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Hyers–Ulam–Rassias stability of impulsive Fredholm integral equations on finite intervals

Rahim Shah[a,*], Natasha Irshad[a], Hajra Imtiaz Abbasi[a]

[a] *Department of Mathematics, Kohsar University Murree, Murree, Pakistan*

Abstract. The main aim of this paper is to establish the Hyers–Ulam–Rassias and Hyers–Ulam stability of certain homogeneous and non–homogeneous impulsive Fredholm integral equations by using a fixed–point method. Both Hyers–Ulam–Rassias stability and Hyers–Ulam stability are obtained for such a class of Fredholm integral equations when considered on a finite interval. Finally four examples are presented to support the usability of our results.

1. Introduction

A functional equation is said to be stable if there is an exact solution close to each approximative solution. A problem regarding the stability of homomorphisms was mentioned by Ulam [35] in 1940. The first answer was then found by Hyers in [5], which motivated the study of the stability problems of functional equations.

Thereafter, this type of stability is called the Hyers–Ulam stability. In 1978, Rassias [42] proved the existence of unique linear mappings near approximate additive mappings that provide a generalization of the Hyers result. By using the notion of Cădariu and Radu [15], Jung [36] applied the fixed–point method to the investigation of the Volterra integral equation. They verified that if a continuous function $y : I \to \mathbb{C}$ satisfies the Volterra integral equation of the second kind such that

$$\left| y(x) - \int_c^x f(t, y(t))\, dt \right| \leq \phi(x)$$

for all $x \in I$, then there exists a unique continuous function $y_0 : I \to \mathbb{C}$ and a constant M such that

$$y_0(x) = \int_c^x f(t, y_0(t))\, dt \quad \text{and} \quad \left| y(x) - y_0(x) \right| \leq M\phi(x) \text{ for all } x \in I.$$

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* Corresponding author: Rahim Shah
Email addresses: rahimshah@kum.edu.pk (Rahim Shah), natashairshad24@gmail.com (Natasha Irshad), hajraabbasi168@gmail.com (Hajra Imtiaz Abbasi)
ORCID iDs: https://orcid.org/0009-0001-9044-5470 (Rahim Shah), https://orcid.org/0009-0008-8166-6520 (Natasha Irshad)

Jung's work [36] has made a valuable contribution to the literature, laying the groundwork for further research on Ulam stabilities of integral equations.

In a subsequent paper, Jung [37] investigated an integral equation in which the unknown function depends on two independent variables. Through this work, Jung established a strong connection between the studied integral equation and the wave equation, thus confirming the generalized Hyers–Ulam stability of the equation, primarily using the fixed–point method.

Jung's book [38] aimed to provide a thorough overview of the stability theory of functional equations. In this text, Jung [38] examined and discussed the stability of various types of functional equations.

In 2009, Castro and Ramos [17] addressed the nonlinear Volterra integral equation given below:

$$y(x) = \int_a^x f(x, t, y(t))\, dt.$$

In [17], they established the Hyers–Ulam and Hyers–Ulam–Rassias stabilities of this integral equation for both finite and infinite interval cases.

In 2011, Akkouchi [26] analyzed the Volterra integral equation represented by

$$y(x) = h(x) + \lambda \int_a^x G(x, t, y(t))\, dt.$$

Utilizing the fixed–point method, Akkouchi derived new results on the Hyers–Ulam and Hyers–Ulam–Rassias stabilities of this Volterra integral equation within Banach spaces.

In 2010, Castro and Ramos [18] examined the Hyers–Ulam and Hyers–Ulam–Rassias stabilities of delay Volterra integral equations using the fixed–point method:

$$y(x) = \int_a^x f(x, t, y(t), y(\alpha(t)))\, dt.$$

In 2013, Castro and Guerra [19] studied a nonlinear Volterra integral equation incorporating a variable delay:

$$y(x) = g(x) + \Psi\left(\int_a^x k(x, t, y(t), y(\alpha(t)))\, dt\right).$$

In [19], they addressed the Hyers–Ulam–Rassias stability of this Volterra integral equation, establishing stability conditions via the Banach fixed–point theorem within a suitable complete metric space, employing the Bielecki metric. Additionally, [19] includes several examples to illustrate their findings.

Janfada and Sadeghi [27], along with Öğrekçi et al. [39], explored the Hyers–Ulam and Hyers–Ulam–Rassias stabilities of the Volterra integral equation given by

$$y(x) = g(x, y(x)) + \int_0^x K(x, t, y(t))\, dt$$

utilizing the fixed–point method.

Nonlinear impulsive differential theory, integral equations, and inclusions have gained significance in some mathematical models of real processes and phenomena examined in the fields of economics, population dynamics, physics, chemical technology, and biotechnology (see [6, 7, 11, 12, 21–23, 41, 45]). In 1993, Guo [4] established some existence theorems of external solutions for nonlinear impulsive Volterra equations on a finite interval with a finite number of moments of impulse effect in Banach spaces, and offered some applications to initial value problems for the first–order impulsive differential equations in Banach spaces. Seeing that many problems in applied mathematics lead to the study of systems of differential or integral equations, the existence of solutions for system of nonlinear impulsive Volterra integral equations on the infinite interval $\mathbb{R}^+$ with an infinite number of moments of impulse effect in Banach spaces is studied.

In recent years, the study of Hyers–Ulam stability of integral equations has gained attention. This concept is particularly valuable in applications such as optimization, numerical analysis, biology, and economics, where finding exact solutions can be challenging. Notably, in 2015, L. Hua et al. [16] explored the Hyers–Ulam stability of specific types of Fredholm integral equations, while Z. Gu and J. Huang [44] investigated the Hyers–Ulam stability of Fredholm integral equations in the same year. The concept of Hyers–Ulam stability has also been applied in various contexts involving differential and integral equations. For further recent studies, see [2, 17, 20, 24, 25, 28, 29, 31–33, 40] (and references therein). The main purpose of this paper is to examine the Hyers–Ulam–Rassias stability and the Hyers–Ulam stability of certain impulsive Fredholm integral equations.

Volterra and Fredholm integral equations play a significant role in various fields including physics, engineering, biology, and economics due to their ability to model complex dynamic systems (see, e.g., earlier studies [1, 3, 8, 9, 13, 14, 30, 34]). Volterra integral equations are often employed to describe systems with memory, allowing them to capture the influence of past states on present behavior. On the other hand, Fredholm integral equations are particularly useful in scenarios with fixed limits of integration, providing solutions for problems involving boundary conditions or interactions over a specified interval. The incorporation of impulsive effects into these frameworks leads to impulsive Fredholm integral equations, which address systems subject to sudden changes or discontinuities at specific moments in time. This extension is particularly relevant in applications such as control theory, where instantaneous alterations in inputs can drastically impact system dynamics, making the study of impulsive Fredholm equations crucial for developing accurate models that reflect both continuous and discrete changes in real–world phenomena.

Motivated by the above ideas, our foremost aim is to study the Hyers–Ulam–Rassias and the Hyers–Ulam stability of the homogeneous impulsive Fredholm integral equation

$$y(x) = \lambda \int_a^b K(x,t)y(t)\,dt + \sum_{a<x_k<b} I_k\Big(y(x_k^-)\Big), \tag{1}$$

and the non–homogeneous impulsive Fredholm integral equation

$$y(x) = x + \lambda \int_a^b K(x,t)y(t)\,dt + \sum_{a<x_k<b} I_k\Big(y(x_k^-)\Big), \tag{2}$$

for all $x, t \in I = [a, b]$, where, for starting, a and b are fixed real numbers, $K : I \times I \to \mathbb{C}$ be a continuous function, $I_k : \mathbb{C} \to \mathbb{C}$, $k = 1, 2, \ldots, m$, $y(x_k^-)$ represents the left limit of $y(x)$ at $x = x_k$ and λ be positive constant.

2. Basic concepts and some preliminary results

This section contains the notations, definitions, and some basic concepts from the literature that will be used in the sequel.

For a nonempty set X, we introduce the definition of a generalized metric on X as follows:

Definition 2.1 ([43]). *A mapping $d : X \times X \to [0, \infty]$ is called a generalized metric on a set X if and only if d satisfies the following conditions:*
(C_1) *$d(x, y) = 0$ if and only if $x = y$;*
(C_2) *$d(x, y) = d(y, x)$ for all $x, y \in X$;*
(C_3) *$d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.*

Next, we recall the main result of Diaz and Margolis [10], which will help us prove our main results.

Theorem 2.2 ([10]). *Let (X, d) be a generalized complete metric space. Assume that $T : X \to X$ is a strictly contractive operator with $L < 1$, where L is a Lipschitz constant. If there exists a nonnegative integer k such that*

$d(T^{k+1}x, T^k x) < \infty$ for some $x \in X$, then the following are true:
(a) The sequence $\{T^n x\}$ converges to a fixed point x^* of T;
(b) x^* is the unique fixed point of T in

$$X^* = \left\{ y \in X \mid d(T^k x, y) < \infty \right\};$$

(c) If $y \in X^*$, then

$$d(y, x^*) \leq \frac{1}{1-L} d(Ty, y).$$

Now, we give the definitions of Hyers–Ulam–Rassias and Hyers–Ulam stability of certain impulsive Fredholm integral equations (1) and (2).

Definition 2.3. *If for each function $y(x)$ satisfying*

$$\left| y(x) - \lambda \int_a^b K(x,t) y(t)\, dt - \sum_{a<x_k<b} I_k(y(x_k^-)) \right| \leq \phi(x),$$

where $\phi(x) \geq 0$ for all $x \in I$, there exists a solution $y_0(x)$ of the homogeneous impulsive Fredholm integral equation (1) and a constant $M > 0$ with

$$\left| y(x) - y_0(x) \right| \leq M\, \phi(x),$$

for all $x \in I$, where M is independent of $y(x)$ and $y_0(x)$, then we say that the homogeneous impulsive Fredholm integral equation (1) has the Hyers–Ulam–Rassias stability. If $\phi(x)$ is a constant function in the above inequalities, we say that the homogeneous impulsive Fredholm integral equation (1) has the Hyers–Ulam stability.

Definition 2.4. *If for each function $y(x)$ satisfying*

$$\left| y(x) - x - \lambda \int_a^b K(x,t) y(t)\, dt - \sum_{a<x_k<b} I_k(y(x_k^-)) \right| \leq \phi(x),$$

where $\phi(x) \geq 0$ for all $x \in I$, there exists a solution $y_0(x)$ of the non–homogeneous impulsive Fredholm integral equation (2) and a constant $M > 0$ with

$$\left| y(x) - y_0(x) \right| \leq M\, \phi(x),$$

for all $x \in I$, where M is independent of $y(x)$ and $y_0(x)$, then we say that the non–homogeneous impulsive Fredholm integral equation (2) has the Hyers–Ulam–Rassias stability. If $\phi(x)$ is a constant function in the above inequalities, we say that the non–homogeneous impulsive Fredholm integral equation (2) has the Hyers–Ulam stability.

In this paper, by using the idea of Cădariu and Radu [15], we shall study the Hyers–Ulam–Rassias and the Hyers–Ulam stability of the homogeneous impulsive Fredholm integral equation (1) and non–homogeneous impulsive Fredholm integral equation (2).

3. Stability results of homogeneous impulsive Fredholm integral equation

In this section, by using the idea of C*ădariu* and Radu [15], we will prove the Hyers–Ulam and Hyers–Ulam–Rassias stability of homogeneous impulsive Fredholm integral equation (1).

3.1. Hyers–Ulam–Rassias stability

In this subsection, we will prove the Hyers–Ulam–Rassias stability of homogeneous impulsive Fredholm integral equation (1).

Theorem 3.1. *Suppose $I = [a, b]$ is given for fixed real numbers a, b with $a < b$ and let M, L_1, L_2 and λ be positive constants with $0 < ML_1\lambda + L_2 < 1$. Let $K : I \times I \to \mathbb{C}$ be a continuous function which satisfies the Lipschitz condition*

$$\left|K(x,t)y_1 - K(x,t)y_2\right| \leq L_1\left|y_1 - y_2\right| \tag{3}$$

for any $x, t \in I$ and $y_1, y_2 \in \mathbb{C}$.
Moreover, $I_k : \mathbb{C} \to \mathbb{C}$ and there exists a constant L_2 such that

$$\left|I_k(y_1) - I_k(y_2)\right| \leq L_2\left|y_1 - y_2\right| \tag{4}$$

for all $y_1, y_2 \in \mathbb{C}$.

Let $y : I \to \mathbb{C}$ be a continuous function such that

$$\left|y(x) - \lambda\int_a^b K(x,t)y(t)dt - \sum_{a<x_k<b} I_k(y(x_k^-))\right| \leq \phi(x) \tag{5}$$

for all $x \in I$, where $I_k : \mathbb{C} \to \mathbb{C}$, $k = 1, 2, ..., m$, $y(x_k^-)$ represents the left limit of $y(x)$ at $x = x_k$, and $\phi : I \to (0, \infty)$ is a continuous function with,

$$\left|\int_a^b \phi(t)dt\right| \leq N\phi(x) \tag{6}$$

for all $x \in I$, then there exists a unique continuous function $y_0 : I \to \mathbb{C}$ such that

$$y_0(x) = \lambda\int_a^b K(x,t)y_0(t)dt + \sum_{a<x_k<b} I_k(y_0(x_k^-)) \tag{7}$$

and

$$\left|y(x) - y_0(x)\right| \leq \frac{1}{1 - (ML_1\lambda + L_2)}\phi(x) \tag{8}$$

for all $x \in I$.

Proof. First, we define a set

$$X = \{h : I \to \mathbb{C} | h \text{ is continuous}\} \tag{9}$$

and introduce a generalized metric on X as follows:

$$d(g,h) = \inf\{C \in [0,\infty] : |g(x) - h(x)| \leq C\phi(x), \text{ for all } x \in I\}. \tag{10}$$

Here, we give a proof for the triangle inequality. Assume $d(g,h) > d(g,k) + d(k,h)$ holds for some $g, h, k \in X$. Then, there should exist an $x_0 \in I$ with

$$|g(x_0) - h(x_0)| > \{d(g,k) + d(k,h)\}\phi(x_0) = d(g,k)\phi(x_0) + d(k,h)\phi(x_0). \tag{11}$$

In view of (10), this inequality would yield

$$\left|g(x_0) - h(x_0)\right| > \left|g(x_0) - k(x_0)\right| + \left|k(x_0) - h(x_0)\right|, \tag{12}$$

which is a contradiction. Our task is to show that (X, d) is a complete metric space. Let $\{h_n\}$ be a Cauchy sequence in (X, d). Then for any $\epsilon > 0$, there exists an integer $N_\epsilon > 0$ such that $d(h_m, h_n) \leq \epsilon$ for all $m, n \geq N_\epsilon$.

In view of (10), we have:

$$\forall \epsilon > 0, \quad \exists N_\epsilon \in \mathbb{N} \quad \forall m, n \geq N_\epsilon \quad \forall x \in I : |h_m(x) - h_n(x)| \leq \epsilon \phi(x). \tag{13}$$

If x is fixed, equation (13) implies that $\{h_n(x)\}$ is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, $\{h_n(x)\}$ converges for each $x \in I$. Thus, we can define a function $h : I \to \mathbb{C}$ by:

$$h(x) = \lim_{n \to \infty} h_n(x). \tag{14}$$

Since ϕ is continuous on the compact interval I, ϕ is bounded. Thus, equation (13) implies that $\{h_n\}$ converges uniformly to h in the usual topology of $\mathbb{C}$. Hence, h is continuous, i.e., $h \in X$. We need to show that $\{h_n\}$ converges to h in (X, d).

Let m increase to infinity; it follows from equation (13) that:

$$\forall \epsilon > 0, \quad \exists N_\epsilon \in \mathbb{N} \quad \forall n \geq N_\epsilon \quad \forall x \in I : |h(x) - h_n(x)| \leq \epsilon \phi(x). \tag{15}$$

By considering (10), we obtain:

$$\forall \epsilon > 0, \quad \exists N_\epsilon \in \mathbb{N}, \quad \forall n \geq N_\epsilon : d(h, h_n) \leq \epsilon. \tag{16}$$

This means that the Cauchy sequence $\{h_n\}$ converges to h in (X, d). Hence, (X, d) is a generalized complete metric space.

Next, we define an operator $T : X \to X$ by

$$(Th)(x) = \lambda \int_a^b K(x, t) h(t) dt + \sum_{a < x_k < b} I_k(h(x_k^-)) \tag{17}$$

for all $h \in X$ and $x \in I$. Next, we will show that the operator T is strictly contractive on the set X. Suppose $g, h \in X$ and let $C_{gh} \in [0, \infty]$ be a constant with $d(g, h) \leq C_{gh}$ for any $g, h \in X$.

From (10), we can write:

$$\left| g(x) - h(x) \right| \leq C_{gh} \phi(x). \tag{18}$$

Then, it follows from (3), (4), (6), (17) and (18) that:

$$\begin{aligned}
|(Tg)(x) - (Th)(x)| &= \left|\lambda \int_a^b K(x,t)g(t)dt + \sum_{a<x_k<b} I_k(g(x_k^-)) \right.\\
&\quad \left. - \lambda \int_a^b K(x,t)h(t)dt - \sum_{a<x_k<b} I_k(h(x_k^-))\right|\\
&= \left|\lambda \int_a^b \{K(x,t)g(t) - K(x,t)h(t)\}dt \right.\\
&\quad \left. + \sum_{a<x_k<b} \{I_k(g(x_k^-)) - I_k(h(x_k^-))\}\right|\\
&\le \left|\lambda \int_a^b \{K(x,t)g(t) - K(x,t)h(t)\}dt\right|\\
&\quad + \left|\sum_{a<x_k<b} \{I_k(g(x_k^-)) - I_k(h(x_k^-))\}\right|\\
&\le \lambda L_1 \int_a^b \left|g(t) - h(t)\right|dt + L_2 \sum_{a<x_k<b} |g(x_k^-) - h(x_k^-)|\\
&\le \lambda L_1 C_{gh} \int_a^b \phi(t)dt + L_2 \sum_{a<x_k<b} |g(x_k^-) - h(x_k^-)|\\
&\le \lambda L_1 M C_{gh}\phi(x) + L_2 C_{gh}\phi(x)\\
&= C_{gh}\phi(x)(\lambda L_1 M + L_2).
\end{aligned}$$

This implies that

$$\left|(Tg)(x) - (Th)(x)\right| \le C_{gh}\phi(x)(\lambda L_1 M + L_2),$$

for all $x \in I$, that is,

$$d(Tg, Th) \le C_{gh}(ML_1\lambda + L_2).$$

Hence, we conclude that $d(Tg, Th) \le (ML_1\lambda + L_2)d(g, h)$ for any $g, h \in X$, where $0 < ML_1\lambda + L_2 < 1$.

Let $h_0 \in X$ (arbitrary) be given. By equations (9) and (17), there exists a constant $C \in [0, \infty]$ such that

$$\begin{aligned}
|Th_0(x) - h_0(x)| &= \left|\lambda \int_a^b K(x,t)h_0(t)dt + \sum_{a<x_k<b} I_k(h_0(x_k^-)) - h_0(x)\right|\\
&\le C\phi(x), \quad \forall x \in I.
\end{aligned}$$

Since K and h_0 are bounded on I and $\min_{x\in I} \phi(x) > 0$, equation (10) implies that

$$d(Th_0, h_0) < \infty. \tag{19}$$

According to Theorem (2.2) (a), there exists a continuous function $y_0 : I \to \mathbb{C}$ such that $T^n h_0 \to y_0$ in (X, d) and $Ty_0 = y_0$, meaning y_0 satisfies (7) for all $x \in I$.

Next, we show that $\{g \in X \mid d(h_0, g) < \infty\} = X$, where h_0 was chosen with the property (19). Let $g \in X$, since g and h_0 are bounded on the closed interval I and $\min_{x\in I} \phi(x) > 0$, a constant $0 < C_g < \infty$ exists such that

$$\left|h_0(x) - g(x)\right| \le C_g\phi(x), \quad \forall x \in I.$$

Thus, we can write that $d(h_0, g) < \infty$ for any $g \in X$. Therefore, we get that $\{g \in X \mid d(h_0, g) < \infty\} = X$.

From Theorem (2.2) (b), we conclude that y_0, given by equation (7), is the unique continuous function.

Finally, Theorem (2.2) (c) implies that:

$$d(y, y_0) \leq \frac{1}{1 - (ML_1\lambda + L_2)} d(Ty, y) \leq \frac{1}{1 - (ML_1\lambda + L_2)}, \tag{20}$$

since inequality (5) means that $d(Ty, y) \leq 1$. In view of (10), we can conclude that inequality (8) holds for all $x \in I$. □

3.2. Hyers–Ulam stability

In this subsection, by using the idea of Cădariu and Radu [15], we will prove the Hyers–Ulam stability of homogeneous impulsive Fredholm integral equation (1).

Theorem 3.2. *Given $a \in \mathbb{R}$ and $q > 0$, suppose that $I(a; q)$ denotes a closed interval $\{b \in \mathbb{R} \mid a - q \leq b \leq a + q\}$ and let $K : I(a; q) \times I(a; q) \to \mathbb{C}$ be a continuous function that satisfies the Lipschitz condition* (3) *for all $x \in I$, $y_1, y_2 \in \mathbb{C}$, where L_1, L_2, and λ are constants with $0 < L_1 q\lambda + L_2 < 1$, and $I_k : \mathbb{C} \to \mathbb{C}$ with constant L_2 satisfies the Lipschitz condition* (4). *If $\sigma \geq 0$ and a continuous function $y : I(a; q) \to \mathbb{C}$ satisfies*

$$\left| y(x) - c - \lambda \int_a^b K(x, t) y(t)\, dt - \sum_{a < x_k < b} I_k(y(x_k^-)) \right| \leq \sigma,$$

for all $x \in I(a; q)$, where c is a complex number, then there exists a unique continuous function $y_0 : I(a; q) \to \mathbb{C}$ such that

$$y_0(x) = c + \lambda \int_a^b K(x, t) y_0(t)\, dt + \sum_{a < x_k < b} I_k(y_0(x_k^-)), \tag{21}$$

and

$$\left| y(x) - y_0(x) \right| \leq \frac{\sigma}{1 - (\lambda L_1 q + L_2)} \tag{22}$$

for all $x \in I(a; q)$.

Proof. Let

$$X = \{h_1 : I(a; q) \to \mathbb{C} \mid h_1 \text{ is continuous}\}$$

be a set, and we introduce a generalized metric on set X as follows:

$$d(g_1, h_1) = \inf\{C \in [0, \infty] \mid |g_1(x) - h_1(x)| \leq C, \text{ for all } x \in I(a; q)\}. \tag{23}$$

We can easily see that (X, d) is a complete generalized metric space (see [36]). Consider the operator $T : X \to X$ defined by

$$(Th_1)(x) = c + \lambda \int_a^b K(x, t) h_1(t)\, dt + \sum_{a < x_k < b} I_k(h_1(x_k^-)), \tag{24}$$

for all $h_1 \in X$ and $x \in I(a; q)$.

Next, we will check that the operator T is strictly contractive on the set X. Suppose that $C_{g_1 h_1} \in [0, \infty]$ is a constant with $d(g_1, h_1) \leq C_{g_1 h_1}$ for any $g_1, h_1 \in X$. We have

$$\left| g_1(x) - h_1(x) \right| \leq C_{g_1 h_1}, \quad \text{for all} x \in I(a; q). \tag{25}$$

By making use of (3), (4), (23), (24) and (25), we deduce

$$\begin{aligned}
\left|(Tg_1)(x) - (Th_1)(x)\right| &= \left|c + \lambda \int_a^b K(x,t)g_1(t)\,dt + \sum_{a<x_k<b} I_k(g_1(x_k^-)) \right.\\
&\quad \left. - c - \lambda \int_a^b K(x,t)h_1(t)\,dt - \sum_{a<x_k<b} I_k(h_1(x_k^-))\right|\\
&= \left|\lambda \int_a^b K(x,t)g_1(t)\,dt - \lambda \int_a^b K(x,t)h_1(t)\,dt \right.\\
&\quad \left. + \sum_{a<x_k<b} I_k(g_1(x_k^-)) - \sum_{a<x_k<b} I_k(h_1(x_k^-))\right|\\
&\leq \lambda\left|\int_a^b \{K(x,t)g_1(t) - K(x,t)h_1(t)\}\,dt\right|\\
&\quad + \left|\sum_{a<x_k<b}\{I_k(g_1(x_k^-)) - I_k(h_1(x_k^-))\}\right|\\
&\leq \lambda L_1 \int_a^b \left|g_1(t) - h_1(t)\right| dt + L_2 \sum_{a<x_k<b}\left|g_1(x_k^-) - h_1(x_k^-)\right|\\
&\leq \lambda L_1 \int_a^b C_{g_1h_1}\,dt + L_2 C_{g_1h_1}\\
&\leq \lambda L_1 C_{g_1h_1}|b-a| + L_2 C_{g_1h_1}\\
&\leq (\lambda L_1 q + L_2) C_{g_1h_1}
\end{aligned}$$

for all $x \in I(a;q)$, i.e., $d(Tg_1, Th_1) \leq (\lambda L_1 q + L_2)C_{g_1h_1}$. Hence, we may conclude that $d(Tg_1, Th_1) \leq (\lambda L_1 q + L_2)d(g_1, h_1)$ for any $g_1, h_1 \in X$, where $0 < \lambda L_1 q + L_2 < 1$.

By applying the same procedure as in Theorem 3.1, we can choose $h_0 \in X$ with $d(Th_0, h_0) < \infty$. Hence, from Theorem 2.2 (a), it follows that there exists a continuous function $y_0 : I(a;q) \to \mathbb{C}$ such that $T^n h_0 \to y_0$ in (X, d) as $n \to \infty$, and such that y_0 satisfies the homogeneous impulsive Fredholm integral equation (21) for any $x \in I(a;q)$.

Next, we will show that $X = \{g_1 \in X \mid d(h_0, g_1) < \infty\}$. By applying a similar argument to the proof of Theorem 3.1 to this case. Therefore, Theorem 2.2 (b) implies that y_0 is a unique continuous function with the property (21).

Furthermore, Theorem 2.2 (c) implies that

$$\left|y(x) - y_0(x)\right| \leq \frac{\sigma}{1 - (\lambda L_1 q + L_2)},$$

for all $x \in X$. □

3.3. Examples

Now, we present two examples which indicate how our results can be applied to concrete problems.

Example 3.3. *Suppose $I = [0,1]$ be given and let M, L_1, L_2, λ be positive constants with $0 < ML_1\lambda + L_2 < 1$ for $\lambda < \frac{4}{3ML_1}$. Let $\phi : [0,1] \to (0,\infty)$ be a continuous function and the kernel $K : [0,1]\times[0,1] \to \mathbb{C}$ defined by $K(x,t) = 1 + x + t$.*

Consider the homogeneous impulsive Fredholm integral equation,

$$y(x) = \lambda \int_0^1 (1 + x + t)y(t)\,dt + \sum_{0<\frac{1}{5}<1} \frac{|y(\frac{1}{5}^-)|}{3 + |y(\frac{1}{5}^-)|}, \tag{26}$$

for any $x \in I$.

Clearly,

$$\left|(1+x+t)y_1 - (1+x+t)y_2\right| \quad \leq \quad L_1\left|y_1 - y_2\right|.$$

Moreover,

$$I_k(y(x_k^-)) = \triangle y|_{x=x_k}.$$

So that,

$$\triangle y|_{x=\frac{1}{5}} = I_k\left(y\left(\frac{1^-}{5}\right)\right) = \frac{|y(\frac{1^-}{5})|}{3+|y(\frac{1^-}{5})|}.$$

Clearly,

$$\begin{aligned}\left|I_k(y_1) - I_k(y_2)\right| &= \left|\frac{y_1}{3+y_1} - \frac{y_2}{3+y_2}\right| \\ &= \left|\frac{y_1(3+y_2) - y_2(3+y_1)}{(3+y_1)(3+y_2)}\right| \\ &= \left|\frac{3y_1 + y_1y_2 - 3y_2 - y_1y_2}{(3+y_1)(3+y_2)}\right| \\ &= \left|\frac{3(y_1-y_2)}{(3+y_1)(3+y_2)}\right| \\ &\leq \frac{1}{3}\left|y_1 - y_2\right|.\end{aligned}$$

Here, we see that $L_2 = \frac{1}{3}$.

Let $y : I \to \mathbb{C}$ be such that:

$$\left|y(x) - \lambda\int_0^1 (1+x+t)y(t)\,dt - \sum_{0<\frac{1}{5}<1} \frac{|y(\frac{1^-}{5})|}{3+|y(\frac{1^-}{5})|}\right| \leq \phi(x) = e^x,$$

for all $x \in [0,1]$.

Clearly, $0 \leq x \leq 1$ and $0 \leq t \leq x$ so that,

$$\left|\int_0^x \phi(t)\,dt\right| = \left|\int_0^x e^t\,dt\right| = (e^x - 1) \leq e^x = \phi(x).$$

It means that,

$$\left|\int_0^x \phi(t)\,dt\right| \leq \phi(x)$$

for all $x \in [0,1]$ and $t \in [0,x]$. From here, we can see that $M = 1$.

Then, Theorem 3.1 assures that there exists a unique continuous function $y_0 : I \to \mathbb{C}$ such that

$$\left|y(x) - y_0(x)\right| \leq \frac{3}{4 - 3M\lambda L_1}e^x, \quad \forall x \in [0,1].$$

Thus, the homogeneous impulsive Fredholm integral equation (26) is Hyers–Ulam–Rassias stable.

Example 3.4. *Consider the above homogeneous impulsive Fredholm integral equation* (26) *for* $\lambda < \frac{1}{2qL_1} = \frac{L_2}{qL_1}$ *and let* $L_2 = \frac{1}{2}$.

Further assume that for some $q > 0$, $\sigma > 0$ *and let* $y : I \to \mathbb{C}$, *we have:*

$$\left| y(x) - \lambda \int_0^1 (1 + x + t) y(t) dt - \sum_{0<\frac{1}{5}<1} \frac{|y(\frac{1^-}{5})|}{3 + |y(\frac{1^-}{5})|} \right| \le \sigma.$$

In the light of Theorem 3.2, there exists a unique continuous function $y_0 : I \to \mathbb{C}$ *that solves* (26) *for* $\lambda < \frac{1}{2qL_1}$ *and*

$$\left| y(x) - y_0(x) \right| \le \frac{2}{1 - 2qL_1\lambda} \sigma,$$

for all $x \in [0, 1]$. *Hence, equation* (26) *is Hyers–Ulam stable.*

4. Stability results of non–homogeneous impulsive Fredholm integral equation

In this section, we will prove the Hyers–Ulam and Hyers–Ulam–Rassias stability of non–homogeneous impulsive Fredholm integral equation (2).

4.1. Hyers–Ulam–Rassias stability

In this subsection, we will prove the Hyers–Ulam–Rassias stability of non–homogeneous impulsive Fredholm integral equation (2).

Theorem 4.1. *Suppose* $I = [a, b]$ *be given for fixed real numbers* a, b *with* $a < b$ *and let* M, L_1, L_2 *and* λ *be positive constants with* $0 < ML_1\lambda + L_2 < 1$. *Let* $K : I \times I \to \mathbb{C}$ *be a continuous function which satisfies the Lipschitz condition* (3) *for any* $x, t \in I$, *and* $y_1, y_2 \in \mathbb{C}$.
Moreover, $I_k : \mathbb{C} \to \mathbb{C}$ *and there exists a constant* L_2 *which satisfies the condition* (4) *for all* $y_1, y_2 \in \mathbb{C}$.

Let $y : I \to \mathbb{C}$ *be a continuous function such that*

$$\left| y(x) - x - \lambda \int_a^b K(x, t) y(t) dt - \sum_{a<x_k<b} I_k(y(x_k^-)) \right| \le \phi(x) \tag{27}$$

for all $x \in I$. *Also* $I_k : \mathbb{C} \to \mathbb{C}$, $k = 1, 2, \ldots, m$ *and* $y(x_k^-)$ *represents the left limit of* $y(x)$ *at* $x = x_k$, *where* $\phi : I \to (0, \infty)$ *is a continuous function with*

$$\left| \int_a^b \phi(t) dt \right| \le N\phi(x) \tag{28}$$

for all $x \in I$. *Then there exists a unique continuous function* $y_0 : I \to \mathbb{C}$ *which is a solution of non–homogeneous impulsive Fredholm integral equation* (2) *such that*

$$y_0(x) = x + \lambda \int_a^b K(x, t) y_0(t) dt + \sum_{a<x_k<b} I_k(y_0(x_k^-)) \tag{29}$$

and

$$\left| y(x) - y_0(x) \right| \le \frac{1}{1 - (ML_1\lambda + L_2)} \phi(x) \tag{30}$$

for all $x \in I$.

Proof. First, we define

$$X = \{h_0 : I \to \mathbb{C} \mid h_0 \text{ is continuous}\} \tag{31}$$

to be a set and introduce a generalized metric on X as follows:

$$d(g, h_0) = \inf\{C \in [0, \infty] \mid |g(x) - h_0(x)| \le C\phi(x), \text{ for all } x \in I\}. \tag{32}$$

We can easily see that (X, d) is a generalized complete metric space (see [36]).

Next, we define an operator $T : X \to X$ by

$$(Th_1)(x) = x + \lambda \int_a^b K(x,t)h_1(t)dt + \sum_{a<x_k<b} I_k(h_1(x_k^-)) \tag{33}$$

for all $h_1 \in X$ and $x \in I$. Next, we will show that the operator is strictly contractive on the set X. Suppose $g_1, h_1 \in X$ and let $C_{g_1h_1} \in [0, \infty]$ be a constant with $d(g_1, h_1) \le C_{g_1h_1}$ for any $g_1, h_1 \in X$.

By equation (32), we can write:

$$\left|g_1(x) - h_1(x)\right| \le C_{g_1h_1}\phi(x) \tag{34}$$

for all $x \in I$.

Then, it follows from (3), (4), (28), (33), and (34)

$$\begin{aligned}
\left|(Tg_1)(x) - (Th_1)(x)\right| &= \Bigg| x + \lambda \int_a^b K(x,t)g_1(t)dt + \sum_{a<x_k<b} I_k(g_1(x_k^-)) \\
&\quad - x - \lambda \int_a^b K(x,t)h_1(t)dt - \sum_{a<x_k<b} I_k(h_1(x_k^-)) \Bigg| \\
&= \Bigg| \lambda \int_a^b \{K(x,t)g_1(t) - K(x,t)h_1(t)\}dt \\
&\quad + \sum_{a<x_k<b} \{I_k(g_1(x_k^-)) - I_k(h_1(x_k^-))\} \Bigg| \\
&\le \left| \lambda \int_a^b \{K(x,t)g_1(t) - K(x,t)h_1(t)\}dt \right| \\
&\quad + \left| \sum_{a<x_k<b} \{I_k(g_1(x_k^-)) - I_k(h_1(x_k^-))\} \right| \\
&\le \lambda L_1 \int_a^b \left|g_1(t) - h_1(t)\right|dt + L_2 \sum_{a<x_k<b} \left|g_1(x_k^-) - h_1(x_k^-)\right| \\
&\le \lambda L_1 \int_a^b C_{g_1h_1}\phi(t)dt + L_2 \sum_{a<x_k<b} \left|g_1(x_k^-) - h_1(x_k^-)\right| \\
&\le \lambda L_1 M C_{g_1h_1}\phi(x) + L_2 C_{g_1h_1}\phi(x) \\
&= C_{g_1h_1}\phi(x)(\lambda L_1 M + L_2).
\end{aligned}$$

This implies that

$$\left|(Tg_1)(x) - (Th_1)(x)\right| \le C_{g_1h_1}\phi(x)(\lambda M L_1 + L_2),$$

for all $x \in I$, that is,

$$d(Tg_1, Th_1) \le C_{g_1h_1}(ML_1\lambda + L_2).$$

Hence, we may conclude that $d(Tg_1, Th_1) \leq (ML_1\lambda + L_2)d(g_1, h_1)$ for any $g_1, h_1 \in X$, where $0 < ML_1\lambda + L_2 < 1$.

Let $h_0 \in X$ (be arbitrary) be given. There exists a constant $0 < C < \infty$, such that

$$\begin{aligned} \left|Th_0(x) - h_0(x)\right| &= \left|x + \lambda \int_a^b K(x,t)h_0(t)dt + \sum_{a<x_k<b} I_k(h_0(x_k^-)) - h_0(x)\right| \\ &\leq C\phi(x), \forall x \in I. \end{aligned}$$

Since K and h_0 are bounded on I and $\min_{x \in I} \phi(x) > 0$.

Thus, we have

$$d(Th_0, h_0) < \infty. \tag{35}$$

So, according to Theorem 2.2 (a), there exists a continuous function $y_0 : I \to \mathbb{C}$ such that $T^n h_0 \to y_0$ in (X, d) and $Ty_0 = y_0$, i.e., y_0 satisfies (29) for all $x \in I$.

Next, we show that $\{g_1 \in X \mid d(h_0, g_1) < \infty\} = X$, where h_0 was chosen with the property (35). Let $g_1 \in X$, since we know that g_1 and h_0 are bounded on the closed interval I and $\min_{x \in I} \phi(x) > 0$, then there exists a constant $0 < C_g < \infty$ such that

$$\left|h_0(x) - g_1(x)\right| \leq C_g \phi(x),$$

for all $x \in I$. Thus, we can write that $d(h_0, g_1) < \infty$ for any $g_1 \in X$. Therefore, we get that $\{g_1 \in X \mid d(h_0, g_1) < \infty\} = X$. From Theorem 2.2 (b), we conclude that y_0, given by equation (29), is the unique continuous function.

Finally, Theorem 2.2 (c) implies that:

$$d(y, y_0) \leq \frac{1}{1 - (ML_1\lambda + L_2)} d(Ty, y) \leq \frac{1}{1 - (ML_1\lambda + L_2)}. \tag{36}$$

Since inequality (27) means that $d(Ty, y) \leq 1$. In view of (32), we can conclude that the inequality (30) holds for all $x \in I$. □

4.2. Hyers–Ulam stability

In this subsection, by using the idea of Că*dariu* and Radu [15], we will prove the Hyers–Ulam stability of non–homogeneous impulsive Fredholm integral equation (2).

Theorem 4.2. *Given $a \in \mathbb{R}$ and $q > 0$, let $I(a; q)$ denote the closed interval $\{b \in \mathbb{R} \mid a - q \leq b \leq a + q\}$. Suppose that $K : I(a; q) \times I(a; q) \to \mathbb{C}$ is a continuous function which satisfies the Lipschitz condition*

$$\left|K(x,t)y_1 - K(x,t)y_2\right| \leq L_1|y_1 - y_2|, \tag{37}$$

for all $x, t \in I(a; q)$ and $y_1, y_2 \in \mathbb{C}$, where L_1, L_2, and λ are constants with $0 < L_1 q\lambda + L_2 < 1$. Additionally, let $I_k : \mathbb{C} \to \mathbb{C}$ be a function with a constant L_2 that satisfies the Lipschitz condition

$$|I_k(y_1) - I_k(y_2)| \leq L_2|y_1 - y_2|, \tag{38}$$

for all $y_1, y_2 \in \mathbb{C}$.

If $\sigma \geq 0$ and a continuous function $y : I(a; q) \to \mathbb{C}$ satisfies

$$\left|y(x) - c - x - \lambda \int_a^b K(x,t)y(t)dt - \sum_{a<x_k<b} I_k(y(x_k^-))\right| \leq \sigma, \tag{39}$$

for all $x \in I(a;q)$, where c is a complex number, then there exists a unique continuous function $y_0 : I(a;q) \to \mathbb{C}$ such that

$$y_0(x) = c + x + \lambda \int_a^b K(x,t) y_0(t) dt + \sum_{a<x_k<b} I_k(y_0(x_k^-)), \tag{40}$$

and

$$|y(x) - y_0(x)| \le \frac{\sigma}{1 - (L_1 q \lambda + L_2)} \tag{41}$$

for all $x \in I(a;q)$.

Proof. Let

$$X = \{h_2 : I(a;q) \to \mathbb{C} \mid h_2 \text{ is continuous}\} \tag{42}$$

be a set, and we introduce a generalized metric on the set X as follows:

$$d(g_2, h_2) = \inf\{C \in [0,\infty] \mid |g_2(x) - h_2(x)| \le C, \text{ for all } x \in I(a;q)\}. \tag{43}$$

We can easily see that (X, d) is a complete generalized metric space, as noted in [36].

Consider the operator $T : X \to X$ defined by

$$(Th_2)(x) = c + x + \lambda \int_a^b K(x,t) h_2(t) dt + \sum_{a<x_k<b} I_k(h_2(x_k^-)) \tag{44}$$

for all $h_2 \in X$ and $x \in I(a;q)$.

Next, we will check that the operator T is strictly contractive on the set X. Suppose that $C_{g_2h_2} \in [0,\infty]$ is a constant such that $d(g_2,h_2) \le C_{g_2h_2}$ for any $g_2, h_2 \in X$. We have

$$|g_2(x) - h_2(x)| \le C_{g_2h_2}, \quad \text{for all } x \in I(a;q). \tag{45}$$

By utilizing the conditions stated in (37), (38), (43), (44) and (45), we deduce

$$\begin{aligned}
|(Tg_2)(x) - (Th_2)(x)| &= \Bigg| c + x + \lambda \int_a^b K(x,t) g_2(t) dt + \sum_{a<x_k<b} I_k(g_2(x_k^-)) \\
&\quad - c - x - \lambda \int_a^b K(x,t) h_2(t) dt - \sum_{a<x_k<b} I_k(h_2(x_k^-)) \Bigg| \\
&= \Bigg| \lambda \int_a^b K(x,t) g_2(t) - \lambda \int_a^b K(x,t) h_2(t) dt \\
&\quad + \sum_{a<x_k<b} I_k(g_2(x_k^-)) - \sum_{a<x_k<b} I_k(h_2(x_k^-)) \Bigg| \\
&\le \left| \lambda \int_a^b \{K(x,t) g_2(t) - (1 + x + t) h_2(t)\} dt \right| \\
&\quad + \left| \sum_{a<x_k<b} \{I_k(g_2(x_k^-)) - I_k(h_2(x_k^-))\} \right| \\
&\le \lambda L_1 \int_a^b \Big| g_2(t) - h_2(t) \Big| dt + L_2 \sum_{a<x_k<b} \Big| g_2(x_k^-) - h_2(x_k^-) \Big| \\
&\le \lambda L_1 \int_a^b C_{g_2h_2} dt + L_2 C_{g_2h_2}
\end{aligned}$$

$$\leq \lambda L_1 C_{g_2h_2}|b-a| + L_2 C_{g_2h_2}$$
$$\leq (\lambda L_1 q + L_2) C_{g_2h_2}$$

for all $x \in I(a;q)$, i.e., $d(Tg_2, Th_2) \leq (\lambda L_1 q + L_2)C_{g_2h_2}$. Hence, we may conclude that $d(Tg_2, Th_2) \leq (\lambda L_1 q + L_2)d(g_2, h_2)$ for any $g_2, h_2 \in X$, where $0 < \lambda L_1 q + L_2 < 1$.

By applying the same procedure as in Theorem 4.1, we can choose $h_0 \in X$ with $d(Th_0, h_0) < \infty$. Hence, from Theorem 2.2 (a), it follows that there exists a continuous function, say $y_0 : I(a;q) \to \mathbb{C}$, such that $T^n h_0 \to y_0$ in (X, d) as $n \to \infty$, and such that y_0 satisfies the non–homogeneous impulsive Fredholm integral equation (40) for any $x \in I(a;q)$.

Next, we will show that $X = \{g_2 \in X \mid d(h_0, g_2) < \infty\}$. By applying a similar argument to the proof of Theorem 4.1 to this case, we conclude that Theorem 2.2 (b) implies that y_0 is a unique continuous function with property (40).

Furthermore, Theorem 2.2 (c) implies that

$$\left|y(x) - y_0(x)\right| \leq \frac{\sigma}{1 - (\lambda L_1 q + L_2)} \tag{46}$$

for all $x \in X$. □

4.3. Examples

Now, we provide illustrative examples that support the above theorems.

Example 4.3. *Suppose* $I = [0,1]$ *be given and let* M, L_1, L_2, λ *be positive constants with* $0 < ML_1\lambda + L_2 < 1$ *for* $\lambda < \frac{8}{5L_1}$. *Let* $\phi : [0,1] \to (0,\infty)$ *be a continuous function and the kernel* $K : [0,1] \times [0,1] \to \mathbb{C}$ *defined by* $K(x,t) = 1 + x + t$.

Consider the non–homogeneous impulsive Fredholm integral equation,

$$y(x) = x + \lambda \int_0^1 K(x,t)y(t)dt + \sum_{0<\frac{1}{10}<1} \frac{|y(\frac{1}{10}^-)|}{5 + |y(\frac{1}{10}^-)|}, \tag{47}$$

for any $x \in I$.

Clearly,

$$\left|(1+x+t)y_1 - (1+x+t)y_2\right| \leq L_1\left|y_1 - y_2\right|.$$

Moreover,

$$I_k(y(x_k^-)) = \triangle y|_{x=x_k}.$$

So that,

$$\triangle y|_{x=\frac{1}{10}} = I_k\left(y\left(\frac{1^-}{10}\right)\right) = \frac{|y(\frac{1^-}{10})|}{5 + |y(\frac{1^-}{10})|}.$$

Clearly,

$$\begin{aligned}
\left|I_k(y_1) - I_k(y_2)\right| &= \left|\frac{y_1}{5+y_1} - \frac{y_2}{5+y_2}\right| \\
&= \left|\frac{y_1(5+y_2) - y_2(5+y_1)}{(5+y_1)(5+y_2)}\right| \\
&= \left|\frac{5y_1 + y_1y_2 - 5y_2 - y_1y_2}{(5+y_1)(5+y_2)}\right| \\
&= \left|\frac{5(y_1 - y_2)}{(5+y_1)(5+y_2)}\right| \\
&\leq \frac{1}{5}\left|y_1 - y_2\right|.
\end{aligned}$$

Here, we see that $L_2 = \frac{1}{5}$.

Let $y : I \to \mathbb{C}$ *be such that:*

$$\left| y(x) - x - \lambda \int_0^1 K(x,t)y(t)dt - \sum_{0<\frac{1}{10}<1} \frac{|y(\frac{1^-}{10})|}{5 + |y(\frac{1^-}{10})|} \right| \leq \phi(x) = e^{2x},$$

for all $x \in [0, 1]$.

Clearly, $0 \leq x \leq 1$ *and* $0 \leq t \leq x$ *so that,*

$$\left| \int_0^x \phi(t)dt \right| = \left| \int_0^x e^{2t}dt \right| = \frac{1}{2}(e^{2x} - 1) \leq \frac{1}{2}(e^{2x}) = \frac{1}{2}\phi(x).$$

It means that,

$$\left| \int_0^x \phi(t)dt \right| \leq \frac{1}{2}\phi(x)$$

for all $x \in [0, 1]$ *and* $t \in [0, x]$. *From here, we can see that* $M = \frac{1}{2}$.

Then, Theorem 4.1 assures that there exists a unique continuous function $y_0 : I \to \mathbb{C}$ *such that*

$$\left| y(x) - y_0(x) \right| \leq \frac{10}{8 - 5L_1\lambda} e^{2x}, \quad \forall x \in [0, 1].$$

Thus, the non–homogeneous impulsive Fredholm integral equation (47) *is Hyers–Ulam–Rassias stable.*

Example 4.4. *Consider the above non–homogeneous impulsive Fredholm integral equation* (47) *for* $\lambda < \frac{2}{3qL_1}$ *and let* $L_2 = \frac{1}{3}$.

Further assume that for some $q > 0$, $\sigma > 0$, *and let* $y : I \to \mathbb{C}$, *we have:*

$$\left| y(x) - x - \lambda \int_0^1 K(x,t)y(t)dt - \sum_{0<\frac{1}{10}<1} \frac{|y(\frac{1^-}{10})|}{5 + |y(\frac{1^-}{10})|} \right| \leq \sigma.$$

In the light of Theorem 4.2, there exists a unique continuous function $y_0 : I \to \mathbb{C}$ *that solves* (47) *for* $\lambda < \frac{2}{3qL_1}$ *and*

$$\left| y(x) - y_0(x) \right| \leq \frac{3}{2 - 3qL_1\lambda}\sigma,$$

for all $x \in [0, 1]$. *Hence, equation* (47) *is Hyers–Ulam stable.*

5. Conclusion

Two kind of novel stability concepts, the Hyers–Ulam–Rassias stability and the Hyers–Ulam stability, of a homogeneous impulsive Fredholm integral equation and a non–homogeneous impulsive Fredholm integral equation are offered. Using Banach's fixed point theorem in a generalized complete metric space, we have proved the Hyers–Ulam–Rassias stability and the Hyers–Ulam stability results on a finite interval. Four examples are offered to show the useability of our obtained results.

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