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A comprehensive study of refined Hermite-Hadamard inequalities and their applications

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Abstract. In this article, we explore a novel class of Hermite-Hadamard inequalities via exponential type convexity. To establish the main results, we primarily use Hölder's inequality, the power mean inequality and some generalized associated inequalities. These inequalities have strong applicability in inequalities and optimization theory. Moreover, we compare the main findings to demonstrate that the Hölder-.Işan inequality can yield more refined results than Hölder's inequality and the improved power mean inequality can offer better refinements of Hermite-Hadamard inequalities than the power mean inequality. Additionally, we provide some applications for generalized means.

1. Introduction

The concept of convexity can be traced back to the ancient Greek mathematician Archimedes (287 BC-212 BC). In his work "On the Sphere and Cylinder", he considers a convex arc as a bent line in a plane that completely lies on one side of the line joining its extreme points [29]. Major developments in convex functions and their applications started more than a century ago. Fundamentally, convexity arises in geometry [26], with vast applications, particularly in physics [12], chemistry [16], optimization problems [5], DC programming [22], convex programming [30] and also in mathematical disciplines such as functional analysis [13], monotone operators [2] and complex analysis [25].

There are many inequalities that find wide application in numerous scientific and mathematical problems, such as Jensen's inequality [21], the power mean inequality [19], the Cauchy-Schwarz inequality [24], Bell's inequality [3], Boole's inequality [?], the Sobolev inequality [14] and Chernoff's inequality [4]. While various types of inequalities exist, convex inequalities play a vital role in this field. The Hermite-Hadamard ($\mathcal{H}-\mathcal{H}$) inequality is one of these well-known inequalities, applicable to a convex function $\psi : [\sigma, \varpi] \to \Re$ and defined by

$$\psi\left(\frac{\sigma+\varpi}{2}\right) \leq \frac{1}{\varpi-\sigma}\int_{\sigma}^{\varpi}\psi(\mu)d\mu \leq \frac{\psi(\sigma)+\psi(\varpi)}{2}.$$

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This inequality has many applications in various areas of mathematics which include convex analysis [9], fuzzy fractional calculus [15], special means [11, 23], physics [10]. It is also helpful in proving other mathematical inequalities. Due to it's usefulness and properties, many researchers have studied this vital inequality. Several refinements in Hermite-Hadamard inequality have been explored, such as it's generalization to n-intervals [27], weighted Hermite-Hadamard inequality [28]. In this paper, we dedicate ourselves to obtaining some new refined inequalities of Hermite-Hadamard inequality using a new convexity. The fundamental definitions and some basic concepts required to maintain the flow of this work are presented below.

Definition 1.1. *[20] A function $\psi : I \to \Re$ is known as a convex function if it satisfies the following inequality*

$$\psi(\eta\mu + (1-\eta)\omega) \leq \eta\psi(\mu) + (1-\eta)\psi(\omega),$$

where $\eta \in [0,1]$ and $\mu, \omega \in I$.

Definition 1.2. *[17] A function $\psi : I \to \Re$ is called as an exponential type convex function if*

$$\psi(\eta\mu + (1-\eta)\omega) \leq (e^{\eta}-1)\psi(\mu) + (e^{1-\eta}-1)\psi(\omega),$$

where $\eta \in [0,1]$ and $\mu, \omega \in I$.

Remark 1.3. *The exponential-type convex functions have a range of $[0,\infty)$. For the proof, we refer the reader to [17].*

Theorem 1.4. *(Hölder integral inequality [6]). Suppose that ψ and Ξ are real-valued mappings defined on closed interval $[\sigma,\varpi]$. If $|\psi|^p$ and $|\Xi|^q$ are integrable over $[\sigma,\varpi]$, then*

$$\int_{\sigma}^{\varpi} \left|\psi(\mu)\Xi(\mu)\right| d\mu \leq \left(\int_{\sigma}^{\varpi} \left|\psi(\mu)\right|^p d\mu\right)^{\frac{1}{p}} \left(\int_{\sigma}^{\varpi} \left|\Xi(\mu)\right|^q d\mu\right)^{\frac{1}{q}},$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.5. *(Power-mean integral inequality [6]). Let ψ and Ξ be the real-valued mappings defined on $[\sigma,\varpi]$. If $|\psi|, |\psi||\Xi|^q$ are integrable on $[\sigma,\varpi]$, then*

$$\int_{\sigma}^{\varpi} \left|\psi(\mu)\Xi(\mu)\right| d\mu \leq \left(\int_{\sigma}^{\varpi} \left|\psi(\mu)\right| d\mu\right)^{1-\frac{1}{q}} \left(\int_{\sigma}^{\varpi} \left|\psi(\mu)\right| \left|\Xi(\mu)\right|^q d\mu\right)^{\frac{1}{q}}$$

The following inequality is the improved form of Hölder inequality known as the Hölder-Iscan inequality presented in [7].

Theorem 1.6. *Let ψ and Ξ be real mappings defined on closed interval $[\sigma,\varpi]$. If $|\psi|^p$ and $|\Xi|^q$ are integrable over $[\sigma,\varpi]$, then*

$$\int_{\sigma}^{\varpi} \left|\psi(\mu)\Xi(\mu)\right| d\mu \leq \frac{1}{\varpi-\sigma}\left[\left(\int_{\sigma}^{\varpi} (\varpi-\mu)\left|\psi(\mu)\right|^p d\mu\right)^{\frac{1}{p}} \left(\int_{\sigma}^{\varpi} (\varpi-\mu)\left|\Xi(\mu)\right|^q d\mu\right)^{\frac{1}{q}} + \left(\int_{\sigma}^{\varpi} (\mu-\sigma)\left|\psi(\mu)\right|^q d\mu\right)^{\frac{1}{p}} \left(\int_{\sigma}^{\varpi} (\mu-\sigma)\left|\Xi(\mu)\right|^q d\mu\right)^{\frac{1}{q}}\right],$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

The following theorem presents an improved form of power-mean inequality, also known as improved power-mean integral inequality given in [18].

Theorem 1.7. *Let ψ and Ξ be the real-valued mappings defined on closed interval $[\sigma, \varpi]$. If $|\psi|, |\psi||\Xi|^q$ are integrable on $[\sigma, \varpi]$, then*

$$\begin{aligned}
&\int_\sigma^\varpi \left|\psi(\mu)\Xi(\mu)\right| d\mu \\
&\leq \frac{1}{\varpi - \sigma}\left[\left(\int_\sigma^\varpi (\varpi - \mu)\left|\psi(\mu)\right| d\mu\right)^{1-\frac{1}{q}} \left(\int_\sigma^\varpi (\varpi - \mu)\left|\psi(\mu)\right| \left|\Xi(\mu)\right|^q d\mu\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_\sigma^\varpi (\mu - \sigma)\left|\psi(\mu)\right| d\mu\right)^{1-\frac{1}{q}} \left(\int_\sigma^\varpi (\mu - \sigma)\left|\psi(\mu)\right| |\Xi|^q d\mu\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where $q \geq 1$.

The following lemma was proved by İscan in [8].

Lemma 1.8. *Let the mapping $\psi : I^0 \subset \Re \to \Re$ be a differentiable mapping on I^0, $\sigma, \varpi \in I^0$ with $\sigma < \varpi$, $\zeta \in N$ and $\daleth \in \{0, 1, ..., \zeta - 1\}$. If $\psi' \in L[\sigma, \varpi]$, then the below equality holds true*

$$\begin{aligned}
&I_\zeta(\psi, \sigma, \varpi) \\
&= \sum_{\daleth=0}^{\zeta-1} \frac{1}{2\zeta}\left[\psi\left(\frac{(\zeta - \daleth)\sigma + \daleth\varpi}{\zeta}\right) + \psi\left(\frac{(\zeta - \daleth - 1)\sigma + (\daleth + 1)\varpi}{\zeta}\right)\right] - \frac{1}{\varpi - \sigma}\int_\sigma^\varpi \psi(\mu) d\mu \\
&= \sum_{\daleth=0}^{\zeta-1} \frac{\varpi - \sigma}{2\zeta^2}\left[\int_0^1 (1 - 2\eta)\psi'\left(\eta\frac{(\zeta - \daleth)\sigma + \daleth\varpi}{\zeta} + (1 - \eta)\frac{(\zeta - \daleth - 1)\sigma + (\daleth + 1)\varpi}{\zeta}\right) d\eta\right].
\end{aligned}$$

Researchers are continuously working in the field of inequalities, which are extensively used in analysis and convexity theory. Inspired by new extensions of these inequalities, this paper focuses on refinements related to the Hermite-Hadamard inequality via exponential-type convexity. This is a first ever novel class of refinements of Hermite-Hadamarad inequalities. The limiting cases of the main findings are presented in terms of corollaries. Practical applications of the explored results are provided in terms of means.

2. Main Results for Exponential Type Convex Functions

In this section, we study novel general inequalities of Hermite-Hadamard type for exponential type convex functions by applying Hölder's inequality, power mean inequality and their improved forms.

Theorem 2.1. *Suppose that the mapping $\psi : I \subset \Re \to \Re$ is differentiable on I^0, $\sigma, \varpi \in I^0$ with $\sigma < \varpi$. If $|\psi'|^q$ is an exponentially convex mapping on $[\sigma, \varpi]$ for a certain $q > 1$, then the inequality*

$$\begin{aligned}
|I_\zeta(\psi, \sigma, \varpi)| &\leq \sum_{\daleth=0}^{\zeta-1} \frac{\varpi - \sigma}{2\zeta^2}\left(\frac{1}{1 + p}\right)^{\frac{1}{p}} (e - 2)^{\frac{1}{q}} \\
&\times \left[\left|\psi'\left(\frac{(\zeta - \daleth)\sigma + \daleth\varpi}{\zeta}\right)\right|^q + \left|\psi'\left(\frac{(\zeta - \daleth - 1)\sigma + (\daleth + 1)\varpi}{\zeta}\right)\right|^q\right]^{\frac{1}{q}} \quad (1)
\end{aligned}$$

is satisfied.

Proof. Applying Lemma 1.8, we can write

$$\begin{aligned}
&|I_\zeta(\psi, \sigma, \varpi)| \\
&\leq \sum_{\daleth=0}^{\zeta-1} \frac{\varpi - \sigma}{2\zeta^2}\int_0^1 \left|(1 - 2\eta)\psi'\left(\eta\frac{(\zeta - \daleth)\sigma + \daleth\varpi}{\zeta} + (1 - \eta)\frac{(\zeta - \daleth - 1)\sigma + (\daleth + 1)\varpi}{\zeta}\right)\right| d\eta.
\end{aligned}$$

Applying Hölder's inequality, we get

$$I_\zeta(\psi,\sigma,\varpi) \le \sum_{\gimel=0}^{\zeta-1} \frac{\varpi-\sigma}{2\zeta^2}\left(\int_0^1 |1-2\eta|^p d\eta\right)^{\frac{1}{p}}$$
$$\left[\int_0^1 \left|\psi'\left(\eta\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta} + (1-\eta)\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^q d\eta\right]^{\frac{1}{q}}.$$

Since, $|\psi'|^q$ is exponential type convex function, so

$$I_\zeta(\psi,\sigma,\varpi) \le \sum_{\gimel=0}^{\zeta-1} \frac{\varpi-\sigma}{2\zeta^2}\left(\int_0^1 |1-2\eta|^p d\eta\right)^{\frac{1}{p}}$$
$$\left[\int_0^1 \left((e^\eta-1)\left|\psi'\left(\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}\right)\right|^q\right.\right.$$
$$\left.\left. +(e^{1-\eta}-1)\left|\psi'\left(\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^q\right)d\eta\right]^{\frac{1}{q}}$$

$$= \sum_{\gimel=0}^{\zeta-1} \frac{\varpi-\sigma}{2\zeta^2}\left(\int_0^1 |1-2\eta|^p d\eta\right)^{\frac{1}{p}}$$
$$\left[\int_0^1 (e^\eta-1)\left|\psi'\left(\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}\right)\right|^q d\eta\right.$$
$$\left. +\int_0^1 (e^{1-\eta}-1)\left|\psi'\left(\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^q d\eta\right]^{\frac{1}{q}}$$

$$= \sum_{\gimel=0}^{\zeta-1} \frac{\varpi-\sigma}{2\zeta^2}\left(\frac{1}{1+p}\right)^{\frac{1}{p}}(e-2)^{\frac{1}{q}}$$
$$\left[\left|\psi'\left(\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}\right)\right|^q + \left|\psi'\left(\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^q\right]^{\frac{1}{q}}.$$

Hence the result is proved. □

Corollary 2.2. *If we apply exponential type convexity of $|\psi'|^q$ in inequality* (1) *again, then we arrive at*

$$|I_\zeta(\psi,\sigma,\varpi)| \le \sum_{\gimel=0}^{\zeta-1} \frac{\varpi-\sigma}{2\zeta^2}\left(\frac{1}{1+p}\right)^{\frac{1}{p}}(e-2)^{\frac{1}{q}}$$
$$\left[\left\{\left(e^{\frac{\zeta-\gimel}{\zeta}}-1\right)+\left(e^{\frac{\zeta-\gimel-1}{\zeta}}-1\right)\right\}\left|\psi'(\sigma)\right|^q\right.$$
$$\left. +\left\{\left(e^{\frac{\gimel}{\zeta}}-1\right)+\left(e^{\frac{\gimel+1}{\zeta}}-1\right)\right\}\left|\psi'(\varpi)\right|^q\right]^{\frac{1}{q}}.$$

Corollary 2.3. *If we set $\zeta = 1$ in Corollary 2.2, we obtain*

$$\left|\frac{\psi(\sigma)+\psi(\varpi)}{2} - \frac{1}{\varpi-\sigma}\int_\sigma^\varpi \psi(\mu)d\mu\right|$$
$$\le \frac{\varpi-\sigma}{2}\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\left[(e-2)(e-1)\left(|\psi'(\sigma)|^q+|\psi'(\varpi)|^q\right)\right]^{\frac{1}{q}}.$$

Corollary 2.4. *If $\zeta = 2$ is set in Corollary 2.2, it reduces to*

$$\left|\frac{1}{2}\left[\frac{\psi(\sigma)+\psi(\varpi)}{2}+\psi(\frac{\sigma+\varpi}{2})\right]-\frac{1}{\varpi-\sigma}\int_{\sigma}^{\varpi}\psi(\mu)d\mu\right|$$
$$\leq \frac{\varpi-\sigma}{8}\left(\frac{1}{1+p}\right)^{\frac{1}{p}}(e-2)^{\frac{1}{q}}\left[\left((e-1)+2(\sqrt{e}-1)\right)\left(|\psi^{'}(\sigma)|^{q}+|\psi^{'}(\varpi)|^{q}\right)\right]^{\frac{1}{q}}.$$

Theorem 2.5. *Suppose that the mapping $\psi : I \subset \Re \to \Re$ is differentiable on I^0, $\sigma, \varpi \in I^0$ with $\sigma < \varpi$. If $|\psi^{'}|^q$ is exponential type convex on $[\sigma, \varpi]$ for a certain $q \geq 1$, then the following inequality holds true*

$$|I_{\zeta}(\psi,\sigma,\varpi)| \leq \sum_{\daleth=0}^{\zeta-1}\frac{\varpi-\sigma}{\zeta^{2}2^{2-\frac{1}{q}}}\left(-e+4\sqrt{e}-\frac{7}{2}\right)^{\frac{1}{q}}$$
$$\left[\left|\psi^{'}\left(\frac{(\zeta-\daleth)\sigma+\daleth\varpi}{\zeta}\right)\right|^{q}+\left|\psi^{'}\left(\frac{(\zeta-\daleth-1)\sigma+(\daleth+1)\varpi}{\zeta}\right)\right|^{q}\right]^{\frac{1}{q}}. \tag{2}$$

Proof. Applying Lemma 1.8 and then using power-mean inequality, we obtain

$$|I_{\zeta}(\psi,\sigma,\varpi)|$$
$$\leq \sum_{\daleth=0}^{\zeta-1}\frac{\varpi-\sigma}{2\zeta^{2}}\left[\int_{0}^{1}\left|(1-2\eta)\psi^{'}\left(\eta\frac{(\zeta-\daleth)\sigma+\daleth\varpi}{\zeta}+(1-\eta)\frac{(\zeta-\daleth-1)\sigma+(\daleth+1)\varpi}{\zeta}\right)\right|d\eta\right]$$
$$\leq \sum_{\daleth=0}^{\zeta-1}\frac{\varpi-\sigma}{2\zeta^{2}}\left(\int_{0}^{1}\left|1-2\eta\right|d\eta\right)^{1-\frac{1}{q}}$$
$$\left[\int_{0}^{1}\left|1-2\eta\right|\left|\psi^{'}\left(\eta\frac{(\zeta-\daleth)\sigma+\daleth\varpi}{\zeta}+(1-\eta)\frac{(\zeta-\daleth-1)\sigma+(\daleth+1)\varpi}{\zeta}\right)\right|^{q}d\eta\right]^{\frac{1}{q}}.$$

Since $|\psi^{'}|^q$ is exponential type convex on $[\sigma, \varpi]$, therefore

$$|I_{\zeta}(\psi,\sigma,\varpi)| \leq \sum_{\daleth=0}^{\zeta-1}\frac{\varpi-\sigma}{2\zeta^{2}}\left(\int_{0}^{1}\left|1-2\eta\right|d\eta\right)^{1-\frac{1}{q}}\left[\int_{0}^{1}\left|1-2\eta\right|\left((e^{\eta}-1)\left|\psi^{'}\left(\frac{(\zeta-\daleth)\sigma+\daleth\varpi}{\zeta}\right)\right|^{q}\right.\right.$$
$$\left.\left.+(e^{1-\eta}-1)\left|\psi^{'}\left(\frac{(\zeta-\daleth-1)\sigma+(\daleth+1)\varpi}{\zeta}\right)\right|^{q}\right)d\eta\right]^{\frac{1}{q}}$$
$$=\sum_{\daleth=0}^{\zeta-1}\frac{\varpi-\sigma}{2\zeta^{2}}\left(\int_{0}^{1}\left|1-2\eta\right|d\eta\right)^{1-\frac{1}{q}}$$
$$\left[\int_{0}^{1}\left|1-2\eta\right|(e^{\eta}-1)\left|\psi^{'}\left(\frac{(\zeta-\daleth)\sigma+\daleth\varpi}{\zeta}\right)\right|^{q}d\eta\right.$$
$$\left.+\int_{0}^{1}\left|1-2\eta\right|(e^{1-\eta}-1)\left|\psi^{'}\left(\frac{(\zeta-\daleth-1)\sigma+(\daleth+1)\varpi}{\zeta}\right)\right|^{q}d\eta\right]^{\frac{1}{q}}$$
$$=\sum_{\daleth=0}^{\zeta-1}\frac{\varpi-\sigma}{2^{2-\frac{1}{q}}\zeta^{2}}\left(-e+4\sqrt{e}-\frac{7}{2}\right)^{\frac{1}{q}}$$

$$\left[\left|\psi'\left(\frac{(\zeta-\beth)\sigma+\beth\varpi}{\zeta}\right)\right|^{q}+\left|\psi'\left(\frac{(\zeta-\beth-1)\sigma+(\beth+1)\varpi}{\zeta}\right)\right|^{q}\right]^{\frac{1}{q}}.$$

Hence the required result is proved. □

Corollary 2.6. *If exponential type convexity of $|\psi'|^q$ is applied again in Theorem 2.5, we obtain*

$$|I_\zeta(\psi,\sigma,\varpi)| \le \sum_{\beth=0}^{\zeta-1}\frac{\varpi-\sigma}{2^{2-\frac{1}{q}}\zeta^2}\left(-e+4\sqrt{e}-\frac{7}{2}\right)^{\frac{1}{q}}$$
$$\left[\left\{\left(e^{\frac{\zeta-\beth}{\zeta}}-1\right)+\left(e^{\frac{\zeta-\beth-1}{\zeta}}-1\right)\right\}\left|\psi'(\sigma)\right|^{q}+\left\{\left(e^{\frac{\beth}{\zeta}}-1\right)+\left(e^{\frac{\beth+1}{\zeta}}-1\right)\right\}\left|\psi'(\varpi)\right|^{q}\right]^{\frac{1}{q}}.$$

Corollary 2.7. *If we set $\zeta = 1$ in Corollary 2.6, we obtain*

$$\left|\frac{\psi(\sigma)+\psi(\varpi)}{2}-\frac{1}{\varpi-\sigma}\int_\sigma^\varpi\psi(\mu)d\mu\right|$$
$$\le\frac{\varpi-\sigma}{2^{2-\frac{1}{q}}}\left(-e+4\sqrt{e}-\frac{7}{2}\right)^{\frac{1}{q}}\left[(e-1)\left(|\psi'(\sigma)|^q+|\psi'(\varpi)|^q\right)\right]^{\frac{1}{q}}.$$

Corollary 2.8. *If we set $\zeta = 2$ in Corollary 2.6, we get*

$$\left|\frac{1}{2}\left[\frac{\psi(\sigma)+\psi(\varpi)}{2}+\psi(\frac{\sigma+\varpi}{2})\right]-\frac{1}{\varpi-\sigma}\int_\sigma^\varpi\psi(\mu)d\mu\right|$$
$$\le\frac{\varpi-\sigma}{2^{4-\frac{1}{q}}}\left(-e+4\sqrt{e}-\frac{7}{2}\right)^{\frac{1}{q}}\left[\left((e-1)+2(\sqrt{e}-1)\right)\left(|\psi'(\sigma)|^q+|\psi'(\varpi)|^q\right)\right]^{\frac{1}{q}}.$$

Theorem 2.9. *Suppose that the mapping $\psi : I \subset \Re \to \Re$ is differentiable on I^0, $\sigma,\varpi \in I^0$ with $\sigma < \varpi$. If $|\psi'|^q$ is exponential type convex on $[\sigma,\varpi]$, then the following inequality*

$$|I_\zeta(\psi,\sigma,\varpi)| \le \sum_{\beth=0}^{\zeta-1}\frac{\varpi-\sigma}{2\zeta^2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{2e-5}{2}\left|\psi'\left(\frac{(\zeta-\beth)\sigma+\beth\varpi}{\zeta}\right)\right|^{q}+\frac{1}{2}\left|\psi'\left(\frac{(\zeta-\beth-1)\sigma+(\beth+1)\varpi}{\zeta}\right)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{1}{2}\left|\psi'\left(\frac{(\zeta-\beth)\sigma+\beth\varpi}{\zeta}\right)\right|^{q}+\frac{2e-5}{2}\left|\psi'\left(\frac{(\zeta-\beth-1)\sigma+(\beth+1)\varpi}{\zeta}\right)\right|^{q}\right)^{\frac{1}{q}}\right] \tag{3}$$

is true, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Applying Lemma 1.8 and Hölder-İscan inequality in Theorem 1.6, we get

$$|I_\zeta(\psi,\sigma,\varpi)|$$
$$\le\sum_{\beth=0}^{\zeta-1}\frac{\varpi-\sigma}{2\zeta^2}\left[\int_0^1\left|(1-2\eta)\psi'\left(\eta\frac{(\zeta-\beth)\sigma+\beth\varpi}{\zeta}+(1-\eta)\frac{(\zeta-\beth-1)\sigma+(\beth+1)\varpi}{\zeta}\right)\right|d\eta\right]$$
$$\le\sum_{\beth=0}^{\zeta-1}\frac{\varpi-\sigma}{2\zeta^2}\left[\left(\int_0^1(1-\eta)|1-2\eta|^pd\eta\right)^{\frac{1}{p}}\right.$$

$$\times\left(\int_0^1(1-\eta)\left|\psi'\left(\eta\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}+(1-\eta)\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^q d\eta\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\eta|1-2\eta|^p d\eta\right)^{\frac{1}{p}}$$
$$\left.\left(\int_0^1\eta\left|\psi'\left(\eta\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}+(1-\eta)\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^q d\eta\right)^{\frac{1}{q}}\right].$$

Using exponential type convexity of $|\psi'|^q$, we get

$$|I_\zeta(\psi,\sigma,\varpi)|\le\sum_{\gimel=0}^{\zeta-1}\frac{\varpi-\sigma}{2\zeta^2}\left[\left(\int_0^1(1-\eta)|1-2\eta|^p d\eta\right)^{\frac{1}{p}}\right.$$
$$\times\left(\int_0^1(1-\eta)\left((e^\eta-1)\left|\psi'\left(\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}\right)\right|^q\right.\right.$$
$$\left.\left.+(e^{1-\eta}-1)\left|\psi'\left(\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^q\right)d\eta\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\eta|1-2\eta|^p d\eta\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\eta\left((e^\eta-1)\left|\psi'\left(\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}\right)\right|^q\right.\right.$$
$$\left.\left.\left.+(e^{1-\eta}-1)\left|\psi'\left(\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^q\right)d\eta\right)^{\frac{1}{q}}\right]$$
$$=\sum_{\gimel=0}^{\zeta-1}\frac{\varpi-\sigma}{2\zeta^2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\int_0^1(1-\eta)(e^\eta-1)\left|\psi'\left(\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}\right)\right|^q d\eta\right.\right.$$
$$\left.+\int_0^1(1-\eta)(e^{1-\eta}-1)\left|\psi'\left(\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^q d\eta\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\eta(e^\eta-1)\left|\psi'\left(\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}\right)\right|^q d\eta\right.$$
$$\left.\left.+\int_0^1\eta(e^{1-\eta}-1)\left|\psi'\left(\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^q d\eta\right)^{\frac{1}{q}}\right]$$
$$=\sum_{\gimel=0}^{\zeta-1}\frac{\varpi-\sigma}{2\zeta^2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{2e-5}{2}\left|\psi'\left(\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}\right)\right|^q+\frac{1}{2}\left|\psi'\left(\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{1}{2}\left|\psi'\left(\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}\right)\right|^q+\frac{2e-5}{2}\left|\psi'\left(\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^q\right)^{\frac{1}{q}}\right].$$

Hence the required inequality is proved. □

Corollary 2.10. *If we apply exponential type convexity of $|\psi'|^q$ in Theorem 2.9 again, it gives*

$$
\begin{aligned}
&|I_\zeta(\psi,\sigma,\varpi)| \leq \sum_{\gimel=0}^{\zeta-1} \frac{\varpi-\sigma}{2\zeta^2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}} \\
&\Bigg[\Bigg\{\left(\frac{2e-5}{2}\left(e^{\frac{\zeta-\gimel}{\zeta}}-1\right)+\frac{1}{2}\left(e^{\frac{\zeta-\gimel-1}{\zeta}}-1\right)\right)\left|\psi'(\sigma)\right|^q \\
&+\left(\frac{2e-5}{2}\left(e^{\frac{\gimel}{\zeta}}-1\right)+\frac{1}{2}\left(e^{\frac{\gimel+1}{\zeta}}-1\right)\right)\left|\psi'(\varpi)\right|^q\Bigg\}^{\frac{1}{q}} \\
&+\Bigg\{\left(\frac{1}{2}\left(e^{\frac{\zeta-\gimel}{\zeta}}-1\right)+\frac{2e-5}{2}\left(e^{\frac{\zeta-\gimel-1}{\zeta}}-1\right)\right)\left|\psi'(\sigma)\right|^q \\
&+\left(\frac{1}{2}\left(e^{\frac{\gimel}{\zeta}}-1\right)+\frac{2e-5}{2}\left(e^{\frac{\gimel+1}{\zeta}}-1\right)\right)\left|\psi'(\varpi)\right|^q\Bigg\}^{\frac{1}{q}}\Bigg].
\end{aligned}
$$

Corollary 2.11. *If we set $\zeta = 1$ in Corollary 2.10, we obtain*

$$
\begin{aligned}
&\left|\frac{\psi(\sigma)+\psi(\varpi)}{2}-\frac{1}{\varpi-\sigma}\int_\sigma^\varpi \psi(\mu)d\mu\right| \leq \frac{\varpi-\sigma}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}(e-1)^{\frac{1}{q}} \\
&\left[\left(\frac{2e-5}{2}\left|\psi'(\sigma)\right|^q+\frac{1}{2}\left|\psi'(\varpi)\right|^q\right)^{\frac{1}{q}}+\left(\frac{1}{2}\left|\psi'(\sigma)\right|^q+\frac{2e-5}{2}\left|\psi'(\varpi)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Corollary 2.12. *If we set $\zeta = 2$ in Corollary 2.10, we get*

$$
\begin{aligned}
&\left|\frac{1}{2}\left[\frac{\psi(\sigma)+\psi(\varpi)}{2}+\psi(\frac{\sigma+\varpi}{2})\right]-\frac{1}{\varpi-\sigma}\int_\sigma^\varpi \psi(\mu)d\mu\right| \leq \frac{\varpi-\sigma}{8}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}} \\
&\Bigg[\Bigg\{\left(\frac{2e-5}{2}(e-1)+(e-2)(\sqrt{e}-1)\right)\left|\psi'(\sigma)\right|^q+ \\
&\left(\frac{1}{2}(e-1)+(e-2)(\sqrt{e}-1)\right)\left|\psi'(\varpi)\right|^q\Bigg\}^{\frac{1}{q}} \\
&+\Bigg\{\left(\frac{1}{2}(e-1)+(e-2)(\sqrt{e}-1)\right)\left|\psi'(\sigma)\right|^q+ \\
&\left(\frac{2e-5}{2}(e-1)+(e-2)(\sqrt{e}-1)\right)\left|\psi'(\varpi)\right|^q\Bigg\}^{\frac{1}{q}}\Bigg].
\end{aligned}
$$

In the upcoming proposition, a proof by comparison is provided to establish the more refined inequality between Theorem 2.1 and Theorem 2.9.

Proposition 2.13. *The inequality* (3) *is the refined form of inequality* (1). *Since the mapping $\kappa : [0,\infty] \to \Re$, $\kappa(\mu) = \mu^\iota$, where $\iota \in (0,1]$, is a concave mapping, so*

$$
\frac{\delta^\iota+\xi^\iota}{2}=\frac{\kappa(\delta)+\kappa(\xi)}{2}\leq \kappa\left(\frac{\delta+\xi}{2}\right)=\left(\frac{\delta+\xi}{2}\right)^\iota, \tag{4}
$$

for all $\delta, \xi \geq 0$.

By substituting the following values in inequality (4)

$$
\delta=\frac{2e-5}{2}\left|\psi'\left(\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}\right)\right|^q+\frac{1}{2}\left|\psi'\left(\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^q,
$$

$$\xi = \frac{1}{2}\left|\psi'\left(\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}\right)\right|^{q} + \frac{2e-5}{2}\left|\psi'\left(\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^{q}$$

and $\iota = \frac{1}{q}$, *it gives*

$$\begin{aligned}
&\frac{1}{2}\left[\frac{2e-5}{2}\left|\psi'\left(\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}\right)\right|^{q} + \frac{1}{2}\left|\psi'\left(\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^{q}\right]^{\frac{1}{q}}\\
&+\frac{1}{2}\left[\frac{1}{2}\left|\psi'\left(\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}\right)\right|^{q} + \frac{2e-5}{2}\left|\psi'\left(\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^{q}\right]^{\frac{1}{q}}\\
&\leq (e-2)^{\frac{1}{q}}\left[\frac{\left|\psi'\left(\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}\right)\right|^{q} + \left|\psi'\left(\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^{q}}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{\gimel=0}^{\zeta-1}\frac{\varpi-\sigma}{2\zeta^{2}}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\\
&\times\left[\left(\frac{2e-5}{2}\left|\psi'\left(\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}\right)\right|^{q} + \frac{1}{2}\left|\psi'\left(\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^{q}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{1}{2}\left|\psi'\left(\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}\right)\right|^{q} + \frac{2e-5}{2}\left|\psi'\left(\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^{q}\right)^{\frac{1}{q}}\right]\\
&\leq\sum_{\gimel=0}^{\zeta-1}\frac{\varpi-\sigma}{2\zeta^{2}}\left(\frac{1}{1+p}\right)^{\frac{1}{p}}(e-2)^{\frac{1}{q}}\\
&\times\left[\left|\psi'\left(\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}\right)\right|^{q} + \left|\psi'\left(\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^{q}\right]^{\frac{1}{q}}.
\end{aligned}$$

Theorem 2.14. *Suppose that the mapping* $\psi : I \subset \mathbb{R} \to \mathbb{R}$ *is differentiable on* I^0, $\sigma, \varpi \in I^0$ *with* $\sigma < \varpi$. *If* $|\psi'|^q$ *is exponential type convex mapping on* $[\sigma, \varpi]$, *then for* $q \geq 1$ *the following inequality*

$$\begin{aligned}
&|I_\zeta(\psi,\sigma,\varpi)| \leq \sum_{\gimel=0}^{\zeta-1}\frac{\varpi-\sigma}{2\zeta^{2}}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\\
&\left[\left\{\frac{-12e+40\sqrt{e}-33}{4}\left|\psi'\left(\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}\right)\right|^{q}\right.\right.\\
&\left.+\frac{8e-24\sqrt{e}+19}{4}\left|\psi'\left(\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^{q}\right\}^{\frac{1}{q}}\\
&+\left\{\frac{8e-24\sqrt{e}+19}{4}\left|\psi'\left(\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}\right)\right|^{q}\right.\\
&\left.\left.+\frac{-12e+40\sqrt{e}-33}{4}\left|\psi'\left(\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^{q}\right\}^{\frac{1}{q}}\right]. \qquad (5)
\end{aligned}$$

is satisfied.

Proof. Applying Lemma 1.8, we get

$$|I_\zeta(\psi,\sigma,\varpi)|$$
$$\leq \sum_{\gimel=0}^{\zeta-1}\frac{\varpi-\sigma}{2\zeta^2}\left[\int_0^1\left|(1-2\eta)\psi'\left(\eta\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}+(1-\eta)\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|d\eta\right].$$

Applying improved power-mean inequality, we arrive at

$$|I_\zeta(\psi,\sigma,\varpi)|\leq \sum_{\gimel=0}^{\zeta-1}\frac{\varpi-\sigma}{2\zeta^2}\left[\left(\int_0^1(1-\eta)|1-2\eta|d\eta\right)^{1-\frac{1}{q}}\right.$$
$$\times\left\{\int_0^1(1-\eta)\left|1-2\eta\right|\left|\psi'\left(\eta\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}+(1-\eta)\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^q d\eta\right\}^{\frac{1}{q}}$$
$$+\left(\int_0^1\eta|1-2\eta|d\eta\right)^{1-\frac{1}{q}}$$
$$\left.\times\left\{\int_0^1\eta\left|1-2\eta\right|\left|\psi'\left(\eta\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}+(1-\eta)\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^q d\eta\right\}^{\frac{1}{q}}\right]$$
$$=\sum_{\gimel=0}^{\zeta-1}\frac{\varpi-\sigma}{2\zeta^2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}$$
$$\left[\left\{\int_0^1(1-\eta)\left|1-2\eta\right|\left|\psi'\left(\eta\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}+(1-\eta)\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^q d\eta\right\}^{\frac{1}{q}}\right.$$
$$\left.+\left\{\int_0^1\eta\left|1-2\eta\right|\left|\psi'\left(\eta\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}+(1-\eta)\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^q d\eta\right\}^{\frac{1}{q}}\right].$$

Since $|\psi'|^q$ is exponential type convex, so it gives

$$|I_\zeta(\psi,\sigma,\varpi)|\leq \sum_{\gimel=0}^{\zeta-1}\frac{\varpi-\sigma}{2\zeta^2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}$$
$$\left[\left\{\int_0^1(1-\eta)\left|1-2\eta\right|\right.\right.$$
$$\left.\left((e^\eta-1)\left|\psi'\left(\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}\right)\right|^q+(e^{1-\eta}-1)\left|\psi'\left(\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^q\right)d\eta\right\}^{\frac{1}{q}}$$
$$+\left\{\int_0^1\eta\left|1-2\eta\right|\right.$$
$$\left.\left.\left((e^\eta-1)\left|\psi'\left(\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}\right)\right|^q+(e^{1-\eta}-1)\left|\psi'\left(\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^q\right)d\eta\right\}^{\frac{1}{q}}\right]$$
$$=\sum_{\gimel=0}^{\zeta-1}\frac{\varpi-\sigma}{2\zeta^2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left[\left\{\int_0^1(1-\eta)\left|1-2\eta\right|(e^\eta-1)\left|\psi'\left(\frac{(\zeta-\gimel)\sigma+\gimel\varpi}{\zeta}\right)\right|^q d\eta\right.\right.$$
$$\left.+\int_0^1(1-\eta)\left|1-2\eta\right|(e^{1-\eta}-1)\left|\psi'\left(\frac{(\zeta-\gimel-1)\sigma+(\gimel+1)\varpi}{\zeta}\right)\right|^q d\eta\right\}^{\frac{1}{q}}$$

$$+\left\{\int_0^1 \eta\left|1-2\eta\right|(e^{\eta}-1)\left|\psi'\left(\frac{(\zeta-\daleth)\sigma+\daleth\varpi}{\zeta}\right)\right|^q d\eta\right.$$

$$\left.+\int_0^1 \eta\left|1-2\eta\right|(e^{1-\eta}-1)\left|\psi'\left(\frac{(\zeta-\daleth-1)\sigma+(\daleth+1)\varpi}{\zeta}\right)\right|^q d\eta\right\}^{\frac{1}{q}}\Bigg]$$

$$=\sum_{\daleth=0}^{\zeta-1}\frac{\varpi-\sigma}{2\zeta^2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}$$

$$\Bigg[\left\{\frac{-12e+40\sqrt{e}-33}{4}\left|\psi'\left(\frac{(\zeta-\daleth)\sigma+\daleth\varpi}{\zeta}\right)\right|^q\right.$$

$$\left.+\frac{8e-24\sqrt{e}+19}{4}\left|\psi'\left(\frac{(\zeta-\daleth-1)\sigma+(\daleth+1)\varpi}{\zeta}\right)\right|^q\right\}^{\frac{1}{q}}$$

$$+\left\{\frac{8e-24\sqrt{e}+19}{4}\left|\psi'\left(\frac{(\zeta-\daleth)\sigma+\daleth\varpi}{\zeta}\right)\right|^q\right.$$

$$\left.+\frac{-12e+40\sqrt{e}-33}{4}\left|\psi'\left(\frac{(\zeta-\daleth-1)\sigma+(\daleth+1)\varpi}{\zeta}\right)\right|^q\right\}^{\frac{1}{q}}\Bigg].$$

Hence the required result is obtained. □

Corollary 2.15. *If we again apply exponential type convexity on Theorem 2.14, it gives*

$$|I_\zeta(\psi,\sigma,\varpi)|\leq\sum_{\daleth=0}^{\zeta-1}\frac{\varpi-\sigma}{2\zeta^2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}$$

$$\Bigg[\left\{\left(\frac{-12e+40\sqrt{e}-33}{4}\left(e^{\frac{\zeta-\daleth}{\zeta}}-1\right)+\frac{8e-24\sqrt{e}+19}{4}\left(e^{\frac{\zeta-\daleth-1}{\zeta}}-1\right)\right)\left|\psi'(\sigma)\right|^q\right.$$

$$\left.+\left(\frac{-12e+40\sqrt{e}-33}{4}\left(e^{\frac{\daleth}{\zeta}}-1\right)+\frac{8e-24\sqrt{e}+19}{4}\left(e^{\frac{\daleth+1}{\zeta}}-1\right)\right)\left|\psi'(\varpi)\right|^q\right\}^{\frac{1}{q}}$$

$$+\left\{\left(\frac{8e-24\sqrt{e}+19}{4}\left(e^{\frac{\zeta-\daleth}{\zeta}}-1\right)+\frac{-12e+40\sqrt{e}-33}{4}\left(e^{\frac{\zeta-\daleth-1}{\zeta}}-1\right)\right)\left|\psi'(\sigma)\right|^q\right.$$

$$\left.+\left(\frac{8e-24\sqrt{e}+19}{4}\left(e^{\frac{\daleth}{\zeta}}-1\right)+\frac{-12e+40\sqrt{e}-33}{4}\left(e^{\frac{\daleth+1}{\zeta}}-1\right)\right)\left|\psi'(\varpi)\right|^q\right\}^{\frac{1}{q}}\Bigg].$$

Corollary 2.16. *If we set $\zeta=1$ in Corollary 2.15, we get*

$$\left|\frac{\psi(\sigma)+\psi(\varpi)}{2}-\frac{1}{\varpi-\sigma}\int_\sigma^\varpi\psi(\mu)d\mu\right|\leq\frac{\varpi-\sigma}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}(e-1)^{\frac{1}{q}}$$

$$\Bigg[\left(\frac{-12e+40\sqrt{e}-33}{4}\left|\psi'(\sigma)\right|^q+\frac{8e-24\sqrt{e}+19}{4}\left|\psi'(\varpi)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{8e-24\sqrt{e}+19}{4}\left|\psi'(\sigma)\right|^q+\frac{-12e+40\sqrt{e}-33}{4}\left|\psi'(\varpi)\right|^q\right)^{\frac{1}{q}}\Bigg].$$

Corollary 2.17. *If we set $\zeta=2$ in Corollary 2.15, we get*

$$\left|\frac{1}{2}\left[\frac{\psi(\sigma)+\psi(\varpi)}{2}+\psi(\frac{\sigma+\varpi}{2})\right]-\frac{1}{\varpi-\sigma}\int_\sigma^\varpi\psi(\mu)d\mu\right|\leq\frac{\varpi-\sigma}{8}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}$$

$$\left[\left\{\left(\frac{-12e+40\sqrt{e}-33}{4}(e-1)+\frac{-2e+8\sqrt{e}-7}{2}(\sqrt{e}-1)\right)\left|\psi'(\sigma)\right|^q\right.\right.$$
$$\left.+\left(\frac{8e-24\sqrt{e}+19}{4}(e-1)+\frac{-2e+8\sqrt{e}-7}{2}(\sqrt{e}-1)\right)\left|\psi'(\varpi)\right|^q\right\}^{\frac{1}{q}}$$
$$+\left\{\left(\frac{8e-24\sqrt{e}+19}{4}(e-1)+\frac{-2e+8\sqrt{e}-7}{2}(\sqrt{e}-1)\right)\left|\psi'(\sigma)\right|^q\right.$$
$$\left.\left.+\left(\frac{-12e+40\sqrt{e}-33}{4}(e-1)+\frac{-2e+8\sqrt{e}-7}{2}(\sqrt{e}-1)\right)\left|\psi'(\varpi)\right|^q\right\}^{\frac{1}{q}}\right].$$

Proposition 2.18. *The inequality* (5) *is the refined form of the inequality* (2). *It can be proved similarly as Proposition 2.13.*

3. Applications to Different Means

Mean is the average number used to summarize the numerical data to a single value. They are widely used in mathematics, statistics, economics and other calculations. In this section, we utilize the new inequalities of Section 2 to obtain the relations for the following means.

Arithmetic mean:
Let $\sigma, \varpi \in \Re$,

$$A = A(\sigma,\varpi) = \frac{\sigma+\varpi}{2}, \quad \sigma,\varpi \geq 0.$$

r-Logarithmic mean:
Let $\sigma, \varpi \in \Re$,

$$L_r(\sigma,\varpi) = \begin{cases} \sigma, & \text{if } \sigma = \varpi; \\ \left(\frac{\varpi^{r+1}-\sigma^{r+1}}{(r+1)(\varpi-\sigma)}\right)^{\frac{1}{r}}, & \text{if } \sigma \neq \varpi, \end{cases} \quad r \in \Re\backslash\{-1,0\},\ \sigma,\varpi > 0.$$

Proposition 3.1. *Suppose that $\sigma, \varpi \in \Re$, $0 < \sigma < \varpi$, $n \in N$ where $n \geq 2$. So for all $q > 1$, the inequality*

$$\left|\sum_{\beth=0}^{\zeta-1}\frac{1}{\zeta}A\left(\left(\frac{(\zeta-\beth)\sigma+\beth\varpi}{\zeta}\right)^n,\left(\frac{(\zeta-\beth-1)\sigma+(\beth+1)\varpi}{\zeta}\right)^n\right)-L_n^n(\sigma,\varpi)\right|$$
$$\leq \sum_{\beth=0}^{\zeta-1}\frac{n(\varpi-\sigma)}{2\zeta^2}\left(\frac{1}{1+p}\right)^{\frac{1}{p}}(e-2)^{\frac{1}{q}}$$
$$\left[\left\{\left(e^{\frac{\zeta-\beth}{\zeta}}-1\right)+\left(e^{\frac{\zeta-\beth-1}{\zeta}}-1\right)\right\}\sigma^{(n-1)q}+\left\{\left(e^{\frac{\beth}{\zeta}}-1\right)+\left(e^{\frac{\beth+1}{\zeta}}-1\right)\right\}\varpi^{(n-1)q}\right]^{\frac{1}{q}}$$

is satisfied.

Proof. If we substitute $\psi(\mu) = \mu^n$ where $\mu \in [\sigma,\varpi]$, $n \in N$ and $n \geq 2$ in Corollary 2.2, the proof follows. □

Proposition 3.2. *Suppose that $\sigma, \varpi \in \Re$, $0 < \sigma < \varpi$, $n \in N$, where $n \geq 2$. So for all $q > 1$, the inequality*

$$\left|\sum_{\beth=0}^{\zeta-1}\frac{1}{\zeta}A\left(\left(\frac{(\zeta-\beth)\sigma+\beth\varpi}{\zeta}\right)^n,\left(\frac{(\zeta-\beth-1)\sigma+(\beth+1)\varpi}{\zeta}\right)^n\right)-L_n^n(\sigma,\varpi)\right|$$
$$\leq \sum_{\beth=0}^{\zeta-1}\frac{n(\varpi-\sigma)}{2^{2-\frac{1}{q}}\zeta^2}\left(-e+4\sqrt{e}-\frac{7}{2}\right)^{\frac{1}{q}}$$

$$\left[\left\{\left(e^{\frac{\zeta-\jmath}{\zeta}}-1\right)+\left(e^{\frac{\zeta-\jmath-1}{\zeta}}-1\right)\right\}\sigma^{(n-1)q} + \left\{\left(e^{\frac{\jmath}{\zeta}}-1\right)+\left(e^{\frac{\jmath+1}{\zeta}}-1\right)\right\}\varpi^{(n-1)q}\right]^{\frac{1}{q}}$$

is satisfied.

Proof. If we substitute $\psi(\mu) = \mu^n$ where $\mu \in [\sigma, \varpi]$, $n \in N$ and $n \geq 2$ in Corollary 2.6, the proof follows. □

Proposition 3.3. *Let* $\sigma, \varpi \in \Re$, $0 < \sigma < \varpi$, $n \in N$ *where* $n \geq 2$. *So for all* $q > 1$, *the inequality*

$$\left|\sum_{\jmath=0}^{\zeta-1}\frac{1}{\zeta}A\left(\left(\frac{(\zeta-\jmath)\sigma+\jmath\varpi}{\zeta}\right)^n,\left(\frac{(\zeta-\jmath-1)\sigma+(\jmath+1)\varpi}{\zeta}\right)^n\right)-L_n^n(\sigma,\varpi)\right|$$
$$\leq \sum_{\jmath=0}^{\zeta-1}\frac{n(\varpi-\sigma)}{2\zeta^2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}$$
$$\left[\left\{\left(\frac{2e-5}{2}\left(e^{\frac{\zeta-\jmath}{\zeta}}-1\right)+\frac{1}{2}\left(e^{\frac{\zeta-\jmath-1}{\zeta}}-1\right)\right)\sigma^{(n-1)q} + \left(\frac{2e-5}{2}\left(e^{\frac{\jmath}{\zeta}}-1\right)+\frac{1}{2}\left(e^{\frac{\jmath+1}{\zeta}}-1\right)\right)\varpi^{(n-1)q}\right\}^{\frac{1}{q}}\right.$$
$$\left. + \left\{\left(\frac{1}{2}\left(e^{\frac{\zeta-\jmath}{\zeta}}-1\right)+\frac{2e-5}{2}\left(e^{\frac{\zeta-\jmath-1}{\zeta}}-1\right)\right)\sigma^{(n-1)q} + \left(\frac{1}{2}\left(e^{\frac{\jmath}{\zeta}}-1\right)+\frac{2e-5}{2}\left(e^{\frac{\jmath+1}{\zeta}}-1\right)\right)\varpi^{(n-1)q}\right\}^{\frac{1}{q}}\right]$$

holds.

Proof. If we substitute $\psi(\mu) = \mu^n$ where $\mu \in [\sigma, \varpi]$, $n \in N$ and $n \geq 2$ in Corollary 2.10, the proof follows. □

4. Conclusions

In this paper, we have established generalized inequalities of the Hermite-Hadamard type in terms of exponential-type convex functions. We compared the results obtained using Hölder's inequality and power-mean inequality with the consequences established via Hölder-Iscan inequality and improved power-mean inequality. By comparing the outcomes, we concluded that the inequalities achieved by the Hölder-Iscan inequality and improved power-mean inequality are more refined than those explored using Hölder's inequality and power-mean inequality. Theorem 2.9 refines the result of Theorem 2.1 and Theorem 2.14 improves upon Theorem 2.5. In short, we concluded that H"older-.Işan inequality can yield more refined results than H"older's inequality and the improved power mean inequality can offer better refinements of Hermite-Hadamard inequalities than the power mean inequality. We also derived special cases of these inequalities. Finally, we demonstrated the relation of these new results to some special means. Furthermore, these inequalities can prove useful in addressing other mathematical problems. Moreover, we hope that these new results will contribute to further refinements using different convex functions.

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References

[1] A. Cambini, L. Martein, *Generalized convexity and optimization: Theory and applications*, Springer, Berlin, 2008.
[2] A. Hasanov, Convexity argument for monotone potential operators and its application, *Nonlinear Anal. Theory Methods Appl.,* **41** (7) (2000), 907-920.
[3] A.Y. Khrennikov, Epr-bohm experiment and Bell's inequality: Quantum physics meets probability theory, *Theor. Math. Phys.,* **157** (2008), 1448-1460.
[4] G.M. Borzadaran, D.N. Shanbhag, Further results based on Chernoff-type inequalities, *Stat. Probab. Lett.,* **39** (2) (1998), 109-117.
[5] H.D. Raedt, K. Hess, K. Michielsen, Extended Boole-Bell inequalities applicable to quantum theory, *J. Comput. Theor. Nanosci.,* **8** (6) (2011), 1011-1039.
[6] H. Kadakal, On refinements of some integral inequalities using improved power-mean integral inequalities, *Numer. Methods Partial Differential Equations,* **36** (6) (2020), 1555-1565.
[7] I. İscan, New refinements for integral and sum forms of HÖlder inequality, *J. Inequal. Appl.,* **2019** (2019), 204.
[8] I. İscan, T. Toplu, F. Yetgin, Some new inequalities on generalization of Hermite-Hadamard and Bullen type inequalities, applications to trapezoidal and midpoint formula, *Kragujev. J. Math.,* **45** (4) (2021), 647-657.
[9] I. Yesilce, G. Adilov, Hermite-Hadamard inequalities for L(j)-convex functions and S(j)-convex functions, *Malaya J. Mat.,* **3** (03) (2015), 346-359.
[10] J. Lu, S. Steinerberger, A dimension-free Hermite-Hadamard inequality via gradient estimates for the torsion function, *Proceedings of the American Mathematical Society,***148** (2) (2020), 673-679.
[11] J. Wang, C. Zhu, Y. Zhou, New generalized Hermite-Hadamard type inequalities and applications to special means, *J. Inequal. Appl.,* **2013** (1) (2013), 1-15.
[12] K.R. Mecke, *Additivity, convexity and beyond. applications of Minkowski functionals in statistical physics. In Statistical Physics and Spatial Statistics: The art of analyzing and modeling spatial structures and pattern formation*, Springer, Berlin, 2000.
[13] L. Asimow, *Convexity theory and its applications in functional analysis*, Elsevier, Netherlands, 2014.
[14] L. Wu, A new modified logarithmic Sobolev inequality for Poisson point processes and several applications, *Probab. Theory. Related. Fields,* **118** (3) (2000), 427-438.
[15] M.B. Khan, P.O. Mohammed, M.A. Noor, Y.S. Hamed, New Hermite-Hadamard inequalities in fuzzy-interval fractional calculus and related inequalities, *Symmetry,* **13** (4) (2021), 673.
[16] M. Hirsch, W. Quapp, Reaction pathways and convexity of the potential energy surface: application of Newton trajectories, *J. Math. Chem.,* **36** (2004), 307-340.
[17] M. Kadakal, I.İscan, Exponential type convexity and some related inequalities, *J. Inequal. Appl.,* **2020** (2020), 1-9.
[18] M. Kadakal, I. İscan, H. Kadakal, K. Bekar, On improvements of some integral inequalities, *Honam Math. J.,* **43** (3) (2021), 441-452.
[19] M.K. Wang, Y.M. Chu, Y.F. Qiu, S. L. Qiu, An optimal power mean inequality for the complete elliptic integrals, *Appl. Math. Lett.,* **24** (6) (2011), 887-890.
[20] M. Vivas-Cortez, M. Mukhtar, I. Shabbir, M. Samraiz, M. Yaqoob, On fractional integral inequalities of Riemann type for composite convex functions and applications, *Fractal fract.,* **7** (5) (2023), 345.
[21] P.D. Smallwood, An introduction to risk sensitivity: The use of Jensen's inequality to clarify evolutionary arguments of adaptation and constraint, *Am. Zool.,* **36** (4) (1996), 392-401.
[22] P.D. Tao, L.H. An, Convex analysis approach to DC programming: theory, algorithms and applications, *Acta Math. Vietnam.,* **22** (1) (1997), 289-355.
[23] P.S. Bullen, *Handbook of means and their inequalities*, Springer Science & Business Media, Berlin, 2003.
[24] R.A. Horn, R. Mathia, Cauchy-Schwarz inequalities associated with positive semidefinite matrices, *Linear Algebra Appl.,* **142** (1990), 63-82.
[25] S.G. Krantz, *Convexity in complex analysis. Several complex variables and complex geometry. proceedings of Symposa in Pure Mathematics*, AMS, Providence, 1991.
[26] S.I. Ohta, Uniform convexity and smoothness and their applications in Finsler geometry, *Math. Ann.,* **343** (2009), 669-699.
[27] S. Steinerberger, The Hermite-Hadamard inequality in higher dimensions, *J. Geom. Anal.,* **30** (1) (2020), 466-483.
[28] S. Wu, On the weighted generalization of the Hermite-Hadamard inequality and its applications, *Rocky Mountain J. Math.,* **2009** (2009), 1741-1749.
[29] W. Fenchel, *Convexity through the ages. In Convexity and its Applications*, Birkhäuser, Basel, 1983.
[30] X. Shen, S. Diamond, M. Udell, Y. Gu, S. Boyd, Disciplined multi-convex programming, *In 2017 29th Chinese control and decision conference (CCDC),* **2017** (2017), 895-900.