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Generalized interval-valued fuzzy metric spaces and their applications

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Abstract. In this paper, we have generalized the interval-valued fuzzy metric $M(x, y, t)$ by allowing it to take the positive interval number instead of ordinary positive real number. In our case, both 't' and the grade value '$M(x, y, t)$' are interval numbers.The underlying topology of this generalized interval-valued fuzzy metric (*GIVF* metric) is studied. Two celebrated fixed point theorems of Banach and Edelstein are extended in this space. Also the problem related to image filter processing is studied.

1. Introduction

Translating the idea of probalistic metric on a nonempty set X, Kramosil and Michalek [8] first introduced a definition of fuzzy metric in 1975. In fact, the statement that the "the probability that the distance between a pair of points x, y is less than t" is replaced by the fuzzy statement "the truth value $M(x, y, t)$ that the distance between a pair of points $x, y \in X$ is less than t". Afterwards systematic investigations in this area rapidly increased. In 1979, starting with the concept of distance between two fuzzy sets, Ercez [2] introduced another form of fuzzy metric. Later in 1982, Deng [1] proposed a fuzzy metric by assigning distance between any two fuzzy elements. In 1984, Kaleva and Seikkala [6] proceeded with a definition of fuzzy metric as a function which, corresponding to every pair of point $x, y \in X$, gives a non-negative fuzzy real number as their distance. In order to induce a Hausdorff topology, George and Veeramani [3] slightly modified the definition of fuzzy metric given by Kramosil and Michalek which extended some results such as 1st countability and Hausdorffness of the underlying topology induced by the fuzzy metric. Also Kočinac [7] investigated Selection properties in fuzzy metric spaces. As a natural generalization of fuzzy metric the concept of intuitionistic metric is developed. In this direction Park [9] first defined Samanta, Vishali fuzzy metric in 2004. Later Sadati et al. [10] defined modified Samanta, Vishali fuzzy metric in 2008. On the other hand, generalizing the concept of fuzzy set, the concept of interval-valued fuzzy sets was introduced by Zadeh [14] in 1975, where each membership value lies in a subinterval of $[0, 1]$ instead of a definite value in $[0, 1]$. Using this concept, in 2012 [12] Yonghong shen et al. introduced the definition of interval-valued fuzzy metric space by generalizing the ordinary fuzzy set $M(x, y, t)$ to an interval-valued fuzzy set. In

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this paper we further generalize this interval-valued fuzzy metric and call it generalized interval-valued fuzzy metric (GIVF metric) by allowing 't' to take positive interval number instead of ordinary positive real number. We study the underlying topology of this GIVF metric, extend the famous fixed point theorems of Banach and Edelstein and finally study image filter processing in this setting. The organisation of the paper is as follows:

Sections 1 and 2 are respectively Introduction and Preliminaries. Section 3, contains the definition of generalized interval-valued fuzzy (GIVF) metric spaces with illustrations. In Subsection 3.1, GIVF-metric topology is studied by defining open ball and the associated topology is found to be Hausforff and 1st countable. In Subsection 3.2 compactness and boundedness of GIVF-metric spaces are observed. Section 4 is dedicated to establish the fixed point theorems of Banach and Edelstein in GIVF-metric spaces. In Section 5, image filtering problem in GIVF metric setting is discussed. Finally in Section 6 conclusion and the future scopes are outlined.

2. Preliminaries

We begin this section by recalling the necessary definitions and conventions.

Definition 2.1. ([11]) A t-norm is a function $*[0,1]\times[0,1]\to[0,1]$ which is commutative, associative, monotonic increasing w.r.t. both the components and $a*1=a, \forall a\in[0,1]$. If in addition, the function $*:[0,1]\times[0,1]\to[0,1]$ is continuous $*$ is named as continuous t-norm.

Definition 2.2. ([5]) A t-conorm is a function $\diamond:[0,1]\times[0,1]\to[0,1]$ which is commutative, associative, monotonic increasing and $a\diamond 0=a$. If in addition, the function $\diamond:[0,1]\times[0,1]\to[0,1]$ is continuous $\diamond$ is named as continuous t-conorm.

Definition 2.3. [5] A t-conorm $\diamond$ is said to be Archimedean if for each $a,b\in(0,1)$ there exists $n\in\mathbb{N}$ such that $a_\diamond^n>b$, where $a_\diamond^n$ denotes $a\diamond a\diamond\cdots\diamond a$ (n times).

Let r^- and r^+ be two real numbers such that $r^-\le r^+$. Then the closed interval $[r^-,\ r^+]$ is denoted by $\tilde{r}$. For any $r\in\mathbb{R}$, the interval $[r,\ r]$ is denoted by $\bar{r}$.

Define $\tilde{p}\le\tilde{q}$ if and only if $p^-\le q^-$ and $p^+\le q^+$;'$\tilde{p}<\tilde{q}$ if and only if $p^-<q^-$, $p^+<q^+$. Also let $[I]=\{\tilde{r}:\bar{0}\le\tilde{r}\le\bar{1}\}$, $(I]=\{\tilde{r}:\bar{0}<\tilde{r}\le\bar{1}\}$ and $(I)=\{\tilde{r}:\bar{0}<\tilde{r}<\bar{1}\}$. A metric Φ can be defined on $\tilde{R}^+$ as $\Phi(\tilde{r},\tilde{s})=|r^--s^-|+|r^+-s^+|$, where $\tilde{R}^+=\{\tilde{r}:\bar{0}<\tilde{r}\}$.
Let $*$ be a t-norm on $[0,1]$. The binary operation $*_I:[I]\times[I]\to[I]$ defined by $\tilde{r}*_I\tilde{s}=[r^-*s^-,\ r^+*s^+]$ satisfies the properties

a. Commutative

b. Associative

c. Monotonic Increasing

d. $\bar{1}*_I\tilde{t}=\tilde{t}$

and is called the induced interval-valued t-norm (induced IV t-norm) on $[I]$.

Similarly, if $\diamond$ is a t-conorm defined on $[0,1]$ then the binary operation $\diamond_I:[I]\times[I]\to[I]$, that is defined by $\tilde{r}\diamond_I\tilde{s}=[r^-\diamond s^-,\ r^+\diamond s^+]$ satisfies the properties

a. Commutative

b. Associative

c. Monotonic Increasing

d. $\tilde{a}\diamond_I\bar{0}=\tilde{a}$,

and is called the induced interval-valued t-conorm (induced IV t-conorm) on $[I]$.
It is to be noted that if $*(\diamond)$ is continuous then the induced $*_I(\diamond_I)$ is also continuous.

Definition 2.4. ([8]) The triplet $(X, M, *)$ is a fuzzy metric space if X is a nonempty set, * is a continuous t-norm and M is a fuzzy set on $(X^2 \times \mathbb{R})$ satisfying for all $x, y, z \in X$ and $t, s \in \mathbb{R}$ the following axioms:
(KM1) $M(x, y, t) = 0, \forall t < 0$;
(KM2) $M(x, y, t) = 1, \forall t > 0 \iff x = y$;
(KM3) $M(x, y, t) = M(y, x, t)$;
(KM4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(KM5) $M(x, y, .) : (0, \infty) \to [0, 1]$ is left continuous and non -decreasing;
(KM6) $\lim_{t\to\infty} M(x, y, t) = 1$.

In [3], George and Veeramani slightly changed some of the above conditions to introduce following definition of a fuzzy metric space whose induced topology is Hausdorff.

Definition 2.5. ([3]) The triplet $(X, M, *)$ is said to be a fuzzy metric space if X is a nonempty set, '*' is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying for all $x, y, z \in X$ and $t, s > 0$ the following axioms:
(GV1) $M(x, y, t) > 0$;
(GV2) $M(x, y, t) = 1 \ \forall t > 0 \iff x = y$;
(GV3) $M(x, y, t) = M(y, x, t)$;
(GV4) $M(x, y, t) * M(y, z, s) \leqslant M(x, z, t + s)$;
(GV5) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous.

As a generalisation of the concept of George and Veeramani, in [12] Yonghong Shen et al. introduced the concept of interval valued fuzzy metric.

Definition 2.6. ([12]) The triplet $(X, M, *_I)$ is said to be an interval-valued fuzzy metric space if X is an arbitrary nonempty set, $*_I$ is a continuous induced IV t-norm on $[I]$ and M is an interval-valued fuzzy set on $X^2 \times (0, \infty)$ satisfying for all $x, y, z \in X$ and $t, s > 0$ the following axioms:

C1 $M(x, y, t) > \overline{0}$;

C2 $M(x, y, t) = \overline{1}$ if and only if $x = y$;

C3 $M(x, y, t) = M(y, x, t)$;

C4 $M(x, y, t) *_I M(y, z, s) \leq M(x, z, t + s)$;

C5 $M(x, y, .) : (0, \infty) \to (I]$ is continuous;

C6 $\lim_{t\to\infty} M(x, y, t) = \overline{1}$.

In the above definition, $M = [M^-, M^+]$ is called an interval-valued fuzzy metric on X. The functions $M^-(x, y, t)$ and $M^+(x, y, t)$ denote the lower nearness degree and upper nearness degree between x and y with respect to t respectively.

3. Generalized interval-valued fuzzy (GIVF) metric spaces

In this section, we introduce a definition of a Generalized interval-valued fuzzy metric space, give examples and study some of its properties.

Definition 3.1. Let $X \neq \varphi$ and let $*_I$ be the continuous induced IV t-norm on $[I]$. A mapping $\mathcal{M} : X \times X \times (\tilde{R}^+) \longrightarrow (I]$ satisfying for all $t, s, r \in X$ and $\tilde{l}, \tilde{m} > \overline{0}$ the following conditions is called a Generalized Interval-Valued Fuzzy Metric (briefly, GIVF-metric) on X:

a $\mathcal{M}(t, s, \tilde{l}) > \overline{0}\ \forall \tilde{l} = [l^-,\ l^+] > \overline{0}$;

b $\mathcal{M}(t, s, \tilde{l}) = \overline{1}\ \forall \tilde{l} > \overline{0}$ if and only if $t = s$;

c $\mathcal{M}(t, s, \tilde{l}) = \mathcal{M}(s, t, \tilde{l})$;

d $\mathcal{M}(t, s, .)$ is monotonically increasing;

e $\mathcal{M}(t, r, \tilde{l}) *_I \mathcal{M}(r, s, \tilde{m}) \leqslant \mathcal{M}(t, s, \tilde{l} + \tilde{m})$;

f $\mathcal{M}(t, s, .) : \tilde{R}^+ \longrightarrow (I]$ is continuous.

g $\lim_{l^- \longrightarrow \infty} \mathcal{M}(t, s, [l^-, l^+]) = \overline{1}$.

Then we call $(X, \mathcal{M}, *_I)$, a GIVF metric space and $\mathcal{M} = [\mathcal{M}^-,\ \mathcal{M}^+]$ is called the GIVF-metric on the set X.

Proposition 3.2. *Let $(X, \mathcal{M}, *_I)$ be a GIVF metric space and let $\mathcal{M}(t, s, \tilde{l}) > \overline{1} - \tilde{m}$ where $\tilde{l} > \overline{0}$ and $\overline{0} < \tilde{m} < \overline{1}$. Then there exists $\overline{0} < \tilde{l}_0 < \tilde{l}$ such that $\mathcal{M}(t, s, \tilde{l}_0) > \overline{1} - \tilde{m}$.*

Proof. It is known that for a given $t, s \in X$, $\mathcal{M}(t, s, .)$ is continuous. Given $\mathcal{M}(t, s, \tilde{l}) > \overline{1} - \tilde{m}$. So $\mathcal{M}^-(t, s, \tilde{l}) > 1 - m^+$ and $\mathcal{M}^+(t, s, \tilde{l}) > 1 - m^-$. Choose $\overline{\epsilon} > \overline{0}$ such that $\epsilon + (1 - m^+) < \mathcal{M}^-(t, s, \tilde{l})$ and $\epsilon + (1 - m^-) < \mathcal{M}^+(t, s, \tilde{l})$. Then $\overline{\epsilon} + (\overline{1} - \tilde{m}) < \mathcal{M}(t, s, \tilde{l})$. For this $\overline{\epsilon} > \overline{0}$, by the continuity of $\mathcal{M}(t, s, .)$, there exists $\tilde{\delta} > \overline{0}$ such that $\Phi(\mathcal{M}(t, s, \tilde{k}),\ \mathcal{M}(t, s, \tilde{l})) < \overline{\epsilon}$ whenever $\Phi(\tilde{k}, \tilde{l}) < \tilde{\delta}$. Choose $\overline{0} < \tilde{l}_0 < \tilde{l}$ such that $\Phi(\tilde{l}_0, \tilde{l}) < \tilde{\delta}$. Then $\Phi(\mathcal{M}(t, s, \tilde{l}), \mathcal{M}(t, s, \tilde{l}_0)) < \overline{\epsilon}$. So, $\mathcal{M}^-(t, s, \tilde{l}) - \mathcal{M}^-(t, s, l_0) < \epsilon$ and $\mathcal{M}^+(t, s, \tilde{l}) - \mathcal{M}^+(t, s, \tilde{l}_0) < \epsilon$ (by the increasing property of $\mathcal{M}(t, s, .)$). So, $\mathcal{M}^-(t, s, \tilde{l}_0) + \epsilon > \mathcal{M}^-(t, s, \tilde{l})$ and $\mathcal{M}^+(t, s, \tilde{l}_0) + \epsilon > \mathcal{M}^+(t, s, \tilde{l})$. Therefore, $\mathcal{M}(t, s, \tilde{l}_0) + \overline{\epsilon} \geq \mathcal{M}(t, s, \tilde{l}) > (\overline{1} - \tilde{m}) + \overline{\epsilon}$. Thus, the result follows. □

In the following examples the induced IV t-norm is given by $\tilde{t} *_I \tilde{s} = [t^-.s^-,\ t^+.s^+]$, where $\tilde{t} = [t^-, t^+]$, $\tilde{s} = [s^-,\ s^+]$.

Example 3.3. Let $\mathcal{M}(p, q, \tilde{l}) = [\mathcal{M}^-(p, q, \tilde{l}),\ \mathcal{M}^+(p, q, \tilde{l})] = [e^{\frac{-|p-q|}{l^-}},\ e^{\frac{-|p-q|}{l^+}}]$.

Proof. It is sufficient to check the condition (e) of Definition 3.1, because others are straightforward. We see that $|x - z| \leq |x - y| + |y - z| \leq (\frac{t^- + s^-}{t^-}|x - y| + \frac{t^- + s^-}{s^-}|y - z|)$. So, $\frac{|x - z|}{(t^- + s^-)} \leq (\frac{|x - y|}{t^-} + \frac{|y - z|}{s^-}) \Longrightarrow e^{-(\frac{|x-y|}{t^-} + \frac{|y-z|}{s^-})} \leq e^{-(\frac{|x-z|}{t^- + s^-})}$. Similarly, we can show that $e^{-(\frac{|x-y|}{t^+} + \frac{|y-z|}{s^+})} \leq e^{-\frac{|x-z|}{t^+ + s^+}}$. So, $\mathcal{M}(p, q, \tilde{l}) *_I \mathcal{M}(q, r, \tilde{m}) \leq \mathcal{M}(p, r, \tilde{l} + \tilde{m})$. □

Example 3.4. $\mathcal{M}(p, q, \tilde{l}) = [\mathcal{M}^-(p, q, \tilde{l}),\ \mathcal{M}^+(p, q, \tilde{l})] = [\frac{k(l^-)^n}{k(l^-)^n + td(p, q)}, \frac{k(l^+)^n}{k(l^+)^n + sd(p, q)}]$ $\forall p, q \in X$, $\tilde{l} > \overline{0}$ and $k, t, n, s \in \mathbb{R}^+$ such that $t \geq s$ and d is a metric on X.

Proof. Note that $\mathcal{M}(p,q,\tilde{l}) *_I \mathcal{M}(q,r,\tilde{m}) =$
$[\frac{k(l^-)^n \times k(m^-)^n}{(k(l^-)^n + td(p,q))(k(m^-)^n + td(q,r))}, \frac{k(l^+)^n \times k(m^+)^n}{(k(l^+)^n + sd(p,q))(k(m^+)^n + sd(q,r))}]$ and
$\mathcal{M}(p,r,\tilde{l}+\tilde{m}) = [\frac{k(l^- + m^-)^n}{k(l^- + m^-)^n + td(p,r)}, \frac{k(l^+ + m^+)^n}{k(l^+ + m^+)^n + sd(p,r)}]$.
Now, $k((l^-)^n(m^-)^n)(k(l^- + m^-)^n + td(p,r)) \leq (l^- + m^-)^n(k^2(l^-)^n(m^-)^n + kt(l^-)^n d(q,r) + kt(m^-)^n d(p,q) + t^2 d(p,q)d(q,r)$ as
$ktd(p,r)(l^-)^n(m^-)^n \leq ktd(p,q)(m^-)^n(l^- + m^-)^n + ktd(q,r)(l^-)^n(l^- + m^-)^n$. Similarly $k(l^+)^n(m^+)^n(k(l^+ + m^+)^n) + sd(p,r) \leq (l^+ + m^+)^n(k^2((l^+)^n(m^+)^n + ks(l^+)^n d(q,r) + ks(m^+)^n d(p,q) + s^2 d(p,q)d(q,r)$. So, $\mathcal{M}(p,q,\tilde{l}) *_I \mathcal{M}(q,r,\tilde{m}) \leq \mathcal{M}(p,r,\tilde{l}+\tilde{m})$. All other properties of being interval-valued fuzzy metric holds trivially. □

3.1. GIVF-metric Topology

Definition 3.5. Let $(X, \mathcal{M}, *_I)$ be a GIVF metric space. Then the open ball centered at $y \in X$ is defined as $\mathcal{B}(y,\tilde{k},\tilde{l}) = \{z \in X : \mathcal{M}(y,z,\tilde{l}) > \overline{1} - \tilde{k}$ where $\tilde{l} > \overline{0}$ and $\tilde{k} \in (I)\}$.

Definition 3.6. Let $(X, \mathcal{M}, *_I)$ be a GIVF metric space and $O \subseteq X$. Then O is said to be open in $(X, \mathcal{M}, *_I)$ if for each $x \in O$ there exists an open ball B centered at x such that $B \subseteq O$.

Theorem 3.7. *Every open ball in $(X, \mathcal{M}, *_I)$ is an open set.*

Proof. Take $x \in X$ and $\tilde{m} \in (I)$ and $\tilde{l} > 0$. Consider the open ball $\mathcal{B}(x,\tilde{m},\tilde{l})$.

Let $y \in \mathcal{B}(x,\tilde{m},\tilde{l})$ and so $\mathcal{M}(x,y,\tilde{l}) > \overline{1} - \tilde{m}$. Then, by Proposition 3.2,there exists $\overline{0} < \tilde{l}_0 < \tilde{l}$ with $\mathcal{M}(x,y,\tilde{l}_0) > \overline{1} - \tilde{m}$. Choose $\tilde{\epsilon} > \overline{0}$ such that $\tilde{l}_0 + \tilde{\epsilon} < \tilde{l}$ and let $\tilde{m}_0 = \mathcal{M}(x,y,\tilde{l}_0)$. Then $\tilde{m}_0 > \overline{1} - \tilde{m}$. Choose $\tilde{s} \in (I)$ such that $\tilde{m}_0 > \overline{1} - \tilde{s} > \overline{1} - \tilde{m}$. For these $\tilde{m}_0$ and $\tilde{s}$ with $\tilde{m}_0 > \overline{1} - \tilde{s}$ there exists $\tilde{m}_1 \in (I)$ such that $\tilde{m}_0 *_I \tilde{m}_1 \geq \overline{1} - \tilde{s}$. Take the open ball $\mathcal{B}(y, \overline{1} - \tilde{m}_1, \tilde{\epsilon})$ and let $z \in \mathcal{B}(y, \overline{1} - \tilde{m}_1, \tilde{\epsilon})$. Then $\mathcal{M}(y,z,\tilde{\epsilon}) > \tilde{m}_1$ and by triangle inequality of $\mathcal{M}$, we have $\mathcal{M}(x,z,\tilde{l}) \geq \mathcal{M}(x,y,\tilde{l}_0) *_I \mathcal{M}(y,z,\tilde{\epsilon}) \geq (\tilde{m}_0 *_I \tilde{m}_1) \geq \overline{1} - \tilde{s} > \tilde{1} - \tilde{m}$. Hence $z \in \mathcal{B}(x,\tilde{m},\tilde{l})$ and so $\mathcal{B}(y, \overline{1} - \tilde{m}_1, \tilde{\epsilon}) \subseteq \mathcal{B}(x,\tilde{m},\tilde{l})$. Therefore $\mathcal{B}(x,\tilde{m},\tilde{l})$ is open. □

Theorem 3.8. *If $(X, \mathcal{M}, *_I)$ is a GIVF-metric space. Define $\tau_M = \{Y \subseteq X : \forall y \in Y$, there exist $\tilde{k} \in (I)$ and $\tilde{t} > \overline{0}$ such that $\mathcal{B}(y,\tilde{k},\tilde{t}) \subseteq Y\}$. Then τ_M is a topology on X induced by the GIVF-metric $\mathcal{M}$.*

Proof. [(*i*)] Clearly $\emptyset, X \in \tau_M$.

[(*ii*)] τ_M is closed under arbitrary unions.

[(*iii*) Let $A_1, A_2 \in \tau_M$ and let $A = A_1 \cap A_2$. Let $p \in A$.

Then there exist $\overline{0} < \tilde{m}_i < \overline{1}$ and $\tilde{l}_i > \overline{0}$ such that $\mathcal{B}(p,\tilde{m}_i,\tilde{l}_i) \subseteq A_i$, $i = 1,2$. Take $\tilde{l}_0 = \tilde{l}_1 \wedge \tilde{l}_2$ and $\tilde{m}_0 = \tilde{m}_1 \wedge \tilde{m}_2$. Then, $\overline{1} - \tilde{m}_0 \geq \tilde{1} - \tilde{m}_1$, $\overline{1} - \tilde{m}_0 \geq \overline{1} - \tilde{m}_2$ and $\tilde{l}_0 \leq \tilde{l}_1$, $\tilde{l}_0 \leq \tilde{l}_2$. So if, $q \in \mathcal{B}(p,\tilde{m}_0,\tilde{l}_0)$ then $\mathcal{M}(p,q,\tilde{l}_1) \geq \mathcal{M}(p,q,\tilde{l}_0) > \tilde{1} - \tilde{m}_0 \geq \tilde{1} - \tilde{m}_1$ and $\mathcal{M}(p,q,\tilde{l}_2) \geq \mathcal{M}(p,q,\tilde{l}_0) > \overline{1} - \tilde{m}_0 \geq \overline{1} - \tilde{m}_2$. Hence $q \in \mathcal{B}(p,\tilde{m}_1,\tilde{l}_1) \cap \mathcal{B}(p,\tilde{m}_2,\tilde{l}_2) \subseteq A_1 \cap A_2$. Thus $\mathcal{B}(p,\tilde{m}_0,\tilde{l}_0) \subseteq A_1 \cap A_2$ and so $A_1 \cap A_2 \in \tau_M$. So, τ_M is a topology on X. □

Remark 3.9. In otherwords the collection of all open sets in a GIVF metric space X is a topology on X.

Remark 3.10. The above discussed topology is found to be first countable as we can define at every $x \in X$, a collection of open sets
$N_x = \{B(x, [\frac{1}{n^2}, \frac{1}{n^2}], [\frac{1}{k^2}, \frac{1}{k^2}]) | \, n,k \in \mathbb{N}\}$ and N_x which serves as a countable neighbourhood basis at x. Hence first countability of $(X, \tau_{\mathcal{M}})$ follows.

Theorem 3.11. *A GIVF-metric space $(X, \mathcal{M}, *_I)$ induces a Hausdorff topology τ_M.*

Proof. Consider a GIVF-metric space $(X, \mathcal{M}, *_I)$ and $x, y \in X$ (x and y are distinct). Then there exists $\tilde{t} > \tilde{0}$ for which $\mathcal{M}(x, y, \tilde{t}) = \tilde{m} \in (I)$. Now, choose $\tilde{m}_0 \in (I)$ such that $\tilde{m} < \tilde{m}_0 < \overline{1}$. Then there exists $\tilde{m}_1 \in (I)$ such that $\tilde{m}_1 *_I \tilde{m}_1 \geq \tilde{m}_0$. Now we consider, $\mathcal{B}(x, \overline{1} - \tilde{m}_1, [\frac{t^-}{2}, \frac{t^+}{2}])$ and $\mathcal{B}(y, \overline{1} - \tilde{m}_1, [\frac{t^-}{2}, \frac{t^+}{2}])$. Suppose that $\mathcal{B}(x, \overline{1} - \tilde{m}_1, [\frac{t^-}{2}, \frac{t^+}{2}]) \cap \mathcal{B}(y, \overline{1} - \tilde{m}_1, [\frac{t^-}{2}, \frac{t^+}{2}]) \neq \emptyset$. Let $z \in \mathcal{B}(x, \overline{1} - \tilde{m}_1, [\frac{t^-}{2}, \frac{t^+}{2}]) \cap \mathcal{B}(y, \overline{1} - \tilde{m}_1, [\frac{t^-}{2}, \frac{t^+}{2}])$. Then $\mathcal{M}(x, z, [\frac{t^-}{2}, \frac{t^+}{2}]) > \tilde{m}_1$ and $\mathcal{M}(y, z, [\frac{t^-}{2}, \frac{t^+}{2}]) > \tilde{m}_1$. So, $\tilde{m} = \mathcal{M}(x, y, \tilde{t}) \geq \mathcal{M}(x, z, [\frac{t^-}{2}, \frac{t^+}{2}]) *_I \mathcal{M}(y, z, [\frac{t^-}{2}, \frac{t^+}{2}]) \geq \tilde{m}_1 *_I \tilde{m}_1 \geq \tilde{m}_0 > \tilde{m}$, which is a contradiction. Hence τ_M is Hausdorff. □

Theorem 3.12. *Let (X, d) be a metric space and $\mathcal{M}(u, v, \tilde{l}) = [\frac{l^-}{l^- + d(u, v)}, \frac{l^+}{l^+ + d(u, v)}], u, v \in X, \tilde{l} > \overline{0}$. Let τ_d and $\tau_{\mathcal{M}}$ be topologies induced by d and $\mathcal{M}$ on X respectively. Then a set O is open in τ_d iff it is open in $\tau_{\mathcal{M}}$.*

Proof. Let $O \in \tau_d$ and $p \in O$. Then there exists $r_0 > 0$ such that $B_d(p, r_0) \subseteq O$. Choose $0 < r < 1$ such that $r < r_0$. Now, consider the open ball $B_{\mathcal{M}}(p, \tilde{m}, \tilde{l})$ in $\tau_{\mathcal{M}}$, where $\tilde{m} \in (I)$ and $\tilde{l} > \overline{0}$ are such that $\tilde{m} = \overline{r}$ and $\tilde{l} = \overline{1 - r}$. If $q \in B_{\mathcal{M}}(p, \overline{r}, \overline{1 - r})$ then $\mathcal{M}(p, q, \overline{1 - r}) > \overline{1 - r} \implies [\frac{1 - r}{(1 - r) + d(p, q)}, \frac{1 - r}{(1 - r) + d(p, q)}] > [1 - r, 1 - r] \implies \frac{1 - r}{(1 - r) + d(p, q)} > 1 - r$. Hence $d(p, q) < r < r_0$ and so $q \in B_d(p, r_0) \implies B_{\mathcal{M}}(p, \overline{r}, \overline{1 - r}) \subseteq B_d(p, r_0) \implies O$ is open in $\tau_{\mathcal{M}}$.
Let $O \in \tau_{\mathcal{M}}$ and let $p \in O$. Then there exist $\tilde{m} \in (I)$ and $\tilde{l} > \overline{0}$ such that $B_{\mathcal{M}}(p, \tilde{m}, \tilde{l}) \subseteq O$. Now $q \in B_{\mathcal{M}}(p, \tilde{m}, \tilde{l}) \iff [\frac{l^-}{l^- + d(p, q)}, \frac{l^+}{l^+ + d(p, q)}] > [1 - m^+, 1 - m^-] \iff \frac{l^-}{l^- + d(p, q)} > 1 - m^+$, and $\frac{l^+}{l^+ + d(p, q)} > 1 - m^- \iff d(p, q) < \frac{l^- m^+}{1 - m^+}$ and $d(p, q) < \frac{l^+ m^-}{1 - m^-}$. Let $r_d = \min\{\frac{l^- m^+}{1 - m^+}, \frac{l^+ m^-}{1 - m^-}\}$. Take an open ball $B_d(p, r_d)$. Then $B_d(p, r_d) \subseteq B_{\mathcal{M}}(p, \tilde{m}, \tilde{l}) \subseteq O \implies O$ is open in τ_d. Therefore, we conclude that τ_d and $\tau_{\mathcal{M}}$ are the same. □

3.2. Compactness and boundedness

Definition 3.13. (I) Let $(X, \mathcal{M}, *_I)$ be a GIVF metric space. Then a subset C of X is said to be compact in $(X, \mathcal{M}, *_I)$ if for any open cover $\{C_i\}_{i \in \Delta}$ of C, $\exists$ a finite subset δ of Δ such that $C \subseteq \bigcup_{i \in \delta} C_i$.

(II) Given $(X, \mathcal{M}, *_I)$ a GIVF-metric space, we call $Y \subseteq X$ as GIVF-bounded if there exist $\tilde{l} > \overline{0}$ and $\tilde{m} \in (I)$ such that, $\mathcal{M}(p, q, \tilde{l}) > \overline{1} - \tilde{m}\ \forall p, q \in Y$.

Definition 3.14. In a GIVF-metric space $(X, \mathcal{M}, *_I)$ a sequence $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n \to \infty} \mathcal{M}(x, x_n, \tilde{l}) = \overline{1}$, for all $\tilde{l} > \overline{0}$.

Theorem 3.15. *Let $(X, \mathcal{M}, *_I)$ be a compact GIVF Metric space. Then every sequence $\{x_n\}$ has a convergent subsequence.*

Proof. Let $\{x_n\}$ be a sequence in $(X, \mathcal{M}, *_I)$. If possible, let $\{x_n\}$ have no convergent subsequence. Then for each $x \in X$ there exist $\tilde{\alpha}(x) > \overline{0}$ and $\overline{0} < \tilde{k}(x) < \overline{1}$ such that $B(x, \tilde{k}(x), \tilde{\alpha(x)})$ contains only finitely many terms of the sequence $\{x_n\}$.
Now, $\{B(x, \tilde{k}(x), \tilde{\alpha}(x)) : x \in X\}$ is an open cover of X. As $(X, \mathcal{M}, *_I)$ is compact so, there exists a finite subset F of X such that $\{B(x, \tilde{k}(x), \tilde{\alpha}(x)) : x \in F\}$ covers X. Then X contains only finitely many terms of $\{x_n\}$, a contradiction. Hence $\{x_n\}$ has a convergent subsequence. □

Theorem 3.16. *Given a GIVF-metric space $(X, \mathcal{M}, *_I)$ induced by metric d, as in Example 3.4, then $Y \subseteq X$ is GIVF-bounded iff Y is bounded in (X, d).*

Proof. Let Y be GIVF-bounded in $(X, \mathcal{M}, *_I)$. Then there exist $\tilde{m} \in (I)$ and $\tilde{l} > \overline{0}$, such that $\mathcal{M}(p, q, \tilde{l}) > \overline{1} - \tilde{m},\ \forall p, q \in Y$.
Then $[\frac{k(l^-)^n}{k(l^-)^n + td(p,q)}, \frac{k(l^+)^n}{k(l^+)^n + sd(p,q)}] > [1 - m^+, 1 - m^-]$.
i.e. $\frac{k(l^-)^n}{k(l^-)^n + td(p,q)} > 1 - m^+$ and $\frac{k(l^+)^n}{k(l^+)^n + sd(p,q)} > 1 - m^- \implies d(p,q) < \frac{k(l^-)^n m^+}{t(1-m^+)}$ and $d(p,q) < \frac{k(l^+)^n m^-}{s(1-m^-)} \implies$ $d(p,q) < K_0,\ \forall\ p, q \in Y$ where $K_0 = \min(\frac{k(l^-)^n m^+}{t(1-m^+)}, \frac{k(l^+)^n m^-}{s(1-m^-)}) > 0$
$\implies$ Y is a bounded set in (X, d). Conversely, suppose that Y is bounded in (X, d). Then there exists $K_0 > 0$ such that, $d(p,q) < K_0\ \forall p, q \in Y$. Let $\bar{l} > \overline{0}$ where $\tilde{l} = [l^-, l^+]$. Then, $\frac{k(l^-)^n}{k(l^-)^n + td(p,q)} > \frac{k(l^-)^n}{k(l^-)^n + tK_0} = 1 - \frac{tK_0}{k(l^-)^n + tK_0}$ where, $0 < \frac{tK_0}{tK_0 + k(l^-)^n} < 1$ and $\frac{k(l^+)^n}{k(l^+)^n + sd(p,q)} > 1 - \frac{sK_0}{k(l^+)^n + sK_0}$, where $0 < \frac{sK_0}{sK_0 + k(l^+)^n} \leq \frac{tK_0}{tK_0 + k(l^-)^n} < 1$.
Choose $\tilde{m} = [\frac{sK_0}{k(l^+)^n + sK_0}, \frac{tK_0}{k(l^-)^n + tK_0}] \in (I)$. Then $\mathcal{M}(p, q, \tilde{l}) > \tilde{1} - \tilde{m},\ \forall p, q \in Y \implies Y$ is GIVF-bounded in $(X, \mathcal{M}, *_I)$. □

Theorem 3.17. *If $A \subseteq X$ and A is compact in $(X, \mathcal{M}, *_I)$ then A is also GIVF-bounded in the space $(X, \mathcal{M}, *_I)$.*

Proof. Given A is compact. Fix some $\tilde{m} \in (I)$ and $\tilde{l} > \overline{0}$ and consider $\{B_{\mathcal{M}}(c, \tilde{m}, \tilde{l}) : c \in A\}$. Clearly $A \subseteq \bigcup_{c \in A} B_{\mathcal{M}}(c, \tilde{m}, \tilde{l})$. By the compactness of A, then there exists a finite subset $\{B_{\mathcal{M}}(p_1, \tilde{m}, \tilde{l}), B_{\mathcal{M}}(p_2, \tilde{m}, \tilde{l}), \cdots, B_{\mathcal{M}}(p_n, \tilde{m}, \tilde{l})\} \subseteq \{B_{\mathcal{M}}(c, \tilde{m}, \tilde{l}) : c \in A\}$ such that, $A \subseteq \bigcup_{i=1}^{n} B_M(p_i, \tilde{m}, \tilde{l})$. Let $p, q \in A$ and so $p \in B_{\mathcal{M}}(p_i, \tilde{m}, \tilde{l})$ and $q \in B_{\mathcal{M}}(p_j, \tilde{m}, \tilde{l})$, for some $1 \leq i, j \leq n$.
Let $\overline{\delta} = [\min_{1 \leq i,j \leq n} \mathcal{M}^-(p_i, p_j, \tilde{l}), \min_{1 \leq i,j \leq n} \mathcal{M}^+(p_i, p_j, \tilde{l})]$. Now, $\mathcal{M}(p, q, [3l^-, 3l^+]) \geq \mathcal{M}(p, p_i, \tilde{l}) *_I \mathcal{M}(p_j, p_i, \tilde{l}) *_I \mathcal{M}(p_j, q, \tilde{l}) \geq (\overline{1} - \tilde{m}) *_I \delta *_I (\overline{1} - \tilde{m}), \forall p, q \in A$. Choose $\tilde{k} \in (I)$ such that $(\overline{1} - \tilde{m}) *_I \delta *_I (\overline{1} - \tilde{m}) > \overline{1} - \tilde{k}$. Then $\mathcal{M}(p, q, [3l^-, 3l^+]) > \tilde{1} - \tilde{k} \forall p, q \in A \implies A$ is GIVF-bounded. □

Theorem 3.18. *A compact set in a GIVF metric space is closed.*

Proof. As every compact set in Hausdorff space is closed and the topology generated by GIVF-metric is Hausdorff so the result follows immediately. □

4. Banach contraction theorem, Edelstein theorem in GIVF metric spaces

Definition 4.1. (Archimedean induced IV t-conorm) An induced IV t-conorm $\diamond_I$ is called Archimedean if for each $\tilde{a}, \tilde{b} \in (I)$ $\exists n$ so that $\tilde{a}^n_{\diamond_I} > \tilde{b}$.

Proposition 4.2. *If $\diamond_1$ and $\diamond_2$ are two Archimedean t-conorms on $[0, 1]$ then the induced IV t-conorm $\diamond_I$ is also Archimedean.*

Proof. Suppose $\tilde{a}, \tilde{b} \in (I)$. Then there exist $m, n \in \mathbb{N}$ such that $(a^-_{\diamond_1})^n > b^-$ and $(a^+_{\diamond_2})^m > b^+$ (as $\diamond_1$, $\diamond_2$ both are Archimedean t-conorms). Let $\max\{n, m\} = M$. Then $(a^-_{\diamond_1})^M > b^-$ and $(a^+_{\diamond_2})^M > b^+$ which implies $\tilde{a}^M_{\diamond_I} > \tilde{b}$. Hence $\Diamond_I$ is Archimedean. □

Proposition 4.3. *For an Archimedean induced IV t-conorm $\Diamond_I$ $\lim_n \tilde{a}^n_{\Diamond_I} = \overline{1}$ where $\tilde{a} \in (I)$.*

Definition 4.4. In a GIVF-metric space $(X, \mathcal{M}, *_I)$ a sequence $\{x_n\}$ is said to be a cauchy sequence if $\lim_{n,m \to \infty} \mathcal{M}(x_n, x_m, \tilde{l}) = \overline{1}$, for all $\tilde{l} > \overline{0}$.

Definition 4.5. A GIVF-metric space $(X, \mathcal{M}, *_I)$ is complete if every cauchy sequence in it is convergent.

Definition 4.6. A mapping $T : (X, \mathcal{M}) \to (X, \mathcal{M})$ is named as $\tilde{k}$-$\diamond_I$-contraction if there exists $\tilde{k} \in (I)$ and a continuous t-conorm $\diamond_I$ satisfying $\forall\, x, y \in X,\ \forall \tilde{t} > \overline{0}$:
$M(T(x), T(y), \tilde{t}) \geq \tilde{k} \diamond_I M(x, y, \tilde{t})$.

Theorem 4.7. *Let $(X, M, *_I)$ be a complete GIVF metric space and let T be a $\tilde{k}$-$\diamond_I$ contraction on X. If $\diamond_I$ is Archimedean, then T has a unique fixed point.*

Proof. Take $x \in X$. Define the sequence $\{x_n\}$ as follows. For $n \geq 2$, define $x_n = T(x_{n-1})$ and $x_1 = T(x)$. Assumption provides that there exists $\tilde{k} \in (I)$ satisfying $\forall\, x, y \in X, \forall\, \tilde{t} > \tilde{0}$ such that $M(Tx, Ty, \tilde{t}) \geq \tilde{k} \diamond_I M(x, y, \tilde{t})$. By Mathematical induction, we will prove that for each $\tilde{t} > \overline{0}$, $M(x_{n+1}, x_n, \tilde{t}) \geq \tilde{k}^n_{\diamond_I}$ for each $n \in \mathbb{N}$ ....(1)
$M(x_2, x_1, \tilde{t}) = M(Tx_1, Tx, \tilde{t}) \geq \tilde{k} \diamond_I M(x_1, x, \tilde{t}) \geq \tilde{k}_{\diamond_I}$. Let us assume that (1) is true for $n = m$ i.e. $M(x_{m+1}, x_m, \tilde{t}) \geq \tilde{k}^m_{\diamond_I}$. Then $M(x_{m+2}, x_{m+1}, \tilde{t}) = M(T(x_{m+1}), T(x_m), \tilde{t}) \geq \tilde{k} \diamond_I M(x_{m+1}, x_m, \tilde{t}) \geq \tilde{k} \diamond_I \tilde{k}^m_{\diamond_I} = \tilde{k}^{m+1}_{\diamond_I}$. So, (1) is true for $n = m + 1$. Hence as $M(x_{n+1}, x_n, \tilde{t}) \geq k^n_{\diamond_I}$ for each $n \in \mathbb{N}$ and each $\tilde{t} > \overline{0}$. So $\wedge_{\tilde{t}>\overline{0}} M(x_{n+1}, x_n, \tilde{t}) \geq k^n_{\diamond_I}$ for each $n \in \mathbb{N}$
$\implies \lim_{n\to\infty} \wedge_{\tilde{t}>\overline{0}} M(x_{n+1}, x_n, \tilde{t}) \geq \lim_{n\to\infty} k^n_{\diamond_I} = \overline{1}$.

Now, we assume that $\{x_n\}$ is not Cauchy. Then $\exists\, \tilde{\epsilon} \in (I)$ and $\tilde{t} > \overline{0}$ such that for each $n \in \mathbb{N}$, $\exists\, m(n) > l(n) + 1 \geq n + 1$ such that $M(x_{m(n)}, x_{l(n)}, \tilde{t}) \not\geq \overline{1} - \tilde{\epsilon}$. Under this assumption, we construct two subsequences $\{x_{m(n)}\}$ and $\{x_{l(n)}\}$ as follows.

Let $n = 1$ $l_1 = l(1)$ and let m_1 be the smallest positive integer greater than l_1 satisfying $M(x_{m_1}, x_{l_1}, \tilde{t}) \not\geq \overline{1} - \tilde{\epsilon}$ and $M(x_{m_1-1}, x_{l_1}, \tilde{t}) \geq \overline{1} - \tilde{\epsilon}$. The subsequent elements of both subsequences are picked recursively as follows. For each $n \in \mathbb{N}$, first take $l_n = l(n)$ and choose $m_n = m(n) > l_n$ be such that $M(x_{m_n}, x_{l_n}, \tilde{t}) \not\geq \overline{1} - \tilde{\epsilon}$ but $M(x_{m_n-1}, x_{l_n}, \tilde{t}) \geq \overline{1} - \tilde{\epsilon}$.

Then for each n and each $\overline{0} < \overline{s} < \tilde{t}$, we have $\overline{1} - \tilde{\epsilon} \not\leq M(x_{m_n}, x_{l_n}, \tilde{t}) \geq (M(x_{m_n}, x_{m_n-1}, \overline{s})) *_I M(x_{m_n-1}, x_{l_n}, \tilde{t} - \overline{s}) \geq \wedge_{\tilde{t}>\overline{0}} M(x_{m_n}, x_{m_n-1}, \tilde{t}) *_I M(x_{m_n-1}, x_{l_n}, \tilde{t} - \overline{s}), \forall \overline{0} < \overline{s} < \tilde{t}$. By the continuity of $M(x, y,)$ for each $x, y \in X$ and for each $n \in \mathbb{N}$, we have by letting $\overline{s} \to \overline{0}$ that $M(x_{m_n}, x_{l_n}, \tilde{t}) \geq (\wedge_{t>0} M(x_{m_n}, x_{m_n-1}, \tilde{t})) *_I M(x_{m_n-1}, x_{l_n}, \tilde{t}) \geq \wedge_{t>0} M(x_{m_n}, x_{m_n-1}, \tilde{t}) *_I (\overline{1} - \tilde{\epsilon})$.

It follows that, $\limsup_{n\to\infty}(M(x_{m_n}, x_{l_n}, \tilde{t}) \geq \limsup_{n\to\infty}(\wedge_{\tilde{t}>\overline{0}} M(x_{m_n}, x_{m_n-1}, \tilde{t})) *_I (\overline{1} - \tilde{\epsilon}) \geq \lim_{n\to\infty} k^{m_n-1}_{\diamond_I} *_I (\overline{1} - \tilde{\epsilon}) = \overline{1} *_I (\overline{1} - \tilde{\epsilon}) = \overline{1} - \tilde{\epsilon}$.

Now, for $\overline{0} < \overline{s} < \tilde{t}$, $M(x_{m_n}, x_{l_n}, \tilde{t}) \geq M(x_{m_n}, x_{m_n+1}, \frac{\overline{s}}{2}) *_I M(x_{m_n+1}, x_{l_n+1}, \tilde{t} - \overline{s}) *_I M(x_{l_n+1}, x_{l_n}, \frac{\overline{s}}{2})$
$\geq M(x_{m_n}, x_{m_n+1}, \frac{\overline{s}}{2}) *_I (\tilde{k} \diamond_I M(x_{m_n}, x_{l_n}, \tilde{t} - \overline{s}) *_I M(x_{l_n+1}, x_{l_n}, \frac{\overline{s}}{2}))$
$\geq \wedge_{\tilde{t}>\overline{0}} M(x_{m_n}, x_{m_n+1}, \tilde{t}) *_I (\tilde{k} \diamond_I M(x_{m_n}, x_{l_n}, \tilde{t} - \overline{s})) *_I \wedge_{\tilde{t}>\overline{0}} M(x_{l_n+1}, x_{l_n}, \tilde{t})$. Taking limit as n tends to ∞, s tends to 0, the continuity of $*$ and $\diamond$ and continuity of $\mathcal{M}(x_{m_n}, x_{n_l}, .)$ ensure
$\limsup_{n\to\infty} M(x_{m_n}, x_{l_n}, \tilde{t}) \geq k \diamond_I \limsup_{n\to\infty} M(x_{m_n}, x_{l_n}, \tilde{t})$. There may be two cases.
Case-1 $\limsup_{n\to\infty} M(x_{m_n}, x_{l_n}, \tilde{t}) \neq \overline{1}$.
Then from above, we have $\limsup_{n\to\infty} M(x_{m_n}, x_{l_n}, \tilde{t}) > \limsup_{n\to\infty} M(x_{m_n}, x_{l_n}, \tilde{t})$, a contradiction.
Case-2 $\limsup_{n\to\infty} M(x_{m_n}, x_{l_n}, \tilde{t}) = \overline{1}$.
Then there exists a subsequence $M(x_{m_{n_i}}, x_{l_{n_i}}, \tilde{t}) > \overline{1} - \tilde{\epsilon}$ for sufficiently large values of i, which is a contradiction.
Therefore in any case we arrive at a contradiction. Thus $\{x_n\}$ is a Cauchy sequence and since $(X, M, *)$ is complete, there exists $x \in X$ such that $\{x_n\}$ converges to x, i.e. $\lim_n M(x_n, x, \tilde{t}) = \overline{1}$ for each $\tilde{t} > \overline{0}$.

Now, for a fixed $\tilde{t} > \overline{0}$ and for each $n \in \mathbb{N}$, we have that $M(x, T(x), \tilde{t}) \geq M(x, x_n, \frac{\tilde{t}}{2}) *_I M(x_n, T(x), \frac{\tilde{t}}{2}) \geq M(x, x_n, \frac{\tilde{t}}{2}) *_I (k \diamond_I M(x_{n-1}, x, \frac{\tilde{t}}{2}))$.

Taking the limit and using continuity of $*$ and $\diamond$, we have, $M(x, T(x), \tilde{t}) \geq \lim_{n\to\infty} M(x, x_n, \frac{t}{2}) *$ $(k \diamond_I \lim_{n\to\infty} M(x_{n-1}, x, \frac{t}{2}) = \overline{1} * (k \diamond_I \overline{1}) = \overline{1}$. As $\tilde{t} > \overline{0}$ is arbitrary, we conclude that $M(x, Tx, \tilde{t}) = \overline{1}$, for each $\tilde{t} > \overline{0}$, which implies that $T(x) = x$. Now we will prove the uniqueness of the fixed point.
Suppose that $T(y) = y$. Then $M(x, y, \tilde{t}) = M(T(x), T(y), \tilde{t}) \geq k \diamond_I M(x, y, \tilde{t})$. Since $\diamond_I$ is Archimedean, we deduce that $M(x, y, \tilde{t}) = \overline{1}$, for each $\tilde{t} > 0$, which implies that $x = y$. □

Lemma 4.8. *If* $\lim \tilde{x_n} = \tilde{x}$ *and* $\lim \tilde{y_n} = \tilde{y}$, *then* $M(x, y, \tilde{t} - \overline{\epsilon}) \leq \liminf M(x_n, y_n, \tilde{t})$ *and* $M(x, y, \tilde{t} + \overline{\epsilon}) \geq \limsup M(x_n, y_n, \tilde{t})$ $\forall t > \overline{0}$ *and* $\overline{0} < \overline{\epsilon} < \tilde{t}$.

Proof. $M(x_n, y_n, \tilde{t}) \geq M(x_n, x, \frac{1}{2}\overline{\epsilon}) * M(x, y, \tilde{t} - \overline{\epsilon}) * M(y, y_n, \frac{1}{2}\overline{\epsilon})$.
So, $\liminf M(x_n, y_n, \tilde{t}) \geq \overline{1} * M(x, y, \tilde{t} - \overline{\epsilon}) * \overline{1} = M(x, y, \tilde{t} - \bar{\epsilon})$. Also, $M(x, y, \tilde{t} + \overline{\epsilon}) \geq M(x, x_n, \frac{1}{2}\overline{\epsilon}) *_I M(x_n, y_n, \tilde{t}) *_I M(y_n, y, \frac{1}{2}\overline{\epsilon})$.
Hence, $M(x, y, \tilde{t} + \overline{\epsilon}) \geq \limsup M(x_n, y_n, \tilde{t})$. □

Remark 4.9. If $M(x, y, .)$ is continuous then $x_n \to x$ and $y_n \to y \Longrightarrow M(x_n, y_n, \tilde{t}) \to M(x, y, \tilde{t})$.

Theorem 4.10. *Let* (X, M) *be a GIVF metric space and* $T : X \to X$ *be a mapping satisfying* $M(Tx, Ty, .) > M(x, y, .)$ *for* $x \neq y \in X$. *If for some* $x \in X$, *the sequence of iterates* $T^n(x)$ *has a convergent subsequence* $T^{n_i}(x)$ *converging to* $\eta \in X$. *Then* η *is the unique fixed point of* T *and* $T^n(x)$ *converges to* η.

Proof. Suppose $x \in X$ and $x_n = T^n(x)$, $n \in \mathbb{N}$. Clearly $x_n \neq x_{n+1}$, for if $x_n = x_{n+1}$. Then $x_n = T(x_n)$, so that x_n is a fixed point of T. If possible let T have two fixed points x and y. Then by the given condition $M(Tx, Ty, .) > M(x, y, .) \Longrightarrow M(x, y, .) > M(x, y, .)$ which is a contradiction.
Also, for $m \neq n$ $x_n \neq x_m$. Otherwise if $x_n = x_m$ for $m > n$ (say), then $M(x_n, x_{n+1}, .) = M(x_m, x_{m+1}, .) > M(x_{m-1}, x_m, .) > M(x_{m-2}, x_{m-1}, .) > \cdots > \cdots M(x_n, x_{n+1}, .)$, which is a contradiction. Thus for $m \neq n$, $x_n \neq x_m$.
Let $\{x_n\}$ have a convergent subsequence $\{x_{n_i}\}$ converging to η. Without loss of generality we can assume that, $x_{n_i} \neq \eta$ for all $i \in \mathbb{N}$. Then $M(T(x_{n_i}), T(\eta), .) > M(x_{n_i}, \eta, .)$, $\forall i$
$\Longrightarrow \limsup_i M(T(x_{n_i}), T(\eta), \tilde{t}) \geq \limsup_i M(x_{n_i}, \eta, \tilde{t}) = M(\eta, \eta, \tilde{t}) = \overline{1} \Longrightarrow \limsup_i M(T(x_{n_i}), T(\eta), \tilde{t}) = \overline{1}$.
Similarly we can show that $\liminf_i M(T(x_{n_i}), T(\eta), \tilde{t}) = \overline{1}$. Thus $\limsup_i M(Tx_{n_i}, T\eta, \tilde{t}) = \liminf_i M(Tx_{n_i}, T\eta, \tilde{t}) = \overline{1}$. Therefore, $\lim T(x_{n_i}) = T(\eta)$.
If for some i, $T(x_{n_i}) = T(\eta) \Longrightarrow T^2(x_{n_i}) = T^2(\eta)$. Now for those i for which $T(x_{n_i}) \neq T(\eta)$, we have $M(T^2(x_{n_i}), T^2(\eta), \tilde{t}) > M(Tx_{n_i}, T(\eta), \tilde{t})$. Then $\liminf M(T^2(x_{n_i}), T^2(\eta), \tilde{t}) = \overline{1} \Longrightarrow \limsup M(T^2(x_{n_i}), T^2\eta, \tilde{t}) = \overline{1}$.
So $\lim M(T^2(x_{n_i}), T^2(\eta), \tilde{t}) = \overline{1}$. Therefore, $\lim T^2(x_{n_i}) = T^2(\eta)$.
$M(x_{n_1}, Tx_{n_1}, \tilde{t}) < M(Tx_{n_1}, T^2x_{n_1}, \tilde{t}) < M(Tx_{n_1+1}, T^2x_{n_1+1}, \tilde{t})$
$< \cdots < M(x_{n_2}, Tx_{n_2}, \tilde{t}) < M(Tx_{n_2}, T^2x_{n_2}, \tilde{t}) = M(x_{n_2+1}, Tx_{n_2+1}, \tilde{t})$
$< \cdots < M(x_{n_i}, Tx_{n_i}, \tilde{t}) < M(Tx_{n_i}, T^2x_{n_i}, \tilde{t}) < M(x_{n_i+1}, Tx_{n_i+1}, \tilde{t})$
$< M(Tx_{n_i+1}, T^2x_{n_i+1}, \tilde{t}) \leq \overline{1}$ $\forall \tilde{t} > \tilde{0}$. (If $x_r = Tx_r$, then x_r is a fixed point.)
Now, $\{M(x_{n_i}, Tx_{n_i}, \tilde{t})\}$ and $\{M(Tx_{n_i}, T^2x_{n_i}, \tilde{t})\}$ are having the same limit.
Now $M(\eta, T\eta, \tilde{t}) = \lim M(x_{n_i}, Tx_{n_i}, \tilde{t}) = \lim M(Tx_{n_i}, T^2x_{n_i}, \tilde{t}) = M(T\eta, T^2\eta, \tilde{t})$.
If $\eta = T\eta$ then the theorem has been proved otherwise $M(T\eta, T^2\eta, \tilde{t}) > M(\eta, T\eta, \tilde{t})$ and therefore we arrive at a contradiction. Thus T has a unique fixed point. Now for $n \in \mathbb{N}, n = n_i + r$, so that $M(T^n(x), \eta, \tilde{t}) = M(T^{n_i+r}(x), T^r(\eta), \tilde{t}) > M(T^{n_i}(x), \eta, \tilde{t})$. As $i \to \infty, M(T^{n_i}(x), \eta, \tilde{t}) \to \overline{1}$ Hence $M(T^n(x), \eta, \tilde{t}) \to \overline{1}$ as $n \to \infty$, Hence $T^n(x)$ converges to η. □

Corollary 4.11. *Let $(X, \mathcal{M})$ be a compact GIVF metric space and suppose $T : X \to X$ be a mapping satisfying $M(Tx, Ty, .) > M(x, y, .)$ for $x \neq y \in X$. Then T has a unique fixed point.*

Proof. The result follows from Theorem 3.15 and Theorem 4.10. □

5. Image filtering using GIVF metrics

Before we discuss the main problem let us take some examples of GIVF metrics to be used in this filtering process.

Example 5.1. Let $X \neq \phi$ and $*_I$ be a continuous induced IV t-norm given by $\tilde{a} *_I \tilde{b} = [a^-.b^-, a^+.b^+]$, where $\tilde{a} = [a^-, a^+]$ and $\tilde{b} = [b^-, b^+]$.
Let $(X, M_1, *_I)$ and $(X, M_2, *_I)$ be two GIVF metric spaces. Let $M(x, y, \tilde{t}) = M_1(x, y, \tilde{t}) *_1 M_2(x, y, \tilde{t})$.
Then $(X, M, *_I)$ is a GIVF metric space. Clearly

(1) $M(x, y, \tilde{t}) > \overline{0}, \forall \tilde{t} > \overline{0}, \forall x, y \in X$;

(2) $M(x, y, \tilde{t}) = M(y, x, \tilde{t}), \forall \tilde{t} > \overline{0}, \forall x, y \in X$

(3) $M(x, y, \tilde{t}) = \overline{1}, \forall \tilde{t} > \overline{0}$ iff $x = y$.

(4) For $x, y, z \in X$ and $\tilde{t} > \overline{0}, \tilde{s} > \overline{0}$,
$M(x, y, \tilde{t}) *_1 M(y, z, \tilde{s}) = (M_1(x, y, \tilde{t}) *_1 M_2(x, y, \tilde{t})) *_1 (M_1(y, z, \tilde{s}) *_1 M_2(y, z, \tilde{s}))$
$= (M_1(x, y, \tilde{t}) *_1 M_1(y, z, \tilde{s})) *_1 (M_2(x, y, \tilde{t}) *_1 M_2(y, z, \tilde{s}))$
$\leq M_1(x, z, \tilde{t} + \tilde{s}) *_1 M_2(x, z, \tilde{t} + \tilde{s})$, since M_1 and M_2 are GIVF metrics on X.
$= M(x, z, \tilde{t} + \tilde{s})$.

(5) Clearly continuity of $M(x, y, .)$ follows from the continuity of $M_1(x, y, .)$ and $M_2(x, y, .)$.

(6) $\lim_{t^- \to \infty} M(x, y, \tilde{t}) = \lim_{t^- \to \infty}(M_1(x, y, \tilde{t}) *_1 M_2(x, y, \tilde{t})$
$= \lim_{t^- \to \infty}[M_1^-(x, y, \tilde{t})M_2^-(x, y, \tilde{t}), M_2^+(x, y, \tilde{t}).M_2^+(x, y, \tilde{t})]$
$= [1.1, 1.1] = \overline{1}$.
Hence $(X, M, *_I)$ is a GIVF metric space.

Example 5.2. Let $X = \mathbb{R}$ and $*_I$ be the continuous induced IV t-norm given by $[a, b] *_1 [c, d] = [a.c, b.d]$.
Let for $x, y \in X$ and $\tilde{t} > \overline{0}$,
$M(x, y, \tilde{t}) = [\frac{min\{x, y\} + kt^-}{max\{x, y\} + kt^-}, \frac{min\{x, y\} + kt^+}{max\{x, y\} + kt^+}]$, where $k > 0$
The function $f(t) = \frac{y + t}{x + t}$ is increasing if $x - y > 0$. So
$\frac{min\{x, y\} + kt^-}{max\{x, y\} + kt^-} \leq \frac{min\{x, y\} + kt^+}{max\{x, y\} + kt^+}$.
Clearly

(1) $M(x, y, \tilde{t}) > \overline{0}, \forall t > \overline{0}, x, y \in X$.

(2) $M(x, y, \tilde{t}) = \overline{1} \forall \tilde{t} > \overline{0}$ iff $x = y$.

(3) $M(x, y, \tilde{t}) = M(y, x, \tilde{t}), \forall \tilde{t} > \overline{0}, \forall x, y \in X$.

(4) For the triangle inequality we have to show that
$M(x, z, \tilde{t} + \tilde{s}) \geq M(x, y, \tilde{t}) *_I M(y, z, \tilde{s})$.

i.e.,$[\frac{\min\{x,z\}+k(t^-+s^-)}{\max\{x,z\}+k(t^-+s^-)}, \frac{\min\{x,z\}+k(t^++s^+)}{\max\{x,z\}+k(t^++s^+)}]$
$\geq [\frac{\min\{x,y\}+kt^-}{\max\{x,y\}+kt^-}, \frac{\min\{x,y\}+kt^+}{\max\{x,y\}+kt^+}] * [\frac{\min\{y,z\}+ks^-}{\max\{y,z\}+ks^-}, \frac{\min\{y,z\}+ks^+}{\max\{y,z\}+ks^+}]$
$= [\frac{(\min\{x,y\}+kt^-)(\min\{y,z\}+ks^-)}{(\max\{x,y\}+kt^-)(\max\{y,z\}+ks^-)}, \frac{(\min\{x,y\}+kt^+)(\min\{y,z\}+ks^+)}{(\max\{x,y\}+kt^+)(\max\{y,z\}+ks^+)}]$.
Firstly we show that
$\frac{min\{x,z\}+k(t^-+s^-)}{max\{x,z\}+(t^-+s^-)} \geq \frac{(min\{x,y\}+kt^-)(min\{y,z\}+ks^-)}{(max\{x,y\}+kt^-)(max\{y,z\}+ks^-)}$.................(*)
Let $x \leq z$. Then we have the following cases:
Case 1: $x \leq y \leq z$; Case 2: $y \leq x \leq z$; Case 3: $x \leq z \leq y$.
Case 1: $x \leq y \leq z$
Then (*) becomes
$\frac{x+kt^-}{y+kt^-}.\frac{y+ks^-}{z+ks^-} \leq \frac{x+k(t^-+s^-)}{z+k(t^-+s^-)}$
Now R.H.S.
$\frac{x+k(t^-+s^-)}{z+k(t^-+s^-)} = \frac{x+k(t^-+s^-)}{y+k(t^-+s^-)}.\frac{y+k(t^-+s^-)}{z+k(t^-+s^-)}$
$\geq \frac{x+kt^-}{y+kt^-}.\frac{y+ks^-}{z+ks^-}(\because x \leq y \leq z)$
= L. H. S.
Case 2: $y \leq x \leq z$.
Then (*) becomes
$\frac{y+kt^-}{x+kt^-}.\frac{y+ks^-}{z+ks^-} \leq \frac{x+k(t^-+s^-)}{z+k(t^-+s^-)}$.
Now R. H. S.
$= \frac{x+k(t^-+s^-)}{z+k(t^-+s^-)} = \frac{x+k(t^-+s^-)}{x+k(t^-+s^-)}.\frac{x+k(t^-+s^-)}{z+k(t^-+s^-)}$
$\geq \frac{y+k(t^-+s^-)}{x+k(t^-+s^-)}.\frac{y+k(t^-+s^-)}{z+k(t^-+s^-)}(\because y \leq x \leq z)$
$\geq \frac{y+kt^-}{x+kt^-}.\frac{y+ks^-}{z+ks^-}(\because y \leq x \leq z)$
= L. H. S.
Similarly Case 3 can be dealt with. Replacing t^- by t^+ and s^- by s^+ we can prove the other relevant inequalities. Hence triangle inequality holds.

(5) As $\tilde{t_n} \to \tilde{t}$ iff $t_n^- \to t^-$ and $t_n^+ \to t^+$, continuity of $M(x, y, .)$ can be easily verified.

(6) $\lim_{t^-\to\infty} M(x,y,t) = [\lim_{t^-\to\infty} \frac{\min\{x,y\}+kt^-}{\max\{x,y\}+kt^-}, \lim_{t^-\to\infty} \frac{\min\{x,y\}+kt^+}{\max\{x,y\}+kt^+}]$
$= \bar{1}$.

Hence $(X, M, *_I)$ is a GIVF metric space.

Scheme of image processing

For every pixel P select a 3×3 window with $P(R, G, B, x, y)$, where R, G, B are the color values of the pixel and (x, y) are the Euclidean coordinates of the pixel. Let F_i, F_j be two pixels with $F_i = (F_i^1, F_i^2, F_i^3, F_i^4, F_i^5)$ and $F_j = (F_j^1, F_j^2, F_j^3, F_j^4, F_j^5)$.
Consider the GIVF metrics R and S defined by
$R(F_i, F_j, \tilde{t}) = \prod_{l=1}^{3}[\frac{\min\{F_i^l, F_j^l\}+kt^-}{\max\{F_i^l, F_j^l\}+kt^-}, \frac{\min\{F_i^l, F_j^l\}+kt^+}{\max\{F_i^l, F_j^l\}+kt^+}]$

(Here $\prod$ denotes IV-product t-norm $*_I$)
and $S(F_i, F_j, \tilde{t}) = [\frac{t^-}{t^- + d(F_i, F_j)}, \frac{t^+}{t^+ + d(F_i, F_j)}]$,
where $d(F_i, F_j)$ denotes the Euclidean distance of the (x, y) coordinates of F_i, F_j. We fix a suitable value of k and calculate
$C(F_i, F_j, \tilde{t}) = R(F_i, F_j, \tilde{t}) *_I S(F_i, F_j, \tilde{t})$.
For each value of F_k in the filter window, an accumulated measure $A_k = \sum_{j \in W, j \neq k} C(F_k, F_j, t)$ to all other vectors in the window is to be calculated. The interval values of $A_k s$ are to be approximated by taking mid values of the intervals and then ordered in the descending sequence. Then the filter output will be the vector F_0 corresponding to the lowest rank in the ordered sequence A_k.

6. Conclusion

In this paper a definition of generalized interval-valued fuzzy metric space is introduced. Some of its topological properties such as Hausdorffness, first countability etc. are studied. Banach and Edelstein fixed point theorems are extended in this setting. Image filtering process using this GIVF metric is also studied. There is a further scope of studying boundedness, total boundedness, completeness and compactness, Baire's category theorem etc.in this setting. There is also a scope for applying this GIVF metric in decision making problems.

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