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Stabilities of Ulam-Hyers type for a class of nonlinear fractional differential equations with integral boundary conditions in Banach spaces

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Abstract. Motivated by the knowledge of the existence of continuous solutions of a certain fractional boundary value problem with integral boundary conditions, we present in here –in a unified manner– new sufficient conditions to conclude the existence and uniqueness of continuously differentiable solutions to this fractional boundary value problem and analyse its stability in the sense of Ulam-Hyers and Ulam-Hyers-Rassias. After presenting the main conclusions, two illustrative examples are provided to verify the effectiveness of the proposed theoretical results.

1. Introduction

In recent years we have witnessed a great growth in the investigation of different types of properties related to fractional differential equations and fractional integral equations (see [1–7, 24, 25, 27, 30, 34, 39]). Much of this development and interest comes directly from different applications where such equations, with this fractional characteristic, play a decisive role. In fact, fractional order derivatives and fractional integration have proven to be able to closely interpret real-life events that can fall under diverse disciplines such as physics, chemistry, mechanics, biology, engineering, etc. (cf., for example, [20, 22, 25, 28, 30, 34, 35]).

Although the origin of fractional calculus is known to date back to 1695, when Leibniz wrote his famous reply letter to L'Hôpital suggesting the possibility to consider a derivative of fractional order, it was not until later that Lacroix (in 1819) introduced the fractional derivative (based on the expression for the nth

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derivative of the power function). Today, the most used fractional derivatives are certainly the Riemann-Liouville and Caputo fractional derivatives, which play an immodest role in the fractional order differential equations area.

In any case, in addition to the definitions in the Riemann-Liouville and Caputo sense, there is also a huge variety of other possibilities for fractional derivatives that have already been introduced and used, such as Grünwald-Letnikov, Caputo-Hadamard, Caputo-Fabrizio, Losada-Nieto, Weyl, Marchaud, Hadamard, Chen, Davidson-Essex, Canavati, Jumarie, Hilfer, Katugampola, Hilfer-Katugampola, Atangana-Baleanu Caputo, Atangana-Baleanu Riemann-Liouville, Sun-Hao-Zhang-Baleanu, Yang et al, or even global generalisations such as Ψ-Caputo and Ψ-Hilfer. It should be noted that, as a rule, for the most typical function spaces to be considered in the domains of the corresponding fractional derivative operators, fractional derivation does not satisfy the additive property in its derivative orders when there are successive compositions of the fractional derivative operator (with different non-integer positive derivative orders). That is, there is no index law property [26], or there is no semigroup property for these fractional derivative operators (at least when some restrictive conditions or the so-called "strict sense criterion" are not considered, cf. [36]). This last circumstance has led to a certain criticism of these fractional derivative operators. However, it is also recognised that sometimes not satisfying the property of the index law opens the door to modelling real phenomena in a more appropriate way and therefore increases the usefulness of these operators in applying mathematics to real-life phenomena. This inherent circumstance in modelling phenomena is undoubtedly one of the reasons why fractional calculus continues to be widely applied and the reason for multiple different definitions, with different nuances in the properties of the derivative operators that are considered in each application. In our case, we chose to work with the Riemann-Liouville case only from a perspective of continuity in relation to work already considered on the same problem. Therefore, there was no crucial motivation here to choose this or avoid another type of fractional operator. Comparisons between the various definitions of fractional derivative operators are now well-known and enable appropriate choices to be made, particularly when the main concern is real-life problems and their modelling (see, for example, [33, 36] and the references cited there).

As a rule, it is rare to find an explicit exact solution of a differential equation (or integral equation) of fractional order. In this way, it is very important to have additional knowledge about approximate solutions and, in particular, about different types of eventual stabilities that those equations may present. That is why recently several methods and problems are being analyzed for additional information in that way (see, for example, [8, 11–19, 23, 29, 31, 37, 38, 40] and the references contained therein).

In this work we will be especially concerned with the study of Ulam-Hyers and Ulam-Hyers-Rassias stabilities [9, 21, 32] for a given fractional boundary value problem (FBVP), involving nonlinear conditions and incorporating integral boundary conditions. Such a problem will be Ulam-Hyers stable if, for each solution of the fractional problem, there is an approximate solution of the problem in question that approaches it (in an appropriate defined distance). It should be noted that Ulam formulated the stability of a functional equation, which was later solved by Hyers [21] using an additive function defined on a Banach space. This result led Aoki [9] and Rassias [32] to study and generalize the concept of stability, establishing what is currently more commonly called Ulam-Hyers-Rassias stability.

Motivated by the above, the main goal of this paper is to investigate a condition not only to the existence but also to the uniqueness of continuously differentiable solutions to the problem in question, and that will be also a sufficient condition to ensure its stabilities of Ulam-Hyers and Ulam-Hyers-Rassias types. This will be mainly based on a certain iterative scheme and a consequent convergence within a fixed point method.

The paper is organized such that in the next section we will formulate the problem in mathematical terms and recall the background materials and preliminaries. In Section 3, focused on the aforementioned method, we investigate the existence and uniqueness of a continuously differentiable solution. Further on, in Section 4, an appropriate deduction is performed to derive the Ulam-Hyers and the Ulam-Hyers-Rassias stabilities of the problem under study for the same conditions exhibited in Section 3. The paper ends with two concrete examples to illustrate the results obtained and a brief conclusion on the work carried out, as well as additional considerations on future possibilities.

2. Formulation of the Problem and Background Material

In this paper we will analyse different types of stabilities for the following fractional boundary value problem (FBVP) with nonlinear integral conditions

$$\begin{cases} {}^{C}D^{\alpha} y(t) = f(t, y(t)), \qquad t \in J = [0, b] \\ y(0) - y'(0) = \int_0^b g(s, y(s))ds \\ y(b) + y'(b) = \int_0^b h(s, y(s))ds \end{cases} \tag{2.1}$$

where $\alpha \in (1, 2)$ is the fractional order of differentiation, ${}^{C}D^{\alpha}$ denotes the Caputo fractional derivative operator, $b \in \mathbb{R}^+$ and $f, g, h : J \times E \to E$ are given E-valued functions, for some Banach space E endowed with a norm $\|\cdot\|_E$.

We must clarify that the existence of solutions of the FBVP (2.1) was already analysed in [12] for the case of $E = \mathbb{R}$ and considering fixed point arguments based on operators that act on the solutions of the FBVP (2.1) when considering as a domain the space $C(J)$ of all real-valued continuous functions defined on J, endowed with the supremum norm. The existence of continuous solutions of the FBVP (2.1) was also studied in [11], now in the more general case of an abstract Banach space $(E, \|.\|_E)$, using mainly techniques associated with measures of noncompactness and a Mönch type fixed point argument, by considering an operator N that acts on the solutions of the FBVP (2.1) in the framework of the following spaces

$$N : C(J, E) \to C(J, E)$$

where $C(J, E)$ denotes the space of E-valued continuous functions, defined on $J = [0, b]$, endowed with the supremum norm

$$\|y\|_{C(J,E)} := \sup_{t\in[0,b]} \|y(t)\|_E.$$

As already mentioned in general terms, the aim of this work is to obtain sufficient conditions to guarantee the stability of the Ulam-Hyers type and the Ulam-Hyers-Rassias type for the FBVP (2.1) and to consider $C^1(J, E)$ as the natural space of solutions to the FBVP (2.1) for which we should guarantee their existence and uniqueness.

Here, $C^1(J, E)$ denotes the space of all E-valued continuous functions whose first derivative is also continuous, defined on $J = [0, b]$, and endowed with the (natural) norm

$$\|y\|_{C^1(J,E)} := \sup_{t\in[0,b]} \|y(t)\|_E + \sup_{t\in[0,b]} \|y'(t)\|_E.$$

In this work, we will often use the well-known Euler Gamma function given by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$, $\alpha > 0$.

We will now recall and introduce several definitions that we will be using in the next sections.

Definition 2.1. *[25] For a continuous function y, given on the interval $(0, \infty)$, the Caputo fractional derivative of order $\alpha \in \mathbb{R}^+$, is defined by*

$${}^{C}D^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

provided the right-hand side is pointwise defined on $(0, \infty)$, where $n \in \mathbb{N}$ is such that $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.2. *[10] Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\gamma : \Omega_E \to [0, \infty]$ defined by*

$$\gamma(B) = \inf\{\varepsilon > 0 : B \subset \Omega_E \text{ can be covered with a finite number of sets of diameter not greater than } \varepsilon\}.$$

Definition 2.3. *(i) We say that the FBVP (2.1) has the Ulam-Hyers stability if for each function y satisfying*

$$y(0) - y'(0) = \int_0^b g(s, y(s))ds$$
$$y(b) + y'(b) = \int_0^b h(s, y(s))ds$$

and

$$\left\| {}^C D^\alpha y(t) - f(t, y(t)) \right\|_E \leq \theta,\ t \in J,\ \theta \geq 0, \tag{2.2}$$

there is a solution y_0 of the FBVP (2.1) and a constant $C > 0$ (independent of y and y_0) such that

$$\left\| y(t) - y_0(t) \right\|_E \leq C\theta,\ t \in J. \tag{2.3}$$

(ii) If instead of θ in (2.2) and (2.3), we have a nonnegative function σ (defined on J), then we say that the FBVP (2.1) is Ulam-Hyers-Rassias stable.

3. Existence and Uniqueness of Solution

From [12] and [11], we already know that the FBVP (2.1) is equivalent to the integral equation

$$y(t) = (Ty)(t), \tag{3.1}$$

where

$$(Ty)(t) = P_y(t) + \int_0^b G(t, s) f(s, y(s))ds, \tag{3.2}$$

with

$$P_y(t) = A(t)\int_0^b g(s, y(s))ds + B(t)\int_0^b h(s, y(s))ds,$$
$$A(t) = \frac{b+1-t}{b+2},$$
$$B(t) = \frac{t+1}{b+2},$$
$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1+t)(b-s)^{\alpha-1}}{(b+2)\Gamma(\alpha)} - \frac{(1+t)(b-s)^{\alpha-2}}{(b+2)\Gamma(\alpha-1)}, & 0 \leq s \leq t \\ -\frac{(1+t)(b-s)^{\alpha-1}}{(b+2)\Gamma(\alpha)} - \frac{(1+t)(b-s)^{\alpha-2}}{(b+2)\Gamma(\alpha-1)}, & t \leq s \leq b \end{cases}. \tag{3.3}$$

It is important to remark that due the presence of the monomial $(b-s)^{\alpha-2}$ (and the range of values of $\alpha \in (1,2)$), in (3.3), we can not guarantee the boundedness of G.

Anyway, when considering

$$t \mapsto \int_0^b |G(t,s)|\, ds$$

we are facing a continuous function on $J = [0, b]$, and so bounded. In view of this, we shall use the notation

$$\widetilde{G} = \sup_{t\in[0,b]} \int_0^b |G(t,s)|\, ds. \tag{3.4}$$

As pointed out before, using mainly the Kuratowski measure of noncompactness γ as a map

$$\gamma : \Omega_E \to [0, \infty]$$

(for bounded subsets Ω_E of E), cf. [10] and [11], the authors of the last paper obtained the following conditions (as sufficient conditions) to ensure the existence of solutions of FBVP (2.1):

(C_1) The maps $f, g, h : [0, b] \times E \to E$ are Carathéodory.

(C_2) There are elements p_f, p_g, p_h in $L^\infty([0, b], \mathbb{R}_+)$ so that

$$\left\|f(t, y)\right\|_E \leq p_f(t)\|y\|_E \quad \text{(a.e. } t \in [0, b] \text{ and for each } y \in E),$$
$$\left\|g(t, y)\right\|_E \leq p_g(t)\|y\|_E \quad \text{(a.e. } t \in [0, b] \text{ and for each } y \in E),$$
$$\left\|h(t, y)\right\|_E \leq p_h(t)\|y\|_E \quad \text{(a.e. } t \in [0, b] \text{ and for each } y \in E).$$

(C_3) For each bounded set $B \in \Omega_E \subset E$ and almost each $t \in [0, b]$, it holds

$$\lim_{k \to 0^+} \gamma(w(J_{t,k} \times B)) \leq p_w(t).\gamma(B)$$

for $w = f, g, h$, and $J_{t,k} = [t - k, t] \cap [0, b]$.

(C_4) $\frac{b(b+1)}{b+2}\left(\left\|p_g\right\|_{L^\infty} + \left\|p_h\right\|_{L^\infty}\right) + \widetilde{G}\left\|p_f\right\|_{L^\infty} < 1.$

Using a different method, we will now obtain other sufficient conditions that ensure the existence and the uniqueness of $C^1([0, b], E)$ solutions for the FBVP (2.1).

Theorem 3.1. *If f, g and h satisfy the Lipschitz conditions*

$$\left\|g(s, y_1(s)) - g(s, y_0(s))\right\|_E \leq L_g \left\|y_1(s) - y_0(s)\right\|_E, \ s \in [0, b], \tag{3.5}$$
$$\left\|h(s, y_1(s)) - h(s, y_0(s))\right\|_E \leq L_h \left\|y_1(s) - y_0(s)\right\|_E, \ s \in [0, b], \tag{3.6}$$
$$\left\|f(s, y_1(s)) - f(s, y_0(s))\right\|_E \leq L_f \left\|y_1(s) - y_0(s)\right\|_E, \ s \in [0, b], \tag{3.7}$$

for some constants L_g, L_h and L_f, and

$$C := \frac{b(b+1)}{b+2}\left(L_g + L_h\right) + \widetilde{G}L_f < 1, \tag{3.8}$$

then the FBVP (2.1) admits one and only one solution $y \in C^1([0, b], E)$.

Proof. Having in mind (3.1), we will introduce an iterative scheme defined by

$$y_{n+1}(t) = (Ty_n)(t), \tag{3.9}$$

for T defined in (3.2), and will evaluate $\|y_{n+1}(t) - y_n(t)\|_E$ in the following way:

$$\begin{aligned} \|y_{n+1}(t) - y_n(t)\|_E &= \Bigg\|A(t)\int_0^b g(s, y_n(s))ds + B(t)\int_0^b h(s, y_n(s))ds \\ &\quad + \int_0^b G(t, s)f(s, y_n(s))ds - A(t)\int_0^b g(s, y_{n-1}(s))ds \\ &\quad - B(t)\int_0^b h(s, y_{n-1}(s))ds - \int_0^b G(t, s)f(s, y_{n-1}(s))ds\Bigg\|_E \\ &\leq \frac{b+1}{b+2}L_g\int_0^b \|y_n(s) - y_{n-1}(s)\|_E ds \\ &\quad + \frac{b+1}{b+2}L_h\int_0^b \|y_n(s) - y_{n-1}(s)\|_E ds \\ &\quad + \widetilde{G}\sup_{t\in[0,b]}\|y_n(t) - y_{n-1}(t)\|_E L_f \\ &\leq C\sup_{t\in[0,b]}\|y_n(t) - y_{n-1}(t)\|_E, \end{aligned} \tag{3.10}$$

with $C = \frac{b(b+1)}{b+2}\left(L_g + L_h\right) + \widetilde{G}L_f$.

Hence,

$$\begin{aligned}\|y_{n+1}(t) - y_n(t)\|_E &\leq C \sup_{t\in[0,b]} \|y_n(t) - y_{n-1}(t)\|_E \\ &\leq C^2 \sup_{t\in[0,b]} \|y_{n-1}(t) - y_{n-2}(t)\|_E \\ &\vdots \\ &\leq C^n \sup_{t\in[0,b]} \|\left(Ty_1\right)(t) - y_1(t)\|_E.\end{aligned}$$

We will now use the generalized Weierstrass M-test for the Banach space $(E, \|\cdot\|_E)$: If a sequence of positive constants $M_1, M_2, M_3, \ldots$ can be found such that in the interval J we have (a) $\|u_n(t)\|_E \leq M_n$, $n = 1, 2, 3, \ldots$, and (b) $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} u_n(t)$ is uniformly and absolutely convergent in the interval.

Considering our sequence $(y_n)_{n\in\mathbb{N}}$, in (3.9), in the framework of the Banach space $C^1([0,b], E)$, and having in mind (3.8), by using the generalized Weierstrass M-test (for elements with images on the Banach space E), we have that

$$\sum_{n=1}^{\infty} \left(y_{n+1}(t) - y_n(t)\right)$$

is absolutely and uniformly convergent on $[0, b]$.

Since y_n can be written as

$$y_n(t) = y_1(t) + \sum_{k=1}^{n-1} \left(y_{k+1}(t) - y_k(t)\right),$$

there exists a unique solution $y \in C^1([0,b], E)$ such that

$$\lim_{n\to\infty} y_n = y.$$

Moreover, taking the limit on both sides of (3.9), we observe that the limit function y is the unique solution $y \in C^1([0,b], E)$ such that $Ty = y$. □

4. Ulam-Hyers and Ulam-Hyers-Rassias Stabilities

We are now in a position to derive the main goal of this work and identify sufficient conditions so that the FBVP (2.1) will admit the above mentioned types of stability.

Theorem 4.1. *If f, g and h satisfy (3.5)-(3.7), and (3.8) holds true, then the FBVP (2.1) has the Ulam-Hyers stability (in the sense of Definition 2.3 (i)).*

Proof. Let us consider a function y satisfying

$$\begin{aligned}y(0) - y'(0) &= \int_0^b g(s, y(s))ds \\ y(b) + y'(b) &= \int_0^b h(s, y(s))ds\end{aligned}$$

and

$$\left\|{}^C D^\alpha y(t) - f(t, y(t))\right\|_E \leq \theta,\ t \in [0, b],\ \theta \geq 0.$$

This means that

$$\left\|(Ty)(t) - y(t)\right\|_E \le \theta,\ t \in [0,b].$$

Under the conditions of the present theorem, we already know from Theorem 3.1 that

$$y_0(t) = \lim_{n\to\infty} (T^n y)(t)$$

is the exact solution of $Ty = y$. Moreover, $T^n y$ converges uniformly to y_0, as $n \to \infty$. Therefore, there is a natural k such that

$$\left\|\left(T^k y\right)(t) - y_0(t)\right\|_E \le \theta,\ t \in [0,b].$$

As a consequence, we obtain for any $t \in [0,b]$,

$$\begin{aligned}
\left\|y(t) - y_0(t)\right\|_E &\le \left\|y(t) - \left(T^k y\right)(t)\right\|_E + \left\|\left(T^k y\right)(t) - y_0(t)\right\|_E \\
&\le \left\|y(t) - (Ty)(t)\right\|_E + \left\|(Ty)(t) - \left(T^2 y\right)(t)\right\|_E + \cdots \\
&\qquad \cdots + \left\|\left(T^{k-1} y\right)(t) - \left(T^k y\right)(t)\right\|_E + \left\|\left(T^k y\right)(t) - y_0(t)\right\|_E \\
&\le \sup_{t\in[0,b]} \left\|y(t) - (Ty)(t)\right\|_E + C \sup_{t\in[0,b]} \left\|y(t) - (Ty)(t)\right\|_E + \cdots \\
&\qquad \cdots + C^{k-1} \sup_{t\in[0,b]} \left\|y(t) - (Ty)(t)\right\|_E + \theta \\
&\le (1 + C + C^2 + \cdots + C^{k-1})\theta + \theta \\
&\le \frac{\theta}{1-C} + \theta \\
&= \frac{2-C}{1-C}\theta.
\end{aligned}$$

□

Theorem 4.2. *If f, g and h satisfy (3.5)-(3.7), and (3.8) holds true, then the FBVP (2.1) has the Ulam-Hyers-Rassias stability (in the sense of Definition 2.3 (ii)).*

Proof. Let us now take y so that

$$\left\|(Ty)(t) - y(t)\right\|_E \le \sigma(t),\ t \in [0,b],$$

for a nonnegative function σ (defined on $J = [0,b]$), and let us also consider y_0 to be the solution of the FBVP (2.1).

It is clear that

$$\begin{aligned}
\left\|y(t) - y_0(t)\right\|_E &\le \left\|y(t) - (Ty)(t)\right\|_E + \left\|(Ty)(t) - y_0(t)\right\|_E \\
&\le \sigma(t) + \left\|(Ty)(t) - y_0(t)\right\|_E,\ t \in [0,b]. \qquad (4.1)
\end{aligned}$$

Now, using the same argument as in (3.10), we realize that

$$\begin{aligned}
\left\|(Ty)(t) - y_0(t)\right\|_E &= \left\|(Ty)(t) - (Ty_0)(t)\right\|_E \\
&\le C \sup_{t\in[0,b]} \left\|y(t) - y_0(t)\right\|_E. \qquad (4.2)
\end{aligned}$$

Thus, from (4.1) and (4.2), we conclude that

$$\left\|y(t) - y_0(t)\right\|_E \le \sigma(t) + C \sup_{t\in[0,b]} \left\|y(t) - y_0(t)\right\|_E.$$

Therefore,

$$(1-C)\sup_{t\in[0,b]}\left\|y(t)-y_0(t)\right\|_E \le \sigma(t),$$

and so

$$\left\|y(t)-y_0(t)\right\|_E \le \frac{1}{1-C}\sigma(t),\ t\in[0,b].$$

□

5. Examples

In this section we will exemplify the previous theory by analysing that the conditions of the above theorems are satisfied in some chosen fractional differential boundary value problems (that belong to the above analysed class of problems).

5.1. First Example

We will first consider the concrete case taken in [12] in order to study there the existence of a solution to a particularization of the general problem mentioned above. Namely, let us analyse

$$\begin{cases} {}^{C}D^{\alpha}y(t) = \frac{e^{-t}|y(t)|}{(9+e^t)(1+|y(t)|)}, & t\in J=[0,1],\ 1<\alpha<2,\\ y(0)-y'(0)=\displaystyle\sum_{i=0}^{\infty} c_i\cdot y(t_i), & \\ y(1)+y'(1)=\displaystyle\sum_{j=0}^{\infty} d_j\cdot y(\widetilde{t_i}), & \end{cases} \tag{5.1}$$

for some $c_i, d_j \in \mathbb{R}^+$ $(i,j=0,1,\dots)$ so that $\sum_{i=0}^{\infty} c_i < \infty$ and $\sum_{j=0}^{\infty} d_j < \infty$ and where $0<t_0<t_1<\cdots<1$ and $0<\widetilde{t_0}<\widetilde{t_1}<\cdots<1$.

Thus, when compared with our general situation in (2.1) we have here $(E,\|\cdot\|_E)=(\mathbb{R},|\cdot|)$, and

$$\begin{aligned} f(t,y(t)) &= \frac{e^{-t}|y(t)|}{(9+e^t)(1+|y(t)|)},\ t\in J=[0,1],\\ \int_0^1 g(s,y(s))ds &= \sum_{i=0}^{\infty} c_i\cdot y(t_i),\\ \int_0^1 h(s,y(s))ds &= \sum_{j=0}^{\infty} d_j\cdot y(\widetilde{t_i}). \end{aligned}$$

Therefore, it is clear that (3.5)-(3.7) hold with

$$L_g=\sum_{i=0}^{\infty} c_i,\quad L_h=\sum_{j=0}^{\infty} d_j \quad\text{and}\quad L_f=\frac{1}{10}$$

just because

$$\begin{aligned} |f(t,y_1(t))-f(t,y_2(t))| &\le \frac{e^{-t}}{9+e^t}|y_1(t)-y_2(t)|\\ &\le \frac{1}{10}|y_1(t)-y_2(t)|,\ t\in J=[0,1]. \end{aligned}$$

Moreover, having in mind the definition of G in (3.3) and C in (3.8), we have in this case

$$\begin{aligned} C &= \frac{2}{3}\left(\sum_{i=0}^{\infty} c_i + \sum_{j=0}^{\infty} d_j\right) + \frac{1}{10}\sup_{t\in[0,1]}\int_0^1 |G(t,s)|\,ds \\ &\leq \frac{2}{3}\left(\sum_{i=0}^{\infty} c_i + \sum_{j=0}^{\infty} d_j\right) + \frac{5+2(\alpha-1)}{30(\alpha-1)\Gamma(\alpha-1)}. \end{aligned} \tag{5.2}$$

Therefore, for $\alpha \in (1,2)$ and $c_i, d_j \in \mathbb{R}^+$ such that

$$\sum_{i=0}^{\infty} c_i + \sum_{j=0}^{\infty} d_j \leq \frac{15(\alpha-1)\Gamma(\alpha-1) - (\alpha-1) - 5/2}{10(\alpha-1)\Gamma(\alpha-1)} \tag{5.3}$$

we have from (5.2) that $C < 1$ and so (for those α, c_i, d_j) the class of fractional boundary value problem (5.1) is Ulam-Hyers-Rassias stable and Ulam-Hyers stable (cf. Theorem 4.2 and Theorem 4.1).

5.2. *Second Example*

Let us now consider a second example of a fractional boundary value problem of the type (2.1), given by

$$\begin{cases} {}^{C}D^{\alpha} y(t) = \frac{1}{27+3e^t} y(t), \qquad t \in J = [0,1],\ 1 < \alpha < 2, \\ y(0) - y'(0) = \displaystyle\int_0^1 \frac{1}{6+4e^{6s}} y(s)ds, \\ y(1) + y'(1) = \displaystyle\int_0^1 \frac{1}{9+e^s} y(s)ds. \end{cases} \tag{5.4}$$

In the notation of the previous sections, we have in here

$$\begin{aligned} &f(t,y(t)) := \frac{1}{27+3e^t} y(t), \\ &g(s,y(s)) := \frac{1}{6+4e^{6s}} y(s), \\ &h(s,y(s)) := \frac{1}{9+e^s} y(s), \\ &b := 1. \end{aligned}$$

Therefore, it is clear that (3.5)-(3.7) are satisfied in this case with

$$L_g = \frac{1}{10}, \quad L_h = \frac{1}{10} \quad \text{and} \quad L_f = \frac{1}{30}.$$

Additionally, from (3.3) and (3.4), we have in this case

$$\begin{aligned}
\widetilde{G} &= \sup_{t\in[0,1]} \int_0^1 |G(t,s)|\,ds \\
&= \sup_{t\in[0,1]} \left\{ \int_0^t \left| \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1+t)(1-s)^{\alpha-1}}{3\Gamma(\alpha)} - \frac{(1+t)(1-s)^{\alpha-2}}{3\Gamma(\alpha-1)} \right| ds \right. \\
&\qquad \left. + \int_t^1 \left| \frac{(1+t)(1-s)^{\alpha-1}}{3\Gamma(\alpha)} + \frac{(1+t)(1-s)^{\alpha-2}}{3\Gamma(\alpha-1)} \right| ds \right\} \\
&\le \sup_{t\in[0,1]} \left\{ \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{(1+t)(1-t)^\alpha}{3\Gamma(\alpha+1)} + \frac{1+t}{3\Gamma(\alpha+1)} + \frac{(1+t)(1-t)^{\alpha-1}}{3\Gamma(\alpha)} + \frac{1+t}{3\Gamma(\alpha)} \right. \\
&\qquad \left. + \frac{(1+t)(1-t)^\alpha}{3\Gamma(\alpha+1)} + \frac{(1+t)(1-t)^{\alpha-1}}{3\Gamma(\alpha)} \right\} \\
&< \frac{7}{3\Gamma(\alpha+1)} + \frac{4}{3\Gamma(\alpha)}.
\end{aligned}$$

Moreover, in this case, we have

$$\begin{aligned}
C &= \frac{b(b+1)}{b+2}\left(L_g + L_h\right) + \widetilde{G}L_f \\
&< \frac{2}{3}\left(\frac{1}{10} + \frac{1}{10}\right) + \left(\frac{7}{3\Gamma(\alpha+1)} + \frac{4}{3\Gamma(\alpha)}\right)\frac{1}{30} \\
&= \frac{2}{15} + \frac{7}{90\Gamma(\alpha+1)} + \frac{4}{90\Gamma(\alpha)}.
\end{aligned}$$

And so, $C < 1$ for all orders $\alpha \in (1,2)$. Thus, all conditions of Theorem 4.1 and Theorem 4.2 hold true and so the fractional boundary value problem (5.4) is Ulam-Hyers-Rassias stable, as well as Ulam-Hyers stable.

6. Conclusions

Guaranteeing the existence of solutions in the most appropriate solution spaces is an important area of research, especially in classes of boundary value problems where derivatives of various kinds occur and it is necessary to ensure that they are well defined. The consideration of inappropriate spaces leads to the existence of ill-posed frameworks – which is obviously undesirable, especially with regard to the existence of solutions and the possible types of stability associated with them.

In this article we present new conditions for the existence of C^1 solutions to the FBVP (2.1) and, in the same framework of function spaces, we determine conditions that guarantee its stability in the sense of Ulam-Hyers and Ulam-Hyers-Rassias.

An important question that requires further research is the determination of the most optimal constants that guarantee the existence of the inequalities that are at the heart of the definitions of the stabilities considered here. Finally, we emphasise that the stability conditions derived here can help in the search for approximate solutions to the problems in question.

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