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Pricing options in new generalized fractional Black-Scholes model

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Abstract. The recent introduction of significant features in conformable integrals and derivatives has paved the way for advancements in the field of fractional calculus [1–4]. This study provides a comprehensive exploration of this emerging area, with a particular focus on the various definitions and distinct fractional derivatives that have been proposed. Of particular interest is the concept of "new conformable derivatives," as introduced in [1], which we thoroughly investigate.

$$(\mathcal{D}^{\alpha}F)(t) = \lim_{l\longrightarrow 0} \frac{F\left(t+le^{(\alpha-1)t}\right)-F(t)}{l},$$

where $\alpha \in (0,1]$, this derivative is explored in terms of its origin, unique characteristics, and how it compares to other conformable fractional derivatives. Furthermore, the study extends its analysis to the practical applications of these derivatives in financial mathematics. Specifically, we examine the construction of a new fractional Black-Scholes option pricing model, highlighting the potential of these mathematical tools in addressing complex problems in finance. This investigation not only enriches the theoretical framework of fractional calculus but also opens up new avenues for applying these concepts in real-world scenarios.

1. Introduction

The theory of fractional derivatives is a branch of mathematics with a history that stretches back nearly as far as classical calculus itself. Its origins can be traced to the late 17th century, a period when Sir Isaac Newton and Gottfried Wilhelm Leibniz were laying down the foundational principles of differential and integral calculus that have become central to modern mathematics. Leibniz, in particular, contributed significantly to this emerging field by introducing the notation $\frac{d^n f}{dt^n}$ to represent the nth derivative of a function f. This notation, which is now a fundamental aspect of calculus, was originally intended to apply where n was a natural number.

However, even in its infancy, the concept of extending these operators to non-integer, or fractional, orders was considered. In a famous letter dated September 30, 1695, Leibniz posed the question to L'Hôpital about the meaning of a derivative of order $\frac{1}{2}$. This seemingly abstract query laid the groundwork for what would

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become the field of fractional calculus, an area of study that explores the generalization of integrals and derivatives to non-integer orders.

While the initial reaction to fractional derivatives was largely theoretical and speculative, the concept gradually gained mathematical rigor and practical significance over the centuries. The development of fractional calculus was not linear but rather sporadic, with contributions made by various mathematicians over different periods. Notable figures such as Euler, Fourier, and Liouville made significant strides in the 18th and 19th centuries, each building upon the work of their predecessors and expanding the understanding of fractional operations.

In the 20th and 21st centuries, fractional calculus has found numerous applications across a wide range of fields, including physics, engineering, finance, biology, and control theory. The fractional derivative offers a more accurate modeling tool for systems that exhibit anomalous behavior or memory effects, where the rate of change is not adequately described by integer-order derivatives. For example, in viscoelastic materials, diffusion processes, and chaotic systems, fractional derivatives can describe phenomena where the current state depends not only on the immediate past but on the entire history of the process.

This growing importance of fractional calculus in applied mathematics has led to the development of various definitions and approaches to fractional differentiation and integration, such as the Riemann-Liouville, Caputo, and Grünwald-Letnikov derivatives, each suited to different types of problems and applications. As research continues to evolve, fractional calculus is being increasingly recognized as a valuable tool in both theoretical investigations and practical applications, providing deeper insights into the behavior of complex systems.

Thus, what began as an abstract question posed by Leibniz has grown into a comprehensive and indispensable field, with fractional derivatives offering a powerful extension to the traditional calculus that has shaped much of modern mathematics and science. As this field continues to develop, it promises to yield even more profound applications and insights, further solidifying its role in both theoretical and applied mathematics.

The first comprehensive formalization of these local operators emerged in 2014, marking a pivotal advancement in the field of fractional calculus. This development introduced a differential operator that significantly transformed non-integer order calculus.

This breakthrough addressed several limitations associated with global operators. In their influential paper "A New Definition of the Fractional Derivative" (refer to [2]), R. Khalil and colleagues introduced the innovative concept of the "Conformable Derivative." This new definition aimed to refine and enhance the understanding and application of fractional derivatives, leading to improved solutions and insights in fractional calculus. The conformable derivative represents a substantial shift in the approach to fractional differentiation, facilitating more accurate and effective handling of various mathematical and engineering problems.

Let $\alpha \in (0,1]$ and $F : (0,+\infty) \to \mathbb{R}$. The conformable derivative of order α at $t_0 > 0$ is defined by

$$\mathcal{D}^{(\alpha)}F(t_0) = \lim_{\varepsilon \to 0} \frac{F\left(t_0 + \varepsilon t_0^{1-\alpha}\right) - F(t_0)}{\varepsilon}, \tag{1}$$

if this limit exists.

Indeed, this novel approach to differentiation maintains all the standard properties of classical differentiation, except for the semigroup property. While R. Khalil et al. are credited as the pioneers of this new concept, T. Abdeljawad played a crucial role in establishing its theoretical framework through his seminal paper, "On the Conformable Fractional Calculus" [3].

Further advances were made by A. A. Abdelhakim in 2019, who demonstrated in his paper [4] that the existence of the limit defining the conformable derivative is equivalent to the conventional notion of differentiability. His work not only reaffirmed the fractional nature of the approach but also highlighted its theoretical significance, as originally established by R. Khalil, T. Abdeljawad, and other contributors.

This ongoing development has sparked considerable debate among researchers. Proponents argue that the conformable fractional derivative provides a more intuitive and applicable framework for fractional calculus, while critics question its generality and applicability compared to traditional fractional calculus methods. This discourse continues to shape the field, reflecting both the potential and the challenges of integrating new concepts into established mathematical theories.

This dissertation examines the article by D. R. Anderson and D. J. Ulness, published in 2015, titled "Newly Defined Conformable Derivatives" [5]. The definition presented in their work builds upon the conventional notion of differentiability and extends the approach originally introduced by R. Khalil et al. In this dissertation, we first review the concept of conformable differentiability as defined in [5]. We then outline the principal calculation rules established in that paper without providing proofs. Following this, we introduce several new properties of conformable derivatives. Finally, the aim of this dissertation is to apply and update the results from [1] to develop a new conformable Black-Scholes model.

2. Conformable Fractional calculus

Definition 2.1. *Let $L : [0, \infty) \to \mathbb{R}$ and $t > 0$. Then the fractional derivative of L of order α is defined by,*

$$\mathcal{D}^{\alpha}(L)(t) = \lim_{\epsilon \to 0} \frac{L\left(te^{\epsilon t^{-\alpha}}\right) - L(t)}{\epsilon}, \tag{2}$$

for $t > 0, \alpha \in (0, 1)$. If L is α-differentiable in some $(0, b), b > 0$, and $\lim_{t \to 0^+} \mathcal{D}^{\alpha}(L)(t)$ exists, then define

$$\mathcal{D}^{\alpha}(L)(0) = \lim_{t \to 0^+} \mathcal{D}^{\alpha}(L)(t). \tag{3}$$

Theorem 2.2. *If a function $L : [0, \infty) \to \mathbb{R}$ is α-differentiable at $b > 0, \alpha \in (0, 1]$, therefore, L is continuous at b.*

Proof. Since $L\left(be^{\epsilon b^{-b}}\right) - L(b) = \frac{L\left(be^{\epsilon b^{-b}}\right) - L(b)}{\epsilon}\epsilon$, we have

$$\lim_{\epsilon \to 0}\left[L\left(be^{\epsilon b^{-b}}\right) - L(b)\right] = \lim_{\epsilon \to 0} \frac{L\left(be^{\epsilon b^{-\alpha}}\right) - L(b)}{\epsilon} \cdot \lim_{\epsilon \to 0} \epsilon.$$

□

Definition 2.3. *[1] Given a function $L : [0, \infty) \longrightarrow \mathbb{R}$, and then the conformable fractional derivative of L order α is defined by*

$$\left(D^{\alpha}L\right)(t) = \lim_{k \longrightarrow 0} \frac{L\left(t + ke^{(\alpha-1)t}\right) - L(t)}{k}, \tag{4}$$

for all $t > 0$, and $\alpha \in (0, 1)$. If L is α differentiable in some $(0, a), a > 0$, and $\lim_{t \to 0^+} \left(D^{\alpha}L\right)(t)$ exists, then define

$$\left(D^{\alpha}L\right)(0) = \lim_{t \longrightarrow 0^+} \left(D^{\alpha}L\right)(t). \tag{5}$$

Theorem 2.4. *[1] If a function $L : [0, +\infty) \longrightarrow \mathbb{R}$ and α differentiable at $t_0 > 0$, then L is continuous at t_0.*

Theorem 2.5. *If L be α differentiable at a point $t > 0$.*

1. $\mathbf{D}^{\alpha}(aL + bL) = a\left(\mathbf{D}^{\alpha}L\right) + b\left(\mathbf{D}^{\alpha}L\right)$, *for all $a, b \in \mathbb{R}$.*
2. $\mathbf{D}^{\alpha}\left(t^{n}\right) = ne^{(\alpha-1)t}t^{p-1}$ *for all $n \in \mathbb{R}$.*
3. $\mathbf{D}^{\alpha}(\beta) = 0$, *for all constant $L(t) = \beta$.*
4. $\left(\mathbf{D}^{\alpha}LH\right) = L\left(\mathbf{D}^{\alpha}H\right) + H\left(\mathbf{D}^{\alpha}L\right)$.
5. $\left(\mathbf{D}^{\alpha}(L/H)\right) = \left(L\left(\mathbf{D}^{\alpha}H\right) + H\left(\mathbf{D}^{\alpha}L\right)\right)/H^{2}$.
6. *If L is differentiable, then* $\left(\mathbf{D}^{\alpha}L\right)(t) = e^{(\alpha-1)t}L'(t)$.

Proof. (1) to (5) see [1], for (6) we have

$(D^{\alpha}L)(t) = \lim_{L\longrightarrow 0} \frac{L(t+he^{(\alpha-1)t})-L(t)}{h}(D^{\alpha}L)(t)$

$= \lim_{\varepsilon\longrightarrow 0} \frac{L(t+\varepsilon)-L(t)}{\varepsilon e^{(1-\alpha)t}}\left(D^{\alpha}L\right)(t)$

$= e^{(\alpha-1)t}\lim_{\varepsilon\longrightarrow 0} \frac{L(t+\varepsilon)-L(t)}{\varepsilon}\left(D^{\alpha}L\right)(t)$

$= e^{(\alpha-1)t}L'(t).$ □

3. Application to fractional Black-Scholes model

Mathematicians have long sought to address the intricate problems presented by the financial sector. These problems often exhibit seemingly erratic behavior, as seen in the volatility and unpredictability of the stock market. Probabilistic models, adept at managing randomness and uncertainty, are particularly effective for analyzing such financial phenomena.

In 1973, Fischer Black and Myron Scholes made a significant contribution with their introduction of a formula for pricing European call options, now known as the Black-Scholes formula. This model transformed financial mathematics by offering a systematic method for option valuation, taking into account variables such as the price of the underlying asset, the strike price of the option, the time remaining until expiration, the risk-free interest rate, and the asset's volatility. The Black-Scholes formula has since become a fundamental element in modern financial theory and practice.

Its impact goes well beyond theoretical realms; the formula is widely employed by traders, financial analysts, and risk managers. The volatility measure derived from the Black-Scholes model has become a key tool in financial markets, used as a standard for evaluating market conditions and pricing various financial instruments.

Over time, the Black-Scholes model has been refined and expanded to address its initial limitations and to integrate additional elements, such as dividends and varying volatility. Despite these advancements, the original formula remains a crucial and influential tool for understanding and forecasting financial market dynamics. The ongoing importance of the Black-Scholes model highlights its role in the evolution of quantitative finance and its lasting influence on financial theory and practice.

3.1. *The Black-Scholes Option Pricing equation (1973):*

It's essential to present the Black-Scholes model by detailing the well-known formula that provides the pricing of European call options in its simplest form, assuming constant parameters. The Black-Scholes model articulates the formula for the call price C and the put price P as follows:

$$C_{\text{call}} = S\phi(D_1) - Xe^{-RT}\phi(D_2), \tag{6}$$

$$P_{\text{put}} = Xe^{-RT}\phi(-D_2) - S\phi(-D_1), \tag{7}$$

where

$$D_1 = \frac{\log(S/X) + \left(R + \sigma^2/2\right)T}{\sigma\sqrt{T}}, \tag{8}$$

$$D_2 = D_1 - \sigma\sqrt{T}. \tag{9}$$

In these equations, S denotes the current price of the underlying asset, while X represents the strike price of the option. The risk-free interest rate is denoted by R, and T stands for the time remaining until the option's expiration. The volatility of the underlying asset is given by σ, which measures the asset's price fluctuations over time.

The function ϕ in the formulas refers to the cumulative distribution function (CDF) of the standard normal distribution, which is used to calculate the probabilities associated with the option's payoff. Specifically, $\phi(D_1)$ and $\phi(D_2)$ represent the probabilities that the option will be in-the-money at expiration, adjusted for the current time and volatility.

These Black-Scholes formulas are crucial for determining the theoretical prices of European call and put options, providing a benchmark for traders and investors. The model's ability to incorporate key factors such as asset price, strike price, interest rates, and volatility into a coherent pricing framework underscores its importance in financial markets. It facilitates informed decision-making by offering a method to estimate option values and assess risk.

The Black-Scholes model has had a profound and lasting impact on quantitative finance, influencing both theoretical research and practical trading strategies. Its widespread use has led to the development of various extensions and modifications to address additional complexities, such as dividends and varying volatility. Despite these advancements, the foundational Black-Scholes formulas continue to be central to option pricing and financial analysis, highlighting the model's enduring relevance and significance.

3.2. *Model of Black-Scholes and Stochastic formula:*

The derivation of the B-S using the stochastic differential equation and Ito's Lemma, defined by equation:

$$\frac{dS}{S} = vdT + \sigma dX \tag{10}$$

is called the stochastic differential equation. And, from (10) we have the Wiener process with the following properties:

$$E(dX) = 0, E(dX)^2 = dt, E(dS) = \sigma^2 S^2 dX^2. \tag{11}$$

Therefore, σ is proportional to

$$\frac{\sqrt{\mathrm{Var}(ds)}}{S}. \tag{12}$$

This final result is called Stochastic model .

3.3. *Derivation of the B-S equation*

The derivation of the Black-Scholes equation using the equation (10) and Ito's Lemma, we can write

$$dH = \left(\mu S \frac{\partial H}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 H}{\partial S^2} + \frac{\partial H}{\partial t}\right) dt + \sigma S \frac{\partial H}{\partial S} dX_t. \tag{13}$$

The B-S idea is first to find this proportion Δ so that the portfolio becomes deterministic. Note that the value of this portfolio is

$$\Pi(t) = H - \Delta S. \tag{14}$$

The change in the value of this portfolio in one time-step dt is

$$d\Pi(t) = dH - \Delta dS. \tag{15}$$

Substituting (10) and (11) into (13), we have

$$d\Pi(t) = \left(\mu S \frac{\partial H}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 H}{\partial S^2} + \frac{\partial H}{\partial t} - \mu \Delta S\right) dt + \sigma S \left(\frac{\partial H}{\partial S} - \Delta\right) dX_t. \tag{16}$$

But if we choose $\Delta = \partial H/\partial S$, then the stochastic term is zero, and (16) becomes

$$d\Pi = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 H}{\partial S^2} + \frac{\partial H}{\partial t}\right)dt. \tag{17}$$

And so the choice

$$\Delta = \frac{\partial H}{\partial S} \tag{18}$$

reduces the stochastic expression into a deterministic expression. Thus, we should have $d\Pi = R\Pi dt$, and hence by (17),

$$R\Pi dt = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 H}{\partial S^2} + \frac{\partial H}{\partial t}\right)dt. \tag{19}$$

Now replace Π in (19) by $H - \Delta S$ as given in , and replace Δ by $\partial H/\partial S$ as given in (16), and then divide both sides by dt. We arrive at

$$\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 H}{\partial S^2} + RS\frac{\partial H}{\partial S} - RH = 0 \tag{20}$$

with the initial condition :

$$H(x.0) = H_x(0),$$

and

$$H(x, T) = x - c, x \geq c. \tag{21}$$

3.4. *Solution of the B-S equation*

The solution of the B.S equation is represented by equation :

$$H(x, t) = xN(D_1) - ce^{R(t-T)}N(D_2), \tag{22}$$

with

$$D_1 := \frac{\ln \frac{\pi}{c} + \left(R + \frac{1}{2}v^2\right)(T - t)}{v\sqrt{(T - t)}}, \tag{23}$$

and

$$D_2 := \frac{\ln \frac{x}{c} + \left(R - \frac{1}{2}v^2\right)(T - t)}{H\sqrt{(T - t)}}, \tag{24}$$

where $N(D)$ is the cumulative Laplace-Gaussian . In today's world, finance plays a very important role and is sometimes the origin of global crises. It then appears important that finance is based on models solid data for assessing risks and prices. From this necessity, the model and formula of Black-Scholes has established itself as a reference since 1973, in option calculation. Despite its flaws, this model is successful because it has many advantages: its simplicity of application and formula, its significant use by market operators but also and above all because it allows you to calculate an important parameter in finance. There variability measures the average variation over time of a financial asset and therefore gives an critical risk information.

The Black-Scholes formula can be demonstrated rigorously if a certain number of conditions are established. We then talk about the Black-Scholes model, or we say that we are in the Black-Scholes case. Financial markets fit this model quite well, but not exactly of course and, in particular, contrary to the central hypothesis of the model, time is not continuous. There is therefore a certain gap between this model and reality, which can become significant when the markets are agitated with frequent price discontinuities.

3.5. Derivation of new conformable Fractional Black-Scholes Model

Taking B-S PDE as an example, we have the following stochastic formula [6]:

$$RSdt + \sigma v(t)(dt)^{\alpha/2} = dS, \quad 0 < \alpha \leq 1, \tag{25}$$

with σ and S : volatility and stock value, R is the risk interest rate, and $v(t)$ represents the Wiener deviation. With the case of constant results (expressed in β), the above formula becomes .

$$(R - \beta)Sdt + \sigma v(t)(dt)^{\alpha/2} = dS, \quad 0 < \alpha \leq 1. \tag{26}$$

According to Jumarie [7], we use some properties, which support the score Jumarie Taylor in [6]:

$$\begin{aligned} &\frac{1}{\Gamma(2-\alpha)} e^{(\alpha-1)t}(dt)^{\alpha} = d^{\alpha}t, \quad 0 < \alpha \leq 1, \\ &\Gamma(\alpha+1)dS = d^{\alpha}S, \quad 0 < \alpha \leq 1, \end{aligned} \tag{27}$$

and

$$\frac{d^{\alpha}S}{(dS)^{\alpha}} = \frac{1}{\Gamma(2-\alpha)} e^{(\alpha-1)S}, \quad 0 < \alpha \leq 1. \tag{28}$$

Combining (26) and (27) gives a formula by which all unit percentages can be converted to unit percentages and vice versa:

$$dS = \frac{e^{(\alpha-1)S}}{\Gamma(\alpha+1)\Gamma(2-\alpha)}(dS)^{\alpha}, \quad 0 < \alpha \leq 1. \tag{29}$$

Assume that $H = H(S, t)$ is the cost of the European value, and we have the equation

$$dH = RHdt. \tag{30}$$

Multiplying both sides of (30) by $\Gamma(1 - \alpha)$ gives us

$$\Gamma(1-\alpha)dH = \Gamma(1-\alpha)RHdt. \tag{31}$$

Now, combining (31) and (27) gives the formula:

$$d^{\alpha}H = \Gamma(\alpha+1)RHdt. \tag{32}$$

Equation (32) together with (29) gives the following dynamic equation :

$$d^{\alpha}H = \frac{RH}{\Gamma(2-\alpha)} e^{(\alpha-1)t}T(dt)^{\alpha}. \tag{33}$$

Since $H(S, t)$ is smooth in S and the α derivative with respect to t remains the same, fractional taylor series is used in $H(S, t)$ of ordre α until the remaining error

$$\frac{1}{\Gamma(\alpha+1)} \frac{\partial^{\alpha}H}{\partial t^{\alpha}}(dt)^{\alpha} + \frac{\partial H}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 = dH. \tag{34}$$

Combining this equation with Ito's lemma and equation (26) results in

$$\frac{1}{\Gamma(\alpha+1)} \frac{\partial^{\alpha}H}{\partial t^{\alpha}}(dt)^{\alpha} + (R-\beta)S\frac{\partial H}{\partial S}dt + \frac{1}{2}S^2\sigma^2\frac{\partial^2 H}{\partial S^2}(dt)^{\alpha} = dH. \tag{35}$$

Using the transformation method (29) but in terms of t replace dt in (35) with .

$$dt = \frac{e^{(\alpha-1)t}(dt)^{\alpha}}{\Gamma(\alpha+1)\Gamma(2-\alpha)}, \tag{36}$$

$$dH = \frac{1}{\Gamma(\alpha+1)}\frac{\partial^{\alpha} H}{\partial t^{\alpha}}(dt)^{\alpha} + \frac{(R-\beta)}{\Gamma(\alpha+1)\Gamma(2-\alpha)} S e^{(\alpha-1)t}\frac{\partial H}{\partial S}(dt)^{\alpha} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 H}{\partial S^2}(dt)^{\alpha}. \tag{37}$$

Multiplying both sides of (37) with $\Gamma(\alpha+1)$ yields

$$\Gamma(\alpha+1)dH = \left(\frac{\partial^{\alpha} H}{\partial t^{\alpha}} + \frac{(R-\beta)}{\Gamma(2-\alpha)} S e^{(\alpha-1)t}\frac{\partial H}{\partial S} + \frac{\Gamma(\alpha+1)}{2}S^2\sigma^2\frac{\partial^2 H}{\partial S^2}\right)(dt)^{\alpha}. \tag{38}$$

Using (37), the left side of (38) it can be rewritten as

$$\begin{aligned}\Gamma(\alpha+1)dH &= d^{\alpha}H \\ &= \frac{RH}{\Gamma(2-\alpha)}e^{(\alpha-1)t}(dt)^{\alpha}.\end{aligned} \tag{39}$$

Using (39), along with (38), yields

$$\frac{RH}{\Gamma(2-\alpha)}e^{(\alpha-1)t} = \frac{\partial^{\alpha} H}{\partial t^{\alpha}} + \frac{(R-\beta)}{\Gamma(2-\alpha)} S e^{(\alpha-1)t}\frac{\partial H}{\partial S} + \frac{\Gamma(\alpha+1)}{2}S^2\sigma^2\frac{\partial^2 H}{\partial S^2}. \tag{40}$$

Equation (40) can be converted to BS-PDE:

$$\frac{\partial^{\alpha} H}{\partial t^{\alpha}} = \left(RH - NS\frac{\partial H}{\partial S}\right)\frac{e^{(\alpha-1)t}}{\Gamma(2-\alpha)} - \frac{\Gamma(\alpha+1)}{2}\sigma^2 S^2\frac{\partial^2 H}{\partial S^2}, \quad N = R-\beta, 0 < \alpha \leq 1. \tag{41}$$

with the terminal condition and following boundary :

$$H(S,0) = Max(K-S,0), \qquad H(0,t) = Ke^{R(t-T)}, \qquad \lim_{S\to\infty} H(S,t) = 0, \tag{42}$$

T is the expiration date and K is the strike price of the European option .

Remark 3.1. *For $\alpha = 1$ we coincide with classical form of B-S formula* (20).

3.6. *The solution of the new conformable Black-Scholes equation:*

3.6.1. *Derivation of a new conformable B-S:*

We present on the solution of the Black-Scholes formula

$$H_t^{(\alpha)}(s,t) = (RH - NSH_s)\frac{e^{(\alpha-1)t}}{\Gamma(2-\alpha)} - \frac{\Gamma(\alpha+1)}{2}\sigma^2 S^2 H_{ss} \tag{43}$$

with the condition $H(s,T)$ defined by (42).

(Step 1) Deleting the rH-term. If $H(s,t)$ is differentiable w.r.t. time , therefore we gets the followig change of variable

$$H(s,t) = \mathrm{e}^{R(t-T)}\tilde{H}(s,t), \tag{44}$$

but if $H(s,t)$ is not differentiable w.r.t time, we will settle down

$$H(s,t) = E_{\alpha}\left(R(t-T))^{\alpha}\right)\tilde{H}(s,t). \tag{45}$$

then, we have the equation

$$H_t^{(\alpha)}(s,t) = \left(D_t^\alpha \mathrm{e}^{R(t-T))}\right)\tilde{H}(s,t) + \mathrm{e}^{R(t-T)}\tilde{H}_t^{(\alpha)}(s,t)$$
$$= R\mathrm{e}^{R(t-T)}\frac{e^{(\alpha-1)t}}{\Gamma(2-\alpha)}\tilde{H}(s,t) + \mathrm{e}^{R(t-T)}\tilde{H}_t^{(\alpha)}(s,t),$$

and substituting into (43), we obtain the form

$$\tilde{H}_t^{(\alpha)}(s,t) = -RS\frac{e^{(\alpha-1)t}}{\Gamma(2-\alpha)}\tilde{H}_s(s,t) - \frac{\Gamma(\alpha+1)}{2}\sigma^2 S^2\tilde{H}_{ss}(s,t). \tag{46}$$

with the terminal condition

$$\tilde{H}(s,T) = H(s,T). \tag{47}$$

The same equation (46) is obtained with the transformation (45), by virtue see [8] we remember the definition of modified Riemann-Liouville's derivative. The solution of the fractional equation

$$z^{(\alpha)}(t) = \lambda z(t), \quad t \geq 0, \quad z(0) = z_0, 0 < \alpha \leq 1, \tag{48}$$

λ constant, is

$$z(t) = z_0 E_\alpha\left(\lambda t^\alpha\right), \tag{49}$$

where $E_\alpha(.)$ is the Mittag-Leffler function.

(Step 2) The conformable derivation PDE with const coefficient. The presence of $S\tilde{H}_s$ and $S^2\tilde{H}_{ss}$ in the Eq. (46) is recommended to change the variables

$$y = \ln s + b, \tag{50}$$

with b denotes a const, and search for a solution in the form

$$\tilde{H}(s,t) \equiv Z(y,t). \tag{51}$$

where the terminal condition

$$Z(y,T) = H(s,T) = H\left(\mathrm{e}^{y-b}, T\right). \tag{52}$$

And indeed, on substituting (50) into (46) yields

$$Z_t^{(\alpha)}(y,t) = \left(\frac{\Gamma(\alpha+1)}{2}\sigma^2 - R\frac{e^{(\alpha-1)t}}{\Gamma(2-\alpha)}\right)Z_y(y,t) - \frac{\Gamma(\alpha+1)}{2}\sigma^2 Z_{yy}(y,t). \tag{53}$$

(Step 3) Corresponds to first-order solutions of fractional partial differential equations. To suggest the solution to (53), we first consider a discrete problem.

$$\tilde{Z}^{(\alpha)}(y,t) + \left(R\frac{e^{(\alpha-1)t}}{\Gamma(2-\alpha)} - \frac{\Gamma(\alpha+1)}{2}\sigma^2\right)\tilde{Z}_y(y,t) = 0. \tag{54}$$

We apply the appropriate extension of the Lagrange characteristics method to obtain its solution that we recently proposed for this type of equations [9]. On the other hand, we consider the linear system associated with (54) which is written

$$\frac{(\mathrm{d}t)^\alpha}{1} = \frac{\mathrm{d}^\alpha y}{R\frac{e^{(\alpha-1)t}}{\Gamma(2-\alpha)} - \frac{\Gamma(\alpha+1)}{2}\sigma^2}. \tag{55}$$

The formula on the right gives the following first integral

$$\tilde{Z}(y,t) = \text{ const.} \tag{56}$$

The formula on the left, rewritten in the form

$$R\frac{e^{(\alpha-1)t}}{\Gamma(2-\alpha)}(\mathrm{d}t)^{\alpha} - \sigma^{2}\frac{\Gamma(\alpha+1)}{2}(\ \mathrm{d}t)^{\alpha} = \mathrm{d}^{\alpha}y, \tag{57}$$

give the second integral

$$y - Rt + (1/2)\sigma^{2}e^{(1-\alpha)t} = \text{ const.} \tag{58}$$

Then, we obtain the general solution

$$\tilde{Z}(y,t) = \psi\left(y - Rt + (1/2)\sigma^{2}e^{(1-\alpha)t}\sigma^{2}\right). \tag{59}$$

(Step 4) Derivation of new fractional equation. This result (59) suggests looking for $Z(y,t)$ in the form

$$Z(y,t) = G(k,t), \tag{60}$$

with

$$k = y - \ln c + R(T-t) - \frac{1}{2}\sigma^{2}e^{(T-t)(1-\alpha)}, \tag{61}$$

effectively, the constant b in (50) is selected in the form $b = -\ln c + RT - \frac{1}{2}\sigma^{2}e^{T(1-\alpha)}$. On substituting (59) into (53), we observe the new conformable fractional equation

$$G_{t}^{(\alpha)}(k,t) = -\gamma^{2}G_{kk}(k,t), \tag{62}$$

with

$$\gamma^{2} = \frac{\Gamma(\alpha+1)}{2}\sigma^{2}, \tag{63}$$

then, with the terminal condition

$$G(k,T) = Z(y,T) = H(s,T), \tag{64}$$

we gets

$$G(k,T) = c\left(\mathrm{e}^{k} - 1\right). \tag{65}$$

3.6.2. Solution of the new fractional B-S:

$$\hat{G}(\xi,t) := \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\xi k}G(k,t)\mathrm{d}k \tag{66}$$

Let Fourier's transform of $G(k,t)$, and taking the Fourier's transform of the Eq. (61), we come across the new conformable differential equation

$$\hat{G}_{t}^{(\alpha)}(\xi,t) = \xi^{2}\gamma^{2}\widehat{G}(\xi,t). \tag{67}$$

The solution to the equation. (67) which explains the terminal condition (65) is

$$\hat{G}(\xi,t) = E_{\alpha}\left(-\gamma^{2}\xi^{2}e^{(T-t)\alpha}\right)\hat{G}(\xi,T), \tag{68}$$

therefore, we have the final resultat :

$$G(k,t) = \int_{-\infty}^{+\infty}\Phi(k-v),(T-t)G(v,T)\mathrm{d}v, \tag{69}$$

where $\Phi(k)$ is defined by the following expression

$$\Phi(k,T-t) = \int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}\xi k}E_{\alpha}\left(\xi^{2}e^{(T-t)\alpha}\right)\mathrm{d}\xi. \tag{70}$$

3.6.3. Comparative solutions for serval values of α:

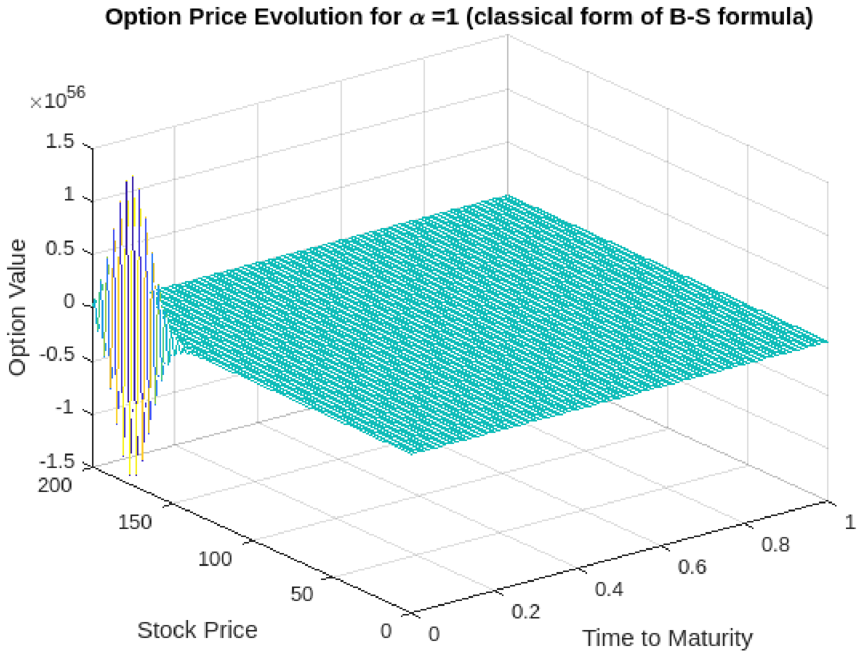

Figure 1

When $\alpha = 1$, the equation reduces to a classical first-order derivative with respect to time, representing the standard case in financial mathematics. Here are the key observations:

- For $\alpha = 1$ the equation becomes a traditional partial differential equation (PDE) without any fractional components. This means there is no memory effect, and the system's behavior at any given time depends solely on the current state, not on the history. The graph reflects this by showing sharper transitions and a more direct response to changes in the variables.

- Impact on Dynamics:
  - The solution $H(S, t)$ for $\alpha = 1$ represents the well-known Black-Scholes PDE, which is commonly used in option pricing. The graph is expected to show a smooth, yet relatively direct, relationship between the variables S (the underlying asset price) and t (time). The surface is typically more linear and predictable compared to fractional cases.

- Temporal Behavior:
  - The term $\frac{e^{(\alpha-1)t}}{\Gamma(2-\alpha)}$ simplifies when $\alpha = 1$, eliminating any exponential decay. This indicates that the graph evolves in a straightforward manner over time, without any additional complexities introduced by fractional derivatives.

- Shape of the Graph:
  - The graph for $H(S, t)$ when $\alpha = 1$ will show a surface that is smooth but not influenced by previous states of the system. The relationship between S and t is more straightforward, with the solution $H(S, t)$ following the standard option pricing formula. The surface is typically convex, with the highest values near the strike price K and decreasing as S moves further away from K.

- Parameter Influence:
  - The parameters R, $N = R - \beta$, and σ^2 directly influence the shape and steepness of the graph. For $\alpha = 1$, their effects are immediate and apparent, with no delay or memory effect. The graph will

show a clear and direct response to changes in these parameters, making the solution sensitive to the current values of S and t.

The graph for $\alpha = 1$ represents a classical PDE solution, where the system's behavior is determined entirely by the current state, with no influence from past states. The surface is expected to be smooth and convex, following the well-known Black-Scholes formula. The graph will show a direct relationship between the underlying asset price S and time t, with predictable and sharp transitions. The lack of a memory effect makes the graph more straightforward and easier to interpret compared to fractional cases with $\alpha < 1$.

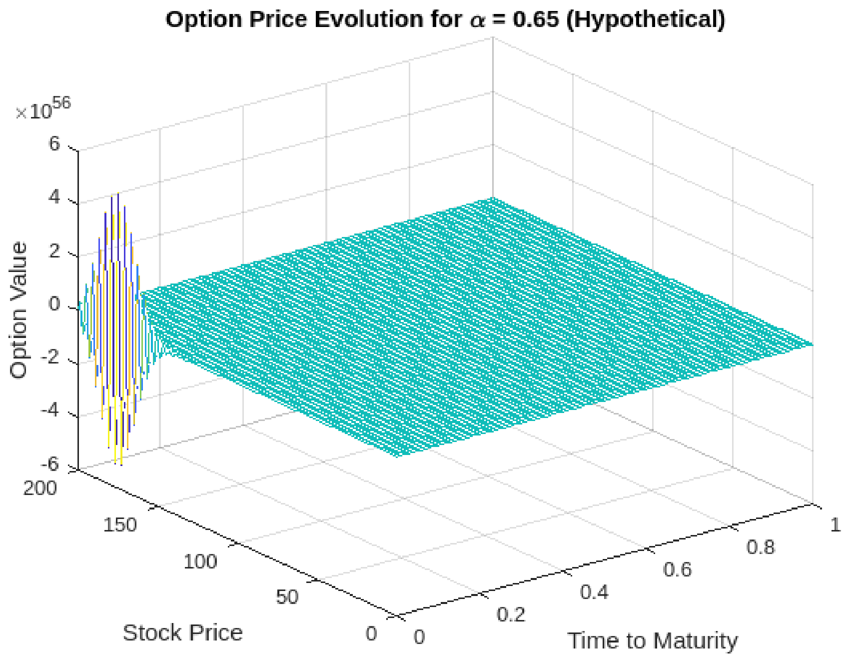

Figure 2

When $\alpha = 0.65$, the fractional derivative introduces a moderate memory effect, making the solution $H(S, t)$ more complex than in the integer-order case. Here are the key observations:

- Memory Effect:
  - For $\alpha = 0.65$, the memory effect is stronger than in the case of $\alpha = 1$, meaning the system's history has a more pronounced influence on its current state. The graph reflects this by showing smoother, more continuous changes over time, with the solution $H(S, t)$ taking into account past states to a greater extent.

- Impact on Dynamics:
  - The fractional order introduces a non-local behavior, meaning the graph does not simply depend on the current state but also on how the system has evolved. This creates a more intricate and possibly more stable solution surface, with the transitions in $H(S, t)$ being more gradual and less sharp. The graph is likely to show a surface that is smoother and less prone to abrupt changes.

- Temporal Decay:
  - The term $\frac{e^{(\alpha-1)t}}{\Gamma(2-\alpha)}$ influences the time decay of the solution. For $\alpha = 0.65$, this factor still introduces a decay, but it is slower than for smaller α, leading to a graph that evolves more gradually over time. The impact of past states lingers longer, resulting in a more extended memory effect in the graph.

- Shape of the Graph:
  - The graph for $H(S,t)$ when $\alpha = 0.65$ will show a surface that is smoother and more continuous, with fewer sharp transitions. The memory effect introduced by the fractional derivative makes the graph less steep and more rounded, indicating that the system's response to changes is more moderated and influenced by its history.

- Parameter Influence:
  - The parameters R, $N = R - \beta$, and σ^2 continue to shape the graph, but their influence is spread out over time due to the memory effect. This results in a solution that responds more gradually to changes in these parameters, making the graph more stable and less sensitive to sudden changes.

The graph for $\alpha = 0.65$ represents a system with a moderate memory effect, where the solution $H(S,t)$ depends on both the current state and the history of the process. The graph is expected to show a smoother, more continuous surface compared to the case with lower α, with gradual transitions and fewer abrupt changes. The memory effect leads to a more complex and stable graph, reflecting the system's non-local behavior and moderated response to changes. The solution evolves in a way that considers past states, resulting in a richer and more nuanced surface.

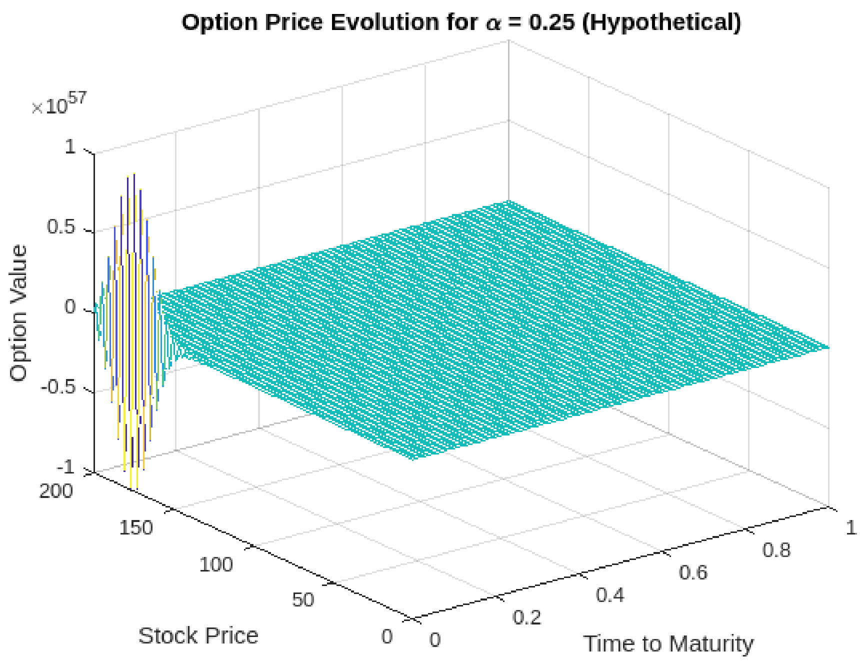

Figure 3

When $\alpha = 0.25$, the fractional derivative introduces a memory effect into the system, making the solution $H(S,t)$ dependent not only on the current state but also on the history of the process. Here are the key observations:

- Memory Effect:
  - With $\alpha = 0.25$, the fractional derivative $\frac{\partial^\alpha H}{\partial t^\alpha}$ introduces a weak memory effect, meaning that the past states of the system influence its current behavior. This makes the solution $H(S,t)$ less straightforward and more complex compared to the case when $\alpha = 0$. The graph will reflect this memory by showing a smoother transition in the evolution of $H(S,t)$ over time.

- Time-Dependent Behavior:

- The term $\frac{e^{(\alpha-1)t}}{\Gamma(2-\alpha)}$ plays a significant role in shaping the solution. For $\alpha = 0.25$, this factor introduces a decaying effect over time, which can cause the influence of past states to diminish as time progresses. However, the influence is still present, leading to a graph that shows gradual changes rather than abrupt transitions.

- Nonlinear Dynamics:
  - The fractional order $\alpha = 0.25$ results in a more nonlinear dynamic system. The interaction between the memory effect and the terms involving R, N, and σ^2 creates a graph with more subtle variations in curvature and slope. The graph will likely exhibit smoother transitions and possibly more intricate patterns in $H(S,t)$ as it evolves.

- Graph Shape :
  - The graph for $H(S,t)$ when $\alpha = 0.25$ will show a more complex surface compared to the $\alpha = 0$ case. The surface is expected to exhibit a more gradual evolution, with the influence of past states leading to smoother curves. The memory effect introduced by the fractional derivative will make the graph less steep and more rounded, reflecting the non-instantaneous response of the system.

- Impact of Parameters:
  - The parameters R, $N = R - \beta$, and σ^2 continue to influence the graph significantly. However, due to the memory effect, their impact may be less immediate and more distributed over time, leading to a less pronounced but more sustained effect on the shape of the graph.

The graph for $\alpha = 0.25$ represents a system with a weak memory effect, where the solution $H(S,t)$ depends on both the current state and the history of the process. The graph is expected to show smoother transitions and more gradual changes compared to the $\alpha = 0$ case. The memory effect leads to a more complex and nuanced surface, with subtler variations in curvature and slope. The solution evolves in a way that reflects the past states' influence, making the graph less straightforward but richer in detail.

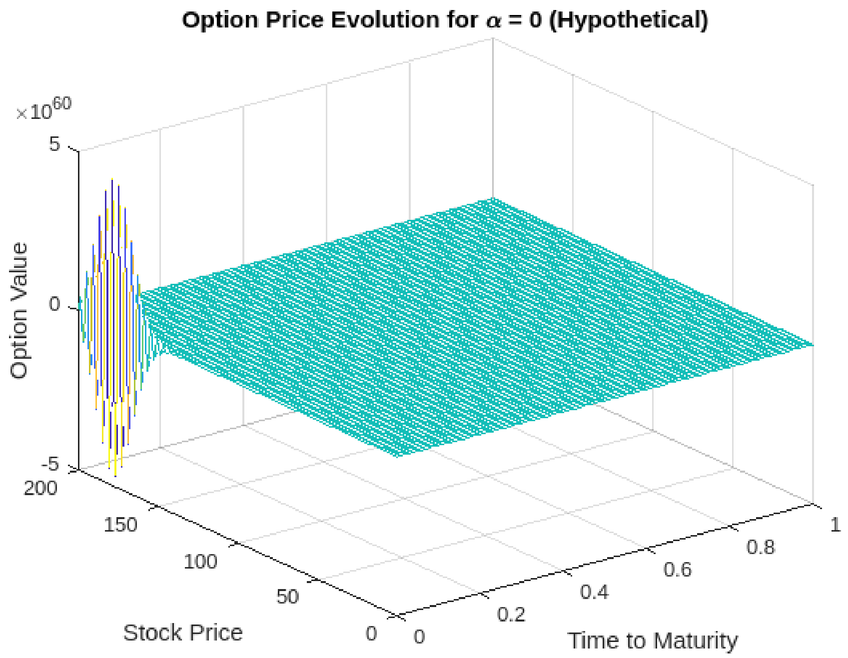

Figure 4

For $\alpha = 0$, the fractional derivative term $\frac{\partial^\alpha H}{\partial t^\alpha}$ becomes a constant, which significantly alters the behavior of the equation compared to cases where $0 < \alpha \leq 1$. Here's how you can interpret the graph for this scenario:

- Fractional Derivative Term:
  - When $\alpha = 0$, the fractional derivative term reduces to a constant. The term $\frac{e^{(\alpha-1)t}}{\Gamma(2-\alpha)}$ also becomes constant, as $\alpha = 0$ simplifies the time dependence.

- Behavior of the Solution:
  - The solution $H(S,t)$ will likely exhibit less sensitivity to the passage of time, as the fractional derivative no longer contributes time-dependent changes to the equation.
  - The impact of volatility (σ) and the drift term involving $NS\frac{\partial H}{\partial S}$ still play crucial roles, but the absence of a time-varying fractional derivative leads to a more stable or flat profile over time.

- Terminal and Boundary Conditions:
  - The terminal condition $H(S,0) = \max(K - S, 0)$ provides the initial profile for H, representing a European put option.
  - Boundary conditions at $S = 0$ and as $S \to \infty$ ensure that the solution aligns with the expected behavior at extreme values of the stock price.

- Graph Characteristics:
  - The graph for $H(S,t)$ when $\alpha = 0$ would likely show a flatter surface over time, especially when compared to higher values of α.
  - Since the equation lacks a time-dependent fractional derivative, the option price (or the function $H(S,t)$) will evolve more predictably, possibly showing less curvature in the t-direction.

The graph for $\alpha = 0$ indicates a scenario where time-dependent memory effects (typically introduced by fractional derivatives) are absent. The resulting behavior of $H(S,t)$ becomes more dominated by the other terms in the equation, leading to a simpler and potentially more stable surface over time. The reduction in complexity for $\alpha = 0$ makes it similar to classical option pricing models, though the specific form of the PDE still introduces unique characteristics that differ from the standard Black-Scholes model.

4. Conclusion remarks

This informative article provides an in-depth exploration of conformable derivatives and their applications in financial mathematics, with a specific emphasis on their role in the Black-Scholes option pricing model. By introducing the concept of the "new conformable derivative," the article highlights a significant advancement in the field. This novel derivative offers a refined and versatile approach to modeling complex financial instruments, paving the way for the development of a new fractional Black-Scholes formula.

The incorporation of the new conformable fractional derivative represents a substantial enhancement over traditional models. Classical Black-Scholes models, while foundational, often face limitations in accurately capturing the nuances of market behavior, especially under conditions of high volatility or irregular price movements. The new conformable derivative addresses these limitations by providing a more flexible framework that can better accommodate the intricate dynamics of financial markets.

This advancement not only improves the accuracy of option pricing but also offers valuable insights into the underlying market processes. The fractional Black-Scholes formula, developed using conformable derivatives, can better reflect the real-world phenomena observed in financial markets, such as fractal-like price changes and complex volatility patterns. This approach has the potential to enhance risk management strategies and improve financial decision-making by offering a more nuanced understanding of market behavior.

Furthermore, the introduction of conformable derivatives into financial mathematics opens new avenues for research and development. It sets the stage for further exploration of fractional calculus in finance, potentially leading to the creation of more sophisticated models and analytical tools. As researchers continue to investigate the applications of conformable derivatives, it is likely that additional breakthroughs will emerge, contributing to a deeper understanding of financial systems and leading to more innovative solutions in quantitative finance.

In summary, the article not only introduces a groundbreaking approach to fractional calculus but also demonstrates its practical significance in financial mathematics. The new conformable derivative offers a promising tool for advancing the Black-Scholes model and improving the precision of financial instrument valuation. As the field progresses, this innovative approach could transform how financial risks and opportunities are assessed, leading to more effective and informed financial strategies.

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