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q-Schröder sequence spaces and Schröder core

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Abstract. This research employs the q-Schröder matrix $\widetilde{S}_q$ to create the sequence spaces $c_0(\widetilde{S}_q)$, $c(\widetilde{S}_q)$, $\ell_\infty(\widetilde{S}_q)$ and $\ell_p(\widetilde{S}_q)$ where $(1 \leqslant p < \infty)$. We demonstrate certain topological features, derive Schauder bases, calculate the alpha, beta and gamma duals of new sequence spaces, build some matrix classes, and finally show some topological properties. In addition, we give Schröder's core of complex valued sequences and define various inclusion theorems for the new core type.

1. Introduction and Preliminaries

The q-calculus is a branch of mathematics that has a wide range of applications in many domains, including approximation theory, combinatorics, hypergeometric functions, operator theory, special functions, quantum algebras, and more.

The q-number $[b]_q$ is defined as follows for $0 < q < 1$,

$$[b]_q = \begin{cases} \sum_{s=0}^{b-1} q^s, & b = 1, 2, 3, \ldots, \\ 0, & b = 0. \end{cases}$$

It is reasonable to assume that $[b]_q \to b$ as $q \to 1^-$. Briefly, we represent $[b]_q$ by $[b]$. The q-binomial coefficient is defined by

$$\begin{bmatrix} b \\ d \end{bmatrix} = \begin{cases} \dfrac{[b]!}{[d]![b-d]!}, & 0 \leqslant d \leqslant b, \\ 0, & \text{otherwise}, \end{cases}$$

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where q-factorial $[b]!$ of b is given by

$$[b]! = \begin{cases} \prod_{s=1}^{b} [s], & b = 1, 2, 3, \ldots, \\ 1, & b = 0. \end{cases}$$

Using the definition of the q-binomial coefficient, we obtain

$$(b+d)_q^r = \sum_{s=0}^{r} \begin{bmatrix} r \\ s \end{bmatrix} q^{\binom{s}{2}} x^{r-s} r^s.$$

The final equation is known as the Gauss q-binomial formula. We strictly cited [30, 48] for information on the q-calculus.

1.1. Sequence Spaces

We can now give some fundamental details about sequence spaces and summability theory. Each Γ subset of ω is referred to as a sequence space, and ω denotes the space of all real or complex sequences. To symbolize the spaces of all bounded, convergent, and null sequences, we shall use the symbols ℓ_∞, c, and c_0. We designate the spaces of all convergent, bounded, absolutely, and p-absolutely convergent series, respectively, by cs, bs, ℓ_1, and ℓ_p, where $1 < p < \infty$.

A sequence space with a linear topology is known as a K-space, where each of the mappings $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all i in $\mathbb{N}$. A K-space that is also a complete linear metric space is known as an FK-space. A BK-space is an FK-space with a normable topology.

When there are real entries in an infinite matrix $A = (a_{rs})$, A_r represents the r_{th} row for each $r \in \mathbb{N}$. If the series is convergent for each $r \in \mathbb{N}$, the A-transform of $u = (u_s) \in \omega$ is given by the equation:

$$(Au)_r = \sum_{s=0}^{\infty} a_{rs} u_s$$

If $Au \in \Psi$, it is stated that A is a matrix transformation from Υ to Ψ for all $u \in \Upsilon$. (Υ, Ψ) denotes the class of all matrices that transform Υ to Ψ. The matrix domain of A in Υ is the set of all vectors $u = (u_s)$ in ω such that $Au \in \Upsilon$. If Υ and Ψ are two sequence spaces, then the multiplier set $\mathfrak{D}(\Upsilon : \Psi)$ is described as

$$\mathfrak{D}(\Upsilon : \Psi) = \{x = (x_s) \in \omega : xu = (x_s u_s) \in \Psi \text{ for all } (u_s) \in \Upsilon\}.$$

In that case, α-, β-, and γ-duals of Υ are described as

$$\Upsilon^\alpha = \mathfrak{D}(\Upsilon : \ell_1), \Upsilon^\beta = \mathfrak{D}(\Upsilon : cs) \text{ and } \Upsilon^\gamma = \mathfrak{D}(\Upsilon : bs).$$

The sequence spaces $(\ell_p)_{N_q}$, $(\ell_p)_{C_1} = X_p$, $(\ell_\infty)_R t = r_\infty^t$, $c_R t = r_c^t$, and $(c_0)_R t = r_0^t$ were introduced by Wang [54], Ng and Lee [42], Malkowsky [40], and Altay and Başar [3], respectively. These sequence spaces can be defined using the Nörlund, arithmetic, Riesz, and Euler means, respectively, for $1 \leqslant p \leqslant \infty$.

Şengönül and Başar [50] conducted research on the sequence spaces $\tilde{c}_0 = (c_0)_{C_1}$ and $\tilde{c} = c_{C_1}$, where C_1 stands for the matrix $C_1 = (c_{rs})$ that is defined by

$$c_{rs} = \begin{cases} \frac{1}{r+1}, & 0 \leqslant s \leqslant r, \\ 0, & s > r, \end{cases}$$

for every $r, s \in \mathbb{N}$. We cite the following publications for relevant literature [2, 6, 10–12, 22, 23, 26, 27, 43, 44, 51] as well as the books [5, 8, 41].

1.2. Core Theorems

Any sequence $f = (f_s)$ with complex entries has a Knopp Core (or $\mathcal{K} - core$), which is defined as the intersection of all R_s, which are the least convex closed regions of the complex plane containing $f_s, f_{s+1}, f_{s+2}, \ldots$ [15]. Additionally, it is understood from [45] that

$$\mathcal{K} - core(f) = \bigcap_{t\in\mathbb{C}} \mathcal{M}_f(t)$$

for any bounded sequence f, where $\mathcal{M}_f(t) = \left\{\varepsilon \in \mathbb{C} : |\varepsilon - t| \leqslant \limsup_s |f_s - t|\right\}$.

A subset N of the set $\mathbb{N}$ of all natural numbers has a natural density defined by

$$\delta(N) = \lim_r \frac{1}{r}|\{s \leqslant r : s \in \mathbb{N}\}|.$$

If $\delta(\{s : |f_s - f_0| \geq \varepsilon\}) = 0$, in that case we say that $f = (f_s)$ is statistically convergent and this situation is denoted by $\mathcal{S} - \lim f = f_0$ [47]. We refer to the space of all statistically convergent sequences as $\mathcal{S}$.

The concept of the statistical core ($\mathcal{S} - core$) of $f = (f_s)$ is acquainted as

$$\mathcal{S} - core(f) = \bigcap_{t\in\mathbb{C}} \Omega_f(t),$$

where $\Omega_f(t) = \{\varepsilon \in \mathbb{C} : |\varepsilon - t| \leqslant \mathcal{S} - \limsup_s |f_s - t|\}$ and f is statistically bounded [24]. Researchers interested in the aforementioned subject can benefit from the studies [1, 14, 16–18].

1.3. Schröder matrix and related sequence spaces

In recent years, special integer sequences such as the Fibonacci sequence, the Lucas sequence, and the Pell sequence have become widely used in the study of sequence spaces. In this context, the first work done is the study with a tag [31] made by Başar and Kara. After this work, some special integer sequences such as Lucas, Padovan, Pell, Leanardo, Catalan, Bell and Schröder were used to define new sequence spaces in summability theory. For relevant literature, we refer the papers [28, 29, 32–36, 52, 53, 56].

The large Schröder numbers S_r and the little Schröder numbers s_r are two different types of Schröder numbers in mathematics. They bear the name Ernst Schröder in honor of the German mathematician. For $0 \leqslant r \leqslant 10$, the first eleven large Schröder numbers are

$$1, 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098, 1037718.$$

The large Schröder numbers S_r were shown to have the generating function

$$G(t) = \frac{1 - t - \sqrt{t^2 - 6t + 1}}{2t}$$

as demonstrated in [9, Theorem 8.5.7]. For $1 \leqslant r \leqslant 11$, the first eleven little Schröder numbers s_r are

$$1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, 518859.$$

The existence of the generating function

$$g(t) = \frac{1 + t - \sqrt{t^2 - 6t + 1}}{4}$$

for the little Schröder numbers s_r was demonstrated in [9, Theorem 8.5.6].

In this article, we will look at large Schröder numbers. Let us move on to the large Schröder numbers. These numbers satisfy the following recursive formula

$$S_{r+1} = S_r + \sum_{s=0}^{r} S_s S_{r-s}, \quad \text{for} \quad r \geq 0, \tag{1}$$

with the initial condition $S_0 = 1$. The Schröder matrix $\widetilde{S} = (\widetilde{S}_{rs})$ [20] is defined by

$$\widetilde{S}_{rs} = \begin{cases} \dfrac{S_s S_{r-s}}{S_{r+1} - S_r}, & 0 \leqslant s \leqslant r, \\ 0, & s > r. \end{cases}$$

Recently, the domains $c_0(\widetilde{S}), c(\widetilde{S}), \ell_p(\widetilde{S})$ and $\ell_\infty(\widetilde{S})$ of the matrix $\widetilde{S}$ in the spaces c_0, c, ℓ_p and ℓ_∞, respectively are studied by Dağlı [19, 20].

In the literature, Schröder numbers have a number of q-analogs. Let $0 < q < 1$. In order to define the Schröder number q-analogs, we use the formula[4]

$$S_{r+1}(q) = S_r(q) + \sum_{s=0}^{r} q^{r-s+1} S_s(q) S_{r-s}(q) \tag{2}$$

where $S_0(q) = 1$. By setting $q = 1$, Schröder's numbers are obtained. For more interesting studies in the q-Schröder number, we strictly refer to [7, 9, 13, 39].

In this paper, we define new q-Schröder sequence spaces. We derive Schauder bases, calculate the alpha, beta, and gamma duals of the new sequence spaces, build some matrices classes, and finally show some topological properties. In addition, we give Schröder's core of complexly valued sequences and define various inclusion theorems for this new core type.

2. q-Schröder Sequence Spaces

In this section, we will talk about q-Schröder sequence spaces' definition and characteristics. The q-Schröder matrix $\widetilde{S}_q = (\widetilde{S}_{rs}(q))$ is defined by the following equation

$$\widetilde{S}_{rs}(q) = \begin{cases} q^{r-s+1} \dfrac{S_s(q) S_{r-s}(q)}{S_{r+1}(q) - S_r(q)}, & 0 \leqslant s \leqslant r, \\ 0, & s > r. \end{cases}$$

It is clear that the q-Schröder matrix $\widetilde{S}_q$ reduces to the Schröder matrix $\widetilde{S}$, when q tends to 1^-. The inverse of $\widetilde{S}_q$ is given by

$$\widetilde{S}^{-1}_{rs}(q) = \begin{cases} (-1)^{r-s} \dfrac{S_{k+1}(q) - S_s(q)}{q^{r-s+1} S_r(q)} P_{r-s}(q), & 0 \leqslant s \leqslant r, \\ 0, & s > r, \end{cases}$$

where $P_r(q)$ is a determinant given by

$$P_r(q) = \begin{vmatrix} S_1(q) & 1 & 0 & \cdots & 0 \\ S_2(q) & S_1(q) & S_0(q) & \cdots & 0 \\ S_3(q) & S_2(q) & S_1(q) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_r(q) & S_{r-1}(q) & S_{r-2}(q) & \cdots & 0 \end{vmatrix}$$

subject to initial condition $P_0(q) = 1$.

Now, we give the definitions of the q-Schröder sequence spaces $c_0(\widetilde{S_q})$, $c(\widetilde{S_q})$, $\ell_p(\widetilde{S_q})$ and $\ell_\infty(\widetilde{S_q})$:

$$\begin{aligned} c_0(\widetilde{S_q}) &= \left\{x=(x_s)\in\omega : \lim_{r\to\infty}\sum_{s=0}^{r} q^{r-s+1}\frac{S_s(q)S_{r-s}(q)}{S_{r+1}(q)-S_r(q)}x_s=0\right\} \\ c(\widetilde{S_q}) &= \left\{x=(x_s)\in\omega : \lim_{r\to\infty}\sum_{s=0}^{r} q^{r-s+1}\frac{S_s(q)S_{r-s}(q)}{S_{r+1}(q)-S_r(q)}x_s \quad \text{exists}\right\} \\ \ell_p(\widetilde{S_q}) &= \left\{x=(x_s)\in\omega : \sum_{r}\left|\sum_{s=0}^{r} q^{r-s+1}\frac{S_s(q)S_{r-s}(q)}{S_{r+1}(q)-S_r(q)}x_s\right|^p<\infty\right\} \\ \ell_\infty(\widetilde{S_q}) &= \left\{x=(x_s)\in\omega : \sup_{r\in\mathbb{N}}\left|\sum_{s=0}^{r} q^{r-s+1}\frac{S_s(q)S_{r-s}(q)}{S_{r+1}(q)-S_r(q)}x_s\right|<\infty\right\}. \end{aligned}$$

We note that when $q\to 1^-$, the spaces $c_0(\widetilde{S_q}), c(\widetilde{S_q}), \ell_p(\widetilde{S_q})$ and $\ell_\infty(\widetilde{S_q})$ decrease to the Schröder sequence spaces $c_0(\widetilde{S}), c(\widetilde{S}), \ell_p(\widetilde{S})$ and $\ell_\infty(\widetilde{S})$, respectively, as investigated by Dağlı [19, 20]. The previously mentioned sequence spaces can be redefined by

$$c_0(\widetilde{S_q})=(c_0)_{\widetilde{S_q}}, \quad c(\widetilde{S_q})=(c)_{\widetilde{S_q}} \tag{3}$$

$$\ell_p(\widetilde{S_q})=(\ell_p)_{\widetilde{S_q}} \quad \text{and} \quad \ell_\infty(\widetilde{S_q})=(\ell_\infty)_{\widetilde{S_q}} \tag{4}$$

using the notation of matrix domain.

The $\widetilde{S_q}$-transform of a sequence $x=(x_r)$ is defined as $y=(y_r)$, where

$$y_r=(\widetilde{S_q}x)_r=\sum_{s=0}^{r} q^{r-s+1}\frac{S_s(q)S_{r-s}(q)}{S_{r+1}(q)-S_r(q)}x_s, \tag{5}$$

for each $r\in\mathbb{N}_0$. The sequences x and y relate to the equation in (5) throughout the rest of the article. Therefore,

$$x_s=\sum_{i=0}^{s}(-1)^{s-i}\frac{S_{i+1}(q)-S_i(q)}{q^{s-i+1}S_s(q)}P_{s-i}(q)y_i, \tag{6}$$

for each $s\in\mathbb{N}_0$.

Theorem 2.1. *The space $\ell_p(\widetilde{S_q})$ is a BK-space with the norm*

$$\|(\widetilde{S_q}x)_r\|_p=\|x\|_{\ell_p(\widetilde{S_q})}=\left(\sum_r\left|(\widetilde{S_q}x)_r\right|^p\right)^{1/p}, \ (1\leqslant p<\infty)$$

and the spaces $\ell_\infty(\widetilde{S_q})$, $c_0(\widetilde{S_q})$ and $c(\widetilde{S_q})$ are BK-spaces with the norm

$$\|(\widetilde{S_q}x)_r\|_{\ell_\infty}=\|x\|_{\ell_\infty(\widetilde{S_q})}=\|x\|_{c_0(\widetilde{S_q})}=\|x\|_{c(\widetilde{S_q})}=\sup_{r\in\mathbb{N}}\left|(\widetilde{S_q}x)_r\right|.$$

Proof. The matrix $\widetilde{S_q}$ is triangular. Then, according to Wilansky's Theorem 4.3.12 of [55, p.63], the spaces $\ell_p(\widetilde{S_q})$ are BK-spaces with the given norms, where $(1\leqslant p\leqslant\infty)$.

Also, the spaces $c_0(\widetilde{S_q})$ and $c(\widetilde{S_q})$ are BK-spaces with the given norms, according to Wilansky's Theorem 4.3.2 of [55, p.61]. □

Theorem 2.2. *The sequence spaces $\ell_p(\widetilde{S_q})$ are isomorphic to the space ℓ_p, where* $(1\leqslant p\leqslant\infty)$.

Proof. For all x in $\ell_p(\widetilde{S}_q)$, define the mapping $\tau : \ell_p(\widetilde{S}_q) \to \ell_p$ by $\tau x = y = \widetilde{S}_q x$. It is obvious that τ is linear and one to one. Assume that $x = (x_s)$ is defined as in (6) for any sequence $y = (y_n)$ in ℓ_p. Then we have

$$\begin{aligned}\|x\|_{\ell_p(\widetilde{S}_q)} &= \left(\sum_r \left|(\widetilde{S}_q x)_r\right|^p\right)^{1/p}\\ &= \left(\sum_r \left|\sum_{s=0}^r q^{r-s+1}\frac{S_s(q)S_{r-s}(q)}{S_{r+1}(q)-S_r(q)}x_s\right|^p\right)^{1/p}\\ &= \left(\sum_r |y_r|^p\right)^{1/p} = \|y\|_p < \infty,\end{aligned}$$

and

$$\|x\|_{\ell_\infty(\widetilde{S}_q)} = \sup_{r\in\mathbb{N}}\left|(\widetilde{S}_q x)_r\right| = \|y\|_\infty < \infty.$$

Consequently, we understand that x is a sequence in $\ell_p(\widetilde{S}_q)$ and the mapping τ is onto, and norm preserving. □

Theorem 2.3. *The sequence spaces $c_0(\widetilde{S}_q)$ and $c(\widetilde{S}_q)$ are isomorphic to the spaces c_0 and c, respectively.*

Proof. Similar to Theorem 2.2, this theorem may be shown. □

Theorem 2.4. *Define the sequence $b^{(s)} = \{b^{(s)}\}_{s\in\mathbb{N}}$ of the elements of the space $\ell_p(\widetilde{S}_q)$ by*

$$b_n^{(s)} = \begin{cases} (-1)^{r-s}\dfrac{S_{s+1}(q)-S_s(q)}{q^{r-s+1}S_r(q)}P_{r-s}(q) &, \quad 0 \leqslant s \leqslant r\\ 0 &, \quad s > r\end{cases}$$

for every fixed $s \in \mathbb{N}$ and $1 \leqslant p < \infty$. The following claims are accurate:

(a) *The sequence $\{b^{(s)}\}_{s\in\mathbb{N}_0}$ is a basis for the spaces $c_0(\widetilde{S}_q)$ and $\ell_p(\widetilde{S}_q)$, and any $x \in c_0(\widetilde{S}_q)$ and $x \in \ell_p(\widetilde{S}_q)$ has a unique representation of the form*

$$x = \sum_s y_s b^{(s)}.$$

(b) *The sequence $\{e, b^{(s)}\}_{s\in\mathbb{N}}$ is a basis for the space $c(\widetilde{S}_q)$, and any $x \in c(\widetilde{S}_q)$ has a unique representation of the form*

$$x = le + \sum_s [y_s - l]b^{(s)},$$

where $y_s = (\widetilde{S}_q(x))_s \to l$, as $s \to \infty$.

(c) *The space $\ell_\infty(\widetilde{S}_q)$ does not have a basis.*

3. Dual Spaces

In this section, we will determine the alpha, beta and gamma duals of the new sequence spaces. From now on, we will denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{N}$ and we assume that p^* is the conjugate of p, i.e., $p^{-1} + p^{*-1} = 1$. Firstly, we give the lemmas used in the proofs in this section.

Lemma 3.1. *[49] The following claims are accurate:*

(i) $A = (a_{rs}) \in (c_0 : \ell_1) = (c : \ell_1) = (\ell_\infty : \ell_1)$ *iff*

$$\sup_{K\in\mathcal{N}} \sum_{r=0}^{\infty} \left| \sum_{s\in K} a_{rs} \right| < \infty. \tag{7}$$

(ii) $A = (a_{rs}) \in (c_0 : c) = (c : c)$ *iff*

$$\exists \alpha_s \in \mathbb{C} \ni \lim_{r\to\infty} a_{rs} = \alpha_s \ \textit{for each} \ s \in \mathbb{N}_0, \tag{8}$$

$$\sup_{r\in\mathbb{N}} \sum_{s=0}^{\infty} |a_{rs}| < \infty. \tag{9}$$

(iii) $A = (a_{rs}) \in (\ell_\infty : c)$ *iff (8) holds, and*

$$\lim_{r\to\infty} \sum_{s=0}^{\infty} |a_{rs}| = \sum_{s=0}^{\infty} |\lim_{r\to\infty} a_{rs}|. \tag{10}$$

(iv) $A = (a_{rs}) \in (c_0 : \ell_\infty) = (c : \ell_\infty) = (\ell_\infty : \ell_\infty)$ *iff (9) holds.*

Lemma 3.2. **(i)** *[49] Let* $1 < p < \infty$*. Then,* $A = (a_{rs}) \in (\ell_p : \ell_\infty)$ *iff*

$$\sup_{r\in\mathbb{N}} \sum_{s=0}^{\infty} |a_{rs}|^{p^*} < \infty. \tag{11}$$

(ii) *[49] Let* $1 < p < \infty$*. Then,* $A = (a_{rs}) \in (\ell_p : c)$ *iff (8) and (11) hold.*

(iii) *[25]* $A = (a_{rs}) \in (\ell_p : \ell_1)$ *iff*

$$\sup_{N\in\mathcal{N}} \sup_{s\in\mathbb{N}} \left| \sum_{r\in N} a_{rs} \right|^p < \infty, \ (0 < p \le 1), \tag{12}$$

$$\sup_{N\in\mathcal{N}} \sum_{s=0}^{\infty} \left| \sum_{r\in N} a_{rs} \right|^{p^*} < \infty, \ (1 < p < \infty). \tag{13}$$

(iv) *[38]Let* $0 < p \le 1$*.* $A = (a_{rs}) \in (\ell_p : \ell_\infty)$ *iff*

$$\sup_{r,s\in\mathcal{N}} \left| a_{rs} \right|^p < \infty. \tag{14}$$

(v) *[38] Let* $0 < p \le 1$*. Then,* $A = (a_{rs}) \in (\ell_p : c)$ *iff (8) and (14) hold.*

Theorem 3.3. *Define the matrix* $T = (t_{rs})$ *by*

$$t_{rs} = \begin{cases} (-1)^{r-s}\dfrac{S_{s+1}(q) - S_s(q)}{q^{r-s+1}S_r(q)}P_{r-s}(q)t_r & , \quad (0 \leqslant s \leqslant r) \\ 0 & , \quad (s > r) \end{cases}$$

for all $s, r \in \mathbb{N}$. *Then,* $\left\{c_0(\widetilde{S}_q)\right\}^{\alpha} = \left\{c(\widetilde{S}_q)\right\}^{\alpha} = \left\{\ell_\infty(\widetilde{S}_q)\right\}^{\alpha} = \mathfrak{c}_1$, *where* $\mathfrak{c}_1$ *defined by*

$$\mathfrak{c}_1 = \left\{t = (t_s) \in w : \sup_{K \in \mathcal{N}} \sum_{r=0}^{\infty} \left|\sum_{s \in K} t_{rs}\right| < \infty.\right\}.$$

Proof. We give the proof only for the sequence $c_0(\widetilde{S}_q)$. Let $t = (t_r) \in \omega$. Thus,

$$t_r x_r = \sum_{s=0}^{r} (-1)^{r-s}\frac{S_{s+1}(q) - S_s(q)}{q^{r-s+1}S_r(q)}P_{r-s}(q)t_r y_s = (Tx)_r, \ \ (r \in \mathbb{N}). \tag{15}$$

It follows from (15), $tx = (t_r x_r) \in \ell_1$ for $x \in c_0(\widetilde{S}_q)$ iff $Ty \in \left\{c_0(\widetilde{S}_q)\right\}$ for $y \in c_0$. Hence, by Lemma 3.1 from (7), it is concluded that $\left\{c_0(\widetilde{S}_q)\right\}^{\alpha} = \mathfrak{c}_1$. □

Theorem 3.4. *Let the sets* $\mathfrak{c}_2$ *and* $\mathfrak{c}_3$ *be as follows:*

$$\mathfrak{c}_2 = \left\{t = (t_s) \in \omega : \sup_{N \in \mathcal{N}} \sup_{s \in \mathbb{N}} \left|\sum_{r \in N} t_{rs}\right|^p < \infty\right\},$$

$$\mathfrak{c}_3 = \left\{t = (t_s) \in \omega : \sup_{N \in \mathcal{N}} \sum_{s=0}^{\infty} \left|\sum_{r \in N} t_{rs}\right|^{p^*} < \infty\right\}.$$

Then, $\left\{\ell_p(\widetilde{S}_q)\right\}^{\alpha} = \begin{cases} \mathfrak{c}_2, & 0 < p \le 1 \\ \mathfrak{c}_3, & 1 < p < \infty. \end{cases}$

Proof. This is accomplished by using the same procedure as in the proof of Theorem 3.3, but substituting the conditions (12) and (13) of Part (iii) of Lemma 3.2 for (7) of Part (i) of Lemma 3.1 with t_{rs} rather than a_{rs}. □

Theorem 3.5. *Consider the definition of* $D = (d_{rj})$ *using the sequence* $a = (a_j)$ *by*

$$d_{rs} = \begin{cases} \displaystyle\sum_{i=s}^{r} (-1)^{s-i}\frac{S_{i+1}(q) - S_i(q)}{q^{s-i+1}S_s(q)}P_{s-i}(q)a_i & , \quad (0 \leqslant s \leqslant r), \\ 0 & , \quad (s > r). \end{cases} \tag{16}$$

and define the folloing sets

$$\mathfrak{b}_1 = \left\{a = (a_s) \in \omega : \sup_{r \in \mathbb{N}} \sum_{s=0}^{\infty} |d_{rs}| < \infty\right\},$$

$$\mathfrak{b}_2 = \left\{a = (a_s) \in \omega : \lim_{r \to \infty} d_{rs} = \alpha_s\right\},$$

$$\mathfrak{b}_3 = \left\{a = (a_s) \in \omega : \lim_{r \to \infty} \sum_{s=0}^{\infty} |d_{rs}| = \sum_{s=0}^{\infty} \left|\lim_{r \to \infty} d_{rs}\right|\right\},$$

$$\mathfrak{b}_4 = \left\{a = (a_s) \in \omega : \lim_{r \to \infty} \sup_{s \in \mathbb{N}} \sum_{s=0}^{\infty} |d_{rs}| < \infty\right\},$$

$$\mathfrak{b}_5 = \left\{a = (a_s) \in \omega : \sup_{r,s \in \mathbb{N}} |d_{rs}|^p < \infty\right\}.$$

Then,

(i) $\left\{c_0(\widetilde{S_q})\right\}^\beta = \mathfrak{b}_1 \cap \mathfrak{b}_2$ *and* $\left\{c_0(\widetilde{S_q})\right\}^\gamma = \mathfrak{b}_1$,

(ii) $\left\{c(\widetilde{S_q})\right\}^\beta = \mathfrak{b}_1 \cap \mathfrak{b}_2$ *and* $\left\{c(\widetilde{S_q})\right\}^\gamma = \mathfrak{b}_1$,

(iii) $\left\{\ell_\infty(\widetilde{S_q})\right\}^\beta = \mathfrak{b}_2 \cap \mathfrak{b}_3$ *and* $\left\{\ell_\infty(\widetilde{S_q})\right\}^\gamma = \mathfrak{b}_1$,

(iv) $\left\{\ell_p(\widetilde{S_q})\right\}^\beta = \begin{cases} \mathfrak{b}_2 \cap \mathfrak{b}_4, & 0 \leqslant p < 1, \\ \mathfrak{b}_2 \cap \mathfrak{b}_5, & 1 \leqslant p < \infty, \end{cases}$

and $\left\{\ell_p(\widetilde{S_q})\right\}^\gamma = \begin{cases} \mathfrak{b}_4, & 0 \leqslant p < 1, \\ \mathfrak{b}_5, & 1 \leqslant p < \infty. \end{cases}$

Proof. We give the proof only for the $\beta-$dual of the sequence $\ell_p(\widetilde{S_q})$. Consider the equation

$$\begin{aligned} \sum_{s=0}^{r} a_s x_s &= \sum_{s=0}^{r}\left[\sum_{i=0}^{s}(-1)^{s-i}\frac{S_{i+1}(q)-S_i(q)}{q^{s-i+1}S_s(q)}P_{s-i}(q)y_i\right]a_s \\ &= \sum_{s=0}^{r}\left[\sum_{i=s}^{r}(-1)^{s-i}\frac{S_{i+1}(q)-S_i(q)}{q^{s-i+1}S_s(q)}P_{s-i}(q)a_i\right]y_s = (Dy)_r \end{aligned}$$

for any $r \in \mathbb{N}_0$. This equation states that if x is an element of $\ell_p(\widetilde{S_q})$, then ax is an element of cs iff Dy is an element of c for x in ℓ_p. This means that D is an element of $(\ell_p : c)$. As a consequence, by Lemma 3.2 from (8) and (11), it is deduced that

$$\left\{\ell_p(\widetilde{S_q})\right\}^\beta = \begin{cases} \mathfrak{b}_2 \cap \mathfrak{b}_4, & 0 \leqslant p < 1, \\ \mathfrak{b}_2 \cap \mathfrak{b}_5, & 1 \leqslant p < \infty. \end{cases}$$

□

4. Matrix transformations

In this section, let $\lambda \in \{c_0(\widetilde{S_q}), c(\widetilde{S_q}), \ell_p(\widetilde{S_q}), \ell_\infty(\widetilde{S_q})\}$ and $\mu \in \{c_0, c, \ell_\infty, \ell_1\}$. We provide necessary and sufficient conditions for matrix mappings from the spaces λ to any one of the spaces μ and from the spaces μ to the space λ.

Theorem 4.1. *Define, for all* $s, r \in \mathbb{N}_0$, $\mathcal{Z}^{(r)} = (z_{ms}^{(r)})$ *and* $\mathcal{Z} = (z_{rs})$ *by*

$$z_{ms}^{(r)} = \begin{cases} \sum_{i=s}^{m}(-1)^{s-i}\frac{S_{i+1}(q)-S_i(q)}{q^{s-i+1}S_s(q)}P_{s-i}(q)a_{ri}, & 0 \leqslant s \leqslant m, \\ 0, & s > m, \end{cases}$$

and

$$z_{rs} = \sum_{i=s}^{\infty}(-1)^{s-i}\frac{S_{i+1}(q)-S_i(q)}{q^{s-i+1}S_s(q)}P_{s-i}(q)a_{ri}.$$

In this case $\mathcal{A} = (a_{rs}) \in (\ell_p(\widetilde{S_q}) : \mu)$ *iff* $\mathcal{Z}^{(r)} \in (\ell_p : c)$ *for all* $r \in \mathbb{N}_0$ *and* $\mathcal{Z} \in (\ell_p : \mu)$.

Proof. Let $\mathcal{A} \in (\ell_p(\widetilde{S_q}) : \mu)$ and $x = (x_s) \in \ell_p(\widetilde{S_q})$. Next, we obtain the equality shown below

$$\sum_{s=0}^{m} a_{rs}x_s = \sum_{s=0}^{m}\left[\sum_{i=s}^{m}(-1)^{s-i}\frac{S_{i+1}(q)-S_i(q)}{q^{s-i+1}S_s(q)}P_{s-i}(q)a_{ri}\right]y_s. \tag{17}$$

Since $\mathcal{A}x$ exists, therefore $Z^{(r)} \in (\ell_p, c)$. Also, we get $\mathcal{A}x = \mathcal{Z}y$ by using $m \to \infty$ again as in (17). Given that $\mathcal{A}x \in \mu$, $\mathcal{Z}y \in \mu$ follows, with the result that $\mathcal{Z} \in (\ell_p, \mu)$.

On the other hand, suppose that $\mathcal{Z}^{(r)} \in (\ell_p, c)$ for all $r \in \mathbb{N}$ and that $\mathcal{Z} \in (\ell_p, \mu)$. Let $x = (x_s) \in \ell_p(\widetilde{S_q})$. Consequently, for each $r \in \mathbb{N}$, $\{z_{rs}\}_{s=0}^{\infty} \in \ell_p^{\beta}$, which means that $\{a_{rs}\}_{s=0}^{\infty} \in (\ell_p(\widetilde{S_q})^{\beta}$ for each $r \in \mathbb{N}$. Again from (17), $\mathcal{A}x = \mathcal{Z}y$ by as $m \to \infty$. This suggests that $\mathcal{A} \in (\ell_p(\widetilde{S_q}) : \mu)$. □

Theorem 4.2. *Let $\mathcal{A} = (a_{rs})$ be an infinite matrix and define the matrix $B = (b_{rs})$ by*

$$b_{rs} = \sum_{i=0}^{r} q^{r-i+1} \frac{S_i(q)S_{r-i}(q)}{S_{r+1}(q) - S_r(q)} a_{is} \tag{18}$$

for all $s, r \in \mathbb{N}$ and μ be a sequence space. Then, $A \in (\mu : \ell_p(\widetilde{S_q}))$ iff $B \in (\mu : \ell_p)$.

Proof. Let $z = (z_s) \in \mu$. Then, we have

$$\begin{aligned} \sum_{s=0}^{\infty} b_{rs} z_s &= \sum_{s=0}^{\infty} \left(\sum_{i=0}^{r} q^{r-i+1} \frac{S_i(q)S_{r-i}(q)}{S_{r+1}(q) - S_r(q)} a_{is} \right) z_s \\ &= \sum_{i=0}^{r} q^{r-i+1} \frac{S_i(q)S_{r-i}(q)}{S_{r+1}(q) - S_r(q)} \left(\sum_{s=0}^{\infty} a_{is} z_s \right). \end{aligned}$$

This yields $(Bz)_r = \left(\tilde{S}_q(\mathcal{A}z)\right)_r$ for all $r \in \mathbb{N}$. Hence, $\mathcal{A}z \in \ell_p(\widetilde{S_q})$ iff $Bz \in \ell_p$. □

Now, combining Theorem 4.1 and the matrix mapping characterization findings presented in Stieglitz and Tietz [49], we arrive at the following conclusions.

Corollary 4.3. *The following claims are accurate:*

(i) *$\mathcal{A} \in (\ell_p(\widetilde{S_q}) : c_0)$ iff*

$$\sup_{m \in \mathbb{N}_0} \sum_{s=0}^{\infty} \left| z_{ms}^{(r)} \right|^{p^*} < \infty, \tag{19}$$

$$\lim_{m \to \infty} z_{ms}^{(r)} \text{ exists for all } s \in \mathbb{N}_0 \tag{20}$$

hold, and

$$\lim_{r \to \infty} z_{rs} = 0 \text{ for all } s \in \mathbb{N}_0$$

also holds.

(ii) *$\mathcal{A} \in (\ell_p(\widetilde{S_q}) : c)$ iff (19) and (20) hold, and*

$$\sup_{r \in \mathbb{N}_0} \sum_{s=0}^{\infty} |z_{rs}|^{p^*} < \infty, \tag{21}$$

$$\lim_{r \to \infty} z_{rs} \text{ exists for all } s \in \mathbb{N}_0$$

also hold.

(iii) *$\mathcal{A} \in (\ell_p(\widetilde{S_q}) : \ell_\infty)$ iff (19), (20) and (21) hold.*

(iv) $\mathcal{A} \in (\ell_p(\widetilde{S}_q) : \ell_1)$ *iff (19) and (20) hold, and*

$$\sup_N \sum_{s=0}^{\infty} \left| \sum_{r \in N} z_{rs} \right|^{p^*} < \infty$$

Then, combining Theorem 4.2 and the matrix mapping characterization findings presented in Stieglitz and Tietz [49], we arrive at the following conclusions:

Corollary 4.4. *The following claims are accurate:*

(i) $\mathcal{A} \in (c_0 : \ell_p(\widetilde{S}_q)) = (c : \ell_p(\widetilde{S}_q)) = (\ell_\infty : \ell_p(\widetilde{S}_q))$ *iff*

$$\sup_K \sum_{r=0}^{\infty} \left| \sum_{s \in K} b_{rs} \right|^{p} < \infty$$

hold.

(ii) $\mathcal{A} \in (\ell_1 : \ell_p(\widetilde{S}_q))$ *iff*

$$\sup_s \sum_{r=0}^{\infty} |b_{rs}|^p < \infty$$

holds.

5. Schröder Core

Knopp was the first one to develop the idea of the core of a sequence [15, p. 137]. So, this initial form of core was known as the $\mathcal{K} - core$ or Knopp core.

$\mathcal{M} = (m_{rs}) \in (c : c)_{reg}$ is a non-negative matrix. In this part, the Schröder core (or $\widetilde{S} - core$) will be defined, and the matrix satisfying $\widetilde{S} - core(\mathcal{M}f) \subseteq \mathcal{K} - core(f)$ and $\widetilde{S} - core(\mathcal{M}f) \subseteq \mathcal{S} - core(f)$ for any bounded sequences f will be described.

Definition 5.1. *[15] The $\widetilde{S} - core$ of f is the intersection of all $\mathcal{H}_s$, where $\mathcal{H}_s$ be the least closed convex hull containing $\widetilde{S}_s(f)$, $\widetilde{S}_{s+1}(f), \ldots$. This can be expressed as*

$$\widetilde{S} - core(f) = \bigcap_{s=1}^{\infty} \mathcal{H}_s.$$

It should be noted that we define $\widetilde{S} - core$ of the function f by $\mathcal{K} - core$ of the sequence $(S_r(f))$.

The following theorem, which is an analogue of $\mathcal{K} - core$, may be created as a result [45]:

Theorem 5.2. *Take into account*

$$\mathcal{G}_f(t) = \left\{ \varepsilon \in \mathbb{C} : |\varepsilon - t| \leqslant \limsup_s |\widetilde{S}_s(f) - t| \right\},$$

for any $t \in \mathbb{C}$. Then, for any $f \in \ell_\infty$,

$$\widetilde{S} - core(f) = \bigcap_{t \in \mathbb{C}} \mathcal{G}_f(t).$$

Here are several lemmas that will help the key findings of this section. Consequently, we must describe the classes $(c : c(\widetilde{S}))_{reg}$ and $(\mathcal{S} \cap \ell_\infty : c(\widetilde{S}))_{reg}$. Now, let us take a matrix $\mathcal{M}' = (m'_{rs})$ in terms of $\mathcal{M} = (m_{rs})$ as

$$m'_{rs} = \sum_{s=0}^{r} \frac{S_s S_{r-s}}{S_{r+1} - S_r} m_{rs} \text{ for all } r, s \in \mathbb{N}.$$

Lemma 5.3. *$\mathcal{M} \in (\ell_\infty : c(\widetilde{S}))$ iff*

$$\|\mathcal{M}'\| = \sup_r \sum_s |m'_{rs}| < \infty, \tag{22}$$

$$\lim_r m'_{rs} = \alpha_s \text{ for each } s, \tag{23}$$

$$\lim_r \sum_s |m'_{rs} - \alpha_s| = 0. \tag{24}$$

Lemma 5.4. *$\mathcal{M} \in (c : c(\widetilde{S}))_{reg}$ iff the conditions (22) and (23) of the Lemma 5.3 hold with $\alpha_s = 0$ for all $s \in \mathbb{N}$ and*

$$\lim_r \sum_s |m'_{rs}| = 1. \tag{25}$$

Lemma 5.5. *$\mathcal{M} \in (\mathcal{S} \cap \ell_\infty : c(\widetilde{S}))_{reg}$ iff $\mathcal{M} \in (c : c(\widetilde{S}))_{reg}$ and*

$$\lim_r \sum_{s \in B} |m'_{rs}| = 0 \tag{26}$$

for every $B \subset \mathbb{N}$ with $\delta(B) = 0$.

Proof. Because of the fact that $c \subset \mathcal{S} \cap \ell_\infty$, $\mathcal{M} \in (c : c(\widetilde{S}))_{reg}$ holds. Now, for any $f \in \ell_\infty$ and a set $B \subset \mathbb{N}$ with $\delta(B) = 0$, let us define the sequence $f' = (f'_s)$ by

$$f'_s = \begin{cases} f_s, & s \in B \\ 0, & s \notin B. \end{cases}$$

Then, since $f' \in \mathcal{S}_0$, $\mathcal{M}f' \in c_0(\widetilde{S})$ and

$$\sum_s m'_{rs} t_s = \sum_{s \in B} m'_{rs} f_s,$$

the matrix $D = (d_{rs})$ defined by

$$d_{rs} = \begin{cases} m'_{rs}, & s \in B \\ 0, & s \notin B \end{cases}$$

is in the class $(\ell_\infty : c(\widetilde{S}))$. The need of (26) therefore derives from Lemma 5.3.

Let the opposite be true, $f \in \mathcal{S} \cap \ell_\infty$ with $\mathcal{S}-\lim f = l$. The set B formed by the equation $B = \{s : |f_s - l| \geq \varepsilon\}$ has density zero and $|f_s - l| \leqslant \varepsilon$ if s is not in the set B. We can now write

$$\sum_s m'_{rs} f_s = \sum_s m'_{rs}(f_s - l) + l \sum_s m'_{rs}. \tag{27}$$

Since

$$\left| \sum_s m'_{rs}(f_s - l) \right| \leqslant \|f\| \sum_{s \in \mathcal{M}} |m'_{rs}| + \varepsilon \cdot \|\mathcal{M}'\|,$$

letting $r \to \infty$ in (27) and using (25) with (26), we have

$$\lim_r \sum_s m'_{rs} f_s = l.$$

This implies that $\mathcal{M} \in (\mathcal{S} \cap \ell_\infty : c(\widetilde{S}))_{reg}$. □

Lemma 5.6. *[[46], Corollary 12] Let $\mathcal{M} = (m_{rs})$ be a matrix satisfying $\sum_s |m_{rs}| < \infty$ and $\lim_r m_{rs} = 0$. Then, there exists an $f \in \ell_\infty$ with $\|f\| \leqslant 1$ such that*

$$\limsup_r \sum_s m_{rs} f_s = \limsup_r \sum_s |m_{rs}|.$$

Theorem 5.7. *Let $\mathcal{M} \in (c : c(\widetilde{S}))_{reg}$. Then, $\widetilde{S} - core(\mathcal{M}f) \subseteq K - core(f)$ for all $f \in \ell_\infty$ iff*

$$\lim_r \sum_s |m'_{rs}| = 1. \tag{28}$$

Proof. The matrix $\mathcal{M} = (m'_{rs})$ satisfies the conditions of Lemma 5.6. So, there exists an $f \in \ell_\infty$ with $\|f\| \leqslant 1$ such that

$$\left\{\varepsilon \in \mathbb{C} : |\varepsilon| \leq \limsup_r \sum_s m'_{rs} f_s\right\} = \left\{\varepsilon \in \mathbb{C} : |\varepsilon| \leqslant \limsup_r \sum_s |m'_{rs}|\right\}.$$

On the other hand, since $\mathcal{K} - core(f) \subseteq \mathcal{M}_1(0)$, by the hypothesis

$$\left\{\varepsilon \in \mathbb{C} : |\varepsilon| \leq \limsup_r \sum_s |m'_{rs}|\right\} \subseteq \mathcal{M}_1(0) = \{\varepsilon \in \mathbb{C} : |\varepsilon| \leqslant 1\}$$

which implies (28).

Conversely, let $\varepsilon \in \widetilde{S} - core(\mathcal{M}f)$. Then, for any given $t \in \mathbb{C}$, we can write

$$\begin{aligned} |\varepsilon - t| &\leq \limsup_r |f_r(\mathcal{M}f) - t| \qquad (29)\\ &= \limsup_r \left| t - \sum_s m'_{rs} f_s \right| \\ &\leq \limsup_r \left| \sum_s m'_{rs}(t - f_s) \right| + \limsup_r |t| \left| 1 - \sum_s m'_{rs} \right| \\ &= \limsup_r \left| \sum_s m'_{rs}(t - f_s) \right|. \end{aligned}$$

Now, let $\limsup_s |f_s - t| = l$. Then, for any $\varepsilon > 0$, $|f_s - t| \leqslant l + \varepsilon$ whenever $s \geq s_0$. Hence, one can write that

$$\begin{aligned} \left| \sum_s m'_{rs}(t - f_s) \right| &= \left| \sum_{s<s_0} m'_{rs}(t - f_s) + \sum_{s \geq s_0} m'_{rs}(t - f_s) \right| \qquad (30)\\ &\leq \sup_s |t - f_s| \sum_{s<s_0} |m'_{rs}| + (l + \varepsilon) \sum_{s \geq s_0} |m'_{rs}| \\ &\leq \sup_s |t - f_s| \sum_{s<s_0} |m'_{rs}| + (l + \varepsilon) \sum_s |m'_{rs}|. \end{aligned}$$

As a result, by using $\limsup_r$ in accordance with the hypothesis and adding (29) to (30), we get

$$|\varepsilon - t| \leqslant \limsup_r \left| \sum_s m'_{rs}(t - f_s) \right| \leqslant l + \varepsilon,$$

which denotes that $\varepsilon \in \mathcal{K} - core(f)$. □

Theorem 5.8. *Let $\mathcal{M} \in (\mathcal{S} \cap \ell_\infty : c(\widetilde{S}))_{reg}$. Then, $\widetilde{S} - core(\mathcal{M}f) \subseteq \mathcal{S} - core(f)$ for all $f \in \ell_\infty$ if and only if (28) holds.*

Proof. Given that for each sequence f, $\mathcal{S} - core(f) \subseteq \mathcal{K} - core(f)$, the conclusion that the condition (28) is necessary derives from Theorem 5.7.

On the other hand, consider $\varepsilon \in \widetilde{S} - core(\mathcal{M}f)$. Then, we may write (29) once more. Now, if $\mathcal{S} - \limsup |f_s - t| = s$, then the set B defined by $B = \{s : |f_s - t| > s + \varepsilon\}$ has $\mathcal{M}$-density zero [21]. We can now write

$$\begin{aligned} \left|\sum_s m'_{rs}(t - f_s)\right| &= \left|\sum_{s\in\mathcal{M}} m'_{rs}(t - f_s) + \sum_{s\notin\mathcal{M}} m'_{rs}(t - f_s)\right| \\ &\leq \sup_s |t - f_s| \sum_{s\in\mathcal{M}} |m'_{rs}| + (s + \varepsilon) \sum_{s\notin\mathcal{M}} |m'_{rs}| \\ &\leq \sup_s |t - f_s| \sum_{s\in\mathcal{M}} |m'_{rs}| + (s + \varepsilon) \sum_{s} |m'_{rs}|. \end{aligned}$$

As a result, by utilizing the operator $\limsup_r$ and the condition (28) with (27), we may deduce that

$$\limsup_r \left|\sum_s m'_{rs}(t - f_s)\right| \leq s + \varepsilon. \tag{31}$$

Finally, combining (29) with (31), we have

$$|\varepsilon - t| \leqslant \mathcal{S} - \limsup_s |f_s - t|$$

which means that $\varepsilon \in \mathcal{S} - core(f)$. □

References

[1] H. S. Allen, T-transformations which leave the core of every bounded sequence invariant, J. London Math. Soc. **19**(1944), 42-46.
[2] B. Altay, F. Başar, Some Euler sequence spaces of nonabsolute type, Ukrainian Math. J., **57**(1) (2005), 1-17.
[3] B. Altay, F.Başar, M. Mursaleen, On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ I, Information Sciences, **176**(2006), 1450-1462.
[4] E. Barcucci, A. Del Lungo, E. Pergola, R. Pinzani, Some combinatorial interpretations of q-analogs of Schröder numbers, Annals of Combinatorics, **3** (1999), 171-190.
[5] F. Başar, Summability Theory and Its Applications, 2^{nd}ed., CRC Press/Taylor Frencis Group: Boca Raton, FL, USA, 2022
[6] F. Başar, M. Kirişçi, Almost convergence and generalized difference matrix, Comput. Math. Appl., **61**(3) (2011), 602-611.
[7] J. Bonin, L. Shapiro, R. Simon, Some q-analogues of the Schröder numbers arising from combinatorial statistics on lattice paths, J. Stat. Planning and Inference, **34** (1993), 35-55.
[8] J. Boos, Classical and Modern Methods in Summability, Oxford Science Publications, Oxford University Press, 2000.
[9] R. A. Brualdi, Introductory combinatorics, 5th ed, Upper Saddle River (NJ): Pearson Prentice Hall, 2010.
[10] M. Candan, Domain of the double sequential band matrix in the classical sequence spaces, J. Inequal. Appl., **281**(1) (2012), 15 pages.
[11] M. Candan, Some new sequence spaces derived from the spaces of bounded, convergent and null sequences, Int. J. Mod. Math. Sci., **12**(2) (2014), 74-87.
[12] M. Candan, E. E. Kara, A study on topological and geometrical characteristics of a new Banach sequence spaces, Gulf J. Math, **3**(4) (2015), 67-84.
[13] H-Q. Cao, H. Pan, A Stern-type congruence for the Schröder numbers, Discrete Math., **340** (2017), 708-712.
[14] J. Connor, J. A. Fridy, C. Orhan, Core equality results for sequences, J. Math. Anal. Appl. **321**(2006), 515-523.
[15] R. G. Cooke, Infinite Matrices and Sequence Spaces, Mcmillan, New York 1950.
[16] H. Çoşkun, C. Çakan, A class of statistical and σ-conservative matrices, Czechoslovak Math. J. **55**(3)(2005), 791-801.
[17] H. Çoşkun, C. Çakan, Mursaleen, On the statistical and σ-cores, Studia Math. **154**(1)(2003), 29-35.
[18] C. Çakan, H. Çoşkun, Some new inequalities related to the invariant means and uniformly bounded function sequences, Appl. Math. Lett. **20**(6) (2007), 605-609.
[19] M. C. Dağlı, Matrix mappings and compact operators for Schröder sequence spaces, Turkish Journal of Mathematics, **46** (2022), 2304-2320.
[20] M. C. Dağlı, A novel conservative matrix arising from Schröder numbers and its applications, Linear and Multilinear Algebra 71 (8) (2023) 1338–1351, https://doi.org/10.1080/03081087.2022.2061401
[21] K. Demirci, $A-$statistical core of a sequence, Demonstr Math. 2000; 33:43-51.
[22] M. Et, On some difference sequence spaces, Turk. J. Math, **17** (1993), 18-24.

[23] M. Et, R. Çolak, On some generalized difference sequence spaces, Soochow J. Math., **21**(4) (1995), 377-386.
[24] J. A. Fridy, C. Orhan, Statistical core theorems, J. Math. Anal. Appl. **208**(1997), 520-527.
[25] K.-G. Grosse-Erdmann, Matrix transformations between the sequence spaces of Maddox. J. Math. Anal. Appl.(1993) 180(1):223–238.
[26] F. Gökçe, M. A. Sarıgöl, Generalization of the space $\ell(p)$ derived by absolute Euler summability and matrix operators, J. Inequal. Appl., **2018**; 2018:133
[27] G. C. Hazar, M. A. Sarıgöl, Absolute Cesàro series spaces and matrix operators, Acta Appl. Math., **154**(1) (2018), 153-165.
[28] M. İlkhan, A new conservative matrix derived by Catalan numbers and its matrix domain in the spaces c and c_0, Linear and Multilinear Algebra, 68 (2) (2020) 417-434. https://doi.org/10.1080/03081087.2019.1635071
[29] M. İlkhan, E. E. Kara, Matrix transformations and compact operators on Catalan sequence spaces, J. Math. Anal. Appl., **498** (2021), 124925.
[30] V. Kaç, P. Cheung, Quantum Calculus, Springer, New York, (2002).
[31] E. E. Kara, M. Başarır, An application of Fibonacci numbers into infinite Toeplitz matrices, Caspian Journal of Mathematical Sciences, **1**(1) (2012), 43-47.
[32] E. E. Kara, Some topological and geometrical properties of new Banach sequence spaces, Journal of Inequalities and Applications, **2013**; 2013:38.
[33] M. Karakaş, H. Karabudak, An application on the Lucas numbers and infinite Toeplitz matrices, Cumhuriyet Sci. J., **38**(3) (2017), 557-562.
[34] M. Karakaş, A. M. Karakaş, New Banach sequence spaces that is defined by the aid of Lucas numbers, Iğdır Univ. J. Inst. Sci. Tech., **7**(4) (2017), 103-111.
[35] M. Karakaş, M. C.Dağlı, Some topological and geometrical properties of new Catalan sequence spaces, Advances in Operator Theory, (2023)8:14
[36] M. Karakaş, On the sequence spaces involving Bell numbers, Linear and Multilinear Algebra, 71 (14) (2023) 2298-2309. https://doi.org/10.1080/03081087.2022.2098225
[37] T. Koshy, Fibonacci and Lucas Numbers with Applications. Wiley, 2001.
[38] C. G. Lascarides, I.J. Maddox, Matrix transformations between some classes of sequences, Proc. Camb. Phil. Soc. **68**(1970), 99-104.
[39] J. C. Liu, Some congruences for Schröder type polynomials, Colloq. Math., **146** (2017), 187-195.
[40] E. Malkowsky, Recent results in the theory of matrix transformations in sequence spaces, Mat. Vesnik **49**(1997), 187-196.
[41] M. Mursaleen, F. Başar, Sequence spaces: Topic in Modern Summability Theory, CRC Press,Taylor Frencis Group, Series:Mathematics and its applications, Boca Raton, London, New York, 2020.
[42] P.-N. Ng, P.-Y. Lee, Cesàro sequences spaces of non-absolute type, Comment Math. Prace Mat. **20**(2)(1978), 429-433.
[43] H. Roopaei, T. Yaying, On Quasi-Cesàro matrix and associated sequence spaces, Turkish J. Math., **45**(1) (2021), 153-166.
[44] M. A. Sarıgöl, Spaces of series summable by absolute Cesàro and matrix operators, Commun. Math. Appl., **7**(1) (2016), 11-22.
[45] A. A. Shcherbakov, Kernels of sequences of complex numbers and their regular transformations, Math. Notes **22**(1977), 948-953.
[46] S. Simons, Banach limits, infinite matrices and sublinear functionals, J. Math. Anal. Appl., 26 (1969), 640–655.
[47] H. Steinhaus, Quality control by sampling, Collog. Math. **2**(1951), 98-108.
[48] H. M. Srivastava, Operators of basic q-calculus and fractional q-calculus and their applications in geometric function theory of complex analysis, Iran J. Sci. Technol. Sci., **44** (2020), 327-344.
[49] M.Stieglitz, H.Tietz, Matrix transformationen von folgenräumen eine ergebnisübersicht, Math. Z. **154** (1977), 1-16.
[50] M. Şengönül, F. Başar, Some new Cesàro sequence spaces of non-absolute type which include the spaces c_0 and c, Soochow J. Math. **31**(1) (2005), 107-119.
[51] T. Yaying, B. Hazarika, M. Ilkhan, M.Mursaleen, Poisson like matrix operator and its application in $p-$summable space, Math. Slovaca, **71**(5) (2021), 1189-1210.
[52] T. Yaying, B. Hazarika, S. A. Mohiuddine, Domain of Padovan q-difference matrix in sequence spaces ℓ_p and ℓ_∞, Filomat, **36**(3) (2022), 905-919.
[53] T. Yaying, B. Hazarika, O. M. Kalthum S. K. Mohamed, A. A. Bakery, On new Banach sequence spaces involving Leonardo numbers and the associated mapping ideal, Journal of Function Spaces, Volume 2022, Article ID: 8269000, 21 Pages.
[54] C.-S.Wang, Nörlund sequence spaces, Tamkang J. Math. **9**(1978), 269-274.
[55] A. Wilansky, Summability through functional analysis, Vol.85, Elsevier, 2000.
[56] P. Zengin Alp, A new paranormed sequence space defined by Catalan conservative matrix, Math. Meth. Appl. Sci., (2020), 1-8.