Filomat 39:2 (2025), 649–657
https://doi.org/10.2298/FIL2502649Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Signless Laplacian spectral radius for a $k$-extendable graph

**Sizhong Zhou[a], Yuli Zhang[b,*]**

[a]*School of Science, Jiangsu University of Science and Technology, Zhenjiang, Jiangsu 212100, China*
[b]*School of Science, Dalian Jiaotong University, Dalian, Liaoning 116028, China*

**Abstract.** Let $k$ and $n$ be two nonnegative integers with $n \equiv 0 \pmod 2$, and let $G$ be a graph of order $n$ with a perfect matching. Then $G$ is said to be $k$-extendable for $0 \leq k \leq \frac{n-2}{2}$ if every matching in $G$ of size $k$ can be extended to a perfect matching. In this paper, we first establish a lower bound on the signless Laplacian spectral radius of $G$ to ensure that $G$ is $k$-extendable. Then we create some extremal graphs to claim that all the bounds derived in this article are sharp.

## 1. Introduction

Graphs discussed in this paper are simple, undirected and connected. Let $G$ be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$, where $|V(G)| = n$ and $\overline{G}$ be the complement of $G$. Denote by $N_G(v)$ the neighbor set of the vertex $v$ in $G$. The degree of the vertex $v$ is $d_G(v) = |N_G(v)|$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of $G$ induced by $S$ and $G - S$ is the subgraph of $G$ induced by $V(G) \setminus S$. Given two vertex-disjoint graphs $G_1$ and $G_2$, the union of $G_1$ and $G_2$ is denoted by $G_1 \cup G_2$ and the join $G_1 \vee G_2$ is obtained from $G_1 \cup G_2$ by joining each vertex of $G_1$ with each vertex of $G_2$ by an edge. Let $K_n$ denote the complete graph of order $n$.

Let $A(G)$ denote the $(0,1)$-adjacency matrix of $G$ and $D(G) = \mathrm{diag}(d_1, d_2, \dots, d_n)$ denote the diagonal degree matrix of $G$, where $d_i = d_G(v_i)$ for $1 \leq i \leq n$. The signless Laplacian matrix $Q(G)$ of $G$ is defined as $Q(G) = D(G) + A(G)$. Obviously, $A(G)$ and $Q(G)$ are real symmetric matrices. The largest eigenvalues of $A(G)$ and $Q(G)$, denoted by $\rho(G)$ and $q(G)$, are called the spectral radius and the signless Laplacian spectral radius of $G$, respectively.

For two positive integers $a$ and $b$ with $a \leq b$, a spanning subgraph $F$ of $G$ is called an $[a,b]$-factor if $a \leq d_F(v) \leq b$ for any $v \in V(G)$. If $a = b = 1$, then an $[a,b]$-factor is a 1-factor (or a perfect matching). Let $G$ be a graph of order $n$ with a perfect matching. Then $G$ is said to be $k$-extendable for $0 \leq k \leq \frac{n-2}{2}$ if every matching in $G$ of size $k$ can be extended to a perfect matching. In particular, $G$ is 0-extendable if and only if $G$ contains a perfect matching.

Many researchers have attempted to find sufficient conditions for the existence of perfect matchings by utilizing various graphic parameters. Tutte [32] obtained a characterization for a graph with a perfect

2020 *Mathematics Subject Classification*. Primary 05C70; Secondary 05C50.
*Keywords*. signless Laplacian spectral radius; perfect matching; extendable graph.
Received: 10 February 2024; Revised: 27 June 2024; Accepted: 15 October 2024
Communicated by Paola Bonacini
* Corresponding author: Yuli Zhang
*Email addresses:* zsz_cumt@163.com (Sizhong Zhou), zhangyuli_djtu@126.com (Yuli Zhang)
ORCID iDs: https://orcid.org/0000-0003-2093-2158 (Sizhong Zhou), https://orcid.org/0009-0002-2139-2149 (Yuli Zhang)

matching. Anderson [3, 4] investigated the relationships between binding numbers and perfect matchings in graphs and presented two binding number conditions for the existence of perfect matchings in graphs. Sumner [31] showed a sufficient condition for a graph to possess a perfect matching. Niessen [24] provided a neighborhood union condition for the existence of perfect matchings in graphs. Enomoto [12] derived a toughness condition for a graph to admit a perfect matching. Plummer [28] first introduced the concept of $k$-extendable graph and posed some properties of $k$-extendable graphs. Up to now, much attention has been paid on various graphic parameters of $k$-extendable graphs, such as binding number [6, 29], connectivity [20, 26], minimum degree [2], independence number [1, 8, 22], distance-regular graph [7], genus [27], eigenvalues [36] and spectral radius [13]. Much effort has been devoted to finding sufficient conditions for the existence of $[1, 2]$-factors (see [10, 11, 14, 16, 17, 19, 40, 41, 46–48, 51]) and $[a, b]$-factors (see [15, 21, 23, 33, 34, 39, 42–45, 49, 50]) in graphs.

The main goal of this paper is to study the existence of $k$-extendable graphs from a spectral perspective. Recall that $G$ is 0-extendable if and only if $G$ has a perfect matching. In the past few years, lots of researchers focused on finding the connections between the spectral radius and perfect matchings in graphs. O [25] provided a spectral radius condition to guarantee that a connected graph has a perfect matching. By imposing the minimum degree of a graph as a parameter, Liu, Liu and Feng [18] extended O's result [25] in a connected graph. Zhang and Lin [37] presented a distance spectral condition to guarantee the existence of a perfect matching in a graph. Zhou [38] established a relationship between signless Laplacian spectral radius and Hamiltonian cycles in graphs. Motivated by O [25], Liu, Liu and Feng [18], Zhang and Lin [37] and Zhou [38], directly, it is natural and interesting to give other sufficient spectral conditions to guarantee that a graph has a perfect matching. Note that the concept of $k$-extendable graph is a generalization of the notation of perfect matching. In this paper, we study the existence of $k$-extendable graphs and obtain a signless Laplacian spectral radius condition for a graph to be $k$-extendable.

**Theorem 1.1.** Let $k$ and $n$ be two positive integers with $n \equiv 0 \pmod 2$, and let $G$ be a connected graph of order $n$ with $n \geq 2k + 4$. Assume that one of the following three conditions holds:

(i) $q(G) > \theta(k, n)$ for $n \notin \{2k + 6, 2k + 8\}$, where $\theta(k, n)$ is the largest root of $x^3 - (3n + 2k - 7)x^2 + (2n^2 + 6kn - 7n - 24k)x - 2(2k + 1)(n - 3)(n - 4) = 0$;

(ii) $q(G) > 3k + 4 + \sqrt{k^2 + 12k + 12}$ for $n = 2k + 6$;

(iii) $q(G) > 3k + 6 + \sqrt{k^2 + 16k + 24}$ for $n = 2k + 8$.

Then $G$ is $k$-extendable unless $G = K_{2k} \vee (K_{n-2k-1} \cup K_1)$.

The proof of Theorem 1.1 will be provided in Section 3.

## 2. Preliminary lemmas

In this section, we put forward some necessary preliminary lemmas, which are very important to the proofs of our main results.

Chen [6] established a necessary and sufficient condition for the existence of $k$-extendable graphs.

**Lemma 2.1** ([6]). Let $k \geq 1$ be an integer. Then a graph $G$ is $k$-extendable if and only if

$$o(G - S) \leq |S| - 2k$$

for any $S \subseteq V(G)$ such that $G[S]$ contains $k$ independent edges, where $o(G - S)$ denotes the number of odd components in $G - S$.

**Lemma 2.2** ([30]). Let $G$ be a connected graph. If $H$ is a subgraph of $G$, then $q(H) \leq q(G)$. If $H$ is a proper subgraph of $G$, then $q(H) < q(G)$.

**Lemma 2.3** ([9]). Let $n \geq 2$ be an integer, and $K_n$ be a complete graph of order $n$. Then $q(K_n) = 2n - 2$.

In what follows, we explain the concepts of equitable matrices and equitable partitions.

**Definition 2.4** ([5]). Let $M$ be a real matrix of order $n$ described in the following block form

$$M=\begin{pmatrix} M_{11} & \cdots & M_{1r} \\ \vdots & \ddots & \vdots \\ M_{r1} & \cdots & M_{rr} \end{pmatrix},$$

where the blocks $M_{ij}$ are $n_i \times n_j$ matrices for any $1 \le i, j \le r$ and $n = n_1 + n_2 + \cdots + n_r$. For $1 \le i, j \le r$, let $b_{ij}$ denote the average row sum of $M_{ij}$, that is, $b_{ij}$ is the sum of all entries in $M_{ij}$ divided by the number of rows. Then $B(M) = (b_{ij})$ (simply by $B$) is called a quotient matrix of $M$. If for every pair $i, j$, $M_{ij}$ admits constant row sum, then $B$ is called an equitable quotient matrix of $M$ and the partition is called equitable.

**Lemma 2.5** ([35]). Let $B$ be an equitable matrix of $M$ as defined in Definition 2.4, and $M$ be a nonnegative matrix. Then $\rho_1(B) = \rho_1(M)$, where $\rho_1(B)$ and $\rho_1(M)$ denote the largest eigenvalues of the matrices $B$ and $M$.

## 3. The proof of Theorem 1.1

In this section, we prove Theorem 1.1, which provides a sufficient condition via the signless Laplacian spectral radius of a connected graph to ensure that the graph is $k$-extendable.

*Proof of Theorem 1.1.* Suppose, to the contrary, that $G$ is not $k$-extendable. Then, according to Lemma 2.1, there exists some nonempty subset $S$ of $V(G)$ such that $|S| \ge 2k$ and $o(G-S) > |S| - 2k$. Since $n$ is even, $o(G-S)$ and $|S|$ possess the same parity. Thus, we deduce

$$o(G-S) \ge |S| - 2k + 2.$$

Select such a connected graph $G$ of order $n$ so that its signless Laplacian spectral radius is as large as possible.

Together with Lemma 2.2 and the choice of $G$, the induced subgraph $G[S]$ and every connected component of $G-S$ are complete graphs, respectively. Furthermore, all components of $G-S$ are odd and $G$ is just the graph $G[S] \vee (G-S)$.

For convenience, let $o(G-S) = q$ and $|S| = s$. Then $q \ge s - 2k + 2$. Assume that $G_1, G_2, \ldots, G_q$ are all the components of $G-S$ with $n_i = |V(G_i)|$ and $n_1 \ge n_2 \ge \cdots \ge n_q$. Then $G = K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_q})$.

**Claim 1.** $n_2 = n_3 = \cdots = n_q = 1$.
*Proof.* If $n_2 \ge 3$, then we let $G' = K_s \vee (K_{n_1+2} \cup K_{n_2-2} \cup K_{n_3} \cup \cdots \cup K_{n_q})$. Note that $o(G'-S) = o(G-S) = q \ge s-2k+2$. Denote the vertex set of $G$ by $V(G) = V(K_s) \cup V(K_{n_1}) \cup V(K_{n_2}) \cup \cdots \cup V(K_{n_q})$. Let $Y$ be the Perron vector of $Q(G)$, and let $Y(v)$ be the entry of $Y$ corresponding to the vertex $v \in V(G)$. By symmetry, it is obvious that all vertices of $K_s$ (resp. $K_{n_1}, K_{n_2}, \cdots, K_{n_q}$) have the same entries in $Y$. Hence, we can suppose $Y(v_0) = y_0$ for every $v_0 \in V(K_s)$, $Y(v_1) = y_1$ for every $v_1 \in V(K_{n_1})$, $Y(v_2) = y_2$ for every $v_2 \in V(K_{n_2}), \cdots, Y(v_q) = y_q$ for every $v_q \in V(K_{n_q})$. Then

$$\begin{cases} q(G)y_1 = sy_0 + (s + 2n_1 - 2)y_1, \\ q(G)y_2 = sy_0 + (s + 2n_2 - 2)y_2. \end{cases} \tag{1}$$

It follows from (1) that

$$(q(G) - s - 2n_1 + 2)y_1 = (q(G) - s - 2n_2 + 2)y_2. \tag{2}$$

Note that $K_{s+n_1}$ and $K_{s+n_2}$ are two proper subgraphs of $G$. Using Lemmas 2.2 and 2.3, we get

$$\begin{aligned} q(G) &> \max\{q(K_{s+n_1}), q(K_{s+n_2})\} \\ &= \max\{2(s+n_1) - 2, 2(s+n_2) - 2\} \\ &> \max\{s + 2n_1 - 2, s + 2n_2 - 2\}. \end{aligned}$$

Together with (2) and $n_1 \geq n_2$, we infer $y_1 \geq y_2$. According to the Rayleigh quotient, we derive

$$\begin{aligned} q(G') - q(G) \geq & Y^T(Q(G') - Q(G))Y \\ =& 2n_1y_1(y_1 + y_2) + 2n_1y_2(y_1 + y_2) - 8(n_2 - 2)y_2^2 \\ \geq & 8n_1y_2^2 - 8(n_2 - 2)y_2^2 \\ =& 8y_2^2(n_1 - n_2 + 2) \\ >& 0. \end{aligned}$$

Hence, $q(G') > q(G)$, which is a contradiction to the choice of $G$. Thus, we deduce $n_2 = 1$.

Recall that $n_2 \geq n_3 \geq \cdots \geq n_q \geq 1$. Combining this with $n_2 = 1$, we infer $n_2 = n_3 = \cdots = n_q = 1$. Claim 1 is proved. □

In what follows, we are to verify $q = s - 2k + 2$. Note that $q \geq s - 2k + 2$, and $q$ and $s$ have the same parity. Consequently, we can suppose $q \geq s-2k+4$. We construct a new graph $G'' = K_s \vee (K_{n_1+2} \cup (q-3)K_1)$. Clearly, $G$ is a proper subgraph of $G''$ and $o(G''-S) = o(G-S)-2 = q-2 \geq s-2k+2$. Together with Lemma 2.2, $q(G'') > q(G)$, which is a contradiction to the choice of $G$. Thus, we infer $q \leq s - 2k + 2$. On the other hand, $q \geq s - 2k + 2$. Hence, we obtain

$$q = s - 2k + 2. \tag{3}$$

By virtue of (3), Claim 1, $n = s + n_1 + n_2 + \cdots + n_q$ and $G = K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_q})$, we have $G = K_s \vee (K_{n_1} \cup (q-1)K_1) = K_s \vee (K_{n_1} \cup (s-2k+1)K_1)$ and $n_1 = n-s-(q-1) = n-2s+2k-1$. If $s = 2k$, then $G = K_{2k} \vee (K_{n-2k-1} \cup K_1)$, which is a contradiction to the condition of this theorem. Hence, $s \geq 2k + 1$. The following proof will be divided into two cases by the value of $n_1$.

**Case 1.** $n_1 \geq 3$.

In this case, $n = n_1+2s-2k+1 \geq 2s-2k+4$. Recall that $G = K_s \vee (K_{n_1} \cup (s-2k+1)K_1)$ and $n_1 = n-2s+2k-1$. Consider the partition $V(G) = V(K_s) \cup V(K_{n_1}) \cup V((s-2k+1)K_1)$. The corresponding quotient matrix of $Q(G)$ equals

$$B_1 = \begin{pmatrix} n+s-2 & n-2s+2k-1 & s-2k+1 \\ s & 2n-3s+4k-4 & 0 \\ s & 0 & s \end{pmatrix}.$$

Then the characteristic polynomial of $B_1$ is

$$\begin{aligned} f_1(x) =& x^3 - (3n - s + 4k - 6)x^2 + (2n^2 + sn + 4kn - 8n - 4s^2 - 4s + 8ks - 8k + 8)x \\ &- 2sn^2 + 4s^2n - 8ksn + 10sn - 2s^3 + 8ks^2 - 10s^2 - 8k^2s + 20ks - 12s. \end{aligned}$$

In view of Lemma 2.5, the largest root, say $q_1$, of $f_1(x) = 0$ equals the signless Laplacian spectral radius of $G$. Consequently, we possess $f_1(q_1) = 0$ and $q(G) = q_1$.

Note that $K_s \vee (n-s)K_1$ is a proper subgraph of $G$. From Lemma 2.2, we infer $q_1 = q(G) > q(K_s \vee (n-s)K_1)$. Consider the partition $V(K_s \vee (n-s)K_1) = V(K_s) \cup V((n-s)K_1)$. The corresponding quotient matrix of $Q(K_s \vee (n-s)K_1)$ has the following form

$$B_2 = \begin{pmatrix} n+s-2 & n-s \\ s & s \end{pmatrix}.$$

Then the characteristic polynomial of $B_2$ equals

$$f_2(x) = x^2 - (n + 2s - 2)x + 2s(s - 1).$$

In terms of Lemma 2.5, the largest root, say $q_2$, of $f_2(x) = 0$ equals $q(K_s \vee (n - s)K_1)$. And so

$$q(K_s \vee (n - s)K_1) = q_2 = \frac{n + 2s - 2 + \sqrt{(n + 2s - 2)^2 - 8s(s - 1)}}{2}. \tag{4}$$

Together with $q_1 = q(G) > q(K_s \vee (n-s)K_1)$, we get

$$q_1 > q_2 = \frac{n + 2s - 2 + \sqrt{(n + 2s - 2)^2 - 8s(s-1)}}{2}. \tag{5}$$

Let $\varphi(x) = x^3 - (3n + 2k - 7)x^2 + (2n^2 + 6kn - 7n - 24k)x - 2(2k+1)(n-3)(n-4)$ and let $\theta(k,n)$ be the largest root of $\varphi(x) = 0$. Note that $f_1(q_1) = 0$. By a direct calculation, we have

$$\varphi(q_1) = \varphi(q_1) - f_1(q_1) = (s - 2k - 1)g_1(q_1), \tag{6}$$

where $g_1(q_1) = -q_1^2+(-n+4s+8)q_1+2n^2-4sn-14n+2s^2-4ks+12s+24$. Utilizing (5) and $n \geq 2s-2k+4 \geq s+5$, we derive

$$-\frac{-n + 4s + 8}{2 \times (-1)} < n + s - 2 < \frac{n + 2s - 2 + \sqrt{(n + 2s - 2)^2 - 8s(s-1)}}{2} < q_1.$$

Consequently, we deduce

$$\begin{aligned}
g_1(q_1) <& g_1\left(\frac{n + 2s - 2 + \sqrt{(n + 2s - 2)^2 - 8s(s-1)}}{2}\right)\\
=& n^2 - 5sn - 7n + 6s^2 - 4ks + 18s + 14\\
&- (n - s - 5)\sqrt{(n + 2s - 2)^2 - 8s(s-1)}\\
\leq& n^2 - 5sn - 7n + 6s^2 - 4ks + 18s + 14 - n(n - s - 5)\\
=& -4sn - 2n + 6s^2 - 4ks + 18s + 14\\
\leq& -4s(2s - 2k + 4) - 2(2s - 2k + 4) + 6s^2 - 4ks + 18s + 14\\
=& -2s^2 + 4ks - 2s + 4k + 6.
\end{aligned} \tag{7}$$

For $s \geq 2k + 2$, it follows from (7) that

$$\begin{aligned}
g_1(q_1) <& -2s^2 + 4ks - 2s + 4k + 6\\
\leq& -2s(2k + 2) + 4ks - 2s + 4k + 6\\
=& -6s + 4k + 6\\
<& 0.
\end{aligned} \tag{8}$$

Recall that $s \geq 2k + 1$. According to (6) and (8), we infer

$$\varphi(q_1) = (s - 2k - 1)g_1(q_1) \leq 0,$$

which yields

$$q(G) = q_1 \leq \theta(k,n),$$

which is a contradiction to $q(G) > \theta(k,n)$ for $n \notin \{2k + 6, 2k + 8\}$.

Let $\varphi'(x)$ denote the derivative of $\varphi(x)$. As for $n = 2k + 6$, one has $\varphi(x) = x^3 - (8k + 11)x^2 + (20k^2 + 46k + 30)x - 16k^3 - 48k^2 - 44k - 12$ and $\varphi'(x) = 3x^2 - 2(8k + 11)x + 20k^2 + 46k + 30$. By a direct calculation, we obtain $\varphi(3k + 4 + \sqrt{k^2 + 12k + 12}) = 6k + 8 + 2\sqrt{k^2 + 12k + 12} > 0$ and $\varphi'(3k + 4 + \sqrt{k^2 + 12k + 12}) = 2k^2 + 24k + 26 + 2(k + 1)\sqrt{k^2 + 12k + 12} > 0$, and so $q(G) = q_1 \leq \theta(k, 2k + 6) < 3k + 4 + \sqrt{k^2 + 12k + 12}$, which contradicts $q(G) > 3k + 4 + \sqrt{k^2 + 12k + 12}$ for $n = 2k + 6$.

As for $n = 2k+8$, one has $\varphi(x) = x^3 - (8k+17)x^2 + (20k^2 + 74k+72)x - 16k^3 - 80k^2 - 116k - 40$ and $\varphi'(x) = 3x^2 - 2(8k+17)x + 20k^2 + 74k + 72$. By a direct computation, we derive $\varphi(3k+6+\sqrt{k^2 + 16k + 24}) = 8k+20 > 0$ and $\varphi'(3k+6+\sqrt{k^2 + 16k + 24}) = 2k^2+32k+48+2(k+1)\sqrt{k^2 + 16k + 24} > 0$, and so $q(G) = q_1 \leq \theta(k, 2k+8) < 3k + 6 + \sqrt{k^2 + 16k + 24}$, which is a contradiction to $q(G) > 3k + 6 + \sqrt{k^2 + 16k + 24}$ for $n = 2k + 8$.

**Case 2.** $n_1 = 1$.

In this case, we possess $G = K_s \vee (s-2k+2)K_1 = K_s \vee (n-s)K_1$ and $n = 2s-2k+2$. By virtue of (4), we obtain

$$q(G) = q(K_s \vee (n-s)K_1) = q_2 = \frac{n+2s-2+\sqrt{(n+2s-2)^2-8s(s-1)}}{2}.$$

Note that $f_2(q_2) = 0$. By a direct computation, we possess

$$\begin{aligned}\varphi(q_2) =& \varphi(q_2) - q_2 f_2(q_2)\\ =& -(2n-2s+2k-5)q_2^2 + (2n^2+6kn-7n-24k-2s^2+2s)q_2\\ & -2(2k+1)n^2 + 14(2k+1)n - 24(2k+1)\\ =& -(s+2k+1)n^2 + (3s+2k+4)sn + 7(2k+1)n - 2s^3 - (20k+8)s - 14(2k+1)\\ & + (-sn+2kn+n+s^2-2ks+4s-10k-5)\sqrt{(n+2s-2)^2-8s(s-1)}. \end{aligned} \tag{9}$$

Recall that $s \geq 2k+1$. If $s = 2k+1$, then $n = 2k+4$ and $q(G) = 3k+2+\sqrt{k^2+8k+4} = \theta(k, 2k+4)$, which contradicts $q(G) > \theta(k,n)$ for $n = 2k+4$. If $s = 2k+2$, then $n = 2k+6$ and $q(G) = 3k+4+\sqrt{k^2+12k+12}$, which contradicts $q(G) > 3k+4+\sqrt{k^2+12k+12}$ for $n = 2k+6$. If $s = 2k+3$, then $n = 2k+8$ and $q(G) = 3k+6+\sqrt{k^2+16k+24}$, which contradicts $q(G) > 3k+6+\sqrt{k^2+16k+24}$ for $n = 2k+8$. In what follows, we consider $s \geq 2k+4$.

Recall that $n = 2s-2k+2$. According to $s \geq 2k+4$, we easily see

$$\begin{aligned}(n+2s-2)^2 - 8s(s-1) =& 8s^2 - 8(2k-1)s + 4k^2\\ \geq& 4s^2 + 4(2k+4)s - 8(2k-1)s + 4k^2\\ =& 4s^2 - 8(k-3)s + 4k^2\\ \geq& 4s^2 - 8(k-2)s + 8(2k+4) + 4k^2\\ =& (2s-2k+4)^2 + 32k + 16\\ >& (2s-2k+4)^2\\ =& (n+2)^2\end{aligned}$$

and

$$-sn+2kn+n+s^2-2ks+4s-10k-5 = -(s-2k-1)(s-2k-3) < 0.$$

Combining these with (9), $s \geq 2k+4$ and $n = 2s-2k+2$, we deduce

$$\begin{aligned}\varphi(q_2) =& -(s+2k+1)n^2 + (3s+2k+4)sn + 7(2k+1)n - 2s^3 - (20k+8)s - 14(2k+1)\\ & + (-sn+2kn+n+s^2-2ks+4s-10k-5)\sqrt{(n+2s-2)^2-8s(s-1)}\\ <& -(s+2k+1)n^2 + (3s+2k+4)sn + 7(2k+1)n - 2s^3 - (20k+8)s - 14(2k+1)\\ & -(s-2k-1)(s-2k-3)(n+2)\\ =& -2s^3 + (8k+6)s^2 - (8k^2+4k-12)s - (2k+1)(8k+16)\\ :=& p(s). \end{aligned} \tag{10}$$

Let $p'(s)$ and $p''(s)$ denote the derivative and the second derivative of $p(s)$, respectively. We easily see

$$p'(s) = -6s^2 + 2(8k+6)s - 8k^2 - 4k + 12$$

and

$$p''(s) = -12s + 2(8k+6).$$

Recall that $s \geq 2k+4$. Then $p''(s) = -12s + 2(8k+6) \leq -12(2k+4) + 2(8k+6) = -8k - 36 < 0$, which implies that $p'(s)$ is decreasing in the interval $[2k+4, +\infty)$. Thus, $p'(s) \leq p'(2k+4) = -6(2k+4)^2 + 2(8k+6)(2k+4) - 8k^2 - 4k + 12 = -12k - 36 < 0$, which yields that $p(s)$ is decreasing in the interval $[2k+4, +\infty)$. Thus, $p(s) \leq p(2k+4) = -2(2k+4)^3 + (8k+6)(2k+4)^2 - (8k^2+4k-12)(2k+4) - (2k+1)(8k+16) = 0$. Together with (10), we infer $\varphi(q_2) < p(s) \leq 0$, which implies $q(G) = q_2 < \theta(k,n)$, a contradiction to the condition. This completes the proof of Theorem 1.1. □

## 4. Concluding remark

In this section, we claim that the bounds derived in Theorem 1.1 are best possible.

**Theorem 4.1.** Let $k$ and $n$ be two nonnegative integers with $n \equiv 0 \pmod 2$, and let $\theta(k,n)$ be the largest root of $x^3 - (3n+2k-7)x^2 + (2n^2+6kn-7n-24k)x - 2(2k+1)(n-3)(n-4) = 0$. Then:

(i) For $n \geq 2k+4$ and $n \notin \{2k+6, 2k+8\}$, we have $q(K_{2k+1} \vee (K_{n-2k-3} \cup 2K_1)) = \theta(k,n)$ and $K_{2k+1} \vee (K_{n-2k-3} \cup 2K_1)$ is not $k$-extendable.

(ii) For $n = 2k+6$, we possess $q(K_{2k+2} \vee 4K_1) = 3k+4+\sqrt{k^2+12k+12}$ and $K_{2k+2} \vee 4K_1$ is not $k$-extendable.

(iii) For $n = 2k+8$, we admit $q(K_{2k+3} \vee 5K_1) = 3k+6+\sqrt{k^2+16k+24}$ and $K_{2k+3} \vee 5K_1$ is not $k$-extendable.

*Proof.* (i) Consider the partition $V(K_{2k+1} \vee (K_{n-2k-3} \cup 2K_1)) = V(K_{2k+1}) \cup V(K_{n-2k-3}) \cup V(2K_1)$. The corresponding quotient matrix of $Q(K_{2k+1} \vee (K_{n-2k-3} \cup 2K_1))$ is equal to

$$B_1 = \begin{pmatrix} n+2k-1 & n-2k-3 & 2 \\ 2k+1 & 2n-2k-7 & 0 \\ 2k+1 & 0 & 2k+1 \end{pmatrix}.$$

Then the characteristic polynomial of the matrix $B_1$ is equal to $x^3 - (3n+2k-7)x^2 + (2n^2+6kn-7n-24k)x - 2(2k+1)(n-3)(n-4)$. In terms of Lemma 2.5, the largest root $\theta(k,n)$ of $x^3 - (3n+2k-7)x^2 + (2n^2+6kn-7n-24k)x - 2(2k+1)(n-3)(n-4) = 0$ equals $q(K_{2k+1} \vee (K_{n-2k-3} \cup 2K_1))$. Namely, $q(K_{2k+1} \vee (K_{n-2k-3} \cup 2K_1)) = \theta(k,n)$. Write $S = V(K_{2k+1})$. Then $o(K_{2k+1} \vee (K_{n-2k-3} \cup 2K_1) - S) = 3 > 1 = (2k+1) - 2k = |S| - 2k$. By virtue of Lemma 2.1, the graph $K_{2k+1} \vee (K_{n-2k-3} \cup 2K_1)$ is not $k$-extendable.

(ii) Consider the partition $V(K_{2k+2} \vee 4K_1) = V(K_{2k+2}) \cup V(4K_1)$. The corresponding quotient matrix of $Q(K_{2k+2} \vee 4K_1)$ equals

$$B_2 = \begin{pmatrix} 4k+6 & 4 \\ 2k+2 & 2k+2 \end{pmatrix}.$$

Then the characteristic polynomial of the matrix $B_2$ is $x^2 - (6k+8)x + (2k+2)(4k+2)$. It follows from Lemma 2.5 that the largest root of $x^2 - (6k+8)x + (2k+2)(4k+2) = 0$ equals $q(K_{2k+2} \vee 4K_1)$. Thus, we possess $q(K_{2k+2} \vee 4K_1) = 3k+4+\sqrt{k^2+12k+12}$. Let $S = V(K_{2k+2})$. Then $o(K_{2k+2} \vee 4K_1 - S) = 4 > 2 = (2k+2) - 2k = |S| - 2k$. Applying Lemma 2.1, the graph $K_{2k+2} \vee 4K_1$ is not $k$-extendable.

(iii) Consider the partition $V(K_{2k+3} \vee 5K_1) = V(K_{2k+3}) \cup V(5K_1)$. The corresponding quotient matrix of $Q(K_{2k+3} \vee 5K_1)$ is equal to

$$B_3 = \begin{pmatrix} 4k+9 & 5 \\ 2k+3 & 2k+3 \end{pmatrix}.$$

Then the characteristic polynomial of the matrix $B_3$ equals $x^2 - (6k+12)x + (2k+3)(4k+4)$. Utilizing Lemma 2.5, the largest root of $x^2 - (6k+12)x + (2k+3)(4k+4) = 0$ is equal to $q(K_{2k+3} \vee 5K_1)$. Thus, we deduce $q(K_{2k+3} \vee 5K_1) = 3k+6+\sqrt{k^2+16k+24}$. Write $S = V(K_{2k+3})$. Then $o(K_{2k+3} \vee 5K_1 - S) = 5 > 3 = (2k+3) - 2k = |S| - 2k$. It follows from Lemma 2.1 that the graph $K_{2k+3} \vee 5K_1$ is not $k$-extendable. □

## Data availability statement

My manuscript has no associated data.

## Declaration of competing interest

The authors declare that they have no conflicts of interest to this work.

## Acknowledgments

The authors would like to thank the anonymous referees and editors for carefully reading the manuscript and providing valuable comments. This work was supported by the Natural Science Foundation of Jiangsu Province (Grant No. BK20241949). Project ZR2023MA078 supported by Shandong Provincial Natural Science Foundation.

## References

[1] N. Ananchuen, L. Caccetta, A note of $k$-extendable graphs and independence number, Australas. J. Combin. 12(1995)59–65.
[2] N. Ananchuen, L. Caccetta, Matching extension and minimum degree, Discrete Math. 170(1997)1–13.
[3] I. Anderson, Perfect matchings of a graph, J. Combin. Theory Ser. B 10(1971)183–186.
[4] I. Anderson, Sufficient conditions for matchings, Proc. Edinb. Math. Soc. (2) 18(1972)129–136.
[5] A. Brouwer, W. Haemers, Spectra of Graphs - Monograph, Springer, 2011.
[6] C. Chen, Binding number and toughness for matching extension, Discrete Math. 146(1995)303–306.
[7] S. Cioaba, J. Koolen, W. Li, Max-cut and extendability of matchings in distance-regular graphs, European J. Combin. 62(2017)232–244.
[8] S. Cioaba, W Li, The extendability of matchings in strongly regular graphs, Electron. J. Combin. 21(2)(2014), Paper 2.34, 23 pp.
[9] D. Cvetković, S. Simić, Towards a spectral theory of graphs based on the signless Laplacian I, Publ. Inst. Math. 99(2009)19–33.
[10] G. Dai, Degree sum conditions for path-factor uniform graphs, Indian J. Pure Appl. Math. DOI: 10.1007/s13226-023-00446-7
[11] G. Dai, Z. Hu, $P_3$-factors in the square of a tree, Graphs Combin. 36(2020)1913–1925.
[12] H. Enomoto, Toughness and the existence of $k$-factors III, Discrete Math. 189(1998)277–282.
[13] D. Fan, H. Lin, Spectral conditions for $k$-extendability and $k$-factors of bipartite graphs, arXiv:2211.09304
[14] W. Gao, W. Wang, Tight binding number bound for $P_{\geq 3}$-factor uniform graphs, Inform. Process. Lett. 172(2021)106162.
[15] W. Gao, W. Wang, Y. Chen, Tight isolated toughness bound for fractional $(k, n)$-critical graphs, Discrete Appl. Math. 322(2022)194–202.
[16] M. Kano, G. Y. Katona, Z. Király, Packing paths of length at least two, Discrete Math. 283(2004)129–135.
[17] A. Kelmans, Packing 3-vertex paths in claw-free graphs and related topics, Discrete Appl. Math. 159(2011)112–127.
[18] W. Liu, M. Liu, L. Feng, Spectral conditions for graphs to be $\beta$-deficient involving minimum degree, Linear Multilinear Algebra 66(4)(2018)792–802.
[19] H. Liu, X. Pan, Independence number and minimum degree for path-factor critical uniform graphs, Discrete Appl. Math. 359(2024)153–158.
[20] D. Lou, Q. Yu, Connectivity of $k$-extendable graphs with large $k$, Discrete Appl. Math. 136(2004)55–61.
[21] X. Lv, A degree condition for graphs being fractional $(a, b, k)$-critical covered, Filomat 37(10)(2023)3315–3320.
[22] P. Maschlanka, L. Volkmann, Independence number for $n$-extendable graphs, Discrete Math. 154(1996)167–178.
[23] H. Matsuda, Fan-type results for the existence of $[a, b]$-factors, Discrete Math. 306(2006)688–693.
[24] T. Niessen, Neighborhood unions and regular factors, J. Graph Theory (19)(1)(1995)45–64.
[25] S. O, Spectral radius and matchings in graphs, Linear Algebra Appl. 614(2021)316–324.
[26] M. Plummer, Extending matchings in claw-free graphs, Discrete Math. 125(1994)301–307.
[27] M. Plummer, Matching extension and the genus of a graph, J. Combin. Theory Ser. B 44(1988)329–337.
[28] M. Plummer, On $n$-extendable graphs, Discrete Math. 31(1980)201–210.
[29] A. Robertshaw, D. Woodall, Binding number conditions for matching extension, Discrete Math. 248(2002)169–179.
[30] Y. Shen, L. You, M. Zhang, S. Li, On a conjecture for the signless Laplacian spectral radius of cacti with given matching number, Linear Multilinear Algebra 65(2017)457–474.
[31] D. Sumner, Graphs with 1-factors, Proc. Amer. Math. Soc. 42(1974)8–12.
[32] W. Tutte, The factorization of linear graphs, J. London Math. Soc. 22 (1947) 107–111.
[33] J. Wu, A sufficient condition for the existence of fractional $(g, f, n)$-critical covered graphs, Filomat 38(6) (2024) 2177–2183.
[34] J. Wu, Characterizing spanning trees via the size or the spectral radius of graphs, Aequationes Math. 98(6) (2024) 1441–1455.
[35] L. You, M. Yang, W. So, W. Xi, On the spectrum of an equitable quotient matrix and its application, Linear Algebra Appl. 577 (2019) 21–40.
[36] W. Zhang, Matching extendability and connectivity of regular graphs from eigenvalues, Graphs Combin. 36(1) (2020) 93–108.
[37] Y. Zhang, H. Lin, Perfect matching and distance spectral radius in graphs and bipartite graphs, Discrete Appl. Math. 304 (2021) 315–322.

[38] B. Zhou, Signless Laplacian spectral radius and Hamiltonicity, Linear Algebra Appl. 432 (2010) 566–570.
[39] S. Zhou, A neighborhood union condition for fractional $(a, b, k)$-critical covered graphs, Discrete Appl. Math. 323 (2022) 343–348.
[40] S. Zhou, Degree conditions and path factors with inclusion or exclusion properties, Bull. Math. Soc. Sci. Math. Roumanie 66(1) (2023) 3–14.
[41] S. Zhou, Some results on path-factor critical avoidable graphs, Discuss. Math. Graph Theory 43(1) (2023) 233–244.
[42] S. Zhou, Q. Bian, Z. Sun, Two sufficient conditions for component factors in graphs, Discuss. Math. Graph Theory 43(3) (2023) 761–766.
[43] S. Zhou, H. Liu, Two sufficient conditions for odd $[1, b]$-factors in graphs, Linear Algebra Appl. 661 (2023) 149–162.
[44] S. Zhou, Q. Pan, L. Xu, Isolated toughness for fractional $(2, b, k)$-critical covered graphs, Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci. 24(1) (2023) 11–18.
[45] S. Zhou, Q. Pan, Y. Xu, A new result on orthogonal factorizations in networks, Filomat 38(20) (2024) 7235–7244.
[46] S. Zhou, Z. Sun, H. Liu, Distance signless Laplacian spectral radius for the existence of path-factors in graphs, Aequationes Math. 98(3) (2024) 727–737.
[47] S. Zhou, Z. Sun, H. Liu, Some sufficient conditions for path-factor uniform graphs, Aequationes Math. 97(3) (2023) 489–500.
[48] S. Zhou, Z. Sun, Y. Zhang, Spectral radius and k-factor-critical graphs, J. Supercomput. 81(3) (2025) 456.
[49] S. Zhou, Y. Xu, Z. Sun, Some results about star-factors in graphs, Contrib. Discrete Math. 19(3) (2024) 154–162.
[50] S. Zhou, Y. Zhang, H. Liu, Some properties of $(a, b, k)$-critical graphs, Filomat 38(16) (2024) 5885–5894.
[51] S. Zhou, Y. Zhang, Z. Sun, The $A_\alpha$-spectral radius for path-factors in graphs, Discrete Math. 347(5) (2024) 113940.