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Oriented diameter of the complete tripartite graph

Jing Liu[a], Guang Rao[b], Jiachen Zhang[a], Hui Zhou[a,*]

[a]*College of Mathematics and Physics, Beijing University of Chemical Technology, Beijing 100029, China*
[b]*Yale-NUS College, National University of Singapore, 16 College Avenue West, Singapore 138527, Singapore*

Abstract. Given a connected and bridgeless graph G, let $\mathbb{D}(G)$ be the set of all strong orientations of G, and define the oriented diameter of G to be

$$f(G) = \min\{\mathrm{diam}(D) \mid D \in \mathbb{D}(G)\}.$$

Rajasekaran and Sampathkumar (Filomat, 2015) conjectured

$$f(\mathsf{K}(2, p, q)) = 3 \text{ when } p \geqslant 5 \text{ and } q > \binom{p}{\lfloor \frac{p}{2} \rfloor}.$$

In this paper, we confirm this conjecture. Combining with the results of Koh and Tan (Graphs and Combinatorics, 1996), the oriented diameter of complete tripartite graph $\mathsf{K}(2, p, q)$ is completely determined.

1. Introduction

Let G be a finite undirected connected graph with vertex set $V(G)$ and edge set $E(G)$. Take $u, v \in V(G)$. The distance $\mathrm{d}_G(u, v)$ is the number of edges in a shortest path connecting u and v in G. The diameter of G is defined to be $\mathrm{diam}(G) = \max\{\mathrm{d}_G(u, v) \mid u, v \in V(G)\}$. An edge $e \in E(G)$ is called a bridge if the resulting graph obtained from G by deleting e is disconnected. A graph is called bridgeless if it has no bridge. An orientation D of G is a digraph obtained from G by assigning a direction to each edge. A digraph is strong (or strongly connected) if for any two vertices u, v, there is a directed path from u to v in this digraph. An orientation D of G is called a strong orientation if the digraph D is strong. Robbins' one-way street theorem [9] states that

a connected graph has a strong orientation if and only if it is bridgeless.

Given a connected graph G which is bridgeless, let $\mathbb{D}(G)$ be the set of all strong orientations of G. Define the oriented diameter of G to be

$$f(G) = \min\{\mathrm{diam}(D) \mid D \in \mathbb{D}(G)\},$$

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* Corresponding author: Hui Zhou
Email addresses: 2022201091@buct.edu.cn (Jing Liu), grant.r@nus.edu.sg (Guang Rao), 2020120182@buct.edu.cn, 1784536225@qq.com (Jiachen Zhang), zhouh06@qq.com, zhouhlzu06@126.com (Hui Zhou)
ORCID iDs: https://orcid.org/0009-0007-2212-955X (Jing Liu), https://orcid.org/0000-0003-4934-5870 (Guang Rao), https://orcid.org/0009-0005-4729-8080 (Jiachen Zhang), 0000-0002-7566-4931 (Hui Zhou)

where diam(D) denote the diameter of D. The problem of evaluating oriented diameter $f(G)$ of an arbitrary connected graph G is very difficult. Chvátal and Thomassen [2] showed that the problem of deciding whether a graph admits an orientation of diameter two is NP-hard.

Given positive integers $n, p_1, p_2, \ldots, p_n$, let K_n denote the complete graph of order n, and let $\mathsf{K}(p_1, p_2, \ldots, p_n)$ denote the complete n-partite graph having p_i vertices in the i-th partite set V_i for each $i \in \{1, 2, \ldots, n\}$. Thus K_n is also a complete n-partite graph $\mathsf{K}(p_1, p_2, \ldots, p_n)$ where $p_1 = p_2 = \cdots = p_n = 1$. The oriented diameter of complete graph K_n was obtained by Boesch and Tindell [1]:

$$f(\mathsf{K}_n) = \begin{cases} 2, & \text{if } n \geqslant 3 \text{ and } n \neq 4; \\ 3, & \text{if } n = 4. \end{cases}$$

The oriented diameter of complete bipartite graph $\mathsf{K}(p, q)$ for $2 \leqslant p \leqslant q$ was obtained by Šoltés [10]:

$$f(\mathsf{K}(p,q)) = \begin{cases} 3, & \text{if } q \leqslant \binom{p}{\lfloor \frac{p}{2} \rfloor}; \\ 4, & \text{if } q > \binom{p}{\lfloor \frac{p}{2} \rfloor}; \end{cases}$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. For $n \geqslant 3$, Plesník [7], Gutin [3], and Koh and Tan [4] obtained independently the following result for oriented diameter of complete n-partite graph:

$$2 \leqslant f(\mathsf{K}(p_1, p_2, \ldots, p_n)) \leqslant 3.$$

They also got some other results on complete multipartite graphs. In a survey by Koh and Tay [6], earlier results were collected in it, for example: lower and upper bounds for some graphs with special parameters, oriented diameters for Cartesian product and extensions of graphs, etc.

Koh and Tay [6] proposed a problem.

Problem 1.1. *Given the graph $G = \mathsf{K}(p_1, p_2, \ldots, p_n)$, classify it according to whether $f(G) = 2$ or $f(G) = 3$.*

We focus on complete tripartite graphs. Koh and Tan [5] proved

$$f(\mathsf{K}(2,p,q)) = 2 \text{ for } 2 \leqslant p \leqslant q \leqslant \binom{p}{\lfloor \frac{p}{2} \rfloor}.$$

Rajasekaran and Sampathkumar [8] proved $f(\mathsf{K}(2,2,q)) = 3$ for $q \geqslant 3$, and $f(\mathsf{K}(2,3,q)) = 3$ for $q \geqslant 4$, and they also got in an unpublished manuscript that $f(\mathsf{K}(2,4,q)) = 3$ for $q \geqslant 7$. Hence they [8] conjectured that

$$f(\mathsf{K}(2,p,q)) = 3 \text{ when } p \geqslant 5 \text{ and } q > \binom{p}{\lfloor \frac{p}{2} \rfloor}.$$

In this paper, we confirm this conjecture. Combining with the results of Koh and Tan [5], the oriented diameter of complete tripartite graph $\mathsf{K}(2,p,q)$ is completely determined: for $2 \leqslant p \leqslant q$,

$$f(\mathsf{K}(2,p,q)) = \begin{cases} 2, & \text{if } q \leqslant \binom{p}{\lfloor \frac{p}{2} \rfloor}; \\ 3, & \text{if } q > \binom{p}{\lfloor \frac{p}{2} \rfloor}. \end{cases}$$

More generally, we prove the following result.

Theorem 1.2. *Suppose $2 \leqslant p \leqslant q$ and $q > \binom{p}{\lfloor \frac{p}{2} \rfloor}$, then $f(\mathsf{K}(2,p,q)) = 3$.*

2. Preliminaries

Let D be a digraph with vertex set $V(D)$. If $u, v \in V(D)$, the distance $\partial_D(u, v)$ is the number of directed edges in a shortest directed path from u to v in D. If D is strongly connected, the diameter of D is defined as $\operatorname{diam}(D) = \max\{\partial_D(u, v) \mid u, v \in V(D)\}$. Let $u, v \in V(D)$, and $U, V \subseteq V(D)$ such that $U \cap V = \emptyset$, we write '$u \to v$' if the direction is from u to v in D, we write '$U \to V$' if $x \to y$ for each $x \in U$ and for each $y \in V$, if $U = \{u\}$ we write '$u \to V$' for $U \to V$, if $V = \{v\}$ we write '$U \to v$' for $U \to V$. All the out-neighbors of u form a set $N_D^+(u) = \{w \in V(D) \mid u \to w\}$, and all the in-neighbors of v form a set $N_D^-(v) = \{w \in V(D) \mid w \to v\}$. For $S \subseteq V(D)$, we use $D[S]$ to denote the subgraph induced by S in D.

Lemma 2.1. *Suppose X is a strongly connected digraph. Let $u, v \in V(X)$ be two vertices of X. If $N_X^+(u) \cap N_X^-(v) = \emptyset$, then $\partial_X(u, v) \neq 2$.*

Proof. We assume $\partial_X(u, v) = 2$, then there exists $w \in V(X)$ such that $u \to w \to v$. So $w \in N_X^+(u) \cap N_X^-(v) \neq \emptyset$, a contradiction. □

3. Proof of Theorem 1.2

The oriented diameter of complete bipartite graph $\mathsf{K}(p, q)$ is crucial in some cases of this proof. The rest of this section is the proof of Theorem 1.2, and we prove it by contradiction. Assuming

$$f(\mathsf{K}(2, p, q)) = 2 \text{ when } 2 \leqslant p \leqslant q \text{ and } q > \binom{p}{\lfloor \frac{p}{2} \rfloor}.$$

Then $\mathsf{K}(2, p, q)$ has a strong orientation D with diameter $\operatorname{diam}(D) = 2$. Let

$$\begin{aligned} V_1 &= \{x_1, x_2\}, \\ V_2 &= \{y_1, y_2, \ldots, y_p\}, \\ V_3 &= \{z_1, z_2, \ldots, z_q\} \end{aligned}$$

be the three partite sets of the vertex set of $\mathsf{K}(2, p, q)$. We consider sets

$$\begin{aligned} N_D^{++} &= N_D^+(x_1) \cap N_D^+(x_2), \\ N_D^{+-} &= N_D^+(x_1) \cap N_D^-(x_2), \\ N_D^{-+} &= N_D^-(x_1) \cap N_D^+(x_2), \\ N_D^{--} &= N_D^-(x_1) \cap N_D^-(x_2). \end{aligned}$$

For $i \in \{2, 3\}$, the following four sets

$$\begin{aligned} V_i^{++} &= V_i \cap N_D^{++}, \\ V_i^{+-} &= V_i \cap N_D^{+-}, \\ V_i^{-+} &= V_i \cap N_D^{-+}, \\ V_i^{--} &= V_i \cap N_D^{--} \end{aligned}$$

form a partition of V_i. By this partition, we have the following properties.

Lemma 3.1. *We use notations as above. Suppose $\{i, j\} = \{2, 3\}$.*

1. *If $V_i^{++} \neq \emptyset$, then $V_i^{++} \to V_j$ and $|V_i^{++}| = 1$;*
   if $V_i^{--} \neq \emptyset$, then $V_j \to V_i^{--}$ and $|V_i^{--}| = 1$.
2. *If $V_i^{++} \neq \emptyset$, then $V_j^{++} = \emptyset$; if $V_i^{--} \neq \emptyset$, then $V_j^{--} = \emptyset$.*

Proof. Suppose $V_i^{++} \neq \emptyset$. Take any $y \in V_i^{++}$ and any $z \in V_j$, we have $\partial_D(y, z) \leqslant 2$. If $z \to y$, then $N_D^+(y) \subseteq V_j \setminus \{z\}$. We know $\partial_D(z', z) \geqslant 2$ for any $z' \in V_j \setminus \{z\}$, so $\partial_D(y, z) \geqslant 3$, a contradiction. Hence $y \to z$. This means $V_i^{++} \to V_j$.

For distinct vertices $y_h, y_k \in V_i^{++}$, we have $N_D^+(y_h) \subseteq V_j$ and $N_D^-(y_k) \subseteq V_1$. $V_1 \cap V_j = \emptyset$ implies $N_D^+(y_h) \cap N_D^-(y_k) = \emptyset$, so by Lemma 2.1 we get $\partial_D(y_h, y_k) \geqslant 3$, a contradiction. Thus $|V_i^{++}| = 1$. The proof for the case $V_i^{--} \neq \emptyset$ is analogous.

Suppose $V_i^{++} \neq \emptyset$ and $V_j^{++} \neq \emptyset$, then $V_i^{++} \to V_j$ and $V_j^{++} \to V_i$, i.e., for $y \in V_i^{++}$ and $z \in V_j^{++}$, we have $y \to z$ and $z \to y$, a contradiction. The proof for the case $V_i^{--} \neq \emptyset$ is analogous. □

Let

$$\mathbb{H} = \{V_2^{++}, V_2^{+-}, V_2^{-+}, V_2^{--}\}.$$

We will divide it into cases according to the number of nonempty sets in $\mathbb{H}$.

3.1. There is exactly one nonempty set in $\mathbb{H}$

Suppose there is exactly one of the four sets in $\mathbb{H}$ that is nonempty. Since V_2 is a partition of the four sets in $\mathbb{H}$, it is exactly the nonempty set, i.e., $V_2 \in \mathbb{H}$.

3.1.1. $V_2 = V_2^{++}$

Suppose $V_2 = V_2^{++}$.

By Lemma 3.1, we have $1 = |V_2^{++}| = |V_2| = p \geqslant 2$, which is a contradiction.

3.1.2. $V_2 = V_2^{--}$

This subcase is the same as in Subsubsection 3.1.1 by reversing directions of all the arcs in D, meanwhile the diameter is preserved.

3.1.3. $V_2 = V_2^{+-}$

Suppose $V_2 = V_2^{+-}$.

We know $x_1 \to V_2 \to x_2$. Take any $z \in V_3$. If $x_1 \to z$, then $N_D^+(z) \subseteq \{x_2\} \cup V_2$. We have $\partial_D(x_2, x_1) \geqslant 2$ and $\partial_D(y, x_1) \geqslant 2$ for any $y \in V_2$. So $\partial_D(z, x_1) \geqslant 3$, a contradiction. Hence we get $z \to x_1$. This means $V_3 \to x_1$. Similarly, we can prove $x_2 \to V_3$.

Take any two vertices $y_h, y_k \in V_2$. We know $\partial_D(y_h, y_k) \leqslant 2$. Since $N_D^+(y_h) \subseteq \{x_2\} \cup V_3$ and $N_D^-(y_k) \subseteq \{x_1\} \cup V_3$, and by Lemma 2.1, then we have $\emptyset \neq N_D^+(y_h) \cap N_D^-(y_k) \subseteq V_3$, i.e., there exists an integer $\delta(h,k)$ and $z_{\delta(h,k)} \in V_3$ such that $y_h \to z_{\delta(h,k)} \to y_k$. Similarly, we can prove that for any $z_i, z_j \in V_3$, there exists an integer $\eta(i,j)$ and $y_{\eta(i,j)} \in V_2$ such that $z_i \to y_{\eta(i,j)} \to z_j$.

Let $F = D[V_2 \cup V_3]$. Then F is an orientation of $\mathsf{K}(p,q)$ where $2 \leqslant p \leqslant q$ and $q > \binom{p}{\lfloor \frac{p}{2} \rfloor}$. Take any distinct vertices $y_h, y_k \in V_2$ and distinct vertices $z_i, z_j \in V_3$. By the above discussion, we get $\partial_F(y_h, y_k) = 2 = \partial_F(z_i, z_j)$. If $y_h \to z_i$, then $\partial_F(y_h, z_i) = 1$ and $N_F^+(z_i) \cap N_F^-(y_h) \subseteq V_2 \cap V_3 = \emptyset$. By Lemma 2.1, we have $\partial_F(z_i, y_h) \geqslant 3$. There is a directed path $z_i \to y_{\eta(i,j)} \to z_{\delta(\eta(i,j),h)} \to y_h$ of length three, so we have $\partial_F(z_i, y_h) = 3$. By the same argument, if $z_i \to y_h$, then $\partial_F(z_i, y_h) = 1$ and $\partial_F(y_h, z_i) = 3$. Thus $\mathrm{diam}(F) = 3 < 4 = f(\mathsf{K}(p,q))$, a contradiction.

3.1.4. $V_2 = V_2^{-+}$

This subcase is the same as in Subsubsection 3.1.3 by interchanging vertices x_1 and x_2 (the diameter of the orientation is also preserved).

3.2. There are exactly two nonempty sets in $\mathbb{H}$

Suppose there are exactly two nonempty sets in $\mathbb{H}$. Since V_2 is a partition of the four sets in $\mathbb{H}$, any possible two sets in $\mathbb{H}$ form a partition of V_2.

3.2.1. $V_2 = V_2^{++} \cup V_2^{+-}$

Suppose $V_2^{++} \neq \emptyset$, $V_2^{+-} \neq \emptyset$, $V_2^{-+} = \emptyset$ and $V_2^{--} = \emptyset$.

By Lemma 3.1, we may assume $V_2^{++} = \{y\}$. So $x_1 \to y \to V_3$, $x_1 \to V_2^{+-} \to x_2 \to y$, $|V_2^{+-}| = p - 1$.

We show that $V_3 \to x_1$. Take any $z_i \in V_3$, we have $N_D^+(z_i) \subseteq V_2^{+-} \cup V_1$, $N_D^-(x_1) \subseteq V_3$. So $N_D^+(z_i) \cap N_D^-(x_1) = \emptyset$. If $x_1 \to z_i$, by Lemma 2.1 we have $\partial_D(z_i, x_1) \geqslant 3$, a contradiction. Hence $z_i \to x_1$.

Since $N_D^+(y) \subseteq V_3$, $N_D^-(x_2) \subseteq V_2^{+-} \cup V_3$ and $\partial_D(y, x_2) \leqslant 2$, we have $\emptyset \neq N_D^+(y) \cap N_D^-(x_2) \subseteq V_3$, i.e., there exists $z \in V_3$ such that $y \to z \to x_2$. Take any $y_k \in V_2^{+-}$, then $N_D^+(y_k) \subseteq \{x_2\} \cup V_3$, $N_D^-(z) \subseteq V_2$, so $N_D^+(y_k) \cap N_D^-(z) = \emptyset$. If $z \to y_k$, by Lemma 2.1 we get $\partial_D(y_k, z) \geqslant 3$, a contradiction. Hence $y_k \to z$. This means $V_2^{+-} \to z$.

Take $z_j \in V_3 \setminus \{z\}$. We have $N_D^+(z) \subseteq V_1$, $N_D^-(z_j) \subseteq \{x_2\} \cup V_2$, and so $N_D^+(z) \cap N_D^-(z_j) \subseteq \{x_2\}$, i.e., $z \to x_2 \to z_j$. This means $x_2 \to V_3 \setminus \{z\}$.

Since $q > \binom{p}{\lfloor \frac{p}{2} \rfloor} \geqslant 2$, we have $q \geqslant 3$ and $|V_3 \setminus \{z\}| = q - 1 \geqslant 2$. Take distinct vertices $z_i, z_j \in V_3 \setminus \{z\}$. We have $N_D^+(z_i) \subseteq \{x_1\} \cup V_2^{+-}$, $N_D^-(z_j) \subseteq \{x_2\} \cup V_2$, and so $N_D^+(z_i) \cap N_D^-(z_j) \subseteq V_2^{+-}$. Since $\partial_D(z_i, z_j) \leqslant 2$, there exists an integer $\eta(i, j)$ and $y_{\eta(i,j)} \in V_2^{+-}$ such that $z_i \to y_{\eta(i,j)} \to z_j$. We know $y_{\eta(i,j)} \neq y_{\eta(j,i)}$, hence $p - 1 = |V_2^{+-}| \geqslant 2$. Take distinct vertices $y_h, y_k \in V_2^{+-}$. We have $N_D^+(y_h) \subseteq \{x_2\} \cup V_3$, $N_D^-(y_k) \subseteq \{x_1\} \cup V_3 \setminus \{z\}$, and so $N_D^+(y_h) \cap N_D^-(y_k) \subseteq V_3 \setminus \{z\}$. Since $\partial_D(y_h, y_k) \leqslant 2$, there exists an integer $\delta(h, k)$ and $y_{\delta(h,k)} \in V_3 \setminus \{z\}$ such that $y_h \to z_{\delta(h,k)} \to y_k$.

Let $F = D[V_2^{+-} \cup V_3 \setminus \{z\}]$. Then F is an orientation of $\mathsf{K}(p - 1, q - 1)$ where $2 \leqslant p - 1 \leqslant q - 1$ and $q - 1 > \binom{p-1}{\lfloor \frac{p-1}{2} \rfloor}$. Take any distinct vertices $y_h, y_k \in V_2^{+-}$ and distinct vertices $z_i, z_j \in V_3 \setminus \{z\}$. By the above discussion, we get $\partial_F(y_h, y_k) = 2 = \partial_F(z_i, z_j)$. If $y_h \to z_i$, then $\partial_F(y_h, z_i) = 1$ and $N_F^+(z_i) \cap N_F^-(y_h) \subseteq V_2 \cap V_3 = \emptyset$. By Lemma 2.1, we have $\partial_F(z_i, y_h) \geqslant 3$. There is a directed path $z_i \to y_{\eta(i,j)} \to z_{\delta(\eta(i,j),h)} \to y_h$ of length three, so we have $\partial_F(z_i, y_h) = 3$. By the same argument, if $z_i \to y_h$, then $\partial_F(z_i, y_h) = 1$ and $\partial_F(y_h, z_i) = 3$. Thus $\mathrm{diam}(F) = 3 < 4 = f(\mathsf{K}(p - 1, q - 1))$, a contradiction.

3.2.2. $V_2 = V_2^{++} \cup V_2^{-+}$

Suppose $V_2^{++} \neq \emptyset$, $V_2^{-+} \neq \emptyset$, $V_2^{+-} = \emptyset$ and $V_2^{--} = \emptyset$.

This subcase is the same as in Subsubsection 3.2.1 by interchanging vertices x_1 and x_2 (the diameter of the orientation is also preserved).

3.2.3. $V_2 = V_2^{+-} \cup V_2^{--}$

Suppose $V_2^{+-} \neq \emptyset$, $V_2^{--} \neq \emptyset$, $V_2^{++} = \emptyset$ and $V_2^{-+} = \emptyset$.

This subcase is the same as in Subsubsection 3.2.2 by reversing directions of all the arcs in D, meanwhile the diemater is preserved.

3.2.4. $V_2 = V_2^{-+} \cup V_2^{--}$

Suppose $V_2^{-+} \neq \emptyset$, $V_2^{--} \neq \emptyset$, $V_2^{++} = \emptyset$ and $V_2^{+-} = \emptyset$.

This subcase is the same as in Subsubsection 3.2.3 by interchanging vertices x_1 and x_2 (the diameter of the orientation is also preserved).

3.2.5. $V_2 = V_2^{++} \cup V_2^{--}$

Suppose $V_2^{++} \neq \emptyset$, $V_2^{--} \neq \emptyset$, $V_2^{+-} = \emptyset$ and $V_2^{-+} = \emptyset$.

By Lemma 3.1, we have $V_3^{++} = \emptyset = V_3^{--}$, $V_3 = V_3^{+-} \cup V_3^{-+}$, and we may assume $V_2^{++} = \{y_+\}$ and $V_2^{--} = \{y_-\}$. So $p = 2$, $q \geqslant 3$, $x_1 \to y_+ \to V_3 \to y_- \to x_1$, $y_- \to x_2 \to y_+$, $x_1 \to V_3^{+-} \to x_2 \to V_3^{-+} \to x_1$. By the pigeonhole principle, we have $|V_3^{+-}| \geqslant 2$ or $|V_3^{-+}| \geqslant 2$. The argument for these two cases are similar, so we may assume $|V_3^{+-}| \geqslant 2$. Take distinct vertices $z_i, z_j \in V_3^{+-}$. We have $N_D^+(z_i) \subseteq \{x_2, y_-\}$ and $N_D^-(z_j) \subseteq \{x_1, y_+\}$, and so $N_D^+(z_i) \cap N_D^-(z_j) = \emptyset$. By Lemma 2.1, we have $\partial_D(z_i, z_j) \geqslant 3$. A contradiction.

3.2.6. $V_2 = V_2^{+-} \cup V_2^{-+}$

Suppose $V_2^{+-} \neq \emptyset$, $V_2^{-+} \neq \emptyset$, $V_2^{++} = \emptyset$ and $V_2^{--} = \emptyset$.

We have $x_1 \to V_2^{+-} \to x_2 \to V_2^{-+} \to x_1$, and $V_3 = V_3^{++} \cup V_3^{+-} \cup V_3^{-+} \cup V_3^{--}$. Since $|V_3| = q \geqslant 3$ and $|V_3^{++} \cup V_3^{--}| \leqslant 2$, we have $V_3^{+-} \neq \emptyset$ or $V_3^{-+} \neq \emptyset$. The argument for these two cases are similar, so we may assume $V_3^{+-} \neq \emptyset$. Take $z_i \in V_3^{+-}$ and $y_h \in V_2^{+-}$.

If $y_h \to z_i$, then $N_D^+(z_i) \subseteq \{x_2\} \cup V_2$, $N_D^-(y_h) \subseteq \{x_1\} \cup V_3$, and so $N_D^+(z_i) \cap N_D^-(y_h) = \emptyset$. By Lemma 2.1, we get $\partial_D(z_i, y_h) \geqslant 3$. A contradiction.

If $z_i \to y_h$, then $N_D^+(y_h) \subseteq \{x_2\} \cup V_3$, $N_D^-(z_i) \subseteq \{x_1\} \cup V_2$, and so $N_D^+(y_h) \cap N_D^-(z_i) = \emptyset$. By Lemma 2.1, we get $\partial_D(y_h, z_i) \geqslant 3$. A contradiction.

3.3. There are exactly three nonempty sets in $\mathbb{H}$

Suppose there are exactly three nonempty sets in $\mathbb{H}$. Since V_2 is a partition of the four sets in $\mathbb{H}$, any possible three sets in $\mathbb{H}$ form a partition of V_2.

3.3.1. $V_2 = V_2^{++} \cup V_2^{+-} \cup V_2^{-+}$

Suppose $V_2^{++} \neq \emptyset$, $V_2^{+-} \neq \emptyset$, $V_2^{-+} \neq \emptyset$ and $V_2^{--} = \emptyset$.

By Lemma 3.1, we have $V_3^{++} = \emptyset$, $V_3 = V_3^{+-} \cup V_3^{-+} \cup V_3^{--}$ where $|V_3^{--}| \leqslant 1$, and we may assume $V_2^{++} = \{y\}$. So $V_1 \to y \to V_3, x_1 \to V_2^{+-} \to x_2 \to V_2^{-+} \to x_1$. We know $|V_3^{+-} \cup V_3^{-+}| \geqslant q-1 \geqslant 2$, and so $V_3^{+-} \neq \emptyset$ or $V_3^{-+} \neq \emptyset$. The proof for the two cases $V_3^{+-} \neq \emptyset$ and $V_3^{-+} \neq \emptyset$ are similar, so we only give the proof of the case $V_3^{+-} \neq \emptyset$.

Suppose $V_3^{+-} \neq \emptyset$, then $x_1 \to V_3^{+-} \to x_2$. Take $y_h \in V_2^{+-}$ and $z_i \in V_3^{+-}$. If $y_h \to z_i$, then $N_D^+(z_i) \subseteq \{x_2\} \cup V_2 \setminus \{y\}$ and $N_D^-(y_h) \subseteq \{x_1\} \cup V_3$. So $N_D^+(z_i) \cap N_D^-(y_h) = \emptyset$, by Lemma 2.1, we have $\partial_D(z_i, y_h) \geqslant 3$. A contradiction. If $z_i \to y_h$, then $N_D^+(y_h) \subseteq \{x_2\} \cup V_3$ and $N_D^-(z_i) \subseteq \{x_1\} \cup V_2$. So $N_D^+(y_h) \cap N_D^-(z_i) = \emptyset$, by Lemma 2.1, we have $\partial_D(y_h, z_i) \geqslant 3$. A contradiction.

3.3.2. $V_2 = V_2^{+-} \cup V_2^{-+} \cup V_2^{--}$

Suppose $V_2^{+-} \neq \emptyset$, $V_2^{-+} \neq \emptyset$, $V_2^{--} \neq \emptyset$ and $V_2^{++} = \emptyset$.

This subcase is the same as in Subsubsection 3.3.1 by reversing directions of all the arcs in D, meanwhile the diemater is preserved.

3.3.3. $V_2 = V_2^{++} \cup V_2^{+-} \cup V_2^{--}$

Suppose $V_2^{++} \neq \emptyset$, $V_2^{+-} \neq \emptyset$, $V_2^{--} \neq \emptyset$ and $V_2^{-+} = \emptyset$.

By Lemma 3.1, we have $V_3^{++} = \emptyset = V_3^{--}$, $V_3 = V_3^{+-} \cup V_3^{-+}$, and we may assume $V_2^{++} = \{y_+\}$ and $V_2^{--} = \{y_-\}$. So $V_1 \to y_+ \to V_3 \to y_- \to V_1, x_1 \to V_2^{+-} \to x_2, x_1 \to V_3^{+-} \to x_2 \to V_3^{-+} \to x_1$.

If $V_3^{+-} = \emptyset$, then $N_D^+(y_+) \subseteq V_3^{-+}$ and $N_D^-(x_2) \subseteq \{y_-\} \cup V_2^{+-}$. So $N_D^+(y_+) \cap N_D^-(x_2) = \emptyset$, by Lemma 2.1, we have $\partial_D(y_+, x_2) \geqslant 3$. A contradiction.

Now suppose $V_3^{+-} \neq \emptyset$. Take $z_i \in V_3^{+-}$ and $y_h \in V_2^{+-}$. If $y_h \to z_i$, then $N_D^+(z_i) \subseteq \{x_2\} \cup V_2 \setminus \{y_+\}$ and $N_D^-(y_h) \subseteq \{x_1\} \cup V_3$. So $N_D^+(z_i) \cap N_D^-(y_h) = \emptyset$, by Lemma 2.1, we have $\partial_D(z_i, y_h) \geqslant 3$. A contradiction. If $z_i \to y_h$, then $N_D^+(y_h) \subseteq \{x_2\} \cup V_3$ and $N_D^-(z_i) \subseteq \{x_1\} \cup V_2 \setminus \{y_-\}$. So $N_D^+(y_h) \cap N_D^-(z_i) = \emptyset$, by Lemma 2.1, we have $\partial_D(y_h, z_i) \geqslant 3$. A contradiction.

3.3.4. $V_2 = V_2^{++} \cup V_2^{-+} \cup V_2^{--}$

Suppose $V_2^{++} \neq \emptyset$, $V_2^{-+} \neq \emptyset$, $V_2^{--} \neq \emptyset$ and $V_2^{+-} = \emptyset$.

This subcase is the same as in Subsubsection 3.3.3 by interchanging vertices x_1 and x_2 (the diameter of the orientation is also preserved).

3.4. There are exactly four nonempty sets in $\mathbb{H}$

Suppose there are exactly four nonempty sets in $\mathbb{H}$, i.e., $V_2^{++} \neq \emptyset$, $V_2^{+-} \neq \emptyset$, $V_2^{-+} \neq \emptyset$ and $V_2^{--} \neq \emptyset$.

By Lemma 3.1, we have $V_3^{++} = \emptyset = V_3^{--}$, $V_3 = V_3^{+-} \cup V_3^{-+}$, and we may assume $V_2^{++} = \{y_+\}$ and $V_2^{--} = \{y_-\}$. So $V_1 \to y_+ \to V_3 \to y_- \to V_1$, $x_1 \to V_2^{+-} \to x_2 \to V_2^{-+} \to x_1$, $x_1 \to V_3^{+-} \to x_2 \to V_3^{-+} \to x_1$. We know $V_3^{+-} \neq \emptyset$ or $V_3^{-+} \neq \emptyset$. The proof for the two cases $V_3^{+-} \neq \emptyset$ and $V_3^{-+} \neq \emptyset$ are similar, so we only give the proof of the case $V_3^{+-} \neq \emptyset$.

Suppose $V_3^{+-} \neq \emptyset$. Take $y_h \in V_2^{+-}$ and $z_i \in V_3^{+-}$. If $y_h \to z_i$, then $N_D^+(z_i) \subseteq \{x_2\} \cup V_2 \setminus \{y_+\}$ and $N_D^-(y_h) \subseteq \{x_1\} \cup V_3$. So $N_D^+(z_i) \cap N_D^-(y_h) = \emptyset$, by Lemma 2.1, we have $\partial_D(z_i, y_h) \geqslant 3$. A contradiction. If $z_i \to y_h$, then $N_D^+(y_h) \subseteq \{x_2\} \cup V_3$ and $N_D^-(z_i) \subseteq \{x_1\} \cup V_2 \setminus \{y_-\}$. So $N_D^+(y_h) \cap N_D^-(z_i) = \emptyset$, by Lemma 2.1, we have $\partial_D(y_h, z_i) \geqslant 3$. A contradiction.

Combining all the proofs in all the subsections in Section 3, the proof of Theorem 1.2 is completed. ■

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