Filomat 39:2 (2025), 629–638
https://doi.org/10.2298/FIL2502629E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence uniqueness and stability of solutions for ψ-Caputo fractional differential iterative equation with boundary value conditions

Mohamed El fadouaki[a,*], Khalid Hilal[a], Ahmed Kajouni[a], Hamid Lmou[a]

[a] *Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, Beni Mellal, Morocco*

Abstract. The subject of this paper revolves around fractional differential equations incorporating a ψ-Caputo fractional derivative, focusing on the Ulam–Hyers stability , the existence and uniqueness of solutions for nonlinear fractional quadratic iterative differential equation by utilizing Schauder's fixed point theorem, reinforced by the Ascoli-Arzelá theorem. Additionally, we present two illustrative examples to buttress our analytical findings.

1. Introduction

The concept of fractional calculus originated in the 17th century during discussions involving Leibniz and L'Hôpital. However, it remained a mathematical curiosity until the 20th and 21st centuries, it is a field that extends classical calculus by introducing non-integer orders of integration and differentiation. This concept has become highly significant in various scientific and engineering fields due to its broad range of applications, such as viscoelastic materials, anomalous diffusion processes, and control systems.

Fractional differential equations provide a powerful and flexible framework for modeling dynamic systems. They allow for the description of complex phenomena that exhibit long-range interactions and memory retention. In recent years, fractional differential equations have gained significant traction in various fields, including physics, engineering, biology, finance, and control theory, see [9, 12, 14, 15, 18]. This ability to describe systems has led to the development of specialized fractional operators, such as the Riemann-Liouville, Caputo, and Hilfer. Recently, Almeida [2] has extended the work of several scientists and proposed a new fractional derivation for the kernel function called ψ-Caputo fractional derivative, for more details for $\psi-$Caputo fractional derivative and their application, we direct readers to the papers [1, 3, 4, 6, 7, 20, 22, 25]. On the other hand, Petuhov[23] introduced the iterative differential equation in 1965 while investigating the existence and uniqueness of solutions to the following equation:

$$z'' = \lambda z(z(t)), \forall t \in [-b, b]$$

2020 *Mathematics Subject Classification*. Primary 26A33 mandatory; Secondary 34A08.
Keywords. ψ-fractional integral, ψ-Caputo fractional derivative, existence, uniqueness, iterative differential equation, Ulam–Hyers stability.
Received: 06 February 2024; Revised: 25 November 2024; Accepted: 29 November 2024
Communicated by Maria Alessandra Ragusa
* Corresponding author: Mohamed El fadouaki
Email addresses: fatouaki56@gmail.com (Mohamed El fadouaki), hilalkhalid2005@yahoo.fr (Khalid Hilal), Ahmed.kajouni@usms.ma (Ahmed Kajouni), hamid.lmou@usms.ma (Hamid Lmou)
ORCID iDs: https://orcid.org/0009-0003-1878-7948 (Mohamed El fadouaki), https://orcid.org/0000-0002-0806-2623 (Khalid Hilal), https://orcid.org/0000-0001-8484-6107 (Ahmed Kajouni), https://orcid.org/0000-0002-8786-2230 (Hamid Lmou)

After that, the existence of a solution for the first order and second order iterative differential equation studied by many scholars we mention here some works [10, 13, 26]. Moreover, much of study has been achieved on fractional iterative terative differential equations, out of which [8, 11, 19], It is worth mentioning that Rui [16] studied the existence and uniqueness of solutions for nonlinear quadratic iterative equations in the sense of the Caputo fractional derivative by using the Leray-Schauder fixed point theorem and topological degree theory. Therefore, it is valuable to contribute to bridging this gap by further exploring in this direction, aiming to enhance and complement the existing literature.
Motivated by the aforesaid work. In this paper, our article centers on investigating the existence and uniqueness, along with the Ulam stability[5], of solutions for a fractional differential equations we consider the following problem:

$$\begin{cases} {}^{C}D^{\alpha,\psi}y(t) = \phi\left(t, y(t), y^{[2]}(t)\right), & t \in [a,b] \\ y(a) = a, \quad y(b) = b. \end{cases} \tag{1}$$

where $y^{[2]}(t) = y(y(t))$, ${}^{C}D^{\alpha,\psi}y(.)$ denotes the $\psi-$Caputo derivative for y with order $1 < \alpha < 2$ (see definition 2.4), and $\phi : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuouns function. We explore the existence and uniqueness of solutions, as well as determine the Ulam–Hyers stability and generalized Ulam–Hyers stability aspect.
In the next sections of this article, we structure the remaining content as follows: We introduce fundamental definitions of fractional calculus, essential lemmas in section 2. In section 3, the addressation of the Leray-Schauder fixed point theorem and Ascoli-Arzelá theorem is employed to investigate the existence and uniqueness of solutions for nonlinear ψ-Caputo fractional quadratic iterative differential equation. Additionally, an illustrative example is provided in Section 4.

2. Preliminaries

In this section, we recall some definition and lemmas results of ψ-fractional derivative and ψ-fractional integral, which will be later employed.see the articles [2, 17] for more details.

Notation 2.1.

- *We denote by* $\mathbb{E} = C([a,b], \mathbb{R})$ *the Banach space of all continuous functions from* $[a,b]$ *into* $\mathbb{R}$ *equipped with the following norm*
$$\|x\| = \sup\{|x(t)|; t \in [a,b]\}.$$

Definition 2.2. *Let* $\alpha > 0$ *,* h *an integral function defined on* $[a,b]$ *and* $\psi \in C^n[a,b]$ *and increasing function such that* $\psi\prime(s) \neq 0$ *for all* $s \in [a,b]$*. The* $\psi-$*Riemann–Liouville fractional integral of* h *of order* α *is defined by*

$$I^{\alpha,\psi}_{a+}h(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} h(\tau) d\tau, \tag{2}$$

where Γ *is the gamma function. Note that Eq. (2) is reduced to the Riemann-Liouville and Hadamard fractional integrals when* $\psi(t) = t$ *and* $\psi(t) = \ln t$*, respectively.*

Definition 2.3. *Let* $n - 1 < \alpha < n, h : [a,b] \to \mathbb{R}$ *be an integrable function and* $\psi \in C^n[a,b]$ *and increassing function such that* $\psi\prime(s) \neq 0$ *for all* $s \in [a,b]$*. The* ψ*-Riemann-Liouville fractional derivative of* h *of order* α *is defined by*

$$D^{\alpha,\psi}_{a^+}h(t) = \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^n I^{n-\alpha,\psi}_{a^+}h(t),$$

where $n = [\alpha] + 1$ *and* $[\alpha]$ *denotes the integer part of the real number* α*.*

Definition 2.4. *Let $n-1<\alpha<n, h\in C^{n-1}[a,b]$ and $\psi \in C^n[a,b]$ and increassing function such that $\psi'(s)\neq 0$ for all $s\in[a,b]$. The ψ-Caputo fractional derivative of function h of order α is determined as*

$$ {}^{C}D_{a^+}^{\alpha,\psi}h(t) = D_{a^+}^{\alpha,\psi}\left[h(t)-\sum_{k=0}^{n-1}\frac{h_{\psi}^{[k]}(a)}{k!}(\psi(t)-\psi(a))^k\right], $$

where $h_{\psi}^{[k]}(t)=\left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^k h(t)$ and $n=[\alpha]+1$ for $\alpha\notin\mathbb{N}, n=\alpha$ for $\alpha\in\mathbb{N}$. Further, if $h\in C^n[a,b]$ and $\alpha\notin\mathbb{N}$, then

$$ \begin{aligned} {}^{C}D_{a^+}^{\alpha,\psi}h(t) &= I_{a^+}^{n-\alpha,\psi}\left[\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right]^n h(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1}h_{\psi}^{[n]}(s)\mathrm{d}s, \end{aligned} \tag{3} $$

Thus, if $\alpha=n\in\mathbb{N}$, one has

$$ {}^{C}D_{a^+}^{\alpha,\psi}h(t) = h_{\psi}^{[n]}(t). $$

Clearly,

$$ {}^{C}D_{a^+}^{\alpha,\psi}c = 0 $$

where c is a constant number.

Remark 2.5. *In particular, the ψ-Caputo fractional derivative is reduced to the the Caputo fractional derivative when $\psi(t)=t$. Moreover, for $\psi(t)=\log(t)$, it gives the Caputo–Hadamard fractional derivative.*

Lemma 2.6. *([2, 17]) Let $\beta,\alpha>0$ and $h:[a,b]\to\mathbb{R}$. The following holds:*
(1) If $h\in C[a,b]$, then ${}^{C}D_{a^+}^{\alpha,\psi}I_{a^+}^{\alpha,\psi}h(t)=h(t)$.
(2) If $h\in C^{n-1}[a,b]$, then $I_{a^+}^{\alpha,\psi}{}^{C}D_{a^+}^{\alpha,\psi}h(t)=h(t)-\sum_{k=0}^{n-1}c_k[\psi(t)-\psi(a)]^k$, where $c_k=\frac{h_{\psi}^{[k]}(a)}{k!}$.
In particular,
if $0<\alpha<1$, we have $I_{a^+}^{\alpha,\psi}{}^{C}D_{a^+}^{\alpha,\psi}h(t)=h(t)-h(a)$.
(3) ${}^{C}D_{a^+}^{\alpha,\psi}[\psi(t)-\psi(a)]^k=0, \forall k\in\{0,1,\dots,n-1\}, n\in\mathbb{N}$.

Theorem 2.7 (Schauder fixed point theorem). *Let B be a compact convex subset of a Banach space $\mathbb{E}$. Assume that $\chi: B\mapsto B$ is a continuous operator. Then χ has at least one fixed point in M.*

3. Existence, uniqueness and stability

3.1. Existence and uniqueness of solution

In this section, Before presenting the existence result for the fractional quadratic iterative problem, it is necessary to establish the following fundamental lemma.

Lemma 3.1. *$y\in\mathbb{E}$ is a solution of the fractional quadratic iterative differential equation (1) if only and if y satisfies the inetgral solution*

$$ y(t) = a+\frac{\psi(t)-\psi(a)}{\psi(b)-\psi(a)}(b-a)+\int_a^b G(t,\tau)\phi\left(\tau,y(\tau),y^{[2]}(\tau)\right)d\tau, \tag{4} $$

where $G(t,\tau)$ is the Green's function defined by

$$ G(t,\tau)=\begin{cases} -\frac{(\psi(t)-\psi(a))\psi'(\tau)}{(\psi(b)-\psi(a))\Gamma(\alpha)}(\psi(b)-\psi(\tau))^{\alpha-1}+\frac{1}{\Gamma(\alpha)}(\psi(t)-\psi(\tau))^{\alpha-1}, & a\leqslant\tau\leqslant t\leqslant b. \\ -\frac{(\psi(t)-\psi(a))\psi'(\tau)}{(\psi(b)-\psi(a))\Gamma(\alpha)}(\psi(b)-\psi(\tau))^{\alpha-1}, & a\leqslant t\leqslant\tau\leqslant b. \end{cases} \tag{5} $$

Proof. Let $y \in \mathbb{E}$ be the solution for the fractional quadratic iterative differential equation(1). then we operate the ψ-fractional integral $I^{\alpha,\psi}_{a+}$ on the both sides of the equation (1), we obtain

$$I^{\alpha,\psi}_{a+} {}^{C}D^{\alpha,\psi} y(t) = I^{\alpha,\psi}_{a+} \phi\left(t, y(t), y^{[2]}(t)\right)$$

Following Lemma (2.6), we get

$$y(t) - y(a) - y^{[1]}_{\psi}(a)(\psi(t) - \psi(a)) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} \phi\left(\tau, y(\tau), y^{[2]}(\tau)\right) d\tau,$$

Then

$$y(t) = y(a) + \frac{y'(a)}{\psi'(a)}(\psi(t) - \psi(a)) + \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} \phi\left(\tau, y(\tau), y^{[2]}(\tau)\right) d\tau. \tag{6}$$

Due to $y(a) = a$, $y(b) = b$, it follows from (6) that

$$y(b) = b = a + \frac{y'(a)}{\psi'(a)}(\psi(b) - \psi(a)) + \frac{1}{\Gamma(\alpha)} \int_a^b \psi'(\tau)(\psi(b) - \psi(\tau))^{\alpha-1} \phi\left(\tau, y(\tau), y^{[2]}(\tau)\right) d\tau,$$

which implies that

$$y'(a) = \frac{\psi'(a)}{\psi(b) - \psi(a)} \left[(b - a) - \frac{1}{\Gamma(\alpha)} \int_a^b \psi'(\tau)(\psi(b) - \psi(\tau))^{\alpha-1} \phi\left(\tau, y(\tau), y^{[2]}(\tau)\right) d\tau\right].$$

Now, substitute $y'(a)$ into (6), and use Green's function to turn problem (6) into the following integral equation:

$$\begin{aligned} y(t) =& a + \frac{\psi(t) - \psi(a)}{\psi(b) - \psi(a)}(b - a) - \frac{\psi(t) - \psi(a)}{(\psi(b) - \psi(a))\Gamma(\alpha)} \int_a^b \psi'(\tau)(\psi(b) - \psi(\tau))^{\alpha-1} \phi\left(\tau, y(\tau), y^{[2]}(\tau)\right) d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} \phi\left(\tau, y(\tau), y^{[2]}(\tau)\right) d\tau \\ =& a + \frac{\psi(t) - \psi(a)}{\psi(b) - \psi(a)}(b - a) + \int_a^b G(t, \tau) \phi\left(\tau, y(\tau), y^{[2]}(\tau)\right) d\tau. \end{aligned}$$

The converse follows by direct computation which completes the proof.
Reciprocally, let $y \in \mathbb{E}$ satisfying (6), then

$$\begin{aligned} y(t) =& a + \frac{\psi(t) - \psi(a)}{\psi(b) - \psi(a)}(b - a) \\ &+ \int_a^t \left[-\frac{(\psi(t) - \psi(a))\psi'(\tau)}{(\psi(b) - \psi(a))\Gamma(\alpha)}(\psi(b) - \psi(\tau))^{\alpha-1} + \frac{1}{\Gamma(\alpha)}(\psi(t) - \psi(\tau))^{\alpha-1}\right] \phi\left(\tau, y(\tau), y^{[2]}(\tau)\right) d\tau \\ &+ \int_t^b \left[-\frac{(\psi(t) - \psi(a))\psi'(\tau)}{(\psi(b) - \psi(a))\Gamma(\alpha)}(\psi(b) - \psi(\tau))^{\alpha-1}\right] \phi\left(\tau, y(\tau), y^{[2]}(\tau)\right) d\tau \\ =& a + \frac{\psi(t) - \psi(a)}{\psi(b) - \psi(a)}(b - a) - \frac{\psi(t) - \psi(a)}{(\psi(b) - \psi(a))\Gamma(\alpha)} \int_a^b \psi'(\tau)(\psi(b) - \psi(\tau))^{\alpha-1} \phi\left(\tau, y(\tau), y^{[2]}(\tau)\right) d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} \phi\left(\tau, y(\tau), y^{[2]}(\tau)\right) d\tau. \end{aligned}$$

It's clear that $y(a) = a$ and $y(b) = b$. by applying the $\psi-$Caputo fractional derivative ${}^{C}D^{\alpha,\psi}$ to both sides of equation (4), we use Lemma (2.6), so we obtain

$$\begin{aligned} {}^{C}D^{\alpha,\psi} y(t) =& {}^{C}D^{\alpha,\psi} \left[\frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \phi\left(\tau, y(\tau), y^{[2]}(\tau)\right) d\tau\right] \\ =& \phi\left(t, y(t), y^{[2]}(t)\right) \end{aligned}$$

This completes the proof. □

Remark 3.2. *The function $t \in [a,b] \mapsto \int_a^b |G(t,\tau)|\, d\tau$ is continuous on $[a,b]$, and hence is bounded. Let*

$$G^\star = sup\left\{\int_a^b |G(t,\tau)|\, d\tau, t \in [a,b]\right\}$$

Theorem 3.3. *Let $\phi : [a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function that meets the following conditions:*
(H_1) $\exists L_1 > 0$ *such that :*

$$\left|\phi(t,w_1,v_1)-\phi(t,w_2,v_2)\right| \le L_1\left(|w_1-w_2|+|v_1-v_2|\right) \tag{7}$$

for any $(t,w_1,v_1),(t,w_2,v_2)\in[a,b]\times\mathbb{R}\times\mathbb{R}$.
(H_2) $\exists K > 0$ *such that :*

$$|\phi(t,w,v)| \le K, \quad \text{for each } (t,w,v)\in[a,b]\times\mathbb{R}\times\mathbb{R}. \tag{8}$$

(H_3) $\exists r > 0$ *such that :*

$$KG^\star \le r. \tag{9}$$

Then the fractional boundary value problem (1) has at least one solution defined on $[a,b]$.

Proof. Let

$$X_r = \{y\in\mathbb{E}, \|y\|\le r\}.$$

The set X_r is a closed, convex and bounded subset of the Banach space $\mathbb{E}$. Now we define the operator ϑ in X_r as follow

$$(\vartheta y)(t) = a + \frac{\psi(t)-\psi(a)}{\psi(b)-\psi(a)}(b-a) + \int_a^b G(t,\tau)\phi\left(\tau, y(\tau), y^{[2]}(\tau)\right)d\tau,$$

where $G(t,\tau)$ is the Green's function given by (5). Then, we can transform problem (1) into a fixed point problem, i.e., $y=\vartheta(y)$.
To prove our results, let's achieve these steps :
Step 1: $\vartheta:\mathbb{E}\to\mathbb{E}$ is continuous
Let y_n be a sequence in $\mathbb{E}$ such that $y_n\to y$ in $\mathbb{E}$, we can conclude that $y_n^{[2]}\to y^{[2]}$.
Then for each $t\in[a,b]$,

$$\begin{aligned}\left|(\vartheta y_n)(t)-(\vartheta y)(t)\right| &\le \int_a^b |G(t,\tau)|\left|\left(\phi(\tau,y_n(\tau),y_n^{[2]}(\tau))-\phi(\tau,y(\tau),y^{[2]}(\tau))\right)\right|d\tau\\ &\le \int_a^b |G(t,\tau)|\,L_1\left(|y_n(\tau)-y(\tau)|+|y_n^{[2]}(\tau)-y^{[2]}(\tau)|\right)d\tau.\end{aligned}$$

The Lebesgue dominated convergence theorem implies that

$$\|\vartheta y_n-\vartheta y\|\to 0 \text{ as } n\to\infty.$$

Step 2: $\vartheta(X_r)$ is uniformly bounded.
Let $y\in X_r$, then we have

$$\begin{aligned}|(\vartheta y)(t)| &\le \left|a+\frac{\psi(t)-\psi(a)}{\psi(b)-\psi(a)}(b-a)\right| + \int_a^b |G(t,\tau)|\left|\phi(\tau,y(\tau),y^{[2]}(\tau))\right|d\tau\\ &\le \left|a+\frac{\psi(b)-\psi(a)}{\psi(b)-\psi(a)}(b-a)\right| + K\int_a^b |G(t,\tau)|\,d\tau.\end{aligned}$$

Thus

$$\|\vartheta y\| \leq KG^{\star}.$$

Step 3: $\vartheta(X_r)$ is equicontinuous.
Let $s_1, s_2 \in [a,b], s_1 \leq s_2$ and $y \in X_r$. Then

$$\begin{aligned}\left|(\vartheta y)(s_1) - (\vartheta y)(s_2)\right| \leq & \int_a^b |G(s_1,\tau) - G(s_2,\tau)| \left|\phi(\tau, y(\tau), y^{[2]}(\tau))\right| d\tau \\ \leq & K \int_a^b |G(s_1,\tau) - G(s_2,\tau)| \, d\tau.\end{aligned}$$

Let $s_1 \to s_2$, and then $\left|(\vartheta y)(s_1) - (\vartheta y)(s_2)\right| \to 0$.
From the step 1-3 and by applying the Ascoli-Arzelá theorem. we conclude that ϑ is completely continuous.
Step 4: $\vartheta(X_r) \subset X_r$.
Let $y \in X_r$, and we prove that $\vartheta(y) \in X_r$. For each $t \in [a,b]$ we have

$$\begin{aligned}|(\vartheta y)(t)| \leq & \left|a + \frac{\psi(t) - \psi(a)}{\psi(b) - \psi(a)}(b-a)\right| + \int_a^b |G(t,\tau)| \left|\phi(\tau, y(\tau), y^{[2]}(\tau))\right| d\tau \\ \leq & K \int_a^b |G(t,\tau)| \, d\tau.\end{aligned}$$

Thus

$$\|\vartheta y\| \leq KG^{\star}.$$

by (9), we have

$$\|\vartheta y\| \leq r.$$

In summary, all of the requirements of the Schauder fixed point theorem are achieved, which means that ϑ has a fixed point in X_r, which is solution of the iterative boundary value problem (1).
□

Now, we need to prove the monotonic increase of ϑ. This result will contribute to establish the uniqueness of the solution in the next theorem.
We consider $t \in [a,b]$, we have

$$\begin{aligned}(\vartheta(y))'(t) = & \frac{\psi'(t)(b-a)}{\psi(b) - \psi(a)} - \frac{\psi'(t)}{(\psi(b) - \psi(a))\Gamma(\alpha)} \int_a^b \psi'(\tau)(\psi(b) - \psi(\tau))^{\alpha-1} \phi\left(\tau, y(\tau), y^{[2]}(\tau)\right) d\tau \\ & + \frac{\psi'(t)}{\Gamma(\alpha)} \int_a^t (\alpha-1)\psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-2} \phi\left(\tau, y(\tau), y^{[2]}(\tau)\right) d\tau. \\ (\vartheta(y))'(t) \geq & \frac{\psi'(t)(b-a)}{\psi(b) - \psi(a)} + \frac{K\psi'(t)}{(\psi(b) - \psi(a))\Gamma(\alpha)} \left[-(\psi(b) - \psi(\tau))^{\alpha}\right]_a^b \\ & - \frac{K\psi'(t)}{\Gamma(\alpha)} \left[-(\psi(t) - \psi(\tau))^{\alpha-1}\right]_a^t \\ \geq & \frac{\psi'(t)(b-a)}{\psi(b) - \psi(a)} + \frac{K\psi'(t)}{\Gamma(\alpha)}(\psi(b) - \psi(a))^{\alpha-1} - \frac{K\psi'(t)}{\Gamma(\alpha)}(\psi(t) - \psi(a))^{\alpha-1} \\ \geq & \frac{\psi'(t)(b-a)}{\psi(b) - \psi(a)} \geq 0, \quad (\phi \text{ is increasing function }).\end{aligned}$$

Thus, ϑ is increasing on $[a,b]$, which implies that $a \leq (\vartheta(y))(t) \leq b$, for all $t \in [a,b]$. We can now demonstrate the uniqueness and stability of the solution of the preblem (1) in the following theorem.

Theorem 3.4. *Suppose that (H1)-(H3) are satisfied. Then the iterative value problem (1) has a unique solution on* $[a,b]$. *Provided that*

$$L_1 G^\star < 1$$

Proof. It should be noted that : $a \leq (\vartheta y)(t) \leq b, \quad \forall t \in [a,b]$.
Let $y_1, y_2 \in \mathbb{E}$, we show that ϑ is a contraction.

$$\begin{aligned}\left|(\vartheta y_1)(t) - (\vartheta y_2)(t)\right| &\leq \int_a^b |G(t,\tau)| \left|\left(\phi(\tau, y_1(\tau), y_1^{[2]}(\tau)) - \phi(\tau, y_2(\tau), y_2^{[2]}(\tau))\right)\right| d\tau \\ &\leq G^\star L_1 \|y_1 - y_2\|.\end{aligned}$$

This yields the inequality

$$\|y_1 - y_2\| \leq G^\star L_1 \|y_1 - y_2\|.$$

Since $G^\star L_1 < 1$, the operator ϑ is a contraction.
Then, ϑ has a fixed point in X_r. Terefore, the problem (1) has a unique solution in S. □

3.2. Ulam–Hyers stability

In this subsection we investigate the Ulam–Hyers and generalized Ulam–Hyer in $\mathbb{E}$ of problem (1). Let $\varepsilon > 0$ and $\Phi : [a,b] \to \mathbb{R}^+$be a continuous function. We consider the following inequality

$$\left|{}^C D^{\alpha,\psi} z(t) - \phi\left(t, z(t), z^{[2]}(t)\right)\right| \leq \varepsilon, \quad t \in [a,b], \tag{10}$$

Definition 3.5. *The problem (1) is Ulam-Hyers stable if there exists a real number* C_ϕ *such that for each* $\varepsilon > 0$ *and for each solution* $z \in \mathbb{E}$ *of the inequality (10), there exists a solution* $x \in \mathbb{E}$ *of the problem (1) such that*

$$|z(t) - x(t)| \leq C_\phi \varepsilon \quad t \in [a,b].$$

Definition 3.6. *The problem (1) is generalized Ulam-Hyers stable if there exists* $\Phi_\phi \in C(\mathbb{R}^+, \mathbb{R}^+)$ *with* $\Phi_\phi(0) = 0$ *such that, for each* $\varepsilon > 0$ *and for each solution* $z \in \mathbb{E}$ *of the inequality (10), there exists a solution* $x \in \mathbb{E}$ *of the problem (1) such that*

$$|z(t) - x(t)| \leq \Phi_\phi(\varepsilon) \quad t \in [a,b].$$

Remark 3.7. *It is clear that Definition (3.5)* $\Rightarrow$ *Definition (3.6)*

Remark 3.8. *A function* $v \in \mathbb{E}$ *is a solution of inequality (10)* $\Longleftrightarrow$ *there exists a function* $g \in \mathbb{E}$ *(which depends on solution* v *), such that*
(1) $|g(t)| \leq \varepsilon, \quad t \in [a,b]$.
(2) ${}^C D^{\alpha,\psi} z(t) = \phi\left(t, z(t), z^{[2]}(t)\right) + g(t), \quad t \in [a,b]$

Currently, we explore the stability of the solution to problem (1) using Ulam–Hyers stability.

Theorem 3.9. *Assume that all conditions of theorem (3.4) are fulfilled, then the fractional boundary value problem (1) is Ulam–Hyers stable on* $[a,b]$ *and consequently generalized Ulam-Hyers stable.*

Proof. Suppose that $z(t) \in \mathbb{E}$ satisfies inequality (10), that is,

$$\left|{}^C D^{\alpha,\psi} z(t) - \phi\left(t, z(t), z^{[2]}(t)\right)\right| \leq \varepsilon, \quad t \in [a,b],$$

and $y \in \mathbb{E}$ is a unique solution of problem (1), so that

$$y(t) = a + \frac{\psi(t) - \psi(a)}{\psi(b) - \psi(a)}(b-a) + \int_a^b G(t,\tau)\phi\left(\tau, y(\tau), y^{[2]}(\tau)\right) d\tau$$

$\forall t \in [a,b]$. By remark (3.8) and lemma (3.1), there exist so $g \in \mathbb{E}$ satisfies the inequality $|g(t)| \leq \varepsilon$ and the equation

$$z(t) = a + \frac{\psi(t) - \psi(a)}{\psi(b) - \psi(a)}(b-a) + \int_a^b G(t,\tau)\phi\left(\tau, z(\tau), z^{[2]}(\tau)\right)d\tau + \int_a^b G(t,\tau)g(\tau)d\tau$$

we have, for each $t \in [a,b]$

$$|z(t) - y(t)| \leq \int_a^b |G(t,\tau)| \left|\phi\left(\tau, z(\tau), z^{[2]}(\tau)\right) - \phi\left(\tau, y(\tau), y^{[2]}(\tau)\right)\right| d\tau + \int_a^b |G(t,\tau)||g(\tau)|d\tau$$

Hence using part (i) of Remark (3.8), and (H2) we can get

$$|z(t) - y(t)| \leq \int_a^b |G(t,\tau)|L_1\left(\left|z(\tau) - y(\tau)\right| + \left|z^{[2]}(\tau) - y^{[2]}(\tau)\right|\right)d\tau + \varepsilon\int_a^b |G(t,\tau)|d\tau$$

In consequence, it follows that

$$\|z - y\| \leq G^\star L_1\|z - y\| + \varepsilon G^\star$$

$$\|z - y\| \leq \varepsilon\frac{G^\star}{1 - L_1 G^\star}$$

By defining $C_\phi = \frac{G^\star}{1-L_1G^\star}$, the Ulam–Hyers stability condition is met, if we consider $\Phi_\phi(\varepsilon) = \frac{G^\star}{1-L_1G^\star}\varepsilon$ with $\Phi_\phi(0) = 0$, the generalized Ulam–Hyers stability condition is also fulfilled. □

4. Illustrated examples

Here are some examples to demonstrate the theoretical outcomes. latex latex

Example 4.1. *Consider the following ψ-Caputo fractional quadratic iterative differential equation given by:*

$$\begin{cases} {}^CD^{3/2,t}y(t) = t + \frac{1}{3}\left(\sin(y(t)) + \frac{y^{[2]}(t)}{y^{[2]}(t)+1}\right), & t \in [0,1] \\ y(0) = 0, \quad y(1) = 1. \end{cases} \tag{11}$$

Here, we have $\alpha = \frac{3}{2}$, $\psi(t) = t$, and the function ϕ defined by $\phi(t, y(t), y^{[2]}(t)) = t + \frac{1}{3}\left(\sin(y(t)) + \frac{y^{[2]}(t)}{y^{[2]}(t)+1}\right)$.

For any $t \in [0,1]$ and $w, v \in \mathbb{R}$, we obtain:

$$\left|\phi(t,w,v)\right| = \left|t + \frac{1}{3}\left(\sin(w) + \frac{v}{v+1}\right)\right| \leq 2,$$

$$\left|\phi(t,w_1,v_1) - \phi(t,w_2,v_2)\right| \leq \frac{1}{3}\left(|w_1 - w_2| + |v_1 - v_2|\right).$$

Here we get $L_1 = \frac{1}{3}$ and $K = 2$. It is evident that the conditions (H_1) and (H_2) are satisfied.

Using $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$, we compute $G^\star = \sup\left\{\int_0^1 |G(t,\tau)|\,d\tau, t \in [0,1]\right\} \leq \frac{8}{3\sqrt{\pi}}$, where

$$G(t,\tau) = \begin{cases} \frac{2}{\sqrt{\pi}}\left[(t-\tau)^{\frac{1}{2}} - t(1-\tau)^{\frac{1}{2}}\right], & 0 \leqslant \tau \leqslant t \leqslant 1, \\ -\frac{2t}{\sqrt{\pi}}(1-\tau)^{\frac{1}{2}}, & 0 \leqslant t \leqslant \tau \leqslant 1. \end{cases}$$

Then by Theorem (3.3), since $KG^\star \leq \frac{16}{3\sqrt{\pi}} = r$, the problem has at least one solution defined on $[0,1]$.

For uniqueness and stability of solution, we verify the fulfillment condition of Theorem (3.4). Specifically, we find that

$$L_1 G^\star \leq \frac{8}{9\sqrt{\pi}} \simeq 0.501 \leq 1.$$

This result establishes the uniqueness of the solution for the problem (11) on the interval $[0,1]$. *Furthermore, we confirm its Ulam–Hyers stability and, as a consequence, its generalized Ulam–Hyers stability.*

Example 4.2. *Consider the following* ψ*-Caputo fractional quadratic iterative differential equation given by:*

$$\begin{cases} {}^{C}D^{\frac{5}{3},t^2} y(t) = \frac{\cos(t)}{2}\left(\frac{e^{y(t)}}{e^{y(t)}+1} + \sin(y^{[2]}(t))\right), \quad t \in [0,1] \\ y(0) = 0, \quad y(1) = 1. \end{cases} \tag{12}$$

Here, we have $\alpha = \frac{5}{3}$, $\psi(t) = t^2$, *and* $\phi(t,w,v) = \frac{\cos(t)}{2}\left(\frac{e^{w}}{e^{w}+1} + \sin(v)\right)$, *for all* $(t,w,v) \in [0,1] \times \mathbb{R} \times \mathbb{R}$.
For any $t \in [0,1]$ *and* $w, v \in \mathbb{R}$, *we obtain:* $|\phi(t,w,v)| \leq 1$, *which implies* $K = 1$.
For any $(t,w_1,v_1), (t,w_2,v_2) \in [0,1] \times \mathbb{R} \times \mathbb{R}$, *we have*

$$\begin{aligned} \left|\phi(t,w_1,v_1) - \phi(t,w_2,v_2)\right| &\leq \left|\frac{\cos(t)}{2}\right| \left[\left|\frac{e^{w_1}}{e^{w_1}+1} - \frac{e^{w_2}}{e^{w_2}+1}\right| + |\sin(v_1) - \sin(v_2)|\right] \\ &\leq \frac{1}{2}\left(|w_1 - w_2| + |v_1 - v_2|\right). \end{aligned}$$

Here we get $L_1 = \frac{1}{2}$. *By careful calculation, we find* $G^\star \simeq 0.9636 \implies KG^\star \leq 1$. *It is evident that the conditions* (H_1), (H_2), *and* (H_3) *are satisfied. Then the problem has at least one solution defined on* $[0,1]$.
Additionally, since $L_1 G^\star < 1$, *by Theorem (3.4), we establish the existence and uniqueness of the solution for the problem (12) on* $[0,1]$. *Furthermore, we confirm its Ulam-Hyers stability.*

5. Conclusions

In this paper, we have established the existence and uniqueness of solutions for the ψ-Caputo fractional quadratic iterative differential equation, as well as the stability of the solution in the sense of the Ulam-Hyers. The main results were proven through the use of the Banach contraction theorem and Schauder's fixed point theorem. Our findings devlop and generalize previous studies that focused on specific cases of the ψ-Caputo fractional differential equation such as the Caputo and Caputo-Hdamard cases. In the end, we demonstrate the practical applications of the obtained results on existence and uniqueness through two illustrative examples.

Acknowledgements

The authors would like to express their sincere appreciation to the referees for their careful reading of the manuscript for providing helpful suggestions and insightful comments, which significantly contributed to the improvement of the quality and presentation of the paper.

Conflict of interest

The authors declare that there are no conflicts of interest.

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