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New triangular $q-$Fibonacci matrix

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Abstract. In this study, we construct a new triangular $q-$analogue of the $q-$Fibonacci matrix $\tilde{f}_q = \left(f_{nk}(q)\right)$ defined by

$$f_{nk}(q) = \begin{cases} \dfrac{q^k f_k(q)}{f_{n+2}(q)-1} & ,1 \leq k \leq n \\ 0 & ,\text{otherwise} \end{cases} .$$

After, we use the analogue to define the sequence spaces $c(\tilde{f}_q)$, $c_0(\tilde{f}_q)$, $\ell_\infty(\tilde{f}_q)$, $\ell_p(\tilde{f}_q)$ $(1 \leq p < \infty)$. Then, we provide some inclusion relations for these spaces and examine a few topological characteristics. Furthermore, we construct a basis for the space $\ell_p(\tilde{f}_q)$, calculate $\alpha-$, $\beta-$, $\gamma-$duals of the same space, characterize certain matrix classes, and look at some geometric properties.

1. Introduction, definitions and preliminaries

Recent years have seen a surge in interest in $q-$calculus, which is particularly prevalent in combinatory analysis, hypergeometric series, partitioning theory, continuous fractions and operator theory.

Specifically, these $q-$generalizations usually enumerate beneficial characteristics of finite dimensional vector spaces over a finite field of order q. This is the reason q is used so often instead of x. Polynomials that take the value of the classical number are all that the $q-$generalizations are when $q = 1$.

Since Euler, Cauchy, Jacobi, and Abel's time, the $q-$series has been in use. The well-known identities of Rogers and Ramanujan, as well as their proof, employed this series. In 1917, Schur [34], who was I.J. Schoenberg's supervisor and one of the founders of the statistical convergence, independently and uninformedly proved these identities. In 1974, Carlitz [8] provided a detailed definition of $q-$Fibonacci and $q-$Lucas numbers using $q-$binomial coefficients. Researchers like Andrews [2], Hirschhorn [20], Cigler [9], Berndt [7], Pan [32], Aytaç [4], Kac and Cheung [23] have all worked on this topic.

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First let us review some notation that will be used in the sequel when we go to work. If k is non-negative, then

$$[k]_q = \begin{cases} k & ,q=1 \\ \dfrac{1-q^k}{1-q} = 1+q+q^2+\ldots+q^{k-1} & ,q \in \mathbb{R}^+ - \{1\} \end{cases}$$

defines the $q-$integer of that value. $q-$factorial and $q-$combination are given as

$$[k]! = \begin{cases} [1][2]...[k] & ,n>0 \\ 1 & ,k=0 \end{cases} \quad \text{and} \quad \begin{bmatrix} k \\ j \end{bmatrix} = \frac{[k]!}{[j]![k-j]!},$$

respectively. With q, the two Pascal rules are specified as

$$\begin{bmatrix} k \\ j \end{bmatrix} = q^{k-j}\begin{bmatrix} k-1 \\ j-1 \end{bmatrix} + \begin{bmatrix} k-1 \\ j \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} k \\ j \end{bmatrix} = \begin{bmatrix} k-1 \\ j-1 \end{bmatrix} + q^j\begin{bmatrix} k-1 \\ j \end{bmatrix},$$

where $0 \leq j \leq k-1$.

The (f_n) Fibonacci sequence was defined by Leonardo Fibonacci in Liber Abaci in 1202 as $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

Schur [34] originally defined the $q-$Fibonacci numbers (polynomials) in 1917 as

$$f_n(q) = \begin{cases} 0 & ,n=0 \\ 1 & ,n=1 \\ f_{n-1}(q) + q^{n-2}f_{n-2}(q) & ,n \geq 2 \end{cases}.$$

$q-$Fibonacci numbers were obtained by Carlitz [8] with the $q-$ binomial coefficient such that:

$$f_{n+1}(q) = \sum_{2k \leq n} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}.$$

The expression $f_n(q)$ for $q-$Fibonacci numbers is given by Andrews [2]:

$$\sum_{k=1}^{n} q^k f_k(q) = f_{n+2}(q) - 1.$$

The $q-$Fibonacci numbers become the ordinary Fibonacci sequence of numbers when $q \to 1$.

Let the set of all sequence spaces be represented by ω. The subspaces of ω that are ℓ_∞, c, c_0, and ℓ_p are characterized as bounded, convergent, null and $p-$absolutely summable sequence space, respectively. The spaces c_0, c, ℓ_∞ are Banach spaces for $k \in \mathbb{N}$ under normed by

$$\|u\|_\infty = \sup_{k \in \mathbb{N}} |u_k|$$

and the space ℓ_p $(1 \leq p < \infty)$ is Banach space normed by

$$\|u\|_p = \left(\sum_k |u_k|^p \right)^{\frac{1}{p}}.$$

Moreover, we designate the spaces of all absolutely convergent series, convergent series, bounded series, and $p-$bounded variation, respectively, by the notations ℓ_1, cs, bs, and bv_p.

If a sequence space U is a complete linear metric space with continuous coordinates $p_n : U \to \mathbb{R}$ $(n \in \mathbb{N})$, where $p_n(x) = x_n$ for every $x = (x_k) \in U$ and every $n \in \mathbb{N}$, then the sequence space is classified as an

$FK-$space. Specifically, a $BK-$space is a Banach space with continuous coordinates, which is equivalent to a normed $FK-$ space.

Let $U, V \subset \omega$ and $B = (b_{nk})$ is a real infinite matrix. The matrix B defines a matrix transformation from U to V if for every sequence $u \in U$,

$$Bu = (B_n(u)) = \left(\sum_{k=1}^{\infty} b_{nk} u_k \right) \in U$$

for each $n \in \mathbb{N}$. (U, V) represents the family of all matrices that map from U to V.

$$U_B = \{u \in \omega : Bu \in U\} \tag{1}$$

is a sequence space that defines the B's matrix domain U_B in a sequence space U.

Moreover, the sequence space U_B is a $BK-$space normed by $\|u\|_{U_B} = \|Bu\|_U$ if B is a triangular matrix and U is a $BK-$space.

Several authors have utilized $q-$numbers in summability theory, including Çınar and Et [10], Demiriz and Şahin [11], Yaying et al. [37–39], Selmanogullari et al. [35], Aktuğlu and Bekar [1], Mursaleen et al. [22], Bekar [6], Atabey et al. [3].

2. Main results

Using a new triangular $q-$analogue of the $q-$Fibonacci matrix with $q-$Fibonacci numbers for $q > 0$, we present the sequence spaces $c_0(\tilde{f_q})$, $c(\tilde{f_q})$, $\ell_\infty(\tilde{f_q})$ and $\ell_p(\tilde{f_q})$ $(1 \le p < \infty)$ in this section. After that, a Schauder basis for $\ell_p(\tilde{f_q})$ will be constructed and some inclusion relations will be shown.

Given a nth Fibonacci number $f_n(q)$ for $n \in \mathbb{N}$ and $q > 0$,

$$\tilde{f_q} = (f_{nk}(q)) = \begin{cases} \dfrac{q^k f_k(q)}{f_{n+2}(q) - 1} & , 1 \le k \le n \\ 0 & , \text{otherwise} \end{cases}$$

$$= \begin{bmatrix} \frac{q f_1(q)}{f_3(q)-1} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{q f_1(q)}{f_4(q)-1} & \frac{q^2 f_2(q)}{f_4(q)-1} & 0 & 0 & 0 & 0 & \cdots \\ \frac{q f_1(q)}{f_5(q)-1} & \frac{q^2 f_2(q)}{f_5(q)-1} & \frac{q^3 f_3(q)}{f_5(q)-1} & 0 & 0 & 0 & \cdots \\ \frac{q f_1(q)}{f_6(q)-1} & \frac{q^2 f_2(q)}{f_6(q)-1} & \frac{q^3 f_3(q)}{f_6(q)-1} & \frac{q^4 f_4(q)}{f_6(q)-1} & 0 & 0 & \cdots \\ \frac{q f_1(q)}{f_7(q)-1} & \frac{q^2 f_2(q)}{f_7(q)-1} & \frac{q^3 f_3(q)}{f_7(q)-1} & \frac{q^4 f_4(q)}{f_7(q)-1} & \frac{q^5 f_5(q)}{f_7(q)-1} & 0 & \cdots \\ \frac{q f_1(q)}{f_8(q)-1} & \frac{q^2 f_2(q)}{f_8(q)-1} & \frac{q^3 f_3(q)}{f_8(q)-1} & \frac{q^4 f_4(q)}{f_8(q)-1} & \frac{q^5 f_5(q)}{f_8(q)-1} & \frac{q^6 f_6(q)}{f_8(q)-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

defines a new triangular $q-$analogue of the $q-$Fibonacci matrix.

For $n \in \mathbb{N}$, the matrix transformation $y_n = (\tilde{f_q})_n(x)$ is denoted by

$$y_n = (\tilde{f_q})_n(x) = \frac{1}{f_{n+2}(q) - 1} \sum_{k=1}^{n} q^k f_k(q) x_k \tag{2}$$

and the sequence spaces $c_0(\tilde{f_q})$, $c(\tilde{f_q})$, $\ell_\infty(\tilde{f_q})$ and $\ell_p(\tilde{f_q})$ $(1 \le p < \infty)$ are defined by

$$c_0(\tilde{f_q}) = \left\{x = (x_k) \in \omega : \lim_{n\to\infty} (\tilde{f_q})_n(x) = 0\right\},$$
$$c(\tilde{f_q}) = \left\{x = (x_k) \in \omega : \lim_{n\to\infty} (\tilde{f_q})_n(x) \text{ exists}\right\},$$
$$\ell_\infty(\tilde{f_q}) = \left\{x = (x_k) \in \omega : \sup_{n\in\mathbb{N}} \left|\frac{1}{f_{n+2}(q) - 1}\sum_{k=1}^{n} q^k f_k(q) x_k\right| < \infty\right\},$$
$$\ell_p(\tilde{f_q}) = \left\{x = (x_k) \in \omega : \sum_n \left|\frac{1}{f_{n+2}(q) - 1}\sum_{k=1}^{n} q^k f_k(q) x_k\right|^p < \infty\right\}.$$

The sequence spaces $\ell_p(\tilde{f_q})$, $\ell_\infty(\tilde{f_q})$, $c_0(\tilde{f_q})$ and $c(\tilde{f_q})$ can be redefined by

$$\ell_p(\tilde{f_q}) = (\ell_p)_{\tilde{f_q}} \ (1 \le p < \infty),\ \ell_\infty(\tilde{f_q}) = (\ell_\infty)_{\tilde{f_q}}, \tag{3}$$
$$c_0(\tilde{f_q}) = (c_0)_{\tilde{f_q}} \text{ and } c(\tilde{f_q}) = (c)_{\tilde{f_q}}. \tag{4}$$

respectively, when (1) notation is considered.

Theorem 2.1. *The space $\ell_p(\tilde{f_q})$ is a BK−space normed by*

$$\|(\tilde{f_q})_n(x)\|_{\ell_p} = \|x\|_{\ell_p(\tilde{f_q})} = \left(\sum_n \left|(\tilde{f_q})_n(x)\right|^p\right)^{\frac{1}{p}}, \quad (1 \le p < \infty)$$

and the spaces $U(\tilde{f_q})$ are BK−spaces normed by

$$\|(\tilde{f_q})_n(x)\|_U = \|x\|_{U(\tilde{f_q})} = \sup_{n\in\mathbb{N}} \left|(\tilde{f_q})_n(x)\right|,$$

where $U \in \{\ell_\infty, c, c_0\}$.

Proof. The matrix $\tilde{f_q}$ is a triangle, and ℓ_∞ and ℓ_p are BK−spaces in terms of their natural norms, because (3) and (4) hold; Theorem 4.3.12 of [40, p. 63] states that the spaces $\ell_p(\tilde{f_q})$ and $\ell_\infty(\tilde{f_q})$ are BK−spaces with the given norms, where $(1 \le p < \infty)$.

The spaces $c_0(\tilde{f_q})$ and $c(\tilde{f_q})$ are BK−spaces with the stated norms, as per [40, p. 61] Theorem 4.3.2 □

Theorem 2.2. *The space $\ell_p(\tilde{f_q})$ $(1 \le p \le \infty)$ is linearly isomorphic to the ℓ_p.*

Proof. To prove that $S : \ell_p(\tilde{f_q}) \to \ell_p$, $(x \to y = Sx = \tilde{f_q}x \in \ell_p)$, is a linear and bijection transformation for $(1 \le p \le \infty)$ is sufficient.

S is obviously linear. In addition, S is implied to be injective as it is evident that $x = 0$ whenever $Sx = 0$.

Let us get $y = (y_n) \in \ell_p$ to show that S is surjective. We have

$$y_n = \frac{1}{f_{n+2}(q) - 1}\sum_{k=1}^{n} q^k f_k(q) x_k$$

and so

$$x_k = \frac{f_{k+2}(q) - 1}{q^k f_k(q)} y_k - \frac{f_{k+1}(q) - 1}{q^k f_k(q)} y_{k-1}.$$

For $(1 \leq p < \infty)$ we consider

$$\begin{aligned}\|x\|_{\ell_p(\tilde{f_q})} &= \left(\sum_n \left|(\tilde{f_q})_n(x)\right|^p\right)^{\frac{1}{p}} = \left(\sum_n \left|\frac{1}{f_{n+2}(q)-1}\sum_{k=1}^{n} q^k f_k(q) x_k\right|^p\right)^{\frac{1}{p}}\\ &= \left(\sum_n \left|\frac{1}{f_{n+2}(q)-1}\sum_{k=1}^{n} q^k f_k(q)\left(\frac{f_{k+2}(q)-1}{q^k f_k(q)}y_k - \frac{f_{k+1}(q)-1}{q^k f_k(q)}y_{k-1}\right)\right|^p\right)^{\frac{1}{p}}\\ &= \left(\sum_n \left|y_n\right|^p\right)^{\frac{1}{p}} = \|y\|_p < \infty\end{aligned}$$

and for $p = \infty$

$$\|x\|_{\ell_\infty(\tilde{f_q})} = \sup_{n\in\mathbb{N}} \left|(\tilde{f_q})_n(x)\right| = \|y\|_\infty < \infty.$$

The proof is now complete. □

Theorem 2.3. *The spaces $c_0(\tilde{f_q})$ and $c(\tilde{f_q})$ are linearly isomorphic to the spaces c_0 and c, respectively.*

Proof. A similar method may be used to prove the theorem using Theorem 2.2. □

Theorem 2.4. *The inclusions $c \subset c(\tilde{f_q})$ and $c_0 \subset c_0(\tilde{f_q})$ strictly hold for $q \geq 1$ and $0 < q < 1$, respectively.*

Proof. For any real number l and each $q \geq 1$, let us get $x \in c$, meaning that $x \to l$. The method $\tilde{f_q}$ is regular since the matrix $\tilde{f_q}$ satisfies the Silverman-Toeplitz criterias;

$$\begin{aligned}&\sup_{n\in\mathbb{N}} \sum_k \left|\tilde{f}_{nk}(q)\right| = \sup_{n\in\mathbb{N}} \left(\left|\frac{1}{f_{n+2}(q)-1}\sum_{k=1}^{n} q^k f_k(q)\right|\right) \leq 1 < \infty,\\ &\lim_{n\to\infty} \tilde{f}_{nk}(q) = 0,\\ &\lim_{n\to\infty} \sum_k \tilde{f}_{nk}(q) = \lim_{n\to\infty}\left(\frac{1}{f_{n+2}(q)-1}\sum_{k=1}^{n} q^k f_k(q)\right) = 1.\end{aligned}$$

Then we can see that $\tilde{f_q}x \to l$. So $x \in c(\tilde{f_q})$. In order to prove the $c_0 \subset c_0(\tilde{f_q})$, $l = 0$ is necessary.

Let us choose $x = (x_k) = \left(\frac{(-1)^k}{q^k f_k(q)}\right)$ to prove the strict of the inclusions for $0 < q < 1$. This sequence is obviously divergent. Therefore, $x \notin c$ and $x \notin c_0$, but

$$\begin{aligned}&\lim_{n\to\infty}\left(\frac{1}{f_{n+2}(q)-1}\sum_{k=1}^{n} q^k f_k(q) x_k\right)\\ &= \lim_{n\to\infty}\left(\frac{1}{f_{n+2}(q)-1}\sum_{k=1}^{n} q^k f_k(q)\left(\frac{(-1)^k}{q^k f_k(q)}\right)\right) = 0 < \infty.\end{aligned}$$

This indicates that $x \in c(\tilde{f_q})$ and $x \in c_0(\tilde{f_q})$. Consequently, $c_0 \subset c_0(\tilde{f_q})$ and $c \subset c(\tilde{f_q})$ are strict inclusions. □

Theorem 2.5. *The inclusion $\ell_p \subset \ell_p(\tilde{f_q})$ holds for $q \geq 1$ and the inclusion is strict for $q < 1$, where $1 \leq p \leq \infty$.*

Proof. Proving a number $K > 0$' s existence is sufficient to demonstrate that, for every $x \in \ell_p$, $\|x\|_{\ell_p(\tilde{f_q})} \leq K\|x\|_p$.

For $(1 < p < \infty)$ and $q \geq 1$, let us get $x \in \ell_p$. Applying From Hölder's inequality for $\forall n \in \mathbb{N}$, we possess

$$\begin{aligned}
\sum_{n=1}^{\infty} \left|(\tilde{f_q})_n(x)\right|^p &= \sum_{n=1}^{\infty} \left|\sum_{k=1}^{n} \frac{q^k f_k(q)}{f_{n+2}(q)-1} x_k\right|^p \\
&\leq \sum_{n=1}^{\infty} \Big(\sum_{k=1}^{n} \frac{q^k f_k(q)}{f_{n+2}(q)-1} |x_k|\Big)^p \\
&\leq \sum_{n=1}^{\infty} \Big(\sum_{k=1}^{n} \frac{q^k f_k(q)}{f_{n+2}(q)-1} |x_k|^p\Big)\Big(\sum_{k=1}^{n} \frac{q^k f_k(q)}{f_{n+2}(q)-1}\Big)^{p-1} \\
&= \sum_{n=1}^{\infty} \Big(\sum_{k=1}^{n} \frac{q^k f_k(q)}{f_{n+2}(q)-1} |x_k|^p\Big) \\
&= \sum_{k=1}^{\infty} |x_k|^p \left(q^k f_k(q) \sum_{n=k}^{\infty} \frac{1}{f_{n+2}(q)-1}\right).
\end{aligned}$$

So this means

$$\|x\|_{\ell_p(\tilde{f_q})} \leq K\|x\|_p, \tag{5}$$

where $K = \sup_{k\in\mathbb{N}} \left(\sum_{n=k}^{\infty} \frac{q^k f_k(q)}{f_{n+2}(q)-1}\right)$. Also for $p = \infty$, we take $x_k \in \ell_\infty$. Then, for all $k \in \mathbb{N}$, there exists a constant $K > 0$ such that $|x_k| \leq K$. Therefore, using the triangle inequality

$$|(\tilde{f_q})_n(x)| \leq \sum_{k=1}^{n} \frac{q^k f_k(q)}{f_{n+2}(q)-1} |x_k| \leq \sum_{k=1}^{n} \frac{q^k f_k(q)}{f_{n+2}(q)-1} K = K.$$

So $x \in \ell_p(\tilde{f_q})$.

Likewise, we skip the details because it is easy to prove the inequality (5) for $p = 1$. Consequently, the inclusion $\ell_p \subset \ell_p(\tilde{f_q})$ holds for $1 \leq p \leq \infty$. □

We give the following two theorems without proof.

Theorem 2.6. *The $\ell_p(\tilde{f_q}) \subset \ell_s(\tilde{f_q})$, if $1 \leq p < s$.*

Theorem 2.7. *For $q > 0$, the inclusion $c_0(\tilde{f_q}) \subset c(\tilde{f_q})$ is strict.*

Theorem 2.8. *For $q > 0$, the inclusion $\ell_p(\tilde{f_q}) \subset \ell_\infty(\tilde{f_q})$ is strict.*

Proof. Let us take $x = (x_n) \in \ell_p(\tilde{f_q})$. Then we have $\tilde{f_q}x \in \ell_p$. Since $\ell_p \subset \ell_\infty$, we can conclude $\tilde{f_q}x \in \ell_\infty$. So $x = (x_n) \in \ell_\infty(\tilde{f_q})$ which means $\ell_p(\tilde{f_q}) \subset \ell_\infty(\tilde{f_q})$. The sequence $x = (x_k) = (1^k)$ be examined for the inclusion's strict. Since

$$\sup_{n\in\mathbb{N}} \left|\sum_{k=1}^{n} \frac{q^k f_k(q)}{f_{n+2}(q)-1}(1^k)\right| = 1 < \infty,$$

we have $x \in \ell_\infty(\tilde{f_q})$. But since

$$\sum_n \left|\sum_{k=1}^{n} \frac{q^k f_k(q)}{f_{n+2}(q)-1}(1^k)\right|^p = \sum_n |1|^p \to \infty$$

we have $x \notin \ell_p(\tilde{f_q})$. □

Theorem 2.9. *The space $\ell_p(\tilde{f_q})$ is not a Hilbert space, where $p \in [1,\infty] - \{2\}$.*

Proof. We use the sequences

$$v = (v_n) = \left(\frac{f_3(q)-1}{qf_1(q)}, \frac{-f_3(q)+f_4(q)}{q^2 f_2(q)}, \frac{-f_4(q)+1}{q^3 f_3(q)}, 0, 0, \ldots\right)$$

and

$$u = (u_n) = \left(\frac{f_3(q)-1}{qf_1(q)}, \frac{-f_3(q)-f_4(q)+2}{q^2 f_2(q)}, \frac{f_4(q)-1}{q^3 f_3(q)}, 0, 0, \ldots\right)$$

for proof. The $\tilde{f_q}$ transformations of these sequences are as follows, respectively:

$\tilde{f_q}v = (1,1,0,0,\ldots)$ and $\tilde{f_q}u = (1,-1,0,0,\ldots)$.

Thus, $\tilde{f_q}(v+u) = (2,0,0,0,\ldots)$ and $\tilde{f_q}(v-u) = (0,2,0,0,\ldots)$ are obtained. Hence, the expression for $p \neq 2$ that results is as follows

$$\|v+u\|^2_{\ell_p(\tilde{f_q})} + \|v-u\|^2_{\ell_p(\tilde{f_q})} = 8 \neq 2^{2+\frac{2}{p}} = 2\left(\|v\|^2_{\ell_p(\tilde{f_q})} + \|u\|^2_{\ell_p(\tilde{f_q})}\right).$$

This implies that the parallelogram equality cannot be satisfied by the norm of the space $\ell_p(\tilde{f_q})$. □

Theorem 2.10. *The space $\ell_p(\tilde{f_q})$ is not absolute type, where $1 \leq p \leq \infty$.*

Proof. To show that it is not an absolute type, let us take a sequence defined by $x = (1,-1,0,0,\ldots)$. Next, we compute transformations $\tilde{f_q}u$ and $\tilde{f_q}|u|$ as the following:

$\tilde{f_q}u = (\frac{qf_1(q)}{f_3(q)-1}, \frac{-q^2 f_2(q)+qf_1(q)}{f_4(q)-1}, \frac{-q^2 f_2(q)+qf_1(q)}{f_5(q)-1}, \ldots)$

and

$\tilde{f_q}|u| = (\frac{qf_1(q)}{f_3(q)-1}, \frac{q^2 f_2(q)+qf_1(q)}{f_4(q)-1}, \frac{q^2 f_2(q)+qf_1(q)}{f_5(q)-1}, \ldots),$

where $|u| = |u_n|$. Since $\|u\|_{\ell_p(\tilde{f_q})} \neq \||u|\|_{\ell_p(\tilde{f_q})}$, the proof is finished. □

For $\ell_p(\tilde{f_q})$ $(1 \leq p < \infty)$, we now provide a basis.

Theorem 2.11. *For $1 \leq p < \infty$ and each fixed $k \in \mathbb{N}$, define a sequence $\xi^{(k)} \in \ell_p(\tilde{f_q})$ as*

$$(\xi^{(k)})_n = \begin{cases} \dfrac{(-1)^{n-k} f_{k+2}(q) - 1}{q^n f_n(q)} & , n-1 \leq k \leq n \\ 0 & , \text{otherwise} \end{cases} \quad (n \in \mathbb{N}). \tag{6}$$

Later, $\{\xi^{(k)}\}_{k\in\mathbb{N}}$ is a Schauder basis for the space $\ell_p(\tilde{f_q})$ and each $u \in \ell_p(\tilde{f_q})$ has a unique representation of the form

$$u = \sum_k (\tilde{f_q})_k(u)\xi^{(k)} \tag{7}$$

for each $k \in \mathbb{N}$.

Proof. Let us consider $1 \leq p < \infty$. Afterward, it is clear by (6) that $(\tilde{f_q})(\xi^{(k)}) = e^{(k)} \in \ell_p$ and hence $\xi^{(k)} \in \ell_p(\tilde{f_q})$.

Let us take $u \in \ell_p(\tilde{f_q})$ and for each non-negative integer m and all $k \in \mathbb{N}$ we put

$$u^{(m)} = \sum_k (\tilde{f_q})_k(u)\xi^{(k)}.$$

Then we can obtain

$$\tilde{f_q}(u^{(m)}) = \sum_{k=0}^{m} (\tilde{f_q})_k(u)(\tilde{f_q})(\xi^{(k)}) = \sum_{k=0}^{m} (\tilde{f_q})_k(u)e^{(k)}$$

and then

$$(\tilde{f_q})_n(u - u^{(m)}) = \begin{cases} 0 & ,(0 \leq n \leq m) \\ (\tilde{f_q})_n(x) & ,(n > m) \end{cases} \quad (n, m \in \mathbb{N}). \tag{8}$$

For any given $\varepsilon > 0$, there is a $m_0 \in \mathbb{N}$ such that

$$\sum_{k=m_0+1}^{\infty} \left|(\tilde{f_q})_n(u)\right|^p = \left(\frac{\varepsilon}{2}\right)^p.$$

As a result, for every $m > m_0$, we acquire

$$\begin{aligned} \|u - u^{(m)}\|_{\ell_p(\tilde{f_q})} &= \left(\sum_{k=m+1}^{\infty} \left|(\tilde{f_q})_n(u)\right|^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{k=m_0+1}^{\infty} \left|(\tilde{f_q})_n(u)\right|^p \right)^{\frac{1}{p}} \leq \frac{\varepsilon}{2} < \varepsilon, \end{aligned}$$

demonstrating that $\lim_{m\to\infty} \|u - u^{(m)}\|_{\ell_p(\tilde{f_q})} = 0$ and as a result, u can be stated as in (7).

To demonstrate the uniqueness of the expression, we assume the existence of another form (7), similar to

$$u = \sum_k (\tilde{g}_q)_k(u)\xi^{(k)}.$$

By using the continuous transform S, we have proved its isomorphism in Theorem 2.2, the equation that follows may be written as

$$(\tilde{f_q})_n(u) = \sum_k (\tilde{g}_q)_k(u)(\tilde{f_q})_n(\xi^{(k)}) = \sum_k (\tilde{g}_q)_k(u)\delta_{nk} = (\tilde{g}_q)_n(u).$$

This proves that the form (7) is unique. This concludes the proof. □

3. $\alpha-$, $\beta-$, $\gamma-$ duals of the space $\ell_p(\tilde{f_q})$

The $\alpha-$, $\beta-$, $\gamma-$ duals of the space $\ell_p(\tilde{f_q})$ are given in this section. Since $p = 1$ can be demonstrated by analogy, we will focus on the case $1 < p \leq \infty$. We serve the lemmas in Stieglitz and Tietz [36] to prove Theorem 3.5 and Theorem 3.6. Many researchers have examined sequence spaces, dual spaces, and matrix transforms utilizing the domain of certain matrices, such as [5, 13, 14, 17, 18, 25–28].

Take note that $(p^{-1} + r^{-1}) = 1$ for $(1 < p \leq \infty)$ and that F represents the family of all finite subsets of $\mathbb{N}$.

Lemma 3.1. $B = (b_{nk}) \in (\ell_p, \ell_1) \Leftrightarrow$

$$\sup_{K\in F} \sum_k \left|\sum_{n\in K} b_{nk}\right|^r < \infty.$$

Lemma 3.2. $B = (b_{nk}) \in (\ell_p, c) \Leftrightarrow$

$$\text{For } (\forall k \in \mathbb{N}) \lim_{n\to\infty} b_{nk} \text{ exists} \tag{9}$$

$$\sup_{n\in\mathbb{N}} \sum_k |b_{nk}|^r < \infty. \tag{10}$$

Lemma 3.3. $B = (b_{nk}) \in (\ell_\infty, c) \Leftrightarrow$ *(9) holds and*

$$\lim_{n\to\infty} \sum_k |b_{nk}| = \sum_k \left| \lim_{n\to\infty} b_{nk} \right|. \tag{11}$$

Lemma 3.4. $B = (b_{nk}) \in (\ell_p, \ell_\infty) \Leftrightarrow$ *(10) holds with* $(1 < p \leq \infty)$.

Theorem 3.5. *The set*

$$D_1(q) = \left\{ b = (b_k) \in \omega : \sup_{K\in F} \sum_k \left| \sum_{n\in K} \frac{(-1)^{n-k} f_{k+2}(q) - 1}{q^n f_n(q)} b_n \right|^r < \infty \right\}$$

is the $\alpha-$*dual of the space* $\ell_p(\tilde{f_q})$, *where* $1 < p \leq \infty$.

Proof. For $1 < p \leq \infty$ and any sequence $b = (b_n) \in \omega$, let us define a matrix G by

$$G = (g_{nk}) = \begin{cases} \dfrac{(-1)^{n-k} f_{k+2}(q) - 1}{q^n f_n(q)} b_n & , n-1 \leq k \leq n \\ 0 & , \text{otherwise} \end{cases}.$$

Furthermore, for each $x = (x_n) \in \omega$, we get $y = \tilde{f_q} x$. After it tracks by (2)

$$b_n x_n = \sum_{k=n-1}^{n} \frac{(-1)^{n-k} f_{k+2}(q) - 1}{q^n f_n(q)} b_n y_k = G_n(y) \quad (n \in \mathbb{N}). \tag{12}$$

Because of (12), we obtain that $bx = (b_n x_n) \in \ell_1$ whenever $x \in \ell_p(\tilde{f_q})$ if and only if $Gy \in \ell_1$ whenever $y \in \ell_p$.

We can see from Lemma 3.1 that

$$\sup_{K\in F} \sum_k \left| \sum_{n\in K} \frac{(-1)^{n-k} f_{k+2}(q) - 1}{q^n f_n(q)} b_n \right|^r < \infty$$

and so $\left(\ell_p(\tilde{f_q})\right)^\alpha = D_1(q)$. □

Theorem 3.6. *Define the following sets* $D_2(q)$, $D_3(q)$, $D_4(q)$ *as:*

$$D_2(q) = \left\{ b = (b_k) \in \omega : \sum_{j=k}^{\infty} \frac{(-1)^{j-k} f_{k+2}(q) - 1}{q^j f_j(q)} b_j \quad \text{exists}, \forall k \in \mathbb{N} \right\},$$

$$D_3(q) = \left\{ b = (b_k) \in \omega : \sup_{n\in\mathbb{N}} \sum_{k=1}^{n} \left| \sum_{j=n-1}^{n} \frac{(-1)^{j-k} f_{k+2}(q) - 1}{q^j f_j(q)} b_j \right|^r < \infty \right\},$$

$$D_4(q) = \left\{ b = (b_k) \in \omega : \lim_{n\to\infty} \sum_{k=1}^{n} \left| \sum_{j=n-1}^{n} \frac{(-1)^{j-k} f_{k+2}(q) - 1}{q^j f_j(q)} b_j \right| = \sum_k \left| \sum_{j=k}^{\infty} \frac{(-1)^{j-k} f_{k+2}(q) - 1}{q^j f_j(q)} b_j \right| < \infty \right\}.$$

Then we have

a) $\left(\ell_p(\tilde{f_q})\right)^{\beta} = D_2(q) \cap D_3(q)$ *and*

b) $\left(\ell_\infty(\tilde{f_q})\right)^{\beta} = D_2(q) \cap D_4(q)$

for $1 < p < \infty$.

Proof. Let us get $b = (b_k) \in \omega$ and look at the equality

$$\begin{aligned}\sum_{k=1}^{n} b_k x_k &= \sum_{k=1}^{n} b_k \left(\sum_{j=n-1}^{n} \frac{(-1)^{k+1} f_{k+2}(q) - 1}{q^j f_j(q)} y_j \right) \\ &= \sum_{k=1}^{n} \left(\sum_{j=n-1}^{n} \frac{(-1)^{k+1} f_{k+2}(q) - 1}{q^j f_j(q)} b_j \right) y_k = D_n(y),\end{aligned} \tag{13}$$

where $D = (d_{nk})$ is determined by

$$d_{nk} = \begin{cases} \sum_{j=n-1}^{n} \dfrac{(-1)^{k+1} f_{k+2}(q) - 1}{q^j f_j(q)} b_j & , n-1 \le k \le n \\ 0 & , otherwise \end{cases}.$$

After, we deduce from Lemma 3.2 using (2) that $Dy \in c$ whenever $y = (y_k) \in \ell_p$ if and only if $bx = (b_k x_k) \in cs$ whenever $x \in \ell_p(\tilde{f_q})$. Therefore, $(b_k) \in \left(\ell_p(\tilde{f_q})\right)^{\beta}$ if and only if $(b_k) \in D_2(q)$ and $(b_k) \in D_3(q)$ are defined by (9) and (10), respectively. Consequently $\left(\ell_p(\tilde{f_q})\right)^{\beta} = D_2(q) \cap D_3(q)$.

An equivalent proof can be formulated when $p = \infty$ by utilizing Lemma 3.3 in place of Lemma 3.2 through analogous approaches. □

Theorem 3.7. $\left(\ell_p(\tilde{f_q})\right)^{\gamma} = D_3(q)$, *for* $1 < p \le \infty$.

Proof. One may utilize (13) to produce the proof by using Lemma 3.4. □

4. Matrix transformations associated with the space $\ell_p(\tilde{f_q})$

The matrix classes $(\ell_p(\tilde{f_q}), U)$ are characterized in this section, where $1 < p \le \infty$ and $U \in \{\ell_\infty, \ell_1, c, c_0\}$. We utilize

$$\tilde{b}_{nk} = \sum_{j=k-1}^{k} \frac{(-1)^{k+1} f_{k+2}(q) - 1}{q^j f_j(q)} b_{nj}$$

in order to achieve brevity.

The following lemma forms the basis of our findings.

Lemma 4.1. (see [29], Theorem 4.1)*) Let μ be an arbitrary subset of ω, U a triangular matrix, V its inverse, and λ a FK–space. Define $H^{(n)} = (h^{(n)}_{mk})$ and $H = (h_{nk})$ by*

$$H^{(n)} = h^{(n)}_{mk} = \begin{cases} \sum_{j=k}^{m} b_{nj} v_{jk} & , 1 \le k \le m \\ 0 & , k > m \end{cases}, \qquad H = (h_{nk}) = \sum_{j=k}^{\infty} b_{nj} v_{jk},$$

respectively. Thus we obtain $H^{(n)} = (h^{(n)}_{mk}) \in (\lambda, c)$ and $H = (h_{nk}) \in (\lambda, \mu)$ if and only if $B = (b_{nk}) \in (\lambda_U, \mu)$ (see Theorem 4.1 of [29]).

The following conditions are now listed:

$$\sup_{m\in\mathbb{N}}\sum_{k=1}^{m}\left|\sum_{j=m-1}^{m}\frac{(-1)^{k+1}f_{k+2}(q)-1}{q^{j}f_{j}(q)}b_{nj}\right|^{r}<\infty, \tag{14}$$

$$\lim_{m\to\infty}\sum_{j=m-1}^{m}\frac{(-1)^{k+1}f_{k+2}(q)-1}{q^{j}f_{j}(q)}b_{nj}=\tilde{b}_{nk}, \qquad \forall n,k\in\mathbb{N}, \tag{15}$$

$$\lim_{m\to\infty}\sum_{k=1}^{m}\left|\sum_{j=m-1}^{m}\frac{(-1)^{k+1}f_{k+2}(q)-1}{q^{j}f_{j}(q)}b_{nj}\right|=\sum_{k}|\tilde{b}_{nk}| \qquad \forall n\in\mathbb{N}, \tag{16}$$

$$\sup_{m\in\mathbb{N}}\sum_{k}|\tilde{b}_{nk}|^{r}<\infty, \tag{17}$$

$$\sup_{N\in F}\sum_{k}\left|\sum_{n\in\mathbb{N}}\tilde{b}_{nk}\right|^{r}<\infty, \tag{18}$$

$$\lim_{n\to\infty}\tilde{b}_{nk}=\tilde{\alpha}_{k}; \quad k\in\mathbb{N}, \tag{19}$$

$$\lim_{n\to\infty}\sum_{k}|\tilde{b}_{nk}|=\sum_{k}|\tilde{\alpha}_{k}|, \tag{20}$$

$$\lim_{n\to\infty}\sum_{k}\tilde{b}_{nk}=0, \tag{21}$$

$$\sup_{n,k\in\mathbb{N}}|\tilde{b}_{nk}|<\infty, \tag{22}$$

$$\sup_{k,m\in\mathbb{N}}\left|\sum_{j=m-1}^{m}\frac{(-1)^{k+1}f_{k+2}(q)-1}{q^{j}f_{j}(q)}b_{nj}\right|<\infty, \tag{23}$$

$$\sup_{k\in\mathbb{N}}\sum_{n}|\tilde{b}_{nk}|<\infty, \tag{24}$$

$$\sup_{N,K\in F}\left|\sum_{n\in N}\sum_{k\in K}\tilde{b}_{nk}\right|<\infty. \tag{25}$$

Thus, utilizing Lemma 4.1 and the findings in [36], we may deduce the following results from the given conditions.

Theorem 4.2.

a) $B=(b_{nk})\in\left(\ell_{1}(\tilde{f_{q}}),\ell_{\infty}\right)\Leftrightarrow(15),(22)$ *and* (23) *hold.*

b) $B=(b_{nk})\in\left(\ell_{1}(\tilde{f_{q}}),c\right)\Leftrightarrow(15),(19),(22)$ *and* (23) *hold.*

c) $B=(b_{nk})\in\left(\ell_{1}(\tilde{f_{q}}),c_{0}\right)\Leftrightarrow(15)$, *with* $\tilde{\alpha}_{k}=0,(19),(22)$ *and* (23) *hold.*

d) $B=(b_{nk})\in\left(\ell_{1}(\tilde{f_{q}}),\ell_{1}\right)\Leftrightarrow(15),(23)$ *and* (24) *hold.*

Theorem 4.3. *For* $1<p<\infty$,

a) $B=(b_{nk})\in\left(\ell_{p}(\tilde{f_{q}}),\ell_{\infty}\right)\Leftrightarrow(14),(15)$ *and* (17) *hold.*

b) $B=(b_{nk})\in\left(\ell_{p}(\tilde{f_{q}}),c\right)\Leftrightarrow(14),(15),(17)$ *and* (19) *hold.*

c) $B=(b_{nk})\in\left(\ell_{p}(\tilde{f_{q}}),c_{0}\right)\Leftrightarrow(14),(15),(17)$ *and with* $\tilde{\alpha}_{k}=0$ (19) *hold.*

d) $B=(b_{nk})\in\left(\ell_{p}(\tilde{f_{q}}),\ell_{1}\right)\Leftrightarrow(14),(15)$ *and* (18) *hold.*

Theorem 4.4.

a) $B = (b_{nk}) \in \left(\ell_\infty(\tilde{f_q}), \ell_\infty\right) \Leftrightarrow (15), (16)$ *and in case* $r = 1$ (17) *hold.*

b) $B = (b_{nk}) \in \left(\ell_\infty(\tilde{f_q}), c\right) \Leftrightarrow (15), (16), (19)$ *and* (20) *hold.*

c) $B = (b_{nk}) \in \left(\ell_\infty(\tilde{f_q}), c_0\right) \Leftrightarrow (15), (16)$ *and* (21) *hold.*

d) $B = (b_{nk}) \in \left(\ell_\infty(\tilde{f_q}), \ell_1\right) \Leftrightarrow (15), (16)$ *and* (25) *hold.*

5. Certain geometric properties of the space $\ell_p(\tilde{f_q})$

One of the most significant properties in functional analysis is the geometric property of Banach spaces. We look at [12, 15, 16, 19, 21, 24, 30, 33] for more details.

Certain geometric properties of the space $\ell_p(\tilde{f_q})$ $(1 < p < \infty)$ are given in this section.

If every bounded sequence (b_n) in U enables a subsequence (s_n) such that the sequence $\{t_k(s)\}$ is convergent in the norm in U, then U is said to satisfy the Banach-Saks property (see [21]), where

$$\{t_k(s)\} = \frac{1}{k+1}(s_0 + s_1 + \ldots + s_k) \quad (k \in \mathbb{N}). \tag{26}$$

A Banach space U has the weak Banach-Saks property for given any weakly null sequence $(b_n) \subset U$ if there exists a subsequence (s_n) of (b_n) such that the $\{t_k(s)\}$ is strongly convergent to zero.

According to García-Falset in [15], the coefficient is as follows:

$$R(U) = \sup\left\{\liminf_{n\to\infty} \|b_n - b\| : (b_n) \subset B(U), b_n \xrightarrow{w} b, b \in B(U)\right\}, \tag{27}$$

where the unit ball of U is indicated by $B(U)$.

Remark 5.1. *A Banach space U possesses the weak fixed point property for* $R(U) < 2$ *[16].*

For $\forall n \in \mathbb{N}$, some $M > 0$ and $1 < p < \infty$, if every weakly null sequence (b_k) possesses a subsequence (b_{k_l}) such that

$$\left\|\sum_{l=1}^{n} b_{k_l}\right\| < Mn^{1/p}, \tag{28}$$

a Banach space possesses the Banach-Saks type p or the property $(BS)_p$ (see [30]).

With $1 < p < \infty$, we can now get the following results from the geometric properties of the space $\ell_p(\tilde{f_q})$.

Theorem 5.2. *The space* $\ell_p(\tilde{f_q})$ $(1 < p < \infty)$ *possesses the Banach-Saks type p.*

Proof. We take (ε_n) sequence such that $(\varepsilon_n) > 0$ for every $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \varepsilon_n \leq \frac{1}{2}$, and moreover we take a weakly null sequence (b_n) in $B(\ell_p(\tilde{f_q}))$. Set $s_0 = b_0 = 0$ and $s_1 = b_{n_1} = b_1$. After, there is a $u_1 \in \mathbb{N}$ such that

$$\left\|\sum_{i=u_1+1}^{\infty} s_1(i)e^{(i)}\right\|_{\ell_p(\tilde{f_q})} < \varepsilon_1. \tag{29}$$

There is an $n_2 \in \mathbb{N}$ such that

$$\left\|\sum_{i=1}^{u_1} b_n(i)e^{(i)}\right\|_{\ell_p(\tilde{f_q})} < \varepsilon_1 \tag{30}$$

when $n \geq n_2$, because (b_n) is a weakly null sequence implies $b_n \to 0$ coordinatewise. Set $s_2 = b_{n_2}$. Then there is an $u_2 > u_1$ such that

$$\left\| \sum_{i=u_2+1}^{\infty} s_2(i) e^{(i)} \right\|_{\ell_p(\tilde{f}_q)} < \varepsilon_2. \tag{31}$$

Considering that $b_n \to 0$ coordinatwise, there is an such that $n_3 > n_2$

$$\left\| \sum_{i=1}^{u_2} b_n(i) e^{(i)} \right\|_{\ell_p(\tilde{f}_q)} < \varepsilon_2, \tag{32}$$

when $n \geq n_3$.

Two increasing subsequences, (u_i) and (n_i), could be obtained when we continue in this way, such that

$$\left\| \sum_{i=1}^{u_j} b_n(i) e^{(i)} \right\|_{\ell_p(\tilde{f}_q)} < \varepsilon_j, \tag{33}$$

for each $n \geq n_{j+1}$ and

$$\left\| \sum_{i=u_j+1}^{\infty} s_j(i) e^{(i)} \right\|_{\ell_p(\tilde{f}_q)} < \varepsilon_j. \tag{34}$$

where $s_j = b_{n_j}$. Thus,

$$\begin{aligned} \left\| \sum_{j=1}^{n} s_j \right\|_{\ell_p(\tilde{f}_q)} &= \left\| \sum_{j=1}^{n} \left(\sum_{i=1}^{u_{j-1}} s_j(i) e^{(i)} + \sum_{i=u_{j-1}+1}^{u_j} s_j(i) e^{(i)} + \sum_{i=u_j+1}^{\infty} s_j(i) e^{(i)} \right) \right\|_{\ell_p(\tilde{f}_q)} \\ &\leq \left\| \sum_{j=1}^{n} \left(\sum_{i=u_{j-1}+1}^{u_j} s_j(i) e^{(i)} \right) \right\|_{\ell_p(\tilde{f}_q)} + 2 \sum_{j=1}^{n} \varepsilon_j. \end{aligned}$$

Alternatively, we can see that $\|x\|_{\ell_p(\tilde{f}_q)} \leq 1$. Hence, we have that

$$\begin{aligned} &\left\| \sum_{j=1}^{n} \left(\sum_{i=u_{j-1}+1}^{u_j} s_j(i) e^{(i)} \right) \right\|_{\ell_p(\tilde{f}_q)}^p = \\ &= \sum_{j=1}^{n} \sum_{i=u_{j-1}+1}^{u_j} \left| \sum_{k=1}^{i} \frac{q^k f_k(q)}{f_{k+2}(q) - 1} s_j(k) \right|^p \\ &\leq \sum_{j=1}^{n} \sum_{i=1}^{\infty} \left| \sum_{k=1}^{i} \frac{q^k f_k(q)}{f_{k+2}(q) - 1} s_j(k) \right|^p \leq n. \end{aligned}$$

Thus, it may be obtained that

$$\left\| \sum_{j=1}^{n} \left(\sum_{i=u_{j-1}+1}^{u_j} s_j(i) e^{(i)} \right) \right\|_{\ell_p(\tilde{f}_q)} \leq n^{\frac{1}{p}}.$$

Making use of the knowledge that $1 \leq n^{\frac{1}{p}}$ for all $n \in \mathbb{N}$ and $1 < p < \infty$, we possess

$$\left\| \sum_{j=1}^{n} s_j \right\|_{\ell_p(\tilde{f_q})} \leq n^{\frac{1}{p}} + 1 \leq 2n^{\frac{1}{p}}.$$

As a consequence, the space $\ell_p(\tilde{f_q})$ possesses the Banach-Saks type p. This ends the proof. □

Remark 5.3. *Because the space $\ell_p(\tilde{f_q})$ is linearly isomorphic to ℓ_p, $R(\ell_p(\tilde{f_q})) = R(\ell_p) = 2^{\frac{1}{p}}$.*

Remarks 5.1 and Remarks 5.3 lead us to the following theorem.

Theorem 5.4. *The space $\ell_p(\tilde{f_q})$ $(1 < p < \infty)$ possesses the weak fixed point property.*

6. Conclusion

The new triangle matrix with $q-$Fibonacci numbers is utilized in this article to define the sequence spaces $c_0(\tilde{f_q})$, $c(\tilde{f_q})$, $\ell_\infty(\tilde{f_q})$ and $\ell_p(\tilde{f_q})$ $(1 \leq p < \infty)$. The variety of q has a major impact on the inclusion links between these spaces. Then, we looked at the topological and certain geometric properties of the space $\ell_p(\tilde{f_q})$.

The $q-$Fibonacci numbers, which play a significant role in algebra, were moved to the area of sequence spaces and summability, which is an invention.

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