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Exact controllability results of non-instantaneous impulsive stochastic integro-differential equations driven by a fractional Brownian motion

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Abstract. This paper discusses the exact controllability of a class of non-instantaneous impulsive stochastic integro-differential equations driven by a fractional Brownian motion with nonlocal conditions in a Hilbert space. The results are based on generalized Darbo's fixed point theorem, utilizing Kuratowskii's measure of non-compactness and a resolvent operator. Examples are given to illustrate the effectiveness of the proposed results.

1. Introduction

This paper is concerned with the existence of mild solution and the exact controllability for a class of stochastic integro-differential equations (SIDEs) driven by a fractional Brownian motion (FBM) accompanied by nonlocal conditions and non-instantaneous impulsive (NII). The system under discussion takes the following form:

$$\begin{cases} dy(t) = Ay(t)dt + \int_0^t h(t-s)y(t)dsdt + \varphi(t, y(t))dt + f(t)dB^H(t), & t \in \cup_{k=0}^N (s_k, t_{k+1}], \\ y(t) = g_k(t, y(t_k^-)), & t \in \cup_{k=1}^N (t_k, s_k], \\ y(0) + \psi(y) = y_0 \in \mathbb{H}. \end{cases} \tag{1}$$

Where A is the infinitesimal generator of a strongly continuous semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ of bounded linear operators on a separable Hilbert space $\mathbb{H}$ with domain $\mathbb{D}(A)$, $h : \mathbb{D}(h) \to \mathbb{H}$ is a closed linear operator on $\mathbb{H}$ with domain $\mathbb{D}(A) \subset \mathbb{D}(h)$, φ, ψ and $f : [0, T] \to \mathcal{L}_2^0(\mathbb{V}, \mathbb{H})$ are appropriate functions, where $\mathcal{L}_2^0(\mathbb{V}, \mathbb{H})$ denotes the space of all Hilbert-Schmidt operators from $\mathbb{V}$ into $\mathbb{H}$. Also, $B^H(t)$ is a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ with

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values on a separable Hilbert space $\mathbb{V}$. Let $0 = s_0 = t_0 < t_1 \leq s_1 < t_2 \leq ... < t_N \leq s_N < t_{N+1} = T$ where $T > 0$ is a constant $g_k : (t_k, s_k] \times \mathbb{H} \to \mathbb{H}$ is called non-instantaneous impulsive function, for all $k = 1, 2, ..., N$. Where the state $y(.)$ takes values on a real separable Hilbert space $\mathbb{H}$ with inner product $(.,.)$ and norm $\|.\|$, y_0 is an $\mathcal{F}_0$-measurable random variable with $\mathbb{E}\|y_0\| < \infty$.

Stochastic differential equations are a topic of great interest to many physicists, mathematicians, engineers, and biologists, as evidenced by numerous studies (see [32, 44]). In this context, several researchers have examined various qualitative aspects of these equations, particularly those driven by fractional Brownian motion (fBm). This concept was introduced by the mathematician Andrey Kolmogorov [24] in 1940 and continues to be a focal point of interest within academic circles (see [5, 8, 21, 30, 37, 41]). The capacity of stochastic differential equations to simulate complex natural phenomena makes them an effective tool for understanding dynamic behaviors across various systems. By employing quantitative techniques and mathematical concepts, scientists can develop models that help explain the random behaviors observed in diverse fields, including biology and physics.

Many real-life phenomena and processes which evolve with respect to time are characterized by abrupt changes in the form of impulses. According to the duration of the change, there are two popular types of impulses:

- Instantaneous Impulses: When the duration of these changes is relatively short compared to the overall duration of the whole process (see [26, 35]).
- Non-instantaneous impulses: When changes start at an arbitrary fixed point and remain active on a finite time interval (see[19]).

Thus, the action of instantaneous impulsive phenomena seen as do not describe some certain dynamics of evolution processes in pharmacotherapy [39]. A well-known application of non-instantaneous impulses is the process of inducing a vaccine and absorption of the drug by the body. The resulting absorption is gradual because it remains active for a finite time interval [28]. This process can be modelled mathematically by non-instantaneous impulsive differential and integro-differential equations. Recently, many authors have established results on non-instantaneous impulse differential equations (see [28, 30, 32]) and references therein.

On the other hand, controllability plays a significant role in various fields such as engineering, physics, robotics to economics and social sciences. The most commonly used types of controllability are exact and approximate controllability. This concept was first proposed by Kalman [22] in 1963, and since then, both the theory of stochastic processes and differential equations have greatly benefited from its application (see references [14, 32, 37] and its allusions). Abid et al. in [1] discussed the approximate controllability of fractional stochastic integro-differential equations driven by mixed fractional Brownian motion. Recently, Jiankang Liu et al. [30] obtained the existence and approximate controllability results for a type of non-instantaneous impulsive stochastic evolution equation (SEE) excited by fractional Brownian motion (FBM). Additionally, Diop et al. [13] investigated the existence and controllability for a class of impulsive stochastic integro-differential equations (ISIDEs) with state-dependent delay in a Hilbert space, Melati et al. [32] discussed the existence and exact controllability of non-instantaneous impulsive stochastic integro-differential equations with nonlocal conditions in a Hilbert space.

However, the study of the exact controllability of non-instantaneous stochastic integro-differential equations driven by fractional brownian has not been discussed in the standard literature. Motivated by the above consideration, the purpose of this paper is to investigate the existence of mild solution and the controllability of a class of non-instantaneous stochastic integro-differential equations driven by fractional brownian motion with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$ and nonlocal Conditions. The present paper is an extension of the work of Melati et al. [32] to the non-instantaneous stochastic integro-differential equations driven by fractional brownian with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$. Using certain assumptions, sufficient conditions are derived using an extended version of Darbo's fixed point theorem, resolvent operator theory and the measure of non-compactness technique to analyze the controllability result.

We offer the following summary of the main contributions of our paper:

- The paper proposes a new class of stochastic integro-differential equations driven by fractional Brownian motion with non-instantaneous impulsive and nonlocal conditions.
- We investigate the existence of mild solutions and the exact controllability for system (1) using measures of non-compactness and applying the generalized Darbo's fixed point theorem.
- We establish a sufficient condition for the existence of mild solutions and the exact controllability of the system (1).
- We reinforce the theoretical results with illustrative examples.

The remainder of this paper is structured as follows: in section 2, we we introduce some necessary preliminary from the fields of fractional stochastic calculus, measure of noncompactness and the fixed point theory. In section 3, we use a generalized Darbo's fixed point theorem to demonstrate the existence of mild solutions to (1). In section 4, the exact controllability of the system (32) is proved. Finally, a conclusion is given in Section 5.

2. Preliminaries

Let $(\mathbb{V}, (.,.)_{\mathbb{V}}, \|.\|_{\mathbb{V}})$, and $(\mathbb{H}, (.,.)_{\mathbb{H}}, \|.\|_{\mathbb{H}})$, be real separable Hilbert spaces. We denote by $L_b(\mathbb{V}, \mathbb{H})$ the space of all bounded linear operators from $\mathbb{V}$ to $\mathbb{H}$ and $L_b(\mathbb{H})$ whenever $\mathbb{V} = \mathbb{H}$, $C(\mathbb{R}^+, \mathbb{V})$ indicate the space of all continuous functions from $[0, +\infty)$ into $\mathbb{V}$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ be a complete probability space with natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$. $\{B^H(t)\}_{t\geq 0}$ are the fractional Brownian motion with Hurst parameter $H \in (0, 1)$.

Let $L^2\Big(\Omega, \mathbb{H}\Big)$ be the space of all $\mathbb{H}$-valued random variable y such that $\mathbb{E}\|y\|^2 = \int_\Omega \|y\|^2 d\mathcal{P} < \infty$. For $y \in L^2\Big(\Omega, \mathbb{H}\Big)$,

$$\|y\|_{L^2\left(\Omega,\mathbb{H}\right)} = \Big(\int_\Omega \|y\|^2 d\mathcal{P}\Big)^{\frac{1}{2}} := \Big(\mathbb{E}\|y\|^2\Big)^{\frac{1}{2}}.$$

It is clear that $L^2\Big(\Omega, \mathbb{H}\Big)$ is a Hilbert space with the norm $\|.\|_{L^2\left(\Omega,\mathbb{H}\right)}$. In the sequel, $L_0^2\Big(\Omega, \mathbb{H}\Big)$ denotes the space of $\mathcal{F}_0$-measurable, $\mathbb{H}$-valued and square integrable stochastic process.

$$L_0^2\Big(\Omega, \mathbb{H}\Big) = \Big\{f \in L^2\Big(\Omega, \mathbb{H}\Big) \ \mid \ f \ \text{ is} \mathcal{F}_0\text{-measurable}\Big\}.$$

Consider the Banach space

$$\begin{aligned}\mathcal{PC}([0,T], \mathbb{H}) = \Big\{&\mathcal{F}_t \text{-adapted } \mathbb{H}\text{-valued process } y(t) \text{ is continuous every where except} \\ &\text{for some } \ t \neq t_k \ \text{ at which } \ y(t_k^-) \ \text{ and } \ y(t_k^+) \ \text{ exist} \\ &\text{and } \ y(t_k^-) = y(t_k), \ \ k = 1, 2, \cdots, N \text{ and} \sup_{0\leq t\leq T} \mathbb{E}\|y(t)\|^2 < \infty\Big\},\end{aligned}$$

with the norm

$$\|y\|_{\mathcal{PC}} = \Big(\sup_{0\leq t\leq T} \mathbb{E}\|y(t)\|^2\Big)^{\frac{1}{2}}.$$

Definition 2.1. *[37][1] A fractional Brownian motion (FBM) $\{B^H(t)\}_{t\geq 0}$ of Hurst parameter $H \in (0, 1)$, is a continuous and centered Gaussian process with covariance function*

$$R_H(t, s) = \mathbb{E}(B^H(t)B^H(s)) := \frac{1}{2}\Big(t^{2H} + s^{2H} - |t - s|^{2H}\Big), \quad \textit{for} \quad t, s \geq 0.$$

Now, we introduce the Wiener integral with respect to the one-dimensional FBM. Let $T > 0$ and denote by Λ the linear space of $\mathbb{R}$- valued step functions on $[0, T]$, that is $\phi \in \Lambda$ if

$$\phi(t) = \sum_{i=1}^{m-1} y_i \mathcal{X}_{[t_i,t_{i+1})}(t), \quad \text{for all} \;\; t \in [0, T]\,,$$

where $\mathcal{X}_{[t_i,t_{i+1})}$ is the indicator function, $y_i \in \mathbb{R}$ and $0 = t_1 < t_2 < \cdots t_m = T$.

For $\phi \in \Lambda$ we define its Wiener integral with respect to β^H by

$$\int_0^T \phi(s) d\beta^H(s) := \sum_{i=1}^{m-1} y_i\Big(\beta^H(t_{i+1}) - \beta^H(t_i)\Big).$$

Let $\mathbb{H}$ be the Hilbert space defined as the closure of Λ with respect to the scalar product

$$\Big(\mathcal{X}_{[0,t]}, \mathcal{X}_{[0,s]}\Big)_{\mathbb{H}} = R_H(t, s).$$

Then the mapping

$$\phi = \sum_{i=1}^{m-1} y_i \mathcal{X}_{[t_i,t_{i+1})} \to \int_0^T \phi(s) d\beta^H(s),$$

is an isometry between Λ and the linear space span $\{\beta^H(t)\}_{t\in[0,T]}$, which can be extended to an isometry between $\mathbb{H}$ and the first Wiener chaos of the fractional Brownian motion $\overline{span}^{L^2(\Omega)}\{\beta^H(t)\}_{t\in[0,T]}$(see [43]). The image of an element $\phi \in \mathbb{H}$ by this isometry is called the Wiener integral of ϕ with respect to β^H. Our next goal is to give an explicit expression for this integral. To this end, we consider the square integrable kernel with $H \in \left(\frac{1}{2}, 1\right)$.

$$K_H(t, s) = C_H s^{\frac{1}{2}-H} \int_s^t (u - s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

where $C_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$, $t > s$ and $\beta(.,.)$ signifies the Beta function.

Observe that by representation for the square integrable kernel $K_H(t, s)$, we obtain

$$\frac{\partial K_H}{\partial t}(t, s) = C_H \Big(\frac{t}{s}\Big)^{\frac{1}{2}-H} (u - s)^{H-\frac{3}{2}}.$$

Now, we present the linear operator $K_H^* : \Lambda \to L^2([0, T])$, which is defined as follows:

$$\Big(K_H^* \Upsilon\Big)(s) = \int_s^T \Upsilon(t) \frac{\partial K_H}{\partial t}(t, s) dt.$$

Then

$$\Big(K_H^* \mathcal{X}_{[0,T]}\Big)(s) = K_H(t, s) \mathcal{X}_{[0,T]}(s),$$

and $\Big(K_H^*\Big)$ is an isometry between Λ and $L^2([0, T])$ that may be extended to Λ (see [3]). Taking $\{B(t)\}_{t\in[0,T]}$ defined by

$$B(t) = \beta^H\Big((K_H^*)^{-1} \mathcal{X}_{[0,T]}\Big),$$

B is a Brownian motion, β^H and has the Wiener integral form shown below

$$\beta^H(t) = \int_0^t K_H(t,s)dB(s).$$

Furthermore, for any $\phi \in \Lambda$,

$$\int_0^T \phi(s)d\beta^H(s) = \int_0^T (K_H^*\phi)(t)dB(t),$$

if and only if $K_H^*\phi \in L^2([0,T])$. Also

$$L^2_{\mathbb{H}}([0,T]) = \Big\{\phi \in \Lambda, \quad K_H^*\phi \in L^2([0,T])\Big\},$$

for all $H > \frac{1}{2}$ we can observe

$$L^{1/H}([0,T]) \subset L^2_{\mathbb{H}}([0,T]).$$

see [34]. Furthermore, the following beneficial finding holds:

Lemma 2.2. *[36] For $\phi \in L^{\frac{1}{H}}([0,T])$*

$$H(2H-1)\int_0^T\int_0^T |\phi(s)||\phi(t)||t-s|^{2H-2}dtds \le c_H\|\phi\|^2_{L^{\frac{1}{H}}([0,T])}. \tag{2}$$

Next we are interested in considering a fBm with values in a Hilbert space and giving the Definition of the corresponding stochastic integral.

Let $\phi \in L_b(\mathbb{V},\mathbb{H})$ be a non-negative self-adjoint operator. Defined by $L^0_\phi(\mathbb{V},\mathbb{H})$ the space of all $\xi \in L_b(\mathbb{V},\mathbb{H})$ such that $\xi\phi^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. The norm is given by

$$|\xi|^2_{L^0_\phi(\mathbb{V},\mathbb{H})} = tr(\xi\phi\xi^*).$$

Let $\{\beta_n^H(t)\}_{n\in\mathbb{N}}$ be a sequence of two-sided one-dimensional standard fractional Brownian motions mutually independent on $(\Omega,\mathcal{F},\mathcal{P})$. When one considers the following series

$$\sum_{i=1}^{\infty}\beta_i^H(t)e_i, \qquad t \ge 0,$$

where $\{e_i\}_{i\in\mathbb{N}}$ is a complete orthonormal basis in $\mathbb{K}$ does not necessarily converge in the space $\mathbb{K}$ Thus, we consider a $\mathbb{H}$-valued stochastic process $B^H(t)$ given formally by the following series:

$$B^H(t) = \sum_{i=1}^{\infty}\beta_i^H(t)\phi^{\frac{1}{2}}e_i, \qquad t \ge 0,$$

which is well-defined as a $\mathbb{V}$-valued ϕ-cylindrical fractional Brownian motion. Let $\phi : [0,T] \mapsto L^0_\phi(\mathbb{V},\mathbb{H})$ such that

$$\sum_{i=1}^{\infty}\|K_H^*\big(\upsilon\phi^{\frac{1}{2}}e_i\big)\|_{L^{\frac{1}{H}}([0,T],\mathbb{H})} < \infty. \tag{3}$$

Definition 2.3. *[8] Let $\upsilon : [0,T] \to L^0_\phi(\mathbb{V},\mathbb{H})$ be a given function, satisfy (3). The stochastic integral of υ with respect to B^H is defined by*

$$\int_0^t \upsilon(s)dB^H(s) := \sum_{i=1}^{\infty}\int_0^t \upsilon(s)\phi^{\frac{1}{2}}e_i d\beta_i^H(s) = \sum_{i=1}^{\infty}\int_0^t \Big(K_H^*(\upsilon\phi^{\frac{1}{2}}e_i)\Big)(s)dB(s).$$

Notice that if

$$\sum_{i=1}^{\infty}\|\upsilon(s)\phi^{\frac{1}{2}}e_i\|_{L^{\frac{1}{H}}(([0,T]),\mathbb{H})} < \infty, \tag{4}$$

then in particular (3) holds, which follows immediately from (2).

Lemma 2.4. *[5] If $H \in (\frac{1}{2},1)$, then for any $\phi : [0,T] \to \mathcal{L}_2^0(\mathbb{V},\mathbb{H})$ satisfies*

$$\int_0^t \|\phi(s)\|^2_{\mathcal{L}_2^0(\mathbb{V},\mathbb{H})}ds < \infty,$$

then the series in (4) is well defined as a $\mathbb{H}-$valued random variable and we have

$$\mathbb{E}\Big\|\int_0^t \phi(s)dB^H(s)\Big\|^2 \le 2Ht^{2H-1}\int_0^t \|\phi(s)\|^2_{\mathcal{L}_2^0(\mathbb{V},\mathbb{H})}ds.$$

Now let us recall some fundamental facts of the notion of Kuratowski measure of non-compactness.

Definition 2.5. *[4] Let $\mathbb{H}$ be a Banach space and $\Omega_{\mathbb{H}}$ the bounded subsets of $\mathbb{H}$. The Kuratowski measure of non-compactness is the map $\alpha : \Omega_{\mathbb{H}} \to [0,+\infty]$ defined by*

$$\alpha(D) = \inf\Big\{\varepsilon > 0 : D \subseteq \bigcup_{i=1}^{n} D_i \text{ and } diam(D_i) \le \varepsilon\Big\}.$$

This measure of non-compactness satisfies some important properties .

Lemma 2.6. *[4] Let $\mathbb{H}$ be a Hilbert space, $B, D \subset \mathbb{H}$ be bounded, then the following properties are satisfied:*

(1) *Regular, if the condition $\alpha(D) = 0 \iff \overline{D}$ is compact,*

(2) *$\alpha(D) = \alpha(\overline{D}) = \alpha(co(D))$, where $co(D)$ means the convex hull of D,*

(3) *$\alpha(\beta D) = |\beta|\alpha(D)$, for any $\beta \in \mathbb{R}$,*

(4) *monotone, if $B \subset D \Longrightarrow \alpha(B) \le \alpha(D)$,*

(5) *algebraically semiadditive, if $\alpha(B + D) \le \alpha(B) + \alpha(D)$,*

(6) *nonsingular, if $\alpha(D + x) = \alpha(D)$, for all $x \in \mathbb{H}$,*

(7) *$\alpha(B \cup D) \le \max\Big\{\alpha(B), \alpha(D)\Big\}$,*

(8) *if the map $\Theta : D(\Theta) \subset \mathbb{H} \longrightarrow \mathbb{H}$ is lipschitz continuous with constant K, then $\alpha(\Theta(D)) \le K\alpha(D)$ for any bounded subset $D \subset D(\Theta)$, and $\mathbb{K}$ is another Hilbert space,*

(9) *if $D \subset \mathcal{PC}([0,T],\mathbb{H})$ is bounded, then $\alpha(D(t)) \le \alpha_{\mathcal{PC}}(D)$ for all $t \in [0,T]$ where $D(t) = \{x(t) : x \in D\} \subseteq \mathbb{H}$ Furthermore, if D is equicontinuous on $[0,T]$, then $D(t)$ is continuous for $t \in [0,T]$, and $\alpha_{\mathcal{PC}}(D) = \sup_{t\in[0,T]} \alpha(D(t))$.*

The notation $\alpha(.)$, $\alpha_C(.)$, $\alpha_{\mathcal{PC}}(.)$ are the Kuratowskii measure of non-compactness on the bounded set of $\mathbb{H}$, $C([0,T],\mathbb{H})$ and $\mathcal{PC}([0,T],\mathbb{H})$, respectively. For more details see [4].

Lemma 2.7. *[27] Let $\mathbb{H}$ be a Banach space, be bounded $D \subset \mathbb{H}$. Then there exist a countable set $D_0 \subset D$, such that*

$$\alpha(D) \leq 2\alpha(D_0). \tag{5}$$

Definition 2.8. *[9] A continuous map $\Theta : D \subset \mathbb{H} \to \mathbb{H}$ is said to be α-contraction if there exists a positive constant $K \in [0,1)$ such that for any bounded set $\Omega \subset D$*

$$\alpha(\Theta(\Omega)) \leq K\alpha(\Omega).$$

We also need the followinga generalization of the Darbo fixed point theorem to prove our theorem. The proof is refined from the proof of the Darbo fixed point theorem . So we don't claim complete originality but include it here for completeness. For more details see [29], [32].

Lemma 2.9. *[29, 42] (Generalized Darbo's fixed point principle) Let I be a closed and convex subset of a real Banach space $\mathbb{H}$. Suppose that $Q : I \to I$ is a continuous operator and $Q(I)$ is bounded, for any bounded subset $C \subset I$*

$$Q^1(C) = Q(C), \quad Q^n(C) = Q\Big(\overline{co}(Q^{n-1}(C))\Big), \ \ n = 2, 3 \cdots N.$$

If there exists a constant $0 \leq \lambda < 1$, and a positive integer n_0 such that for any bounded subset $C \subset I$.

$$\alpha\Big(Q^{n_0}(C)\Big) \leq \lambda\alpha\Big(Q(C)\Big). \tag{6}$$

Then Q has at least one fixed point in C.

In this part, we introduce some basic notions about resolvent operators for integro-differential equations, we seek from the reader to go to [16]. Let A and $h(t)$ are closed linear operators on $\mathbb{H}$ and $\mathbb{V}$ represents the Banach space $\mathcal{D}(A)$ equipped with the graph norm defined by

$$\|y\|_{\mathbb{V}} = \|Ay\|_{\mathbb{H}} + \|y\|_{\mathbb{H}}, \ \ y \in \mathbb{V}.$$

Let us consider the following Cauchy problem

$$\begin{cases} y'(t) = Ay(t) + \int_0^t h(t-s)y(s)ds \quad t \geq 0, \\ y(0) = y_0 \in \mathbb{H}. \end{cases} \tag{7}$$

Definition 2.10. *[16] A resolvent operator for (7) is a bounded linear operator valued function $\mathcal{R}(t) \in L_b(\mathbb{H})$ for $t \geq 0$, which satisfies the following properties*

(i). *$\mathcal{R}(0) = \mathcal{X}$ (The Identity operator of $\mathbb{H}$) and $\|\mathcal{R}(t)\| \leq Me^{\alpha t}$ for some constants $k > 0$ and $\alpha \in \mathbb{R}$.*

(ii). *For each $y \in \mathbb{H}$, $\mathcal{R}(t)y$ is strongly continuous for $t \geq 0$.*

(iii). *For $y \in \mathbb{V}$, $\mathcal{R}(.)y \in C^1(\mathbb{R}^+, \mathbb{H}) \cap C(\mathbb{R}^+, \mathbb{V})$ and*

$$\begin{aligned} \mathcal{R}'(t)y =& A\mathcal{R}(t)y + \int_0^t h(t-s)\mathcal{R}(s)y\,ds \\ =& \mathcal{R}(t)Ay + \int_0^t \mathcal{R}(t-s)h(s)y\,ds, \quad \text{for} \ \ t \geq 0. \end{aligned} \tag{8}$$

The resolvent operators have a great importance in obtaining variation of constants formula for nonlinear systems and in studying the existence of solutions, see [11, 16].

Now, we make the following assumptions:

(H_1) The operator A is the infinitesimal generator of a strongly continuous semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ on $\mathbb{H}$.

(H_2) For all $t \geq 0$, $h(t)$ is a closed linear operator from $\mathbb{D}(h)$ to $\mathbb{H}$ and $h(t) \in L_b(\mathbb{V}, \mathbb{H})$. For any $y \in \mathbb{V}$ the map $t \to h(t)y$ is bounded, differentiable and the derivative $t \to h'(t)y$ is bounded and uniformly continuous on $\mathbb{R}^+$.

Lemma 2.11. *[16] Assume that (H_1) and (H_2) hold. Then there exists a unique resolvent operator to the Cauchy problem(7).*

As stated below, we establish certain results for the existence of solutions to the following integro-differential equation.

$$\begin{cases} y'(t) = Ay(t) + \int_0^t h(t-s)y(s)ds + \sigma(t) & t \geq 0, \\ y(0) = y_0 \in \mathbb{H}. \end{cases} \tag{9}$$

where $\sigma : [0, +\infty[\to \mathbb{H}$ is a continuous function.

Definition 2.12. *[16] A continuous function $y : \mathbb{R}^+ \to \mathbb{H}$ is said to be a strict solution of (9) if*

(i) $y \in C^1(\mathbb{R}^+, \mathbb{H}) \cap C(\mathbb{R}^+, \mathbb{V})$,

(ii) *y satisfies (9) for $t \geq 0$.*

Lemma 2.13. *[16] Assume that (H_1)-(H_2) hold . If x is a strict solution of (9), then*

$$y(t) = \mathcal{R}(t)y_0 + \int_0^t \mathcal{R}(t-s)\sigma(s)ds \quad \text{for } t \geq 0. \tag{10}$$

Definition 2.14. *A continuous function $y : \mathbb{R}^+ \to \mathbb{H}$ is said to be a strict solution of (9) if $y \in C^1(\mathbb{R}^+, \mathbb{H}) \cap C(\mathbb{R}^+, \mathbb{H})$ and satisfies (10) for $t \geq 0$.*

From Definition 2.14 , we deduce that the function $h(t-s)y(s)$ is integrable for all $t \geq 0$ and $s \in \mathbb{R}^+$.

Definition 2.15. *[44] A semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ in $\mathbb{H}$ is said to be equicontinuous if the operator $\mathcal{T}(t)$ is uniformly continuous by operator norm for every $t > 0$.*

Lemma 2.16. *[14] Let A be the infinitesimal generator of a C_0-semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ and let $\{h(t)\}_{t\geq 0}$ satisfy (H_2). Then the resolvent operator $\{\mathcal{R}(t)\}_{t>0}$ is operator norm continuous (or continuous in the uniform operator topology) for $t > 0$ if and only if $\{\mathcal{T}(t)\}_{t\geq 0}$ is operator norm continuous for $t \geq 0$.*

3. Existence of mild solution

In this section, we prove the existence of mild solutions for the system (1). We now introduce the concept of mild solution of (1), we present the following definitions

Definition 3.1. *A $\mathcal{F}_t$-adapted stochastic process $y(t) : [0, T] \to \mathbb{H}$ is called a mild solution of(1) if $y \in \mathcal{PC}([0, T], \mathbb{H})$, $y(0) + \psi(y) = y_0$ and*

$$y(t) = \begin{cases} \mathcal{R}(t)\Big(y_0 - \psi(y)\Big) + \int_0^t \mathcal{R}(t-s)\varphi\Big(s, y(s)\Big)ds \\ \qquad + \int_0^t \mathcal{R}(t-s)f\Big(s\Big)dB^H(s), & t \in [0, t_1], \\ g_k(t, y(t_k^-)), & t \in \cup_{k=1}^N (t_k, s_k], \\ \mathcal{R}(t-s_k)g_k(s_k, y(t_k^-)) + \int_{s_k}^t \mathcal{R}(t-s)\varphi\Big(s, y(s)\Big)ds \\ \qquad + \int_{s_k}^t \mathcal{R}(t-s)f\Big(s\Big)dB^H(s), & t \in \cup_{k=1}^N (s_k, t_{k+1}]. \end{cases} \tag{11}$$

The following assumptions will be needed throughout the paper:

(H_3) The resolvent operator $\mathcal{R}(t)$, $t \geq 0$ is continuous in operator norm topology, and there exists a constant $M > 0$ such that

$$\|\mathcal{R}(t)\| \leq M.$$

(H_4) The nonlinear function $\varphi : I \times \mathcal{PC}([0,T], \mathbb{H}) \to \mathbb{H}$ satisfies

(1) For each $y \in \mathcal{PC}([0,T], \mathbb{H})$, $\varphi(., y)$ is measurable and for any $t \in I$, $\varphi(t, .)$ is continuous.

(2) For some positive number $r > 0$, there exists a constant $k_1 > 0$, function $\omega \in L^1(I, \mathbb{R}^+)$ and a continuous non-decreasing function $\pi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\mathbb{E}\|\varphi(t,y)\|^2 \leq \omega(t)\pi_\varphi\Big(\mathbb{E}\|y\|^2\Big), \qquad \liminf_{r\to+\infty} \frac{\pi_\varphi(r)}{r} = k_1 < +\infty.$$

(3) There exists a positive constant k_2 such that for any bounded set $M \subset \mathbb{H}$

$$\alpha\Big(\varphi(t,M)\Big) \leq k_2\alpha(M).$$

(H_5) The function $f : [0,T] \to \mathcal{L}_2^0(\mathbb{V}, \mathbb{H})$, satisfying the following condition

$$\int_0^t \|f(s)\|^2_{\mathcal{L}_2^0} ds < \infty.$$

(H_6) The nonlocal function $\psi : \mathcal{PC}([0,T], \mathbb{H}) \to \mathbb{H}$ is continuous and compact, and there exists a constant $C_\psi > 0$, such that

$$\mathbb{E}\|\psi(y)\|^2 \leq C_\psi.$$

(H_7) The impulsive function $g_k : (t_k, s_k] \times \mathbb{H} \to \mathbb{H}$ is continuous and compact, and there exist constants $C_{g_k} > 0$, $k = 1, 2, \cdots, N$, such that

$$\mathbb{E}\|g_k(t,y)\|^2 \leq C_{g_k}\mathbb{E}\|y\|^2.$$

where we have used the notation

$$C_g = \max_{k=1,2,\cdots,N} C_{g_k}, \qquad \eta = \max_{k=1,2,\cdots,N} \|\omega\|_{L^1[s_k,t_{k+1}]}.$$

Remark 3.2. *The function f is independent of $y(t)$, $t \in [0,T]$. From the functional point of view, we know that*

$$\alpha\Big(\int_0^t \mathcal{R}(t-s)f\big(s\big)dB^H(s)\Big) = 0.$$

Theorem 3.3. *Suppose that* (H_1)-(H_7) *are satisfied, then the problem (1) has at least one mild solution provided that*

$$L := \max_{0\leq k\leq N} \Big\{3M^2\Big(C_g + (t_{k+1} - s_k)\,\eta k_1\Big)\Big\} < 1. \tag{12}$$

Proof. Consider the operator $\Pi : \mathcal{PC}([0,T],\mathbb{H}) \to \mathcal{PC}([0,T],\mathbb{H})$ defined by

$$\Pi y(t) = \begin{cases} \mathcal{R}(t)\Big(y_0 - \psi(y)\Big) + \displaystyle\int_0^t \mathcal{R}(t-s)\varphi\Big(s, y(s)\Big)ds \\ \qquad + \displaystyle\int_0^t \mathcal{R}(t-s) f\Big(s\Big) dB^H(s), & t \in [0, t_1], \\ g_k(t, y(t_k^-)), & t \in \cup_{k=1}^N (t_k, s_k], \\ \mathcal{R}(t-s_k) g_k(s_k, y(t_k^-)) + \displaystyle\int_{s_k}^t \mathcal{R}(t-s)\varphi\Big(s, y(s)\Big)ds \\ \qquad + \displaystyle\int_{s_k}^t \mathcal{R}(t-s) f\Big(s\Big) dB^H(s), & t \in \cup_{k=1}^N (s_k, t_{k+1}]. \end{cases} \tag{13}$$

Further, finding the solution of the operator equation $\Pi y(t) = y(t)$ leads us to find a solution to problem (1).

Now, we will show that by using the generalized Darbo's fixed point theorem, the operator Π has a fixed point. Obviously, the fixed point of $\Pi y(t)$ is the solution of the problem (1). For each finite constant $r > 0$, let

$$\Omega_r = \Big\{ y \in \mathcal{PC}([0,T],\mathbb{H}) : \|y\|^2_{\mathcal{PC}} \le r \Big\}.$$

It is clear that Ω_r is a bounded closed and convex set in $\mathcal{PC}([0,T],\mathbb{H})$.

The proof falls naturally into four steps.

Step 1. We claim that there exists a positive number r such that $\Pi(\Omega_r) \subset \Omega_r$. If this is not true, then, for each positive integer r, there exists $y_r \in \Omega_r$ such that for $t \in [0,T], t$ may depending upon r. However, on the other hand, we consider three cases.

Case I. For $t \in [0, t_1]$ by (13) and assumptions

$$\begin{aligned} r <& \mathbb{E}\Big\|\Pi y_r(t)\Big\|^2 \\ \le& 3\mathbb{E}\Big\|\mathcal{R}(t)\Big(y_0 - \psi(y_r)\Big)\Big\|^2 + 3\mathbb{E}\Big\|\int_0^t \mathcal{R}(t-s)\varphi\Big(s, y_r(s)\Big)ds\Big\|^2 \\ &+ 3\mathbb{E}\Big\|\int_0^t \mathcal{R}(t-s) f\Big(s\Big) dB^H(s)\Big\|^2, \end{aligned} \tag{14}$$

by using assumptions (H_3)-(H_6), Lemma 2.4 and Hölder's inequality, we obtain

$$\begin{aligned} r < \mathbb{E}\|\Pi y_r(t)\|^2 \le& 3M^2\mathbb{E}\Big\|\Big(y_0 - \psi(y_r)\Big)\Big\|^2 + 3M^2 t_1 \int_0^t \mathbb{E}\Big\|\varphi\Big(s, y_r(s)\Big)\Big\|^2 ds \\ &+ 6M^2 H t_1^{2H-1} \int_0^t \|f(s)\|^2_{\mathcal{L}_2^0} ds \\ \le& 3M^2\Big(\mathbb{E}\|y_0\|^2 + \mathbb{E}\|\psi(y_r)\|^2\Big) + 3M^2 t_1 \int_0^t \omega(s)\pi_\varphi\Big(\mathbb{E}\|y_r\|^2\Big)ds \\ &+ 6M^2 H t_1^{2H-1} \int_0^t \|f(s)\|^2_{\mathcal{L}_2^0} ds \\ \le& 3M^2\Big(\mathbb{E}\|y_0\|^2 + C_\psi\Big) + 3M^2 t_1 \|\omega\|_{L^1[0,t_1]}\pi_\varphi(r) + 6M^2 H t_1^{2H-1} \int_0^t \|f(s)\|^2_{\mathcal{L}_2^0} ds. \end{aligned} \tag{15}$$

Case II. For $t \in (t_k, s_k], k = 1 \cdots, N$. By assumption (H_7), we get

$$\begin{aligned} r < \mathbb{E}\|\Pi y_r(t)\|^2 \leq& \mathbb{E}\|g_k(t, y_r(t_k^-))\|^2 \\ \leq& C_{g_k}\mathbb{E}\|y_r(t)\|^2 \\ \leq& C_g r. \end{aligned} \tag{16}$$

Case III. For $t \in (s_k, t_{k+1}], k = 1 \cdots, N$. By Lemma 2.4, and using assumptions (H_3)-(H_5), we have

$$\begin{aligned} r <& \mathbb{E}\Big\|\Pi y_r(t)\Big\|^2 \\ \leq& 3\mathbb{E}\Big\|\mathcal{R}(t-s_k)g_k(s_k, y_r(t_k^-))\Big\|^2 + \mathbb{E}\Big\|\int_{s_k}^t \mathcal{R}(t-s)\varphi\Big(s, y_r(s)\Big)ds\Big\|^2 \\ &+ 3\mathbb{E}\Big\|\int_{s_k}^t \mathcal{R}(t-s)f\Big(s\Big)dB^H(s)\Big\|^2 \\ \leq& 3M^2\mathbb{E}\Big\|g_k(s_k, y_r(t_k^-))\Big\|^2 + 3M^2\,(t_{k+1}-s_k)\int_{s_k}^t \mathbb{E}\Big\|\varphi\Big(s, y_r(s)\Big)\Big\|^2 ds \\ &+ 6M^2H\,(t_{k+1}-s_k)^{2H-1}\int_{s_k}^t \|f(s)\|^2_{\mathcal{L}_2^0}ds \end{aligned}$$

$$\begin{aligned} \leq& 3M^2\mathbb{E}\Big\|g_k(s_k, y_r(t_k^-))\Big\|^2 + 3M^2\,(t_{k+1}-s_k)\int_{s_k}^t \omega(s)\pi_\varphi\Big(\mathbb{E}\|y_r\|^2\Big)ds \\ &+ 6M^2H\,(t_{k+1}-s_k)^{2H-1}\int_{s_k}^t \|f(s)\|^2_{\mathcal{L}_2^0}ds \\ \leq& 3M^2C_g r + 3M^2\,(t_{k+1}-s_k)\,\|\omega\|_{L^1[s_k,t_{k+1}]}\pi_\varphi(r) + 6M^2H\,(t_{k+1}-s_k)^{2H-1}\int_{s_k}^t \|f(s)\|^2_{\mathcal{L}_2^0}ds, \end{aligned} \tag{17}$$

from (15), (16) and (17), we divide by r and take the lower bound as $r \to +\infty$, we have

$$1 < \mathbb{E}\Big\|\Pi y_r(t)\Big\|^2 \leq L,$$

with

$$1 < L := \max_{0\leq k\leq N}\Big\{3M^2\Big(C_g + (t_{k+1}-s_k)\,\eta k_1\Big)\Big\}.$$

Which contradict with condition (12), hence $\Pi(\Omega_r) \subset \Omega_r$.

Step 2. We prove that the operator Π is continuous in Ω_r.

Let us consider a sequence $(y_n)_{n=1}^{+\infty} \subset \mathcal{PC}([0,T], \mathbb{H})$ such that $\lim_{n\to+\infty} y_n = y \in \mathcal{PC}([0,T], \mathbb{H})$.

By Hölder's inequality, Lemma 2.4 and using (H_3)-(H_7), we have

$$\lim_{n\to+\infty} \varphi(s, y_n(s)) = \varphi(s, y(s)), \tag{18}$$

$$\lim_{n\to+\infty} \psi(y_n) = \psi(y), \tag{19}$$

$$\lim_{n\to+\infty} g_k(s, y_n(t_k^-)) = g_k(s, y(t_k^-)). \tag{20}$$

By assumption (H_4), for a.e $s \in [0,T]$, we obtain

$$\mathbb{E}\Big\|\varphi(s, y_n(s)) - \varphi(s, y(s))\Big\|^2 \leq 2\mathbb{E}\|\varphi(s, y_n(s))\|^2 + 2\mathbb{E}\|\varphi(s, y(s))\|^2 \leq 4\omega(s)\pi_\varphi\Big(r\Big). \tag{21}$$

Case I. For $t \in [0, t_1]$, using the fact that the function $s \to \Phi(s)\pi_f(r)$ is Lebesgue integrable for $s \in [0, T]$ and $t \in [0, t_1]$ so by (18)-(19), 21, Lemma 2.4 and the Lebesgue dominated convergence theorem, we see that

$$\mathbb{E}\|\Pi y_n(t) - \Pi y(t)\|^2 \leq 2M^2\mathbb{E}\|\psi(y_n) - \psi(y)\|^2 + 2M^2 \int_0^t \mathbb{E}\Big\|\varphi(s, y_n(s)) - \varphi(s, y(s))\Big\|^2 ds$$
$$\longrightarrow 0 \quad \text{as} \quad n \to +\infty.$$

Case II. For $t \in (t_k, s_k]$, $k = 1, 2, \cdots, N$, by (20), we get

$$\mathbb{E}\|\Pi y_n(t) - \Pi y(t)\|^2 \leq \mathbb{E}\|g_k(s, y_n(t_k^-)) - g_k(s, y(t_k^-))\|^2 \longrightarrow 0 \quad \text{as} \quad n \to +\infty.$$

Case III. For $t \in (s_k, t_{k+1}]$, $k = 1, 2, \cdots, N$, by Lemma 2.4,(18), (20), 21 and the Lebesgue dominated convergence theorem, we can deduce that

$$\mathbb{E}\|\Pi y_n(t) - \Pi y(t)\|^2 \leq 2M^2\mathbb{E}\|g_k(s, y_n(t_k^-)) - g_k(s, y(t_k^-))\|^2 + 2M^2 \int_{s_k}^t \mathbb{E}\Big\|\varphi(s, y_n(s)) - \varphi(s, y(s))\Big\|^2 ds$$
$$\longrightarrow 0 \quad \text{as} \quad n \to +\infty.$$

Thus

$$\mathbb{E}\|\Pi y_n(t) - \Pi y(t)\|^2 \longrightarrow 0 \quad \text{as} \quad n \to +\infty.$$

Therefore Π is continuous in Ω_r.

Step 3. We now establish the equicontinuous of the operator $\Pi : \Omega_r \longrightarrow \Omega_r$. Since the impulsive function g_k is compact, then $\Pi(\Omega_r)$ is equicontinuous on $(t_k, s_k]$, $k = 1, 2, \cdots, N$.

Case I. Let r_1, $r_2 \in [0, t_1]$, $r_1 < r_2$ and $x \in \Omega_r$, using Lemma 2.4, hypotheses (H_3)-(H_6) and by Hölder's inequality, we got

$$\begin{aligned}
\mathbb{E}\Big\|\Pi y(r_2) - \Pi y(r_1)\Big\|^2 \leq & 5\mathbb{E}\Big\|\Big(\mathcal{R}(r_2) - \mathcal{R}(r_1)\Big)\Big(y_0 - \psi(y)\Big)\Big\|^2 \\
& + 5\mathbb{E}\Big\|\int_{r_1}^{r_2} \mathcal{R}(r_2 - s)\varphi(s, y(s))ds\Big\|^2 \\
& + 5\mathbb{E}\Big\|\int_0^{r_1} \Big(\mathcal{R}(r_2 - s) - \mathcal{R}(r_1 - s)\Big)\varphi(s, y(s))ds\Big\|^2 \\
& + 5\mathbb{E}\Big\|\int_{r_1}^{r_2} \mathcal{R}(r_2 - s)f\Big(s\Big)dB^H(s)\Big\|^2 \\
& + 5\mathbb{E}\Big\|\int_0^{r_1} \Big(\mathcal{R}(r_2 - s) - \mathcal{R}(r_1 - s)\Big)f\Big(s\Big)dB^H(s)\Big\|^2 \\
\leq & 5\Big\|\mathcal{R}(r_2) - \mathcal{R}(r_2)\Big\|^2\Big(\mathbb{E}\|y_0\|^2 + \mathbb{E}\|\psi(y)\|^2\Big) \\
& + 5M^2\,(r_2 - r_1)\int_{r_1}^{r_2} \mathbb{E}\|\varphi(s, y(s))\|^2 ds \\
& + 5M^2 \int_0^{r_1} \Big\|\mathcal{R}(r_2 - s) - \mathcal{R}(r_1 - s)\Big\|^2 ds \int_0^{r_1} \mathbb{E}\|\varphi(s, y(s))\|^2 ds \\
& + 10M^2 H t_1^{2H-1}\,(r_2 - r_1)\int_{r_1}^{r_2} \|f(s)\|^2_{\mathcal{L}_2^0} ds \\
& + 10H t_1^{2H-1} \int_0^{r_1} \Big\|\mathcal{R}(r_2 - s) - \mathcal{R}(r_1 - s)\Big\|^2 ds \int_0^{r_1} \|f(s)\|^2_{\mathcal{L}_2^0} ds \\
\leq & \sum_{i=1}^{5} p_i.
\end{aligned}$$

For the purpose of proving that $\mathbb{E}\left\|\Pi y(r_2) - \Pi y(r_1)\right\|^2 \longrightarrow 0 \quad \text{as} \quad r_2 - r_1 \to 0$ we only need to check independently of $y \in \Omega_r$ when $r_2 - r_1 \to 0$.

For $p_1, \cdots p_5$, since the resolvent operator is continuous in operator norm topology for $t \geq 0$, the nonlocal function ψ is compact and taking use of the function $s \to \Phi(s)\pi_f(r)$ is Lebesgue integrable, we can easily see that

$$p_1 := \left\|\mathcal{R}(r_2) - \mathcal{R}(r_1)\right\|^2 \Big(\mathbb{E}\|y_0\|^2 + C_\psi\Big) \longrightarrow 0 \quad \text{as} \quad r_2 - r_1 \to 0,$$

$$p_2 := 5M^2 (r_2 - r_1)\, \pi_\varphi(r) \int_{r_1}^{r_2} \omega(s)ds \longrightarrow 0 \quad \text{as} \quad r_2 - r_1 \to 0,$$

$$p_3 := 5M^2 \pi_\varphi(r) \int_0^{r_1} \left\|\mathcal{R}(r_2 - s) - \mathcal{R}(r_1 - s)\right\|^2 ds \int_0^{r_1} \omega(s)ds \longrightarrow 0 \quad \text{as} \quad r_2 - r_1 \to 0,$$

$$p_4 := 10M^2 H t_1^{2H-1} (r_2 - r_1) \int_{r_1}^{r_2} \|f(s)\|^2_{\mathcal{L}_2^0} ds \quad \text{as} \quad r_2 - r_1 \to 0,$$

and

$$p_5 := 10H t_1^{2H-1} \int_0^{r_1} \left\|\mathcal{R}(r_2 - s) - \mathcal{R}(r_1 - s)\right\|^2 ds \int_0^{r_1} \|f(s)\|^2_{\mathcal{L}_2^0} ds \quad \text{as} \quad r_2 - r_1 \to 0.$$

Consequently, $\mathbb{E}\left\|\Pi y(r_2) - \Pi y(r_1)\right\|^2 \longrightarrow 0 \quad \text{as} \quad r_2 - r_1 \to 0$ independently of $y \in \Omega_r$ when $r_2 - r_1 \to 0$, it follows that $\Pi(\Omega_r)$ is equicontinuous on $[0, t_1]$.

Case II. For any $x \in \Omega_r$ and $r_1, r_2 \in (s_k, t_{k+1}]$, $k = 1, 2, \cdots, N$, $r_1 < r_2$ by Lemma 2.4, and hypotheses (H_3)-(H_6), we have

$$\begin{aligned}
\mathbb{E}\left\|\Pi y(r_2) - \Pi y(r_1)\right\|^2 \leq & 5\mathbb{E}\left\|\Big(\mathcal{R}(r_2 - s) - \mathcal{R}(r_1 - s)\Big) g_k(s_k, y(t_k^-))\right\|^2 \\
& + 5\mathbb{E}\left\| \int_{r_1}^{r_2} \mathcal{R}(r_2 - s)\varphi(s, y(s))ds \right\|^2 \\
& + 5\mathbb{E}\left\| \int_{s_k}^{r_1} \Big(\mathcal{R}(r_2 - s) - \mathcal{R}(r_1 - s)\Big)\varphi(s, y(s))ds \right\|^2 \\
& + 5\mathbb{E}\left\| \int_{r_1}^{r_2} \mathcal{R}(r_2 - s) f\big(s\big) dB^H(s) \right\|^2 \\
& + 5\mathbb{E}\left\| \int_{s_k}^{r_1} \Big(\mathcal{R}(r_2 - s) - \mathcal{R}(r_1 - s)\Big) f\big(s\big) dB^H(s) \right\|^2 \\
\leq & 5\mathbb{E}\left\|\Big(\mathcal{R}(r_2 - s) - \mathcal{R}(r_1 - s)\Big) C_g r\right\|^2 \\
& + 5M^2 (r_2 - r_1) \int_{r_1}^{r_2} \mathbb{E}\|\varphi(s, y(s))\|^2 ds \\
& + 5M^2 \int_{s_k}^{r_1} \left\|\mathcal{R}(r_2 - s) - \mathcal{R}(r_1 - s)\right\|^2 ds \int_{s_k}^{r_1} \mathbb{E}\|\varphi(s, y(s))\|^2 ds \\
& + 10M^2 H (t_{k+1} - s_k)^{2H-1} (r_2 - r_1) \int_{r_1}^{r_2} \|f(s)\|^2_{\mathcal{L}_2^0} ds \\
& + 10H (t_{k+1} - s_k)^{2H-1} \int_{s_k}^{r_1} \left\|\mathcal{R}(r_2 - s) - \mathcal{R}(r_1 - s)\right\|^2 ds \int_{s_k}^{r_1} \|f(s)\|^2_{\mathcal{L}_2^0} ds.
\end{aligned}$$

We observe that $\mathbb{E}\left\|\Pi y(r_2) - \Pi y(r_1)\right\|^2 \longrightarrow 0$ independently of $y \in \Omega_r$ when $r_2 - r_1 \to 0$, under the same reasoning as in **Case I** and the fact that g_k is compact. Which implies that $\Pi(\Omega_r)$ is equicontinuous on $(s_k, t_{k+1}]$ for $k = 1, 2, \cdots, N$.

Thus, $\mathbb{E}\left\|\Pi y(r_2) - \Pi y(r_1)\right\|^2 \longrightarrow 0$ at every interval on $[0, T]$. Thus, we determine $\Pi(\Omega_r)$ is equicontinuous on each $[0, T]$.

Step 4. Denote $I = \overline{co}\,\Pi(\Omega_r)$. Where $\overline{co}$ is the closure of convex hull, it can be shown that the map $\Pi : I \to I$ is equicontinuous on each interval, and $I \subset \Omega_r$ is also equicontinuous.

In what follows we will prove that there exists a constant $0 \leq \lambda < 1$ and a positive integer n_0 such that for any bounded and nonprecompact subset $C \subset I$

$$\alpha_{\mathcal{PC}}(\Pi^{n_0}\Big(C)\Big) \leq \lambda \alpha_{\mathcal{PC}}(C). \tag{22}$$

For any $C \subset I$ by the definition of operator Π^n and the equicontinuity of I, we get that $\Pi^n \subset \Omega_r$ is also equicontinuous. It follows by Lemma 2.6, that

$$\alpha_{\mathcal{PC}}\Big(\Pi^n(C)\Big) = \max_{t\in[0,T]} \alpha(\Pi^n\Big(C)\,(t)\Big), \qquad n = 1, 2, \cdots, N. \tag{23}$$

And there exists a countable sequence $C_1 = \{x_N^1\} \subset C$ such that

$$\alpha(\Pi\Big(C)\,(t)\Big) \leq 2\alpha(\Pi\Big(C_1)\,(t)\Big). \tag{24}$$

Furthermore, for any bounded set $C_1, C_2 \subset C$ by Lemma 2.4 and (H_4) we can deduce that

$$\begin{aligned}
&\left\| \int_{s_k}^{t} \mathcal{R}(t-s)\varphi\Big(s, C_1(s)\Big)ds - \int_{s_k}^{t} \mathcal{R}(t-s)\varphi\Big(s, C_2(s)\Big)ds \right\| \\
&\qquad = \left(\int_{s_k}^{t} \left\| \Big(\mathcal{R}(t-s)\Big[\varphi\Big(s, C_1(s)\Big) - \varphi\Big(s, C_2(s)\Big)\Big]ds\Big) \right\|^2 \right)^{\frac{1}{2}} \\
&\qquad \leq M \left(\int_{s_k}^{t} \|\varphi\Big(s, C_1(s)\Big) - \varphi\Big(s, C_2(s)\Big)\|^2 ds \right)^{\frac{1}{2}}.
\end{aligned}$$

Then, by Theorem 2.6 - **8.**, we get

$$\alpha\left(\int_{s_k}^{t} \mathcal{R}(t-s)\varphi\Big(s, C(s)\Big)ds \right) \leq M \left(\int_{s_k}^{t} \alpha\Big(\varphi\Big(s, C(s)\Big)\Big)^2 ds \right)^{\frac{1}{2}}. \tag{25}$$

Therefore, by Lemma 2.4, Theorem 2.6, (23), (24), (25), condition (H_3)-(H_7), we get for $t \in [0, t_1]$ that

$$\begin{aligned}
\alpha\Big(\Pi^1\,(C)\,(t)\Big) &= \alpha\Big(\Pi\,(C)\,(t)\Big) \leq 2\alpha\Big(\Pi\,(C_1)\,(t)\Big) \\
&\leq 2\alpha\Big(\mathcal{R}(t)\Big(y_0 - \psi(y_N^1)\Big)\Big) + 2\alpha\left(\int_0^t \mathcal{R}(t-s)\varphi\Big(s, y_N^1(s)\Big)ds \right) \\
&\qquad\qquad + 2\alpha\left(\int_0^t \mathcal{R}(t-s) f\Big(s\Big)dB^H(s) \right) \\
&\leq 2\alpha\left(\int_0^t \mathcal{R}(t-s)\varphi\Big(s, y_N^1(s)\Big)ds \right) \\
&\leq 2M \left(\int_0^t \Big[k_2 \alpha\Big(C_1(s)\Big)\Big]^2 ds \right)^{\frac{1}{2}} \\
&\leq 2Mk_2\sqrt{t_1}\,\alpha_{\mathcal{PC}}\Big(C\Big).
\end{aligned}$$

For every $t \in (t_k, s_k]$, $k = 1, 2, \cdots N$, and since $g_k\Big(s_k, x(t_k^-)\Big)$ is compact according to assumption (H_7), we have

$$\begin{aligned}\alpha\Big(\Pi^1 C(t)\Big) =&\alpha\Big(\Pi C(t)\Big) \leq 2\alpha\Big(\Pi C_1(t)\Big)\\ \leq&2\alpha\Big(g_k\Big(s_k, y_N^1(t_k^-)\Big)\Big),\end{aligned}$$

at the moment, we obtain

$$\alpha\Big(\Pi^1 C(t)\Big) = 0.$$

And similarly, for $t \in (s_k, t_{k+1}]$, $k = 1, 2, \cdots, N$, we have

$$\begin{aligned}\alpha\Big(\Pi^1 C(t)\Big) =&\alpha\Big(\Pi C(t)\Big) \leq 2\alpha\Big(\Pi C_1(t)\Big)\\ \leq&2\alpha\Big(\mathcal{R}(t-s)g_k\Big(s_k, y_N^1(t_k^-)\Big)\Big) + 2\alpha\Big(\int_{s_k}^t \mathcal{R}(t-s)\varphi\Big(s, y_N^1(s)\Big)ds\Big)\\ &+ 2\alpha\Big(\int_{s_k}^t \mathcal{R}(t-s)f\Big(s\Big)dB^H(s)\Big)\\ \leq&2\alpha\Big(\int_{s_k}^t \mathcal{R}(t-s)\varphi\Big(s, y_N^1(s)\Big)ds\Big)\\ \leq&2M\Big(\int_{s_k}^t \Big[k_2\alpha\Big(C_1(s)\Big)\Big]^2 ds\Big)^{\frac{1}{2}}\\ \leq&2Mk_2\sqrt{(t_{k+1}-s_k)}\alpha_{\mathcal{PC}}\Big(C\Big).\end{aligned}$$

Moreover, there exists a countable set $C_2 = \{x_N^2\} \subset \overline{co}\,\Pi^1(C)$ such that

$$\alpha\Big(\Pi\Big(\overline{co}\,\Pi^1(C)\Big)(t)\Big) \leq 2\alpha\Big(\Pi C_2(t)\Big). \tag{26}$$

Hence, by Lemma 2.4, (26) and (H_4), (H_5), (H_7), for $t \in (s_k, t_{k+1}]$, $k = 1, 2, \cdots, N$, we obtain

$$\begin{aligned}\alpha\Big(\Pi^2(C)(t)\Big) =&\alpha\Big(\Pi\Big(\overline{co}\,\Pi^1(C)\Big)(t)\Big) \leq 2\alpha\Big(\Pi C_2(t)\Big)\\ \leq&2\alpha\Big(\mathcal{R}(t-s)g_k\Big(s_k, y_N^2(t_k^-)\Big)\Big) + 2\alpha\Big(\int_{s_k}^t \mathcal{R}(t-s)\varphi\Big(s, y_N^2(s)\Big)ds\Big)\\ &+ 2\alpha\Big(\int_{s_k}^t \mathcal{R}(t-s)f\Big(s\Big)dB^H(s)\Big)\\ \leq&2M\Big(\int_{s_k}^t \alpha\Big(\varphi\Big(s, C_N^2(s)\Big)\Big)^2 ds\Big)^{\frac{1}{2}}\\ \leq&2M\Big(\int_{s_k}^t \Big[\alpha\Big(C_2(s)\Big)\Big]^2 ds\Big)^{\frac{1}{2}}\\ \leq&2M\Big(\int_{s_k}^t \Big[\alpha\Big(\overline{co}\,\Pi^1(C)\Big)\Big]^2 ds\Big)^{\frac{1}{2}}\\ \leq&2M\Big(\int_{s_k}^t \Big(2Mk_2(t_{k+1}-s_k)^{\frac{1}{2}}\alpha_{\mathcal{PC}}\Big(C\Big)\Big)^2 ds\Big)^{\frac{1}{2}}\alpha_{\mathcal{PC}}\Big(C\Big)\\ \leq&\Big(2Mk_2\Big)^2\sqrt{\frac{(t_{k+1}-s_k)^2}{2}}\alpha_{\mathcal{PC}}\Big(C\Big).\end{aligned}$$

By using an iterative process for all $t \in [0, T]$, we obtain

$$\alpha\Big(\Pi^n(C)(t)\Big) \leq \Big(2Mk_2\Big)^n \sqrt{\frac{(t_{k+1} - s_k)^n}{n!}} \alpha_{\mathcal{PC}}\Big(C\Big).$$

Therefore

$$\alpha\Big(\Pi^n(C)\Big) \leq \Big(2Mk_2\Big)^n \sqrt{\frac{T^n}{n!}} \alpha_{\mathcal{PC}}\Big(C\Big).$$

It has been found that

$$\Big(2Mk_2\Big)^n \sqrt{\frac{T^n}{n!}} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty.$$

Then, there exists a large enough positive integer n_0 such that

$$\Big(2Mk_2\Big)^{n_0} \sqrt{\frac{T^{n_0}}{n_0!}} = \lambda < 1.$$

As a result, we demonstrated that (22) is met when $0 \leq \lambda < 1$ and a positive integer n_0 exist. The operator has at least one fixed point, which is a mild solution of (1), according to Theorem 3.3. □

3.1. Example

The following example is given to illustrate the the proposed theory. Let us consider the non-instantaneous impulsive stochastic integro-differential equation driven by a fractional Brownian motion as follows

$$\begin{cases} \dfrac{\partial}{\partial t} z(t,x) = \dfrac{\partial^2}{\partial x^2} z(t,x) + \displaystyle\int_0^t K(t-s) \dfrac{\partial^2}{\partial x^2} z(t,x) ds & \\ \qquad\qquad + \varphi\Big(t, z(t,x)\Big) + F(t) dB^H(t), & t \in [0,1] \cup (2,3],\ x \in [0,\pi], \\ z(t,x) = g_1\Big(t, z(1^-, x)\Big) & t \in (1,2], x \in [0,\pi], \\ z(t,0) = z(t,\pi) = 0, & t \in [0,1] \cup (2,3], \\ z(0,x) + \psi(z) = 0 & x \in [0,\pi]. \end{cases} \tag{27}$$

where B^H denotes a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$.

Let $\mathbb{V} = \mathbb{H} := L^2\Big([0,\pi], \mathbb{R}\Big)$. be the Hilbert space with the scalar product $(u, v) = \int_0^\pi u(x)v(x)dx$. We define the operator $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ by $Au = \frac{\partial z^2}{\partial x^2}$. with domain

$$D(A) = \Big\{z \in \mathbb{H}, \frac{\partial z}{\partial x}, \frac{\partial z^2}{\partial x^2} \in \mathbb{H} \text{ and } z(0) = z(\pi) = 0\Big\}.$$

Then, it is well known that $Au = \sum_{n=1}^{\infty} n^2 (u, e_n) e_n, u \in \mathbb{H}$, where $e_n(u) = \left(\frac{2}{\pi}\right)^{1/2} \sin(nu)$, $n = 1, 2, \cdots$ and A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$ on $\mathbb{H}$, which is given by $T(t)u = \sum_{n=1}^{\infty} e^{-n^2 t}(u, e_n)e_n, u \in \mathbb{H}$ and $e_n(u) = \left(\frac{2}{\pi}\right)^{1/2} \sin(nu)$, $n = 1, 2, \cdots$, is the orthogonal set of eigenvectors of A. It is well known that $\{T(t)\}_{t\geq 0}$ is compact, such that $\|T(t)\|^2 \leq 1$ In order to define the operator $Q : \mathbb{H} \to \mathbb{H}$, we choose a sequence $\{\sigma_n\}_{n\geq 1}$, set $Qe_n = \sigma_n e_n$, and assume that

$$tr(Q) = \sum_{n=1}^{\infty} \sqrt{\sigma_n} < \infty.$$

Define the process $B_Q^H(s)$ by

$$B_Q^H(s) = \sum_{n=1}^{\infty} \sqrt{\sigma_n}\beta_n^H e_n,$$

where $H \in \left(\frac{1}{2}, 1\right)$, and $\{\beta_n^H(t)\}_{n\in\mathbb{N}}$ is a sequence of two-sided one-dimensional fractional Brownian motions mutually independent.

Assume that K is a bounded function in $C^1(\mathbb{R}^+, \mathbb{H})$, and that K' is both bounded and uniformly continuous. As a result, assumptions (H_1) and (H_2) hold. Consequently, we can conclude that Equation 27, has a resolvent operator $h(t)$ for $t \geq 0$, which is norm continuous for $t > 0$, as established by Theorems 2.13 and 2.16.

Let $t \in [0, 3]$ and $\mathcal{PC} := \mathcal{PC}\Big([0, 3], \mathbb{H}\Big)$. For z, we refer to the segment solution defined in the standard manner $z(.,.) : [0, 3] \times [0, \pi] \longrightarrow \mathbb{H}$, with

$$y(t)(z) = z(t, x), \quad t \in [0, T] \quad x \in [0, \pi].$$

By the definition of f, g, φ, ψ one easily verify that assumptions (H_3)-(H_7) hold with the following functions $\varphi : ([0, 1] \cup (2, 3]) \times \mathcal{PC}\Big([0, 3], \mathbb{H}\Big) \to \mathbb{H}$, $g_1 : (1, 2] \times \mathbb{H} \to \mathbb{H}$ and the nonlocal function $\psi : \mathcal{PC}\Big([0, 3], \mathbb{H}\Big) \to \mathbb{H}$ and $f : [0, 3] \to L_2^0(\mathbb{V}, \mathbb{H})$, defined by

$$\varphi\Big(t, z(t, x)\Big) = \frac{t^{\frac{1}{2}} \sin(z(t, x))}{e^7 (1 + \|z\|_2)}\Big(z(t, x)\Big) \qquad t \in [0, 1] \cup (2, 3], \quad x \in [0, \pi], \tag{28}$$

$$g_1\Big(t, z(1^-, x)\Big) = \int_0^{\pi}\int_1^t \zeta(s, x) \frac{z(1^-, w)}{3e^4\Big(1 + \|z(1^-, w)\|_2\Big)} ds\, dw \qquad t \in (1, 2], \quad x \in [0, \pi], \tag{29}$$

$$\psi(z) = \int_0^{\pi}\int_0^3 b(s, x) \cos(z(s, w) ds\, dw, \tag{30}$$

$$f(t) = F(t) \qquad t \in [0, 1] \cup (2, 3], \tag{31}$$

where $\zeta, b : [0, T] \times [0, \pi] \to \mathbb{R}^+$ are continuous functions such that $\zeta(t, \pi) = b(t, \pi) = 0$.

We now present Lemma 3.4 to prove the compactness of a class of functions.

Lemma 3.4. *[32] Let $g : \mathcal{PC}\Big([0, T], \mathbb{H}\Big) \to \mathbb{H}$ be a operator defined by*

$$g(z)(x) = \int_0^{\pi}\int_0^T \varpi(s, x)\vartheta(z(s, w)) ds\, dw,$$

where $\varpi : [0, T] \times [0, \pi] \to \mathbb{R}$ and $\vartheta : \mathbb{H} \to \mathbb{H}$ are continuous functions where ϑ satisfies

$$\|\vartheta(z)\|^2 \leq C\Big(\|z\|^2 + 1\Big), \quad \text{for all } z \in \mathcal{PC}\Big([0, T], \mathbb{H}\Big), \quad \text{for } C > 0.$$

Then, g is a compact.

Proof. Let $\mathbb{B} \subset C\Big([0, T], \mathbb{H}\Big)$ a bounded set, then there exists $L > 0$ such that

$$\|z\|_\infty \sup_{t\in[0,T]} \|z\,(t, .)\|_{L^2(\Omega,\mathbb{H})} \leq l.$$

Let $z \in \mathbb{B}$, applying Hölder inequality and Fubini's theoerem, we have

$$\begin{aligned}\left|g(z)(x)\right|^2 =& \left|\int_0^\pi \int_0^T \varpi(s,x)\vartheta(z(s,w))ds\,dw\right|^2\\ \leq& \left\|\varpi\right\|_\infty^2 \left|\int_0^\pi \int_0^T \vartheta(z(s,w))ds\,dw\right|^2\\ \leq& \pi T\left\|\varpi\right\|_\infty^2 \int_0^T \left\|\vartheta(z(s,.))\right\|_{\mathbb{H}}^2 ds\\ \leq& \pi T^2\left\|\varpi\right\|_\infty^2 C\left(l^2+1\right).\end{aligned}$$

We conclude

$$\left\|g(z)\right\|_{\mathbb{H}}^2 \leq \pi T^2\left\|\varpi\right\|_\infty^2 C\left(l^2+1\right).$$

Consequently, g is bounded on $\mathbb{B}$.

Next, we will show that the operator $g(\mathbb{B})$ satisfied the "integral" equicontinuity condition. Let $x, \xi \in [0,\pi]$, we have

$$\begin{aligned}\int_0^\pi \left|g(z)(x+\xi) - g(z)(x)\right|^2 dx =& \int_0^\pi \left|\int_0^\pi \int_0^T \Big(\varpi(s,x+\xi) - \varpi(s,x)\Big)\vartheta(z(s,w))ds\,dw\right|^2 dx\\ \leq& \pi C T^2\left(l^2+1\right)\int_0^\pi \int_0^T \left|\varpi(s,x+\xi)-\varpi(s,x)\right|^2 ds\,dx.\end{aligned}$$

Thus,

$$\|\tau_\xi \psi(z) - \psi(z)\|_{\mathbb{H}}^2 \to 0 \ \text{ as } \ \xi \to 0,$$

where, $\tau_\xi\psi(z) = g(z)(x+\xi)$. We deduce, from Kolmogorov-Riesz-Fréchet theorem [[6], Theorem 4.26], that $\varphi(\mathbb{B})$ is relatively compact in $\mathbb{H}$. □

Corollary 3.5. *[32] Let $\Upsilon : [0,T]\times\mathbb{H} \to \mathbb{H}$ be a operator defined by*

$$\Upsilon(z)(t,x) = \int_0^\pi \int_0^t \varpi(s,x)\vartheta(z(s,w))ds\,dw,$$

where $\varpi : [0,T]\times[0,\pi] \to \mathbb{R}$ and $\vartheta : \mathbb{H} \to \mathbb{H}$ are continuous functions where ϑ satisfies

$$\|\vartheta(z)\|^2 \leq C\Big(\|z\|^2+1\Big), \quad \text{for all } z \in \mathcal{PC}\Big([0,T],\mathbb{H}\Big), \quad \text{for } C>0.$$

Then, for all bounded $\mathbb{B} \subset \mathbb{H}$, $\Upsilon : [0,T]\times\mathbb{B} \to \mathbb{H}$ is a relatively compact.

Proof. In the same way as the proof of Lemma 3.4, We prove this. □

Assumptions (H_3)-(H_7) are readily verified by using the definitions of f, g_1, φ, and ψ, hold with

$$\omega(t) = t, \qquad \eta = 1, \qquad \pi_\varphi\Big(\left\|z\right\|_2^2\Big) = \frac{\|z\|_2^2}{e^{14}}, \qquad k_1 = \frac{1}{e^{14}}.$$

Corollary 3.5 and Lemma 3.4 hold that ψ and g_1 are compact. As a result, (H_6) and (H_7) are satisfied. Thus, the totality of assumptions in Theorem 3.3 is satisfied. This implies that, the system (27) on $[0,T]$ has a mild solution.

4. Controllability Result

In this section, we formulate sufficient conditions for the exact controllability of the non-instantaneous impulsive stochastic integro-differential equations with noncompact semigroups driven by a fractional Brownian motion of the form:

$$\begin{cases} dy(t) = \Big[Ay(t) + \int_0^t h(t-s)y(t)ds\Big]dt \\ \qquad\qquad +\varphi(t, y(t))dt + Bu(t)dt + f(t)dB^H(t), \quad t \in \cup_{k=0}^{N}(s_k, t_{k+1}], \\ y(t) = g_k(t, y(t_k^-)), \qquad\qquad t \in \cup_{k=1}^{N}(t_k, s_k], \\ y(0) + \psi(y) = y_0 \in \mathbb{H}. \end{cases} \tag{32}$$

The functions φ, ψ, f, and g_k are functions previously defined. The control function $u(.)$ takes values in $L^2([0,T], \mathbb{U})$ of admissible control functions for a separable Hilbert space $\mathbb{U}$, B is a linear bounded operator from $\mathbb{U}$ to $\mathbb{H}$.

Definition 4.1. *A $\mathcal{F}$-adapted stochastic process $y(t) : [0,T] \to \mathbb{H}$ is called a mild solution of (32) if $y(0) + \psi(y) = y_0 \in \mathbb{H}$ and for each $t \in [0,T]$*

$$y(t) = \begin{cases} \mathcal{R}(t)\big(y_0 - \psi(y)\big) + \int_0^t \mathcal{R}(t-s)\varphi\big(s, y(s)\big)ds \\ \qquad + \int_0^t \mathcal{R}(t-s)Bu\big(s\big)ds + \int_0^t \mathcal{R}(t-s)f\big(s\big)dB^H(s), \quad t \in [0, t_1], \\ g_k(t, y(t_k^-)), \qquad t \in \cup_{k=1}^{N}(t_k, s_k], \\ \mathcal{R}(t-s_k)g_k(s_k, y(t_k^-)) + \int_{s_k}^t \mathcal{R}(t-s)\varphi\big(s, y(s)\big)ds \\ \qquad + \int_{s_k}^t \mathcal{R}(t-s)Bu\big(s\big)ds + \int_{s_k}^t \mathcal{R}(t-s)f\big(s\big)dB^H(s), \quad t \in \cup_{k=1}^{N}(s_k, t_{k+1}]. \end{cases} \tag{33}$$

Definition 4.2. *The stochastic control system (32) is said to be exact controllable on $[0,T]$ if for every initial state $y_0, y_1 \in \mathbb{H}$ if the reachable set $\Re(T)$ is dense in the space $u \in L^2\Big([0,T], \mathbb{U}\Big)$ such that the mild solution of (32) satisfies $y(T) + \psi(y) = y_1$, where y_1 is a preassigned terminal state.*

To prove the controllability result, the following hypotheses are necessary:

(H_8) The linear operator $\mathcal{G}$ from $L^2([0,T], \mathbb{U})$ into $L^2([0,T], \mathbb{H})$ defined by

$$\mathcal{G}_u = \int_{s_k}^T \mathcal{R}(T-s)Bu\big(s\big)ds,$$

has an inverse operator $\mathcal{G}^{-1}$ that takes values in $L^2([0,T], \mathbb{U}) / \ker \mathcal{G}$, where

$$\ker \mathcal{G} = \Big\{y \in L^2([0,T], \mathbb{U}), \mathcal{G}_y = 0\Big\}.$$

(1) There exists two positive constants C_B, $C_{\mathcal{G}}$ such that

$$\|B\|^2 \le C_B, \qquad \|\mathcal{G}^{-1}\|^2 \le C_{\mathcal{G}}.$$

(2) There exists $K_B \in \mathbb{R}^+$, $K_{\mathcal{G}} \in L^1([0,T], \mathbb{R}^+)$ such that for any bounded set $D_1 \subset U$, $D_2 \subset \mathbb{H}$

$$\alpha\Big(B(D_1)\Big) \leq K_B \alpha(D_1), \qquad \alpha\Big(\mathcal{G}^{-1}(D_2(t))\Big) \leq K_{\mathcal{G}}(t)\alpha(D_2(t)).$$

The main result of this paper is given in the next theorem

Theorem 4.3. *Suppose that* (H_1)*-*(H_8) *hold. Then, the stochastic integro-differential system (32) is controllable on* $[0.T]$ *. provided that*

$$4M^2\Big(C_g + T\eta k_1\Big)\Big(1 + 5C_{\mathcal{G}}C_B T^2\Big) < 1. \tag{34}$$

Proof. To prove our result, we transform (32) into a fixed point problem. Consider the operator $\Pi \in \mathcal{PC}\Big([0,T], \mathbb{H}\Big)$ defined by

$$\Pi y(t) = \begin{cases} \mathcal{R}(t)\Big(y_0 - \psi(y)\Big) + \int_0^t \mathcal{R}(t-s)\varphi\Big(s, y(s)\Big)ds \\ \qquad + \int_0^t \mathcal{R}(t-s)Bu\Big(s\Big)ds + \int_0^t \mathcal{R}(t-s)f\Big(s\Big)dB^H(s), & t \in [0, t_1], \\ g_k(t, y(t_k^-)), & t \in \cup_{k=1}^N (t_k, s_k], \\ \mathcal{R}(t-s_k)g_k(s_k, y(t_k^-)) + \int_{s_k}^t \mathcal{R}(t-s)\varphi\Big(s, y(s)\Big)ds \\ \qquad + \int_{s_k}^t \mathcal{R}(t-s)Bu\Big(s\Big)ds + \int_{s_k}^t \mathcal{R}(t-s)f\Big(s\Big)dB^H(s), & t \in \cup_{k=1}^N (s_k, t_{k+1}]. \end{cases} \tag{35}$$

Using the hypothesis (H_8) for an arbitrary function $y(.)$, define the stochastic control

$$\begin{aligned} u_y(t) = \mathcal{G}^{-1}\Big(&y_1 - \mathcal{R}(T)\Big(y_0 - \psi(y)\Big) - \mathcal{R}(T-s_k)g_k(s_k, y(t_k^-)) \\ &- \int_{s_k}^T \mathcal{R}(T-s)\varphi\Big(s, y(s)\Big)ds - \int_{s_k}^T \mathcal{R}(T-s)f\Big(s\Big)dB^H(s)\Big)(t), \end{aligned} \tag{36}$$

for $u_y \in \Omega_r$, using Lemma 2.4, (H_3)-(H_7) and (H_8), we obtain the following result

$$\begin{aligned} \mathbb{E}\Big\|u_y\Big\|^2 \leq & 5C_{\mathcal{G}}\Big(\mathbb{E}\|y_1\|^2 + 2M^2\Big(\mathbb{E}\|y_0\|^2 + \mathbb{E}\|\psi(y)\|^2\Big) + M^2\mathbb{E}\Big\|g_k(s_k, y(t_k^-))\Big\|^2 \\ & + M^2(T-s_k)\int_{s_k}^T \mathbb{E}\Big\|\varphi\Big(s, y(s)\Big)\Big\|^2 ds + 2M^2 H(T-s_k)^{2H-1}\int_{s_k}^T \|f(s)\|^2_{\mathcal{L}_2^0} ds\Big) \\ \leq & 5C_{\mathcal{G}}\Big(\mathbb{E}\|y_1\|^2 + 2M^2\Big(\mathbb{E}\|y_0\|^2 + C_\psi\Big) + M^2 C_{g_k}\mathbb{E}\|(s_k, y(t_k^-))\|^2 \\ & + M^2(T-s_k)\int_{s_k}^T \omega(s)\pi_\varphi\Big(\mathbb{E}\|y\|^2\Big)ds + 2M^2 H(T-s_k)^{2H-1}\int_{s_k}^T \|f(s)\|^2_{\mathcal{L}_2^0} ds\Big) \\ \leq & 5C_{\mathcal{G}}\Big(\mathbb{E}\|y_1\|^2 + 2M^2\Big(\mathbb{E}\|y_0\|^2 + C_\psi\Big) + M^2 C_g r \\ & + M^2(T-s_k)\|\omega\|_{L^1[s_k,T]}\pi_\varphi(r) + 2M^2 H(T-s_k)^{2H-1}\int_{s_k}^T \|f(s)\|^2_{\mathcal{L}_2^0} ds\Big). \end{aligned}$$

Hence

$$\mathbb{E}\left\|u_y\right\|^2 \leq 5C_{\mathcal{G}}\Big(\mathbb{E}\|y_1\|^2 + 2M^2\left(\mathbb{E}\|y_0\|^2 + C_\psi\right) + M^2C_g r + M^2(T-s_k)\|\omega\|_{L^1[s_k,T]}\pi_\varphi(r) + 2M^2H(T-s_k)^{2H-1}\int_{s_k}^{T}\|f(s)\|^2_{\mathcal{L}_2^0}ds\Big). \tag{37}$$

step 1 The proof is similar as in problem (1). Here, we merely demonstrate the existence of a constant $r > 0$ such that $\Pi(\Omega_r) \subset \Omega_r$. Let's assume that this is untrue. Then for each $r > 0$, there would exist $y_r \in \Omega_r$ and $t_r \in [0,T]$ such that $\mathbb{E}\left\|\Pi\left(y_r\right)(t_r)\right\|^2 > r$.

Case I. For $t \in [0,t_1]$, using Lemma2.4, (37), and hypotheses (H_3)- (H_6) and (H_8), we have

$$\begin{aligned}
\mathbb{E}\left\|\Pi\left(y_r\right)(t)\right\|^2 \leq & 4M^2\Big(\mathbb{E}\|y_0+\psi(y)\|^2 + t_1\int_0^{t_r}\mathbb{E}\left\|\varphi\left(s,y_r(s)\right)\right\|^2ds \\
& + C_B t_1\int_0^{t_r}\mathbb{E}\left\|u_y(s)\right\|^2ds + 2Ht_1^{2H-1}\int_0^{t_r}\|f(s)\|^2_{\mathcal{L}_2^0}ds\Big)\\
\leq & 4M^2\left(\mathbb{E}\|y_0\|^2 + C_\psi\right) + 4M^2t_1\int_0^{t_r}\omega(s)\pi_\varphi\left(\mathbb{E}\|y_r\|^2\right)ds \\
& + 20M^2C_{\mathcal{G}}C_Bt_1\int_0^{t_r}\Big(\mathbb{E}\|y_1\|^2 + 2M^2\left(\mathbb{E}\|y_0\|^2 + C_\psi\right) + M^2C_gr \\
& + M^2(T-s_k)\|\omega\|_{L^1[s_k,T]}\pi_\varphi(r) + 2M^2H(T-s_k)^{2H-1}\int_{s_k}^{T}\|f(s)\|^2_{\mathcal{L}_2^0}ds\Big)ds \\
& + 8M^2Ht_1^{2H-1}\int_0^{t_r}\|f(s)\|^2_{\mathcal{L}_2^0}ds \\
\leq & 4M^2\left(\mathbb{E}\|y_0\|^2 + C_\psi\right) + 4M^2t_1\|\omega\|_{L^1[0,t_1]}\pi_f(r) \\
& + 20M^2C_{\mathcal{G}}C_Bt_1^2\Big(\mathbb{E}\|y_1\|^2 + 2M^2\left(\mathbb{E}\|y_0\|^2 + C_\psi\right) + M^2C_gr \\
& + M^2(T-s_k)\|\omega\|_{L[s_k,T]}\pi_\varphi(r) + 2M^2H(T-s_k)^{2H-1}\int_{s_k}^{T}\|f(s)\|^2_{\mathcal{L}_2^0}ds\Big) \\
& + 8M^2Ht_1^{2H-1}\int_0^{t_r}\|f(s)\|^2_{\mathcal{L}_2^0}ds,
\end{aligned}$$

consequently, we have

$$\begin{aligned}
\mathbb{E}\left\|\Pi\left(y_r\right)(t)\right\|^2 \leq & 4M^2\Big(\mathbb{E}\|y_0\|^2 + C_\psi + t_1\|\omega\|_{L^1[0,t_1]}\pi_f(r)\Big) \\
& + 20M^2C_{\mathcal{G}}C_Bt_1^2\Big(\mathbb{E}\|y_1\|^2 + 2M^2\left(\mathbb{E}\|y_0\|^2 + C_\psi\right) + M^2C_gr \\
& + M^2(T-s_k)\|\omega\|_{L^1[s_k,T]}\pi_\varphi(r) + 2M^2H(T-s_k)^{2H-1}\int_{s_k}^{T}\|f(s)\|^2_{\mathcal{L}_2^0}ds\Big) \\
& + 8M^2Ht_1^{2H-1}\int_0^{t_r}\|f(s)\|^2_{\mathcal{L}_2^0}ds.
\end{aligned} \tag{38}$$

Case II. For $t_r \in (t_k, s_k]$, $k = 1,2,\cdots,N$, using assumption (H_7), we obtain

$$\begin{aligned}
\mathbb{E}\|\Pi y_r(t)\|^2 \leq & \mathbb{E}\|g_k(t, y_r(t_k^-))\|^2 \\
\leq & C_{g_k}\mathbb{E}\|y_r\|^2 \\
\leq & C_g r.
\end{aligned} \tag{39}$$

Case III. For $t_r \in (s_k, t_{k+1}]$, $k = 1, 2, \cdots, N$,by Lemma2.4, (37), and assumptions (H_3)- (H_5), (H_7) and (H_8), we get the following results

$$\begin{aligned}\mathbb{E}\|\Pi y_r(t)\|^2 \leq& 4M^2\Big(\mathbb{E}\|g_k(s_k, y_r(t_k^-))\|^2 + (t_{k+1} - s_k)\int_{s_k}^{t_r}\mathbb{E}\Big\|\varphi\big(s, y_r(s)\big)\Big\|^2 ds\\&+ C_B\,(t_{k+1} - s_k)\int_{s_k}^{t_r}\mathbb{E}\Big\|u_y\,(s)\,\Big\|^2 ds + 2H\,(t_{k+1} - s_k)^{2H-1}\int_{s_k}^{t_r}\|f(s)\|^2_{\mathcal{L}_2^0}ds\Big)\end{aligned}$$

$$\begin{aligned}\leq& 4M^2\Big(\mathbb{E}\|g_k(s_k, y_r(t_k^-))\|^2 + (t_{k+1} - s_k)\int_{s_k}^{t}\mathbb{E}\Big\|\varphi\big(s, y_r(s)\big)\Big\|^2 ds\\&+ 5C_{\mathcal{G}}C_B\,(t_{k+1} - s_k)\int_{s_k}^{t_r}\Big(\mathbb{E}\|y_1\|^2 + 2M^2\left(\mathbb{E}\|y_0\|^2 + C_\psi\right) + M^2C_gr\\&\quad+ M^2\,(T - s_k)\,\|\omega\|_{L^1[s_k,T]}\pi_\varphi(r) + 2M^2H\,(T - s_k)^{2H-1}\int_{s_k}^{T}\|f(s)\|^2_{\mathcal{L}_2^0}ds\Big)ds\\&+ 2H\,(t_{k+1} - s_k)^{2H-1}\int_{s_k}^{t_r}\|f(s)\|^2_{\mathcal{L}_2^0}ds\Big)\\\leq& 4M^2C_gr + 4M^2\,(t_{k+1} - s_k)\int_{s_k}^{t_r}\omega(s)\pi_\varphi\left(\mathbb{E}\|y_r\|^2\right)ds\\&+ 20M^2C_{\mathcal{G}}C_B\,(t_{k+1} - s_k)^2\Big(\mathbb{E}\|y_1\|^2 + 2M^2\left(\mathbb{E}\|y_0\|^2 + C_\psi\right) + M^2C_gr\\&\quad+ M^2\,(T - s_k)\,\|\omega\|_{L^1[s_k,T]}\pi_\varphi(r) + 2M^2H\,(T - s_k)^{2H-1}\int_{s_k}^{T}\|f(s)\|^2_{\mathcal{L}_2^0}ds\Big)\\&+ 8M^2H\,(t_{k+1} - s_k)^{2H-1}\int_{s_k}^{t_r}\|f(s)\|^2_{\mathcal{L}_2^0}ds,\end{aligned}$$

we obtain

$$\begin{aligned}\mathbb{E}\|\Pi y_r(t)\|^2 \leq& 4M^2\Big(C_gr + (t_{k+1} - s_k)\,\|\omega\|_{L^1[s_k,T]}\pi_\varphi\,(r)\Big)\\&+ 20M^2C_{\mathcal{G}}C_B\,(t_{k+1} - s_k)^2\Big(\mathbb{E}\|y_1\|^2 + 2M^2\left(\mathbb{E}\|y_0\|^2 + C_\psi\right) + M^2C_gr\\&\quad+ M^2\,(T - s_k)\,\|\omega\|_{L^1[s_k,T]}\pi_\varphi(r) + 2M^2H\,(T - s_k)^{2H-1}\int_{s_k}^{T}\|f(s)\|^2_{\mathcal{L}_2^0}ds\Big)\\&+ 8M^2H\,(t_{k+1} - s_k)^{2H-1}\int_{s_k}^{t_r}\|f(s)\|^2_{\mathcal{L}_2^0}ds.\end{aligned}\tag{40}$$

Combining the three cases (38), (39), (40), we obtain

$$\begin{aligned}r < \mathbb{E}\|\Pi x_r(t)\|^2 \leq& 4M^2\Big(\mathbb{E}\|y_0\|^2 + C_\psi + C_gr + T\eta\pi_\varphi\,(r)\Big)\\&+ 20M^2C_{\mathcal{G}}C_BT^2\Big(\mathbb{E}\|y_1\|^2 + 2M^2\left(\mathbb{E}\|y_0\|^2 + C_\psi\right)\\&\quad+ M^2C_gr + M^2T\eta\pi_\varphi(r) + 2M^2HT^{2H-1}\int_{s_k}^{T}\|f(s)\|^2_{L_2^0}ds\Big)\\&+ 8M^2H\,(t_{k+1} - s_k)^{2H-1}\int_{s_k}^{t_r}\|f(s)\|^2_{\mathcal{L}_2^0}ds.\end{aligned}$$

Dividing both sides by r and taking the lower limit as $r \to +\infty$, we have

$$\begin{aligned}1 < \mathbb{E}\|\Pi x_r(t)\|^2 \leq & 4M^2\Big(C_g + T\eta k_1\Big) + 20M^4 C_{\mathcal{G}} C_B T^2 \Big(C_g + T\eta k_1\Big)\\ 1 \leq & 4M^2\Big(C_g + T\eta k_1\Big)\Big(1 + 5M^2 C_{\mathcal{G}} C_B T^2\Big),\end{aligned}$$

which is contradicted with (34), hence, there exists a constant $r > 0$ such that $\Pi(\Omega_r) \subset \Omega_r$.

Using the same method as in problem 13, we show that the operator Π is continuous in Ω_r and equicontinuous for each $t \in [0, T]$.

$$\begin{aligned}\alpha\left(u_C(t)\right) \leq & \alpha\Bigg(\mathcal{G}^{-1}\Big(y_1 - \mathcal{R}(T)\Big(y_0 - \psi(C)\Big) - \mathcal{R}(T - s_k) g_k(s_k, C(t_k^-))\\ & - \int_{s_k}^{T} \mathcal{R}(T-s)\varphi\Big(s, C(s)\Big)ds - \int_{s_k}^{T} \mathcal{R}(T-s) f\Big(s\Big)dB^H(s)\Big)(t)\Bigg)\\ \leq & K_{\mathcal{G}}(t)\alpha\Big(y_1 - \mathcal{R}(T)\Big(y_0 - \psi(C)\Big) - \mathcal{R}(T - s_k) g_k(s_k, C(t_k^-))\\ & - \int_{s_k}^{T} \mathcal{R}(T-s)\varphi\Big(s, C(s)\Big)ds - \int_{s_k}^{T} \mathcal{R}(T-s) f\Big(s\Big)dB^H(s)\Big)\end{aligned}$$

$$\begin{aligned}\leq & \|K_{\mathcal{G}}\|_{L^1[0,T]}\Big(2\alpha\left(y_1 - \mathcal{R}(T)\Big(y_0 - \psi(C)\Big)\right) + 2\alpha\left(\mathcal{R}(T - s_k) g_k(s_k, C(t_k^-))\right)\\ & + 2\alpha\left(\int_{s_k}^{T} \mathcal{R}(T-s)\varphi\Big(s, C(s)\Big)ds\right) + 2\alpha\left(\int_{s_k}^{T} \mathcal{R}(T-s) f\Big(s\Big)dB^H(s)\right)\Big)\\ \leq & 2\|K_{\mathcal{G}}\|_{L^1[0,T]}\alpha\left(\int_{s_k}^{T} \mathcal{R}(T-s)\varphi\Big(s, C(s)\Big)ds\right).\end{aligned}$$

Also that, there exists a large enough positive integer n_0 such that

$$\Big(2Mk_2\Big)^{n_0}.\Big(1 + MK_B\|K_{\mathcal{G}}\|_{L^1[0,T]}\Big)^{n_0}\sqrt{\frac{T^{n_0}}{n_0!}} = \lambda < 1,$$

where $0 \leq \lambda < 1$. Thus, condition (6) is satisfied. According to Theorem 4.3, the operator 4.3 has at least one fixed point. Hence, the system is exactly controllable on the interval $[0, T]$. □

4.1. Example

In this part, we present an example to illustrate our analytical result concerning the controllability of the system. Let us consider the following the non-instantaneous impulsive stochastic integro-differential equation driven by a fractional Brownian motion:

$$\begin{cases}\dfrac{\partial}{\partial t}z(t,x) = \dfrac{\partial^2}{\partial x^2}z(t,x) + \displaystyle\int_0^t K(t-s)\dfrac{\partial^2}{\partial x^2}z(t,x)ds & \\ \qquad\qquad +\varphi\Big(t, z(t,x)\Big) + v(t,x) + F(t)dB^H(t), & t \in [0,1] \cup (2,3],\ x \in [0,\pi]\,, \\ z(t,x) = g_1\Big(t, z(1^-, x)\Big) & t \in (1,2], x \in [0,\pi]\,, \\ z(t,0) = z(t,\pi) = 0, & t \in [0,1] \cup (2,3], \\ z(0,x) + \psi(z) = 0 & x \in [0,\pi]\,. \end{cases} \tag{41}$$

Let $\mathbb{U} = \mathbb{V} = \mathbb{H} := L^2\Big([0,\pi],\mathbb{R}\Big)$. Our presumptions are as follows:

We define $\varphi : ([0,1] \cup (2,3]) \times \mathcal{PC}\Big([0,3],\mathbb{H}\Big) \to \mathbb{H}$, $g_1 : (1,2] \times \mathbb{H} \to \mathbb{H}$, $\psi : \mathcal{PC}\Big([0,3],\mathbb{H}\Big) \to \mathbb{H}$ and $f : [0,3] \to L_2^0(\mathbb{V},\mathbb{H})$, defined by

$$
\begin{aligned}
\varphi\Big(t,z(t,x)\Big) =& \frac{t^{\frac{1}{2}} \sin\Big(z(t,x)\Big)}{e^7\Big(1+\|z\|_2\Big)}\Big(z(t,x)\Big) \quad t \in [0,1] \cup (2,3], \quad x \in [0,\pi],\\
g_1\Big(t,z(1^-,x)\Big) =& \int_0^\pi \int_1^t \zeta(s,x) \frac{z(1^-,x)}{3e^4\Big(1+\|z(1^-,x)\|_2\Big)} ds\, dx \quad t \in (1,2], \quad x \in [0,\pi],\\
\psi(z) =& \int_0^\pi \int_0^3 b(t,x)\cos(z(s,x) ds\, dx,\\
f(t) =& F(t) \quad t \in [0,1] \cup (2,3],
\end{aligned}
$$

where $\zeta, b : [0,T] \times [0,\pi] \to \mathbb{R}^+$ are continuous functions such that $\zeta(t,\pi) = b(t,\pi) = 0$.

Define the bounded linear operator $B : \mathbb{U} \to \mathbb{H}$ by

$$
Bu(t)(x) = v(t,x), \quad x \in [0,\pi], \quad u \in L^2([0,T],\mathbb{U}).
$$

The operator : $L^2([0,T],\mathbb{U}) \to L^2([0,T],\mathbb{H})$ defined by

$$
\mathcal{G}_u(x) = \int_{s_k}^T \mathcal{R}(T-s)v(s,x)ds,
$$

has an inverse $\mathcal{G}_u^{-1}$ and satisfies condition (H_8). Lemma 3.4 and Corollary 3.5 establish that ψ and g_1 are compact.

Next, it shows us that all requirements of Theorem 4.3 are satisfied in the above example. Therefore, the system corresponding to (41) is exact controllable.

5. Conclusions

In this work, we investigated a class of non-instantaneous impulsive stochastic integrate-differential equations driven by a fractional Brownian motion with nonlocal conditions in a Hilbert space. The existence of mild solutions and the exact controllability of the system are studied using a generalized Darbo's fixed point theorem, Kuratowskii measure of non-compactness and the resolvent operator. In the future, we will study the existence of solutions and the stability of the fractional stochastic integro-differential equation driven by fractional brownian motion with non-instantaneous impulsive.

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