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Separation and connectedness in the category of constant filter convergence spaces

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Abstract. The purpose of this work is to introduce two notions of closure in the category **ConFCO** of constant filter convergence spaces with continuous maps and investigate whether they satisfy the idempotency, productivity, (weak) hereditariness, and (full) additiveness as well as examine how they are related to each other. Moreover, we characterize each of T_i, $i = 1, 2$ spaces with respect to these closures and examine epimorphisms in the subcategories of **ConFCO**. Furthermore, we give the characterization of connected constant filter convergence spaces and investigate some invariance properties of them. Finally, we compare our results with results in some other topological categories.

1. Introduction

Since the category **Top** of topological spaces and continuous maps has no natural function space structures, it is not convenient in topological algebra, homotopy theory, and functional analysis etc. Consequently, there have been many attempts to replace **Top** by supercategories which have the desired properties.

The categories **FCO** of filter convergence spaces (resp. **Lim** of limit spaces) and continuous maps which are supercategories of **Top** have natural function space structures [21, 22, 26]. Limit spaces with compatible vector space structures are used in an important way to develop a calculus for vector spaces, without norm [21]. In 1979, Schwarz [26] introduced the full subcategory **ConFCO** of **FCO** which is bireflective and have natural function space structures.

Closure operators have been used to generalize separation and connectedness as well as they are used to characterize the epimorphisms of subcategories of a topological category [6, 15–19, 23, 25].

The main objectives of this paper are stated as follows:

(1) We introduce two notions of closures in **ConFCO** and show whether they satisfy the idempotency, productivity, (weak) hereditariness, and (full) additiveness.

(2) We characterize T_i, $i = 1, 2$ constant filter convergence spaces with respect to these closure operators.

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(3) We characterize the epimorphism in the subcategories of **ConFCO**.

(4) We give the characterization of connected constant filter convergence spaces and investigate invariance properties of it.

(5) We compare our findings with ones in other topological categories.

2. Preliminaries

Let $B \neq \emptyset$ and $F(B)$ be the set of filters on B. $\alpha \in F(B)$ is said to be improper (resp. proper) iff $\emptyset \in \alpha$ (resp. $\emptyset \notin \alpha$). We denote by $\alpha \cup \beta$ the smallest filter containing both β and α for $\alpha, \beta \in F(B)$ and $[U] = \{V \subset B : U \subset V\}$.

If the map $L : B \to P(F(B))$ satisfies,

(i) $[\{x\}] = [x] \in L(x)$ for $\forall x \in B$,

(ii) if $\alpha \in L(x)$ and $\alpha \subset \beta$, then $\beta \in L(x)$,

then (B, L) is called a filter convergence space [22]. If L is a constant function, then (B, L) is called a constant filter convergence space [26]. Let (B, L) and (C, S) be constant filter convergence spaces. A map $f : (B, L) \to (C, S)$ is said to be continuous if $f(\alpha) \in S$ for $\forall \alpha \in L$.

We denote **ConFCO** by the category of constant filter convergence spaces and continuous maps, which is a cartesian closed [26].

Proposition 2.1. (1) *Let $\{(B_i, L_i), i \in I\}$ be a class of constant filter convergence spaces, B be a nonempty set, and $\{f_i : B \to B_i, i \in I\}$ be a source in the category* **Set** *of all sets and functions. A source $\{f_i : (B, L) \to (B_i, L_i), i \in I\}$ in* **ConFCO** *is an initial lift iff $\alpha \in L$ precisely when $f_i(\alpha) \in L_i$ for $\forall i \in I$.*

(2) *Let $\{f_i : B_i \to B, i \in I\}$ be a sink in* **Set**. *An epi sink $\{f_i : (B_i, L_i) \to (B, L), i \in I\}$ in* **ConFCO** *is a final lift iff $\alpha \in L$ implies that there exist $i \in I$ and $\beta_i \in L_i$ with $f_i(\beta_i) \subset \alpha$.*

Let B be a set, $x \in B$, and the infinity wedge $\bigvee_x^\infty B$ (resp. $B^2 \bigvee_\Delta B^2$) be taking countably many disjoint copies of B and identifying them at the point x (resp. two distinct copies of B^2 identified along the diagonal Δ) [2].

The principal axis map $A : B^2 \vee_\Delta B^2 \to B^3$ is given by $A(a, b)_1 = (a, b, a)$ and $A(a, b)_2 = (a, a, b)$ and the Skewed axis map $S : B^2 \vee_\Delta B^2 \to B^3$ is given by $S(a, b)_1 = (a, b, b)$ and $S(a, b)_2 = (a, a, b)$. The fold map $\nabla : B^2 \bigvee_\Delta B^2 \to B^2$ is given by $\nabla((a, b)_i) = (a, b)$ for $i = 1, 2$.

$A_x^\infty : \bigvee_x^\infty B \to B^\infty$ is given by $A_x^\infty(a_i) = (x, ..., x, a, x, x, ...)$, where a_i is the i-th component of $\bigvee_x^\infty B$ and $B^\infty = B \times B \times ...$ is the countable cartesian product of B, and $\nabla_x^\infty : \bigvee_x^\infty B \longrightarrow B$ is given by $\nabla_x^\infty(a_i) = a$ for all $i \in I$, where I is the index set $\{i : a_i$ is the i-th component of $\bigvee_x^\infty B\}$ [2].

Definition 2.2. ([2, 8, 9]) Let $U : \mathcal{E} \to$ **Set** be topological [1], $A \in Ob(\mathcal{E})$ with $x \in U(A) = B$, and $Z \subset A$.

(1) If the initial lift of the U-source $\{A_x^\infty : \vee_x^\infty B \to B^\infty = U(A^\infty)$ and $\nabla_x^\infty : \vee_x^\infty B \to UD(B) = B\}$ is discrete, then $\{x\}$ is said to be closed, where D is the discrete functor.

(2) If $\{*\}$, the image of Z, is closed in $Z = \emptyset$ or A/Z, then Z is said to be closed, where A/Z is the final lift of the epi U-sink $Q : U(A) = B \to B/Z = \{*\} \cup (B \backslash Z)$, identifying Z with a point $*$.

(3) If the complement Z^c of Z is closed, then Z is said to be open.

For **Top**, openness (resp. closedness) coincides with the usual openness (resp. closedness) [2, 8].

Theorem 2.3. ([4]) *Let $(B, L) \in$* **ConFCO**, *$\emptyset \neq M \subset B$, and $x \in B$.*

(1) $\{x\}$ *is closed iff* (B, L) *is* T_1 *at* x *iff* $[y] \cap [x] \notin L$ *for all* $y \in B$ *with* $y \neq x$.

(2) M *is closed iff* $\alpha \not\subset [b]$ *or* $\alpha \cup [M]$ *is improper for every* $\alpha \in L$ *and every* $b \in B$ *with* $b \notin M$.

Theorem 2.4. *Let* $(B, L) \in$ **ConFCO**.

(1) $M \subset B$ *is open iff* $\alpha \not\subset [b]$ *or* $\alpha \cup [M^c]$ *is improper for every* $\alpha \in L$ *and every* $b \in B$ *with* $b \in M$.

(2) *If for each* $i \in I$, $M_i \subset B$ *is closed, then* $\bigcap_{i \in I} M_i$ *is closed.*

Proof. (1) It follows from Definition 2.2 and Theorem 2.3.

(2) Suppose $b \in B$, $b \notin \bigcap_{i \in I} M_i$, and $\alpha \in L$. Then $\exists k \in I$ with $b \notin M_k$. Since M_k is closed, by Theorem 2.3, $\alpha \not\subset [b]$ or $\alpha \cup [M_k]$ is improper. If $\alpha \cup [M_k]$ is improper, then $\alpha \cup [\bigcap_{i \in I} M_i]$ is improper since $\alpha \cup [M_k] \subset \alpha \cup [\bigcap_{i \in I} M_i]$. Consequently, by Theorem 2.3, $\bigcap_{i \in I} M_i$ is closed. □

Example 2.5. (1) Theorems 2.3 and 2.4, all subsets of the discrete constant filter convergence space (B, L) are both closed and open.

(2) Let $(B, F(B))$ be the indiscrete constant filter convergence space, i.e., $L = F(B)$ with $cardB \geq 2$. By Theorem 2.4, the only closed (open) subset of B are $\emptyset$ and B.

(3) Let $B = \{m, n, p, r\}$ and define constant filter convergence structures L_i for $i = 1, 2, 3$ as follows:
$L_1 = \{[p], [m], [n], [r], [n] \cap [m], [\emptyset]\}$.
$L_2 = \{[p], [m], [n], [r], [n] \cap [p], [p] \cap [r], [r] \cap [n], [\emptyset]\}$.
$L_3 = \{[p], [m], [n], [r], [n] \cap [m], [p] \cap [n], [r] \cap [p], [m] \cap [p], [n] \cap [m] \cap [p], [\emptyset]\}$.
By Theorems 2.3 and 2.4, the only closed (open) subsets with respect to L_1 are $\{r\}$, $\{p\}$, $\{n, m\}$, $\{r, p\}$, $\{n, m, p\}$, $\{n, r, m\}$, B, and $\emptyset$. The only closed (open) subsets with respect to L_2 are $\{m\}$, $\{n, p, r\}$, B, and $\emptyset$. The only closed (open) subsets with respect to L_3 are B and $\emptyset$.

(4) Let $B = \{m, n, p, r, s\}$ and define L as $L = \{[m], [n], [p], [r], [s], [m] \cap [n], [m] \cap [p], [n] \cap [p], [m] \cap [n] \cap [p], [\emptyset]\}$. By Theorems 2.3 and 2.4, the only closed (open) subsets with respect to L are $\{r\}$, $\{s\}$, $\{r, s\}$, $\{n, m, p\}$, $\{n, m, p, s\}$, $\{n, m, p, r\}$, B, and $\emptyset$.

Theorem 2.6. *Let* $(A, S), (B, L) \in$ **ConFCO**.

(1) *If* $N \subset A$ *and* $M \subset N$ *are open, then* $M \subset A$ *is open.*

(2) *If* $f : (A, S) \longrightarrow (B, L)$ *is continuous and* $M \subset B$ *is open, then* $f^{-1}(M) \subset A$ *is open.*

Proof. **(1)** Suppose $N \subset A$ and $M \subset N$ are open, $\alpha \in S$ and for each $b \in A$ with $b \notin M$. Let S_N be a subspace structure on N induced by the inclusion map $i : N \to (A, S)$. Note that $i^{-1}(\alpha) = \alpha \cup [M] \in S_N$. Since $M \subset N$ is open, by Theorem 2.3, $\alpha \cup [N] \cup [N \backslash M] = \alpha \cup [N \backslash M]$ is improper or $\alpha \cup [N] \not\subset [b]$ for every $\alpha \in S$ and every $b \in N$ with $b \notin M$.

If $\alpha \cup [N] \not\subset [b]$, then $\alpha \not\subset [b]$ since $[N] \subset [b]$. Suppose $\alpha \cup [N \backslash M]$ is improper. Since $N \subset A$ is open, by Theorem 2.4, $\alpha \not\subset [b]$ or $\alpha \cup [A \backslash N]$ is improper. We need to show $\alpha \cup [A \backslash M]$ is improper. Since $\alpha \cup [A \backslash N]$ is improper and $\alpha \cup [N \backslash M]$, $\exists\, V, U \in \alpha$ with $U \cap (N \backslash M) = \emptyset$ and $V \cap (A \backslash N) = \emptyset$. Note that $U \cap V \in \alpha$ and we will show that $(U \cap V) \cap (A \backslash M) = \emptyset$. Suppose $(U \cap V) \cap (A \backslash M) \neq \emptyset$. $\exists\, x \in (U \cap V) \cap (A \backslash M)$ with $x \in V$, $x \in U$, and $x \in A \backslash M$.

Suppose $x \in N$. If $x \in M$, then $x \notin A \backslash M$, a contradiction.
If $x \notin M$, then $x \in N \backslash M$ and $x \in U \cap (N \backslash M)$, a contradiction, hence $U \cap (N \backslash M) = \emptyset$.
If $x \notin N$, then $x \in A \backslash N$. Since $x \in V$, $x \in V \cap (A \backslash N)$, a contradiction.

Hence, $(U \cap V) \cap (A \backslash M) = \emptyset$, i.e., $\alpha \cup [A \backslash M]$ is improper. By Theorem 2.4, $M \subset A$ is open.

(2) Suppose $M \subset B$ is open, $b \in A$ with $b \in f^{-1}(M)$, and $\alpha \in S$. Note that $f(b) \in M$, $f(\alpha) \in L$ and by Theorem 2.4, $f(\alpha) \not\subset [f(b)]$ or $f(\alpha) \cup [B \backslash M]$ is improper since $M \subset B$ is open.

Suppose $f(\alpha) \cup [B \backslash M]$ is improper. By Lemma 2.1 of [13],

$$f(\alpha \cup [f^{-1}(B \backslash M)]) = f(\alpha \cup [A \backslash f^{-1}(M)]) \supset f(\alpha) \cup [f f^{-1}(B \backslash M)] \supset f(\alpha) \cup [B \backslash M].$$

Since $f(\alpha) \cup [B\backslash M]$ is improper then $f(\alpha \cup [A\backslash f^{-1}(M)])$ is improper and consequently, $\alpha \cup [A\backslash f^{-1}(M)]$ is improper.

Suppose $f(\alpha) \not\subset [f(b)]$. If $\alpha \subset [b]$, then $f(\alpha) \subset [f(b)]$, contradiction. Thus, $\alpha \not\subset [b]$. By Theorem 2.4, $f^{-1}(M) \subset A$ is open. □

3. Closure operators in constant filter convergence spaces

In this section, we introduce two closure operators of **ConFCO** and investigate their properties as well as examine how they are related to each other. Finally, we characterize each of T_i, $i = 1, 2$ spaces with respect to these closures.

Definition 3.1. Let $(B, L) \in$ **ConFCO** and $Z \subset B$. The closure $cl_B(Z)$ of Z is the intersection of all closed subsets of B containing Z. The quasi-component closure $Q_B(Z)$ of Z is the intersection of all open and closed subsets of B containing Z.

Definition 3.2. ([19]) Let c be a closure operator of a topological category $\mathcal{E}$.

(1) $T_1(c) = \{X \in \mathcal{E} : c_X(\{a\}) = \{a\}$, for each $a \in X$ $\}$.

(2) $\triangle(c) = \{X \in \mathcal{E} : c_{X^2}(\Delta) = \Delta$, the diagonal $\}$.

(3) $\nabla(c) = \{X \in \mathcal{E} : c_{X^2}(\Delta) = X^2$ $\}$.

Let $\mathcal{E} =$ **Top**, K be the ordinary closure and Q be the quasi-component closure. Then $T_1(K)$, $\triangle(K)$, $\nabla(Q)$, and $T_1(Q)$ are the class of T_1-spaces, T_2-spaces, connected spaces, and totally disconnected spaces, respectively [19].

Let $(B, L) \in$ **ConFCO** and $M \subset B$.

$K(M) = \{x \in B : \exists \alpha \in L$ with $\alpha \cup [M]$ is proper $\}$.

$K^*(M) = \{x \in B : K(\{x\}) \cap M \neq \emptyset\}$

$= \{x \in B$: there exists a proper filter $\alpha \in L$ with $\alpha \subset [x]$ $\}$.

Theorem 3.3. (1) *cl is idempotent, weakly hereditary, productive, and finitely additive but it is not hereditary.*

(2) *Q is idempotent, weakly hereditary, finitely productive, and additive but it is not hereditary.*

(3) $cl = \widehat{K \wedge K^*}$*, the hull of* $K \wedge K^*$*.*

Proof. **(1)** By Theorem 3.4 of [20] and by Exercise 2.D of [19], *cl* is idempotent and weakly hereditary closure operator.

Suppose M_i is closed in (B_i, L_i) for $\forall i \in I$, $a = (a_1, a_2, ...) \in B = \prod_{i\in I} B_i$ with $a \notin M = \prod_{i\in I} M_i$ and $\alpha \in L$. By Proposition 2.1, $\pi_i\alpha \in L_i$ for $\forall i \in I$ and $\exists\, k \in I$ with $a_k \notin M_k$. Since M_k is closed, by Theorem 2.3, $\pi_k\alpha \cup [M_k]$ is improper or $\pi_k\alpha \not\subset [a_k]$. If $\pi_k\alpha \cup [M_k]$ is improper, then $\exists U \in \pi_k\alpha$ such that $U \cap M_k = \emptyset$ and hence $\exists W \in \alpha$ with $\pi_k W \subset U$ and $\pi_k W \cap M_k = \emptyset$. Let $\sigma = \bigcup_{i\in I} \pi_i^{-1}\pi_i\alpha$, by Corollary 3.3 of [3], $\pi_i\sigma = \pi_i\alpha$ for all $i \in I$ and $\sigma \subset \alpha$. Let $Z = \prod_{i\in I}(\pi_i W)$. Then $Z \in \sigma$ and

$$Z \cap M = \prod_{i\in I}(\pi_i W) \cap \prod_{i\in I} M_i = \prod_{i\in I}(\pi_i W \cap M_i) = \emptyset.$$

Consequently, $\sigma \cup [M]$ is improper and hence $\alpha \cup [M]$ is improper.

If $\pi_k\alpha \not\subset [a_k]$, then $\sigma \not\subset [a]$, otherwise, $\pi_i\sigma = \pi_i\alpha \subset [a_i]$ for each $i \in I$. This is a contradiction. By Theorem 2.3, M is closed, i.e., *cl* is productive.

To see cl is finitely additive, suppose M_i is closed subset of B for $i = 1, 2, ..., n$, $a \in B$, $a \notin M = \bigcup_{i=1}^n M_i$, $\alpha \in L$. Since M_i is closed, by Theorem 2.3, $\alpha \cup [M_i]$ is improper for $i = 1, 2, ..., n$ or $\alpha \not\subset [a]$. If $\alpha \cup [M_i]$ is improper, then there exists $U_i \in \alpha$ such that $U_i \cap M_i = \emptyset$.

Let $U = \bigcap_{i=1}^n U_i$. Since $U \in \alpha$, α is a filter, and

$$M \cap U = (\bigcup_{i=1}^{n} M_i) \cap (\bigcap_{i=1}^{n} U_i) = \bigcup_{i=1}^{n} (M_i \cap U_i) = \emptyset.$$

Hence, $\alpha \cup [M]$ is improper and by Theorem 2.3, M is closed and consequently, cl is finitely additive.

Let (B, L_3) be as in Example 2.5 (3) with $A = \{m, r\}$ and $M = \{m\}$. Let L_A be structure on A induced by $i : A \to (B, L_3)$.

$$L_A = \{[r], [m], [\emptyset]\}.$$

By Theorem 2.3, the closed subsets respect to L_A are $\{m\}$, $\{r\}$, $\emptyset$, and A. Note that $cl_A(M) = \{m\}$, $cl_B(M) = B$ and $cl_A(M) = \{m\} \neq \{m, r\} = cl_B(M) \cap A$. Hence, cl is not hereditary.

(2) By Theorem 3.4 of [20], by Exercise 2.D of [19] and Theorem 2.6, Q is idempotent and weakly hereditary closure operator.

To see Q is finitely productive, let $M_i \subset B_i$ be open and closed for all $i = 1, 2, ..., n$, $a = (a_1, a_2, ...a_n) \in B = \prod_{i=1}^n B_i$ with $a \in M = \prod_{i=1}^n M_i$ and $\alpha \in L$, the product structure on B. For each $i = 1, 2, ..., n$, $a_i \in M_i$ and $\pi_i\alpha \in L_i$. Since $M_i \subset B_i$ are open for all $i = 1, 2, ..., n$, by Theorem 2.4, $\pi_i\alpha \not\subset [a_i]$ or $\pi_i\alpha \cup [B_i \backslash M_i]$ is improper. Let $\sigma = \bigcup_{i=1}^n \pi_i^{-1}\pi_i\alpha$, by Corollary 3.3 of [3], $\pi_i\sigma = \pi_i\alpha \in L_i$ for all $i = 1, 2, ..., n$ and by Proposition 2.1, $\sigma \in L$.

If $\alpha_i \not\subset [a_i]$ for each $i = 1, 2, ..., n$, $\sigma \not\subset [a]$.

Suppose $\pi_i\alpha \cup [B_i \backslash M_i]$ are proper for all $i = 1, 2, ..., n$. Then $\exists W_i \in \alpha$ such that $\pi_i W_i \cap (B_i \backslash M_i) = \emptyset$ for each $i = 1, 2, ..., n$.

Let $W = \bigcap_{i=1}^n W_i$ and $Z = \prod_{i=1}^n (\pi_i W_i)$. Since α is a filter, $W \in \alpha$ and $Z \in \sigma$. We show that $\sigma \cup [M^c] = \sigma \cup [B \backslash M]$ is improper. Note that

$$B \backslash M = (\prod_{i=1}^{n} B_i) \backslash (\prod_{i=1}^{n} M_i) = ((B_1 \backslash M_1) \times B_2 \times B_3 \times ... \times B_n) \cup (B_1 \times (B_2 \backslash M_2) \times B_3 \times ... \times B_n) \cup ... \cup (B_1 \times B_2 \times ... \times (B_n \backslash M_n))$$

$$\begin{aligned} Z \cap M^c &= (\prod_{i=1}^{n} \pi_i W_i) \cap (((B_1 \backslash M_1) \times B_2 \times ... \times B_n) \cup (B_1 \times (B_2 \backslash M_2) \times ... \times B_n) \cup ... \cup (B_1 \times B_2 \times ... \times (B_n \backslash M_n)) \\ &= ((\pi_1 W_1 \cap (B_1 \backslash M_1)) \times \pi_2 W_2 \times ... \times \pi_n W_n) \cup (\pi_1 W \times (\pi_2 W \cap (B_2 \backslash M_2) \times ... \times \pi_n W_n) \cup ... \\ &\quad \cup (\pi_1 W_1 \times \pi_2 W_2 \times ... \times \pi_{n-1} W_{n-1} \times (\pi_n W_n \cap (B_n \backslash M_n))) \\ &= \emptyset \end{aligned}$$

Hence, $\alpha \cup [M^c]$ is improper and by Theorem 2.4, M is open in B. By Part (1), M is closed and by Definition 2.4, Q is finitely productive.

To see Q is additive, let $M_i \subset B$ be open for each $i = 1, 2, ..., n$ and $a \in B$ with $a \in M = \bigcup_{i=1}^n M_i$. By Theorem 2.4, M is open and by Part (1), M is closed. Consequently, by Definition 3.2, Q is additive.

Finally, we will show that Q is not hereditary. Let (B, L_3) be as in Example 2.5(3) with $M = \{r\}$ and $A = \{p, r\}$. Then L_A induced by $i : A \to (B, L_3)$, i.e.,

$$L_A = \{[p], [r], [\emptyset]\}$$

Note that $Q_B(M) = B$, $Q_A(M) = M$, and $Q_B(M) \cap A = B \cap A \neq M = Q_A(M)$. Hence, Q is not hereditary.

(3) Follows from Theorem 2.4. □

Theorem 3.4. (1) *$(B, L) \in T_1(cl)$ iff for any $y, x \in B$ with $y \neq x$, $[x] \cap [y] \notin L$.*

(2) $(B, L) \in T_1(Q)$ *iff for any* $y, x \in B$ *with* $y \neq x$, $[x] \cap [y] \notin L$ *and exists a subset of* M *of* B *with* $x \in M$, $y \notin M$, $\alpha \cup [M]$ *is improper or* $\alpha \cup \beta$ *is improper or* $\beta \cup [M^c]$ *is improper for every proper filters* $\alpha, \beta \in L$.

(3) $(B, L) \in \Delta(cl)$ *iff for every* $y, x \in B$ *with* $y \neq x$ *and for every proper filters* $\alpha, \beta \in L$, $\alpha \not\subset [x]$ *or* $\beta \not\subset [y]$ *or* $\alpha \cup \beta$ *is improper.*

(4) *If* $(B, L) \in \Delta(Q)$, *then for every* $y, x \in B$ *with* $y \neq x$ *and for every proper filters* $\beta, \alpha \in L$, $\alpha \not\subset [x]$ *or* $\beta \not\subset [y]$ *or* $\alpha \cup \beta$ *is improper.*

Proof. **(1)** Suppose $(B, L) \in T_1(cl)$. By Definition 3.2, for $\forall x \in B$, $cl_B(\{x\}) = \{x\}$. Hence, by Theorem 2.4, $\{x\}$ is closed and by Theorem 2.3, $[x] \cap [y] \notin L$ for every $y, x \in B$ with $y \neq x$.

Conversely, suppose for every $y, x \in B$ with $y \neq x$, $[x] \cap [y] \notin L$. By Theorem 2.3, $\{x\}$ is closed and by Definition 3.1, $cl_B(\{x\}) = \{x\}$, i.e., by Definition 3.2, $(B, L) \in T_1(cl)$.

(2) Suppose $(B, L) \in T_1(Q)$, $y, x \in B$ with $y \neq x$. By Theorem 2.4, $\{x\} = Q_B(\{x\})$ is closed and by Theorem 2.3, $[x] \cap [y] \notin L$ for $\forall y, x \in B$ with $x \neq y$. $y \notin \{x\} = Q_B(\{x\})$ and by Definition 3.1, there exists a open and closed $M \subset B$ containing x but not y. Since M is closed (resp. open), by Theorem 2.3 (resp. by Theorem 2.4), $\alpha \cup [M]$ (resp. $\beta \cup [M^c]$) is improper or $\alpha \not\subset [y]$ (resp. $\beta \not\subset [x]$) for proper filters $\alpha, \beta \in L$.

If $\beta \cup [M^c]$ and $\alpha \cup [M]$ are improper, then $\exists V \in \beta$ and $\exists U \in \alpha$ with $U \cap M = \emptyset = V \cap M^c$. It follows that $U \cap V \subset M^c \cap M = \emptyset$. Consequently, $\emptyset = U \cap V$, i.e., $\alpha \cup \beta$ is improper.

If $\alpha \cup [M]$ (resp. $\beta \cup [M^c]$) is improper, then $\exists\, U \in \alpha$ (resp. $V \in \beta$) with $U \cap M$ (resp. $V \cap M^c$) $= \emptyset$. Hence, $\forall a \in M$ (resp. $\forall b \in M^c$), $a \notin U$ (resp. $b \notin V$) and $\alpha \not\subset [x]$ (resp. $\beta \not\subset [y]$).

Since $\{y\}$ and $\{x\}$ are closed, the case $\beta \not\subset [x]$ and $\alpha \not\subset [y]$ always holds. Therefore, we must have $\alpha \cup \beta$ is improper or $\alpha \cup [M]$ is improper or $\beta \cup [M^c]$ is improper for some $M \subset B$ with $y \notin M$, $x \in M$.

Conversely, suppose the conditions hold, α, β are proper filters in L, and $y, x \in B$ with $x \neq y$. Since $[x] \cap [y] \notin L$, by Theorem 2.3, $\{x\}$ is closed. Let $M = \{x\}$. By assumption, $\alpha \cup [x]$ is improper or $\beta \cup [\{x\}^c]$ is improper or $\alpha \cup \beta$ is improper.

If $\alpha \cup \beta$ is improper, then $\exists\, V \in \beta$ and $U \in \alpha$ with $U \cap V = \emptyset$. If $U \cap M = \emptyset$, then $x \notin U$ and $\alpha \not\subset [x]$. If $U \cap M \neq \emptyset$, then $\beta \not\subset [x]$ and $x \notin V$.

If $\alpha \cup [x]$ is improper, then $\alpha \not\subset [x]$.

If $\beta \cup [\{x\}^c]$ is improper, then $\exists U \in \beta$ with $U \cap \{x\}^c = \emptyset$ and so, $U = \emptyset$ or $U = \{x\}$. Since β is proper, $\{x\} = U \in \beta$ and $\beta = [x] \not\subset [a]$ for every $a \neq x$. By Theorem 2.4, $\{x\}$ is open and by Definition 3.1, for every $x \in B$, $Q_B(\{x\}) = \{x\}$, i.e.,$(B, L) \in T_1(Q)$.

(3) Suppose $(B, L) \in \Delta(cl)$, $y, x \in B$, $y \neq x$, and $\beta \subset [y]$ and $\alpha \subset [x]$ for all $\beta, \alpha \in L$. We show $\alpha \cup \beta$ is improper. Suppose $\alpha \cup \beta$ is proper. Let $\sigma = \pi_1^{-1}\alpha \cup \pi_2^{-1}\beta$. By Corollary 3.3 of [3], $\pi_1\sigma = \alpha \in L$, $\pi_2\sigma = \beta \in L$. Hence, $\sigma \in L^2$ and $\sigma \subset [(x, y)]$. $\alpha \cup \beta$ is proper implies for every $U_2 \in \beta$ and $U_1 \in \alpha$, $U_1 \cap U_2 \neq \emptyset$ and thus,

$$(U_1 \times U_2) \cap \Delta \neq \emptyset.$$

Consequently, $\sigma \cup [\Delta]$ is proper. By Theorem 2.3, Δ is not closed and hence, $\alpha \cup \beta$ is improper.

Suppose for every $y, x \in B$, $y \neq x$, and $\forall \beta, \alpha \in L$ if $\beta \subset [y]$ and $\alpha \subset [x]$, then $\alpha \cup \beta$ is improper. We show $(B, L) \in \Delta(cl)$, i.e., by Theorem 2.3, $\forall (x, y) \notin \Delta$ and $\forall \alpha \in L^2$, $\alpha \cup [\Delta]$ is improper or $\alpha \not\subset [(x, y)]$. Let $\sigma = \pi_1^{-1}\pi_1\alpha \cup \pi_2^{-1}\pi_2\alpha$. By Corollary 3.3 of [3], $\sigma \subset \alpha$, $\pi_1\sigma = \pi_1\alpha \in L$ and $\pi_2\sigma = \pi_2\alpha \in L$. By assumption, $\pi_1\sigma \not\subset [x]$ or $\pi_2\sigma \not\subset [y]$ or $\pi_1\alpha \cup \pi_2\alpha = \pi_1\sigma \cup \pi_2\sigma$ is improper. If $\pi_1\alpha = \pi_1\sigma \not\subset [x]$ or $\pi_2\alpha = \pi_2\sigma \not\subset [y]$, then $\alpha \not\subset [(x, y)]$.

Suppose $\pi_1\sigma \cup \pi_2\alpha$ is improper. There exist $V \in \pi_1\sigma = \pi_1\alpha$ and $U \in \pi_2\sigma = \pi_2\alpha$ such that $V \cap U = \emptyset$. Hence, $\Delta \cap (U \times V) = \emptyset$, i.e., $\sigma \cup [\Delta]$ is improper and thus $\alpha \cup [\Delta]$ is improper. Consequently, by Theorem 2.3, Δ is closed and by Definition 3.2, $cl_{B^2}(\Delta) = \Delta$, i.e., $(B, L) \in \Delta(cl)$.

(4) Suppose $(B, L) \in \Delta(Q)$, $y, x \in B$, $y \neq x$, $(x, y) \notin \Delta = Q_{B^2}(\Delta)$. By Definition 3.1, there exists closed and open M set with $M \supset \Delta$ with $(x, y) \notin M$. Since M is closed, by Theorem 2.3, $\forall \alpha \in L^2$, $\alpha \not\subset [(x, y)]$ or

$\alpha \cup [M]$ is improper. If $\alpha \cup [M]$ is improper, then $\alpha \cup [\Delta]$ is improper and hence $\pi_1\alpha \cup \pi_2\alpha$ is improper. If $\alpha \not\subset [(x, y)]$, then $\pi_1\alpha \not\subset [x]$ or $\pi_2\alpha \not\subset [y]$. Since M is open, by Theorem 2.4, $\forall\alpha \in L^2$ and $\forall(a, b) \in M$, $\alpha \not\subset [(a, b)]$ or $\alpha \cup [M^c]$ is improper. If $\alpha \not\subset [(a, b)]$, then $\pi_2\alpha \not\subset [b]$ or $\pi_1\alpha \not\subset [a]$. Suppose $\alpha \cup [M^c]$ is improper. Let $\sigma = \pi_1^{-1}\pi_1\alpha \cup \pi_2^{-1}\pi_2\alpha$. If $\sigma \cup [M^c]$ is improper, then there exist $U_1 \in \pi_1\alpha$ and $U_2 \in \pi_2\alpha$ such that

$$(U_1 \times U_2) \cap M^c = \emptyset.$$

If $\sigma \cup [M]$ is improper, then $\exists\ V_2 \in \pi_2\alpha$ and $V_1 \in \pi_1\alpha$ such that

$$(V_1 \times V_2) \cap M = \emptyset.$$

Therefore, $U_1 \cap V_1 \in \pi_1\alpha$ and $U_2 \cap V_2 \in \pi_2\alpha$. Thus,

$$((U_1 \cap V_1) \times (U_2 \cap V_2)) \cap M^c = \emptyset.$$

By Theorem 2.4, M is not open, i.e., we get $Q_{B^2}(\Delta) = \Delta$, a contradition. □

Theorem 3.5. (1) $T_1(Q) \subset T_1(cl)$.

(2) $T_1(Q) \subset \Delta(cl)$.

(3) $\Delta(cl) \subset T_1(cl)$.

Proof. **(1)** Suppose $(B, L) \in T_1(Q)$ and $y, x \in B$ with $y \neq x$. Since $(B, L) \in T_1(Q)$, $y \notin \{x\} = Q_B(\{x\})$ and by Theorem 2.4, $Q_B(\{x\}) = \{x\}$ is closed. By Theorem 2.3, $[x] \cap [y] \notin L$ and by Theorem 3.4, $(B, L) \in T_1(cl)$.

(2) Suppose $(B, L) \in T_1(Q)$, $\alpha, \beta \in L$, $x, y \in B$ with $y \neq x$. $y \notin \{x\} = Q_B(\{x\})$ and by Definition 3.1, there exists a open and closed subset M of B containing x but not y. Since M is closed, by Theorem 2.3, $\alpha \cup [M]$ is improper or $\alpha \not\subset [y]$ and M is open by Theorem 2.4, $\beta \cup [M^c]$ is improper or $\beta \not\subset [x]$ for all proper filters $\beta, \alpha \in L$.

If $\beta \cup [M^c]$ and $\alpha \cup [M]$ are improper, then $\exists V \in \beta$ and $U \in \alpha$ with $U \cap M = \emptyset = V \cap M^c$. It follows that $U \cap V \subset M^c \cap M = \emptyset$. Consequently, $\emptyset = U \cap V$, i.e., $\alpha \cup \beta$ is improper.

If $\alpha \cup [M]$ (resp. $\beta \cup [M^c]$) is improper, then $\exists U \in \alpha$ (resp. $V \in \beta$) with $U \cap M$ (resp. $V \cap M^c = \emptyset$). Hence $\forall a \in M$ (resp. $\forall b \in M^c$), $a \notin U$ (resp. $b \notin V$). $\alpha \not\subset [x]$ (resp. $\beta \not\subset [y]$). Hence by Theorem 3.4, $(B, L) \in \Delta(cl)$.

(3) Suppose $(B, L) \in \Delta(cl)$ and $[x] \cap [y] \in L$ for some $x, y \in B$ with $y \neq x$. Let $\beta = [x] \cap [y] = \alpha$. Note that $\beta, \alpha \in L$, $\beta \cup \alpha$ is proper, $\beta \subset [y]$ and $\alpha \subset [x]$, a contradiction since $(B, L) \in \Delta(cl)$. Hence, $[x] \cap [y] \notin L$ for $\forall x, y \in A$ with $x \neq y$ and by Theorem 3.4, $(B, L) \in T_1(cl)$. □

Theorem 3.6. *If B is finite, then the following are equivalent:*

(1) $(B, L) \in T_1(Q)$,

(2) $(B, L) \in T_1(cl)$,

(3) $(B, L) \in \Delta(Q)$,

(4) $(B, L) \in \Delta(cl)$.

Proof. Suppose B is a finite set, $(B, L) \in T_1(cl)$, and α is any proper filter on B. Then $\alpha = [Z]$ for some $Z \subset B$. If $\alpha \in L$ and $y, x \in Z$ with $y \neq x$, then $\alpha \subset [x] \cap [y]$ and $[x] \cap [y] \in L$, a contradiction since $(B, L) \in T_1(cl)$. Hence Z must be a one-point set. Note also that if $H \subset B$, then H is both open and closed. Indeed, if $H = \emptyset$, then by Definition 2.2, H is closed and open. If $H = \{a_1\}$, a one-point set, then H is closed since $(B, L) \in T_1(cl)$ and

$$H^c = B\backslash\{a_1\} = \{a_2, a_3, ..., a_n\} = \{a_1\} \cup \{a_2\} \cup ... \cup \{a_n\}$$

is closed by Theorem 3.3. Hence, $H = \{a_1\}$ is open. If

$$H = \{a_1, a_2, ..., a_k\} = \{a_1\} \cup \{a_2\} \cup ... \cup \{a_k\},$$

then H is both open and closed. The result follows from Theorems 3.4 and 3.5. □

Let $B \neq \emptyset$, $M \subset B$, $B \coprod_M B$ denote the quotient of the coproduct $B \coprod B = B \times \{0,1\}$ obtained by identifying each $(m,0)$, $m \in M$ with $(m,1)$. Let $q : B \coprod B \to B \coprod_M B$ be the quotient map. The maps $k_i : B \to B \coprod_M B$, $s : B \coprod_M B \to B \coprod_M B$ and $p : B \coprod_M B \to B$ are respectively defined by $k_i(x) = q(x,i)$, $s(q(x,0)) = q(x,1)$, $s(q(x,1)) = q(x,0)$ and $p(q(x,i)) = x$, for $i = 0,1$ and $x \in X$. If $M = \{p\}$, then $B \coprod_M B = B \bigvee_p B$ in [2] and if $M = \Delta$, then $B^2 \coprod_M B^2 = B^2 \bigvee_\Delta B^2$ in [2].

Lemma 3.7. ([18]) *Let δ be a quotient-reflective subcategory of containing a space with at least two points and let $X \in \delta$. A subset $M \subset X$ is δ-closed iff $X \coprod_M X$ belongs to δ.*

Theorem 3.8. *Let $(B, L) \in$* **ConFCO** *and $M \subset B$ with at least two point.*

(1) *If $(B, L) \in T_1(cl)$, then $(B \coprod B, L') \in T_1(cl)$, where L' is the final lift of $i_1, i_2 : (B, L) \to (B \coprod B, L')$.*

(2) *If $(B, L) \in T_1(cl)$ and M is cl-closed, (i.e., closed), then $(B \coprod_M B, L'') \in T_1(cl)$, where L'' is the final lift of $k_1, k_2 : (B, L) \to (B \coprod_M B, L'')$.*

(3) *If $(B, L) \in \Delta(cl)$, then $(B \coprod B, L') \in \Delta(cl)$.*

(4) *If $(B, L) \in \Delta(cl)$ and M is cl-closed, then $(B \coprod_M B, L'') \in \Delta(cl)$.*

Proof. **(1)** Suppose $(B, L) \in T_1(cl)$ and for some $(y, j) \neq (x, n)$ in $B \coprod B$, $n, j = 0, 1$, $[(y, j)] \cap [(x, n)] \in L'$, where L' is the final lift of the U-sink $\{i_0, i_1 : (B, L) \to B \coprod B\}$ (i_0 and i_1 are the canonical injections). By Proposition 2.1, $\exists\, \alpha \in L$ such that $[(x, n)] \cap [(y, j)] \supset i_k\alpha$ for some $k = 0$ or 1. This holds iff $[x] \cap [y] \in L$ and $n = j = k$, a contradiction since $(B, L) \in T_1(cl)$, if $y = x$, then $n = 0$ and $j = 1$ (resp. $n = 1$ and $j = 0$). Since $[(x, 0)] \cap [(x, 1)] \supset i_k\alpha$ for some $k = 0$ or 1, then $B \times \{k\} \in [(x, 0)] \cap [(x, 1)]$, a contradiction. Hence, by Theorem 3.4, $(B \coprod B, L') \in T_1(cl)$.

(2) Suppose $(B, L) \in T_1(cl)$, $M \subset B$ is cl-closed, and $q' : (B \coprod B, L') \to (B \coprod_M B, L'')$ is the quotient map defined as above. Suppose for some $z = q'((y, j)) \neq q'((x, n) = z'$ for $j, n = 0, 1$, in $B \coprod_M B$, $[z] \cap [z'] \in L''$. By Proposition 2.1, there exixts $\alpha \in L'$ such that $[z] \cap [z'] \supset q'(\alpha)$. If $y, x \notin M$, then $[(y, j)] \cap [(x, n)] \supset \alpha$ and consequently, $[(y, j)] \cap [(x, n)] \in L'$, a contradiction, since $(B \coprod B, L') \in T_1(cl)$ by Theorem 3.4 and by Part (1).

If $y \notin M$ (resp. $y \in M$ and $x \notin M$) and $x \in M$, then $\alpha \cup [\{(x, 1), (x, 0)\}]$ (resp. $\alpha \cup [\{(y, 1), (y, 0)\}]$) is proper and consequently $\alpha \subset [(x, 0)]$ or $\alpha \subset [(x, 1)]$ or $\alpha \subset [\{(x, 1), (x, 0)\}]$ (resp. $\alpha \subset [(y, 0)]$ or $\alpha \subset [(y, 1)]$ or $\alpha \subset [\{(y, 1), (y, 0)\}]$).

If $\alpha \subset [(x, k)]$ for $k = 0, 1$, then $\alpha \subset [(x, 1)] \cap [(x, 0)]$ and thus $[(x, 1)] \cap [(x, 0)] \in L'$, since $\alpha \in L'$, a contradiction.

If $\alpha \subset [(y, k)]$ for $k = 0, 1$, then $\alpha \subset [(y, 1)] \cap [(y, 0)]$ and thus $[(y, 1)] \cap [(y, 0)] \in L'$, since $\alpha \in L'$, a contradiction.

If $x, y \in M$, then $\alpha \cup [\{(x, 1), (x, 0), (y, 1), (y, 0)\}]$ is proper and consequently, $[(x, n)] \cap [(y, j)] \in L'$, contradiction since $(B \coprod B, L') \in T_1(cl)$.

(3) Suppose $(B, L) \in \Delta(cl)$ and there exist $(y, j), (x, n)$ in $(B \coprod B, L')$ with $(x, n) \neq (y, j)$ and proper filter $\beta, \alpha \in L$ with $\beta \cup \alpha$ is proper and $\beta \subset [(y, j)]$, $\alpha \subset [(x, n)]$. By Proposition 2.1, there exist proper filters $\beta_1, \alpha_1 \in L$ such that $\alpha \supset i_n(\alpha_1)$, $\beta \supset i_j(\beta_1)$, and $i_n(x) = (x, n)$, $i_j(y) = (y, j)$ for some $n, j = 0, 1$. It follows that $i_n^{-1}(\alpha) \supset \alpha_1$, $i_j^{-1}(\beta) \supset \beta_1$, $i_n^{-1}(\alpha) \cup i_j^{-1}(\beta) \supset \alpha_1 \cup \beta_1$, $i_n^{-1}(\alpha) \subset [x]$, and $i_j^{-1}(\beta) \subset [y]$. Since $\alpha \cup \beta$ is proper, $i_n^{-1}(\alpha) \cup i_j^{-1}(\beta)$ is proper and consequently, $\alpha_1 \cup \beta_1$ is proper, $\alpha_1 \subset [x]$ and $\beta_1 \subset [y]$, a contradiction since $(B, L) \in \Delta(cl)$. If $y = x$, then $n = 0$ and $j = 1$ ($n = 1$ and $j = 1$). Since $\alpha \supset i_0(\alpha_1)$ and $\beta \supset i_1(\beta_1)$,

$$\emptyset = (B \times \{1\}) \cap (B \times \{0\}) \in \beta \cup \alpha,$$

i.e., $\beta \cup \alpha$ is improper, a contradiction. Hence, by Theorem 3.4, $(B \coprod B, L'') \in \Delta(cl)$.

(4) Suppose $(B, L) \in \Delta(cl)$, $\beta \cup \alpha$ is proper, $\alpha \subset [z]$ and $\beta \subset [z']$ for some $z = q'(x, n)$, $z' = q'(y, j)$ for some proper filter $\alpha, \beta \in L''$. Suppose $x \neq y$. By Proposition 2.1, $\exists \alpha_1, \beta_1 \in L'$ with $\alpha \supset q'(\alpha_1)$ and $\beta \supset q'(\beta_1)$.

Since $\alpha_1, \beta_1 \in L'$, by Proposition 2.1, $\exists \alpha_2, \beta_2 \in L$ such that $\alpha_1 \supset i_n(\alpha_2)$ and $\beta_1 \supset i_j(\beta_2)$ for some $n, j = 0, 1$ and $i_n(x) = z$, $i_j(y) = z'$. It follows that

$$(q'oi_n)^{-1}\alpha \supset \alpha_2, \quad (q'oi_j)^{-1}\beta \supset \beta_2, \quad \alpha_2 \subset [x], \quad \beta_2 \subset [y].$$

Since $\alpha \cup \beta$ is proper, $\alpha_2 \cup \beta_2$ is proper, a contradiction since $(B, L) \in \Delta(cl)$. Suppose $x = y$, $n = 0$, b and $y = 1$. Note that $(q'oi_0)(\alpha_2) \cup (q'oi_1)(\beta_2)$ is proper and consequently $\alpha_2 \cup \beta_2 \cup [M]$ is proper. Since $\alpha_2 \cup \beta_2$ is proper, $\alpha_2 \cup \beta_2 \in L$, $\alpha_2 \cup \beta_2 \subset [x]$, and M is closed, by Theorem 2.3, $x \in M$, a contradiction since $z = (x, 0) \neq (x, 1) = z'$. Hence, by Theorem 3.4, $(B \coprod_M B, L'') \in \Delta(cl)$. □

Theorem 3.9. *The subcategories* $\mathbf{T_1(cl)}$ *and* $\mathbf{\Delta(cl)}$ *are quotient-reflective in* **ConFCO**.

Proof. One can easily show $\mathbf{T_1(cl)}$ and $\mathbf{\Delta(cl)}$ are isomorphism-closed, full, closed under final structures and closed under formation of subspaces. We show that they are closed under products.

Let $(B_i, L_i) \in \mathbf{T_1(cl)}$ for $\forall i \in I$ and $B = \prod_{i \in I} B_i$. We show $(B, L) \in \mathbf{T_1(cl)}$. Suppose $(B, L) \notin \mathbf{T_1(cl)}$. Hence, by Theorem 3.4, $[y] \cap [x] \in L$ for some $y = (y_1, y_2, ...)$, $x = (x_1, x_2, ...)$ in B with $y \neq x$. Then there exists $j \in I$ such that $x_j \neq y_j$ in B_j and by Proposition 2.1 and by Lemma 2.1 of [13],

$$\pi_j([x] \cap [y]) = \pi_j([x]) \cap \pi_j([y]) = [x_j] \cap [y_j] \in L_j,$$

a contradiction since $(B_i, L_i) \in \mathbf{T_1(cl)}$. Hence, $(B, L) \in \mathbf{T_1(cl)}$.

Suppose $(B_i, L_i) \in \mathbf{\Delta(cl)}$ for all $i \in I$ but $(B, L) \notin \mathbf{\Delta(cl)}$. By Theorem 3.4, $\beta \cup \alpha$ is proper, $\beta \subset [y]$ and $\alpha \subset [x]$ for some proper filters $\beta, \alpha \in L$ and $y, x \in B$ with $y \neq x$. Hence, $\exists\, j \in I$ with $y_j \neq x_j$ in B_j, $\pi_j(\beta), \pi_j(\alpha)$ are proper filters in L_j, $\pi_j(\beta) \cup \pi_j(\alpha)$ is proper (since $\pi_j(\beta \cup \alpha) \supset \pi_j(\beta) \cup \pi_j(\alpha)$ and $\beta \cup \alpha$ is proper), $\pi_j(\alpha) \subset [x_j]$ and $\pi_j(\beta) \subset [y_j]$, a contradiction since $(B_j, L_j) \in \mathbf{\Delta(cl)}$. Hence, by Theorem 3.4, $(B, L) \in \mathbf{\Delta(cl)}$. □

Theorem 3.10. *If* $(B, L) \in \mathbf{T_1(cl)}$ *(resp.* $\mathbf{\Delta(cl)}$*) and* $M \subset B$ *is closed (i.e., cl-closed), then the epimorphism in* $\mathbf{T_1(cl)}$ *(resp.* $\mathbf{\Delta(cl)}$*) are onto. In particular,* $\mathbf{T_1(cl)}$ *and* $\mathbf{\Delta(cl)}$ *are co-well-powered categories.*

Proof. By Theorem 3.9, $\mathbf{\Delta(cl)}$ and $\mathbf{T_1(cl)}$ are quotient-reflective and by Theorem 3.8, $(B \coprod_M B, L'') \in \mathbf{\Delta(cl)}$, L'' is the quotient structure on $B \coprod_M B$. By Lemma 1.1 of [18], M is $\mathbf{T_1(cl)}$ (resp. $\mathbf{\Delta(cl)}$) closed and the result follows. □

4. Connected constant filter convergence spaces

Definition 4.1. ([8, 16, 23]) Let B be an object in a topological category $\mathcal{E}$.

(1) B is called strongly connected if the only subsets of B both open and closed are B and $\emptyset$.

(2) B is called D-connected if any morphism from B to discrete object is constant.

(3) B is called c-connected if $B \in \nabla(c)$, where c is a closure operator of $\mathcal{E}$.

In **Top**, Q-connectedness, strong connectedness, and D-connectedness coincides with the usual connectedness [8, 16].

Theorem 4.2. *The following are equivalent:*

(*a*) *A constant filter convergence space* (B, L) *is strongly connected.*

(*b*) *For any nonempty proper subset M of B either the condition (I) or (II) holds.*

  (*I*) *There exists a proper filter* $\alpha \in L$ *such that* $\alpha \cup [M]$ *is proper and* $\alpha \subset [a]$ *for some* $a \in M^c$.

  (*II*) *There exists a proper filter* $\alpha \in L$ *such that* $\alpha \cup [M^c]$ *is proper and* $\alpha \subset [a]$ *for some* $a \in M$.

(*c*) (B, L) *is D-connected.*

Proof. By Theorems 2.3 and 2.4 and Definition 4.1, we get $(a) \Rightarrow (b)$.

$(b) \Rightarrow (c)$: Suppose the condition (I) holds. Let $f : (B, L) \to (A, S)$ be a continuous map with (A, S) is the discrete constant filter convergence space. If $cardA = 1$, then f is continuous. Suppose $cardA > 1$ and f is not constant. There exist $b, a \in B$ with $b \neq a$ and $f(b) \neq f(a)$. Let $M = \{a\}$. By assumption, there exists a proper filter $\alpha \in L$ with $\alpha \cup [M] = \alpha \cup [a]$ is proper and $\alpha \subset [c]$ for some $c \in M^c$. It follows that $f(\alpha) \subset [f(c)]$ and $f(\alpha) \subset [f(a)]$. In particular, $f(\alpha) \subset [f(b)] \cap [f(c)]$. If $f(\alpha) \in S$, then $[f(c)] \cap [f(a)] \in S$, a contradiction since S is the discrete structure on A and $f(c) \neq f(a)$. Hence, $f(\alpha) \notin S$, i.e., f is not continuous, a contradiction.
If the condition (II) holds, by the similar argument the result follows. Hence, by Definition 4.1, (B, L) is D-connected.

$(c) \Rightarrow (a)$: Suppose (B, L) is D-connected but it is not strongly connected, i.e., there is a nonempty proper open and closed $M \subset B$. Let (A, S) be discrete constant filter convergence space with $cardA > 1$. Define $f : (B, L) \to (A, S)$ by

$$f(x) = \begin{cases} c, & if \quad x \in M \\ d, & if \quad x \notin M \end{cases}$$

Let $\alpha \in L$. Since M is closed, $\alpha \not\subset [a]$ or $\alpha \cup [M]$ is improper for $\forall a \in B$ with $a \notin M$.
If $\alpha \not\subset [a]$, then $\exists V \in \alpha$ with $a \notin V$. $f(V) = \{c\} \in f(\alpha)$ i.e., $f(\alpha) = [c] \in S$.
If $\alpha \cup [M]$ is improper, then $\exists U \in \alpha$ with $M \cap U = \emptyset$. If $\alpha = [\emptyset]$, then $f(\alpha) = [\emptyset] \in S$.
Suppose $\alpha \neq [\emptyset]$ and $W \in f(\alpha)$. Then $\exists\ V \in \alpha$ with $W \supset f(V)$. Note that $V \cap U \in \alpha$, $U \subset M^c$, and $f(V \cap U) = \{d\} \in f(\alpha)$ and hence, $f(\alpha) = [d] \in S$.
If $\alpha \cup [M^c]$ is improper, then by similar argument $f(\alpha) = [\emptyset]$ or $[c]$. Hence, f is continuous but it is not constant. This is a contradiction. □

Theorem 4.3. *A constant filter convergence space (B, L) is cl-connected if and only if for every $y, x \in B$ with $y \neq x$ there exist proper filters $\beta, \alpha \in L$ such that $\beta \cup \alpha$ is proper, $\beta \subset [y]$ and $\alpha \subset [x]$.*

Proof. Suppose (B, L) is *cl*-connected and $y, x \in B$ with $y \neq x$. $(x, y) \in B^2 = cl_{B^2}(\Delta)$ since (B, L) is *cl*-connected. By Theorem 2.3, $\exists\, \sigma \in L^2$ with $\sigma \cup [\Delta]$ is proper and $\sigma \subset [(x, y)]$ since $cl(\Delta)$ is closed, L^2 is the product structure on B^2. Let $\theta = \pi_1^{-1}\pi_1\sigma \cup \pi_2^{-1}\pi_2\sigma$ and note that by Corollary 3.3 of [3], $\theta \subset \sigma$, $\pi_1\theta = \pi_1\sigma \in L$, $\pi_2\theta = \pi_2\sigma \in L$, $\pi_1\theta \subset [x]$ and $\pi_2\theta \subset [y]$. Since $\sigma \cup [\Delta]$ is proper, $\theta \cup [\Delta]$ is proper. Hence, $\pi_1\theta \cup \pi_2\theta$ is proper. We have also $\pi_1\theta \subset [x]$ and $\pi_2\theta \subset [y]$.
Suppose for $\forall y, x \in B$ with $y \neq x$ and there exist proper filters $\alpha, \beta \in L$ with $\beta \cup \alpha$ is proper, $\beta \subset [y]$ and $\alpha \subset [x]$. Let $\sigma = \pi_1^{-1}\beta \cup \pi_2^{-1}\alpha$. By Corollary 3.3 of [3], $\pi_2\sigma = \beta \in L$, $\pi_1\sigma = \alpha \in L$, $\sigma \subset [(x, y)]$, and $\sigma \in L^2$. Since $\pi_1\sigma \cup \pi_2\sigma = \alpha \cup \beta$ is proper, $\sigma \cup [\Delta]$ is proper. By Theorem 2.3, $(x, y) \in cl(\Delta)$, which shows $cl_{B^2}(\Delta) = B^2$ and by Definition 4.1, (B, L) is *cl*-connected. □

Example 4.4. (1) By Theorems 4.2 and 4.3, the indiscrete space $(B, F(B))$ is both strongly connected and *cl*-connected.

(2) By Theorems 4.2 and 4.3, the discrete constant filter convergence space with at least two elements is neither strongly connected nor *cl*-connected.

(3) (B, L) discrete constant filter convergence space is strongly connected (*cl*-connected) iff $cardB \leq 1$.

(4) Let $B = \{m, n, p, r\}$ and define constant filter convergence structure L as follows:

$$L = \{[p], [n], [m], [r], [n] \cap [m], [p] \cap [n], [p] \cap [r], [\emptyset]\}.$$

By Theorem 2.3 and 2.4, the only closed and open subsets of B are B and $\emptyset$ and consequently, by Theorem 4.2, (B, L) is strongly connected and D-connected.

Theorem 4.5. *If (B, L) is cl-connected, then (B, L) is strongly connected.*

Proof. Suppose (B,L) is *cl*-connected and M is a nonempty proper subset of B. Let $x \in M$ and $y \in M^c$. Since (B,L) is *cl*-connected, by Theorem 4.3, there exist proper filters $\alpha, \beta \in L$ with $\alpha \cup \beta$ is proper, $\alpha \subset [x]$ and $\beta \subset [y]$. Note that $\alpha \cup [M]$ is proper since $x \in M$ and $\alpha \subset [x]$. Since $\alpha \cup \beta$ is proper, it folllows that $\alpha \subset [a]$ for $a \in M^c$. Hence, condition (I) in Theorem 4.2 holds and by Theorem 4.2, (B,L) is strongly connected.

By Example 4.4(4), (B,L) is strongly connected. For $m, r \in B$, there do not exist proper filter $\alpha, \beta \in L$ with $\alpha \cup \beta$ is proper, $\alpha \subset [r]$ and $\beta \subset [m]$. By Theorem 4.3, (B,L) is not *cl*-connected. □

Theorem 4.6. *Let $f : (B,L) \to (A,S)$ be continuous. If (B,L) is strongly connected (resp. cl-connected), then $f(B)$ is strongly connected (resp. cl-connected).*

Proof. Let M be a nonempty proper subset of $f(B)$. $f^{-1}(M)$ is a nonempty proper subset of B. Since (B,L) is strongly connected, either condition (I) or (II) of Theorem 4.2 holds. If the condition (I) in Theorem 4.2 holds, then there exists a proper filter $\alpha \in K$ with $[f^{-1}(M)] \cup \alpha$ is proper and $\alpha \subset [a]$ for some $a \in f^{-1}(M^c) = (f^{-1}(M))^c$. Since $\alpha \in L$ and f is continuous, $f(\alpha) \in S$ and by Remark 3.4 of [11],

$$f(\alpha) \cup [M] \subset f(\alpha) \cup [ff^{-1}(M)]) \subset f(\alpha \cup [f^{-1}(M)])$$

and consequently, $f(\alpha) \cup [M]$ is proper. Since $\alpha \subset [a]$, $f(\alpha) \subset [f(a)]$ for $f(a) \in M^c$. Similarly, if the condition (II) in Theorem 4.2 holds, $f(\alpha) \subset [f(a)]$ for some $f(a) \in M$ and $[M^c] \cup f(\alpha)$ is proper. Hence, by Theorem 4.2, $f(B)$ is strongly connected.

Let $y, x \in f(B)$ with $y \neq x$. Then $\exists b, a \in B$ with $b \neq a$ such that $f(a) = x$, $f(b) = y$. Since (B,L) is *cl*-connected, by Theorem 4.3, there exist proper filters $\beta, \alpha \in L$ with $\beta \cup \alpha$ is proper and $\beta \subset [b]$ and $\alpha \subset [a]$. Since f is continuous, $f(\alpha), f(\beta) \in S$ and by Remark 3.4 of [11], $f(\alpha \cup \beta) \supset f(\alpha) \cup f(\beta)$. Since $\beta \cup \alpha$ is proper, $f(\beta \cup \alpha)$ is proper and thus $f(\alpha) \cup f(\beta)$ is proper. Also, $f(\beta) \subset [y]$, $f(\alpha) \subset [x]$ and consequently, by Theorem 4.3, $f(B)$ is *cl*-connected. □

Theorem 4.7. *A product of strongly connected constant filter convergence spaces is strongly connected.*

Proof. Let (B_i, L_i) be strongly connected constant filter convergence spaces for $\forall i \in I$ with $B_i \neq \emptyset$. Let $(B = \prod_{i \in I} B_i, L)$ be product space and M be nonempty proper subset of B. We assume without loss of generality, each of $\pi_i M$ is proper nonempty subset of B_i (otherwise, there is always a subset M' of M with $\pi_i M'$ is a nonempty proper subset of B for each $i \in I$). Since (B_i, L_i) is strongly connected, by Theorem 4.2, there exits a proper filter $\alpha_i \in L_i$ such that $\alpha_i \cup [\pi_i(M)]$ is proper and $\alpha_i \subset [a_i]$ for some $a_i \in (\pi_i(M))^c$ or $\alpha_i \cup [(\pi_i(M))^c]$ is proper and $\alpha_i \subset [b_i]$ for some $b_i \in \pi_i(M)$.

Suppose $[(\pi_i(M))] \cup \alpha_i$ is proper and $\alpha_i \subset [a_i]$ for some $a_i \in (\pi_i(M))^c$. Let $\sigma = \bigcup_{i \in I} \pi_i^{-1} \alpha_i$, $b = (b_1, b_2, ...)$ and $a = (a_1, a_2, ...)$, $N = \prod_{i \in I} \pi_i M \supset M$. Since $\alpha_i \in L_i$ for every $i \in I$ and by Corollary 3.3 of [3], $\pi_i \sigma = \alpha_i$ and by Proposition 2.1, $\sigma \in L$. Note that $\sigma \cup [N]$ is proper since $\alpha_i \cup [\pi_i M]$ is proper for each $i \in I$ and for $U \in \sigma$,

$$U \cap N = (U_1 \times U_2 \times ...) \cap \prod_{i \in I} (\pi_i(M)) = (U_1 \cap \pi_1 M) \times (U_2 \cap \pi_2 M) \times ... \neq \emptyset$$

where $U_i \in \alpha_i$. Hence, $\alpha \cup [N]$ is proper and $\alpha \subset [a]$ for $a \in \prod_{i \in I} (\pi_i M)^c \subset N^c$.

Suppose $\alpha_i \cup [(\pi_i(M))^c]$ is proper and $\alpha_i \subset [b_i]$ for some $b_i \in \pi_i(M)$. Let $\sigma = \bigcup_{i \in I} \pi_i^{-1} \alpha_i$ and $N = \prod_{i \in I} (B_i \backslash \pi_i(M))$. Note that $\sigma \in L$ and $\sigma \cup [N]$ is proper and $\alpha \subset [b]$ for $b \in M$. $N \subset M^c$ implies $\sigma \cup [M^c]$ is proper and by Theorem 4.2, (B,L) is strongly connected. □

Theorem 4.8. *A product of cl-connected constant filter convergence spaces is cl-connected.*

Proof. Let (B_i, L_i) be *cl*-connected constant filter convergence spaces for $\forall i \in I$ with $B_i \neq \emptyset$, $(B = \prod_{i \in I} B_i, L)$, $b = (b_1, b_2, ...), a = (a_1, a_2, ...) \in B$ with $b \neq a$. There exists $k \in I$ with $a_k \neq b_k$. We consider the following cases.

(1) $a_i = b_i$ for each $i \in I$ with $k \neq i$.

(2) $a_i \neq b_i$ for all $i \in I$.

(3) $a_m = b_m$ for $m \in J$ and $a_n \neq b_n$ for $n \in I\backslash J$.

Suppose case (i) holds and $a_k \neq b_k$. Since (B_k, L_k) is *cl*-connected by Theorem 4.3, there exist proper filters $\alpha_k, \beta_k \in L_k$ such that $\alpha_k \cup \beta_k$ is proper, $\alpha_k \subset [a_k]$ and $\beta_k \subset [b_k]$. Let

$$\alpha = [(a_1, a_2, ...a_{k-1})] \times \alpha_k \times [(a_{k+1}, a_{k+2}, ...)]$$

and

$$\beta = [(a_1, a_2, ...a_{k-1})] \times \beta_k \times [(a_{k+1}, a_{k+2}, ...)]$$

Note that α and β are proper and by Proposition 2.1, $\alpha, \beta \in L$ since $\pi_i\alpha, \pi_i\beta \in L$ for all $i \in I$. Since $\alpha_k \cup \beta_k$ is proper, $\beta \cup \alpha$ is proper, $\alpha \subset [a]$ and $\beta \subset [b]$.

Suppose case (ii) holds, i.e., $a_i \neq b_i$ for all $i \in I$. Since each (B_i, L_i) is *cl*-connected , by Theorem 4.3, there exists proper filters $\alpha_i, \beta_i \in L_i$ with $\alpha_i \cup \beta_i$ is proper, $\alpha_i \subset [a_i]$ and $\beta_i \subset [b_i]$. Let $\alpha = \bigcup_{i\in I} \pi_i^{-1}\alpha_i$ and $\beta = \bigcup_{i\in I} \pi_i^{-1}\beta_i$. $\pi_i\alpha = \alpha_i \in L_i$, $\pi_i\beta = \beta_i \in L_i$, by Proposition 2.1, $\alpha, \beta \in L$. Let $V \in \beta$ and $U \in \alpha$. Then there exist $U_i \in \alpha_i$ and $V_i \in \beta_i$ with $V_1 \times V_2 \times ... \subset V$ and $U_1 \times U_2 \times ... \subset U$. Hence,

$$(U_1 \cap V_1) \times (U_2 \cap V_2) \times ... \subset U \cap V.$$

Since each $\alpha_i \cup \beta_i$ is proper, $U_i \cap V_i \neq \emptyset$ for all $i \in I$, and consequently, $U \cap V \neq \emptyset$. Hence, $\alpha \cup \beta$ is proper. Since $\alpha_i \subset [a_i]$ and $\beta_i \subset [b_i]$ for all $i \in I$, $\alpha \subset [a]$ and $\beta \subset [b]$.

Suppose case (iii) holds. Since (B_n, L_n) are *cl*-connected for all $n \in I\backslash J$, by Theorem 4.3, there exists proper filters $\alpha_n, \beta_n \in L_n$ such that $\alpha_n \cup \beta_n$ is proper, $\alpha_n \subset [a_n]$ and $\beta_n \subset [b_n]$. Let

$$\alpha = [(a_{i_m}, a_{i_{m+1}}, ...)] \cup (\bigcup_{k\in I\backslash J} \pi_k^{-1}\alpha_k)$$

and

$$\beta = [(a_{i_m}, a_{i_{m+1}}, ...)] \cup (\bigcup_{k\in I\backslash J} \pi_k^{-1}\beta_k)$$

Note that β, α are proper, $[b] \supset \beta$, $[a] \supset \alpha$ and $\alpha \cup \beta$ is proper since each $\alpha_k \cup \beta_k$ is proper for $k \in I\backslash J$. Hence, by Theorem 4.3, (B, L) is *cl*-connected. □

5. Comparative evaluation

We compare our findings with ones in other topological categories. We can infer results:

(1) In **Top**,

(*i*) By Remark 5.2 of [7] and Theorem 2.2.11 of [2], $T_1(Q) = T_2Top \subset T_1(cl) = \Delta(cl) = T_1Top \subset T_0Top$.

(*ii*) By [8], strong connectedness $\Longleftrightarrow$ *D*-connectedness $\Longleftrightarrow$ the usual connectedness.

(2) In **ConFCO**,

(*i*) By Theorem 2.1 of [5] and Theorem 3.4, $\mathbf{T_0ConFCO}$ and $\mathbf{T_1(cl)}$ are isomorphic categories and they are closed under formations of subspaces and products.

(*ii*) By Theorem 3.5, $T_1(Q) \subset \Delta(cl) \subset T_1(cl)$.

(*iii*) By Theorem 3.6, If (B, L) is finite, then
$(B, L) \in \Delta(cl) \Longleftrightarrow (B, L) \in \Delta(Q) \Longleftrightarrow (B, L) \in T_1(Q) \Longleftrightarrow (B, L) \in T_1(cl)$.

(*iv*) By Theorem 3.5, the subcategories $\mathbf{T_1(cl)}$ and $\mathbf{\Delta(cl)}$ are quotient-reflective in **ConFCO**.

(*v*) By Theorems 4.2 and 4.5,
cl-connectedness $\Longrightarrow$ strong connectedness $\Longleftrightarrow$ D-connectedness.

(3) In **psqMet** (the category of extended pseudo-quasi-semi metric spaces and non-expansive maps),

(*i*) By Theorem 3.10 of [12], $\Delta(cl) = T_1(cl) = \nabla(cl) = T_1(Q)$.

(*ii*) By Theorem 4.9 of [12],
strong connectedness $\Longrightarrow$ D-connectedness.
Moreover, if a space is in $T_1 PsqMet$, then by Theorems 3.13 and 3.14 of [14],
strong connectedness $\Longleftrightarrow$ cl-connectedness $\Longleftrightarrow$ D-connectedness.

(4) In **PBorn** (the category of prebornological spaces and bounded maps),

(*i*) By Corollary 3.11 of [4] and Theorem 3.7 of [10], $T_1(cl) = \Delta(cl) = \nabla(cl) = \Delta(Q)$.

(*ii*) By Remarks 4.8 and 5.4 of [10], if a space is NT_2, then,
strong connectedness $\Longleftrightarrow$ cl-connectedness $\Longrightarrow$ D-connectedness.

(5) In **FCO**,

(*i*) By Theorem 2.9 of [6] and Corollary 3.15 of [4], $\Delta(cl) = T_1(cl)$.

(*ii*) By Remark 4.13, Theorems 4.12, 4.9 of [8], and Theorem 3.2 of [4],
strong connectedness $\Longrightarrow$ cl-connectedness $\Longrightarrow$ D-connectedness.

(6) In **RRel** (the category of reflexive relation spaces and relation preserving functions),

(*i*) By Theorem 3.7 of [10], $\Delta(cl) = \Delta(Q) \subset T_1(cl)$.

(*ii*) By Theorems 4.3 and Remark 4.8 of [10], strong connectedness $\Longrightarrow$ D-connectedness and if a space is NT_2, then
strong connectedness $\Longleftrightarrow$ D-connectedness $\Longleftrightarrow$ cl-connectedness.

(7) In **CP** (the category of pairs and functions),

(*i*) By [10] and by Corollary 3.13 of [4], $T_1(cl) = T_1(Q) = \Delta(cl) = \Delta(Q)$.

(*ii*) By [10] and by Theorem 3.8 of [4],
strong connectedness $\Longleftrightarrow$ cl-connectedness $\Longleftrightarrow$ D-connectedness.

(8) In **CApp** (the category of approach spaces and contraction maps), by Theorems 4.8, 4.9, 4.12, and 4.13 of [24],

$\nabla(cl) \subset \Delta(cl) \subset T_1(cl)$.

6. Conclusion

In this work, we introduced two closure operators of **ConFCO** denoted by cl and Q. Then we showed these closure operators are idempotent, weakly hereditary, productive, and finitely additive but they are not hereditary. Furthermore, we obtained $\mathbf{T_0}$**ConFCO** and $\mathbf{T_1(cl)}$ are isomorphic and they are quotient-reflective in **ConFCO**.

Also, we showed that $T_1(Q) \subset \Delta(cl) \subset T_1(cl)$ and if (B, L) is finite, then

$$(B, L) \in \Delta(Q) \iff (B, L) \in \Delta(cl) \iff (B, L) \in T_1(cl) \iff (B, L) \in T_1(Q).$$

Finally, we investigated how D-connectedness, strong connectedness, and cl-connectedness are related to each other and we presented a comparison with our findings and ones in other topological categories.

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