\overline{h_i^n}\colon A_n \to A'_{n+1}$ be an unique morphism extending the restriction of the function:

$$\overline{h_i^n} = \partial_i^n f_n + \partial_i^n g_n - h_i^n$$

to B_n see the diagram below:



Lemma 3.7. *We have that $h_i^n \circledast \overline{h_i^n} = \partial_i^n f_n$ and $\overline{h_i^n} \circledast h_i^n = \partial_i^n g_n$. Therefore if $\mathbf{h} : f \simeq g$ then $\overline{\mathbf{h}} : g \simeq f$.*

Theorem 3.8. *Let $\mathcal{A}$ and $\mathcal{A}'$ be two arbitrary but fixed simplicial algebras, where $\mathcal{A}$ is free. The relation between the simplicial morphisms $f, g\colon \mathcal{A} \to \mathcal{A}'$:*

$$''f \simeq g \Leftrightarrow \text{there exists a simplicial homotopy } \mathbf{h} \text{ of } f \text{ onto } g''$$

is an equivalence relation.

Now, we obtain a groupoid structure with this homotopy. It must be shown that the $\circledast$ operation is associative. This is possible if $\mathcal{A}$ and $\mathcal{A}'$ are 1-truncated simplicial algebras.

Lemma 3.9. *Let $\mathcal{A}$ and $\mathcal{A}'$ be two 1-truncated simplicial algebras and $\mathcal{A}$ be free. Let $f, g, u, v\colon \mathcal{A} \to \mathcal{A}'$ be simplicial morphisms and $\mathbf{h} : f \simeq g$, $\mathbf{k} : g \simeq u$ and $\mathbf{l} : u \simeq v$. Then the operation $\circledast$ is associative, i.e.,*

$$(h_i^n \circledast k_i^n) \circledast l_i^n = h_i^n \circledast (k_i^n \circledast l_i^n).$$

Proof. For all $a \in A_n$, we get:

$$\begin{aligned}
[(h_i^n \circledast k_i^n) \circledast l_i^n](a) &= [h_i^n(a) + k_i^n(a) - \partial_i^n g_n(a) + \delta_{n+2}^{n+2} w_i^{n(h,k)}(a)] \\
&\quad + l_i^n(a) - \partial_i^n u_n(a) + \delta_{n+2}^{n+2} w_i^{n(h\circledast l,l)}(a) \\
&= h_i^n(a) + k_i^n(a) + l_i^n(a) - \partial_i^n g_n(a) - \partial_i^n u_n(a) + \delta_{n+2}^{n+2}(w_i^{n(h,k)}(a) + w_i^{n(h\circledast l,l)}(a)).
\end{aligned}$$

where $\mathcal{A}'$ is a 1-truncated simplicial algebra, then

$$\delta_{n+2}^{n+2}(w_i^{n(h,k)}(a) + w_i^{n(h\circledast l,l)}(a)) = 0.$$

Thus we get;

$$[(h_i^n \circledast k_i^n) \circledast l_i^n](a) = h_i^n(a) + k_i^n(a) + l_i^n(a) - \partial_i^n(g_n(a) + u_n(a)).$$

Similarly;

$$\begin{aligned}
[h_i^n \circledast (k_i^n \circledast l_i^n)](a) &= h_i^n(a) + (k_i^n \circledast l_i^n)(a) - \partial_i^n g_n(a) + \delta_{n+2}^{n+2} w_i^{n(h,k\circledast l)}(a) \\
&= h_i^n(a) + (k_i^n(a) + l_i^n(a) - \partial_i^n u_n(a) + \delta_{n+2}^{n+2} w_i^{n(k,l)}(a)) - \partial_i^n g_n(a) + \delta_{n+2}^{n+2} w_i^{n(h,k\circledast l)}(a) \\
&= h_i^n(a) + k_i^n(a) + l_i^n(a) - \partial_i^n(u_n(a) + g_n(a)) + \delta_{n+2}^{n+2}(w_i^{n(k,l)}(a) + w_i^{n(h,k\circledast l)}(a)) \\
&= h_i^n(a) + k_i^n(a) + l_i^n(a) - \partial_i^n(u_n(a) + g_n(a)).
\end{aligned}$$

means:

$$(h_i^n \circledast k_i^n) \circledast l_i^n = h_i^n \circledast (k_i^n \circledast l_i^n).$$

Thus, we get that the $\circledast$ operation is associative. □

We have now proved the main theorem:

Theorem 3.10. *Let $\mathcal{A}$ and $\mathcal{A}'$ be 1-truncated simplicial algebras. Suppose that $\mathcal{A}$ is free 1-truncated simplicial algebra. We have a groupoid, whose objects are the simplicial morphisms $\mathcal{A} \to \mathcal{A}'$, the morphisms being the simplicial homotopies between them. The groupoid operations are the concatenations and inverses of homotopies described in* (3.1) *and* (3.2).

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