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Characterization of mixed triple derivations on incidence algebras

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Abstract. Let $\Re$ be a 2-torsion free commutative unital ring and $I(S, \Re)$ be the incidence algebra of a locally finite preordered set S. In the present paper, we show that if an $\Re$-linear map $\mathfrak{T} : I(S, \Re) \to I(S, \Re)$ satisfies

$$\mathfrak{T}([t \circ u, v]) = [\mathfrak{T}(t) \circ u, v] + [t \circ \mathfrak{T}(u), v] + [t \circ u, \mathfrak{T}(v)],$$

for all $t, u, v \in I(S, \Re)$, then $\mathfrak{T} = \Psi + \phi$, where $\Psi : I(S, \Re) \to I(S, \Re)$ is a derivation and $\phi : I(S, \Re) \to \mathcal{Z}(I(S, \Re))$ is an $\Re$-linear map.

1. Introduction

Let $\Re$ be a commutative unital ring and $\mathfrak{A}$ an associative algebra over $\Re$ with center $\mathcal{Z}(\mathfrak{A})$. For $t, u \in \mathfrak{A}$, $t \circ u = tu + ut$, $[t, u] = tu - ut$ represent Jordan product and Lie product of t and u respectively. So $(\mathfrak{A}, \circ)$ and $(\mathfrak{A}, [.,.])$ are Jordan algebra and Lie algebra respectively. An $\Re$-linear map $\mathfrak{T} : \mathfrak{A} \to \mathfrak{A}$ is said to be a derivation if $\mathfrak{T}(tu) = \mathfrak{T}(t)u + t\mathfrak{T}(u)$ for all $t, u \in \mathfrak{A}$ and is known as a Jordan (resp. Lie) derivation if $\mathfrak{T}(t^2) = \mathfrak{T}(t)t + t\mathfrak{T}(t)$ (resp. $\mathfrak{T}([t, u]) = [\mathfrak{T}(t), u] + [t, \mathfrak{T}(u)]$) for all $t, u \in \mathfrak{A}$. Also an $\Re$-linear map $\mathfrak{T} : \mathfrak{A} \to \mathfrak{A}$ is called a Lie triple derivation if

$$\mathfrak{T}([[t, u], v]) = [[\mathfrak{T}(t), u], v] + [[t, \mathfrak{T}(u)], v] + [[t, u], \mathfrak{T}(v)]$$

for all $t, u, v \in \mathfrak{A}$. Let Ψ be a derivation of $\mathfrak{A}$ and ϕ be an $\Re$-linear map from $\mathfrak{A}$ into $\mathcal{Z}(\mathfrak{A})$. Then $\Psi + \phi$ is a Lie triple derivation if and only if ϕ vanishes at all second commutators $[[t, u], v]$. Throughout this paper, we call an $\Re$-linear map $\mathfrak{T} : \mathfrak{A} \to \mathfrak{A}$, a mixed triple derivation if

$$\mathfrak{T}([t \circ u, v]) = [\mathfrak{T}(t) \circ u, v] + [t \circ \mathfrak{T}(u), v] + [t \circ u, \mathfrak{T}(v)]$$

for all $t, u, v \in \mathfrak{A}$. Let Ψ be a derivation of $\mathfrak{A}$ and ϕ be an $\Re$-linear map from $\mathfrak{A}$ into $\mathcal{Z}(\mathfrak{A})$. Clearly $\Psi + \phi$ is a mixed triple derivation if and only if

$$\phi([t \circ u, v]) = 2\{[t\phi(u), v] + [\phi(t)u, v]\}$$

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for all $t, u, v \in \mathfrak{A}$. A mixed triple derivation of the form $\Psi + \phi$, where Ψ is a derivation and ϕ, a central valued map will be called a proper mixed triple derivation else, an improper mixed triple derivation.

Let's now review the concept of incidence algebras, with which this paper deals. Let $(S, \leq)$ represent a locally finite preordered set. This implies that "$\leq$" is a reflexive and transitive binary relation on the set S and for every $t \leq u$, there are only finite number of elements $v \in S$ satisfying $t \leq v \leq u$. The incidence algebra $I(S, \mathfrak{R})$ of S over $\mathfrak{R}$ is defined as the set

$$I(S, \mathfrak{R}) = \{\gamma : S \times S \to \mathfrak{R} \mid \gamma(t, u) = 0 \text{ if } t \nleq u\}$$

with algebraic operations given as

$$(\gamma + \eta)(t, u) = \gamma(t, u) + \eta(t, u),$$
$$(r\gamma)(t, u) = r\gamma(t, u),$$
$$(\gamma\eta)(t, u) = \sum_{t \leq v \leq u} \gamma(t, v)\eta(v, u)$$

for all $\gamma, \eta \in I(S, \mathfrak{R}), r \in \mathfrak{R}$ and $t, u, v \in S$. The product $\gamma\eta$ is usually called convolution in function theory. Obviously the full matrix algebra $M_n(\mathfrak{R})$ and upper (or lower) triangular matrix algebras $T_n(\mathfrak{R})$ are special examples of incidence algebras.

The problem of identifying a class of algebra in which every mixed triple derivation is proper has its origin in the Herstein's Lie-type mapping research program [8]. Firstly, Ward [20] considered incidence algebras to be a generalized algebra of arithmetic functions. Rota and Stanley developed incidence algebras as the fundamental structures of enumerative combinatorial theory and allied areas of arithmetic function theory (see [18]). Motivated from Stanley's work [17], automorphisms and other algebraic mappings of incidence algebras have been extensively studied (see [2, 3, 5, 10, 11, 15, 16] and references therein). On the other hand, in the theory of operator algebras, the incidence algebra $I(S, \mathfrak{R})$ of a finite poset S is referred to as a digraph algebra or a finite dimensional CSL algebra. The operator algebras on which every derivation is proper include von Neumann algebras [14], certain CSL algebras [12], nest algebras [6] and C^*-algebras [13]. Miers proved that if $\mathfrak{A}$ is a von Neumann algebra with no central abelian summands, then every Lie triple derivation of $\mathfrak{A}$ is proper [14, Theorem 1]. Bresar [4] extended this result to prime rings and also Miers result was extended to Lie n-derivations for linear and nonlinear cases in [1] and [7] respectively. Furthermore, Xiao [21] and Khrypchenko [9], characterize Jordan derivations of incidence algebras and finite incidence rings respectively. Zhang et al. [22] proved that if S is a locally finite preordered set and $\mathfrak{R}$ is a 2-torsion free commutative unital ring, then any Lie derivation on $I(S, \mathfrak{R})$ is proper and Wang et al. [19] similarly proved that every Lie triple derivation is proper while taking same $I(S, \mathfrak{R})$. Inspired by the preceding results, in this paper, we first characterize mixed triple derivations on $I(S, \mathfrak{R})$ and based on such characterizations, we then prove that every mixed triple derivation of $I(S, \mathfrak{R})$ is proper provided that $\mathfrak{R}$ is 2-torsion free.

2. Finite case

We will use some notations that will be used throughout this paper. For any $t \in S$, let L_t and R_t be defined as:

$$L_t = \{i \in S \mid i \leq t, i \neq t\} \quad \text{and} \quad R_t = \{j \in S \mid t \leq j, j \neq t\}.$$

Since S is a preordered set, $L_t \cap R_t$ may not be empty. Also we denote $\{B^{ij}_{tu} \mid i \leq j,\ t \leq u\}$, the constants in $\mathfrak{R}$ satisfying $\mathfrak{T}(e_{ij}) = \sum_{t \leq u} B^{ij}_{tu} e_{tu}$ with the convention that $B^{ij}_{tu} = 0$, for $t \nleq u$. The identity element δ of $I(S, \mathfrak{R})$ is given as $\delta(t, u) = \delta_{tu}$ for $t \leq u$, where $\delta_{tu} \in \{0, 1\}$ is the Kronecker delta. Let $\mathcal{Z}(I(S, \mathfrak{R}))$ denote the centre of $I(S, \mathfrak{R})$. For $t < u$ or $u > t$, we mean $t \leq u$ and $t \neq u$. For $t, u \in S$ with $t \leq u$, e_{tu} be defined as the function on $S \times S$ by

$$e_{tu}(x, y) = \begin{cases} 1, & (t, u) = (x, y) \\ 0, & (t, u) \neq (x, y). \end{cases}$$

Clearly by definition of convolution, the product $e_{tu}e_{xy} = \delta_{ux}e_{ty}$ holds on $I(S,\mathfrak{R})$. Moreover, the set $\mathfrak{B} = \{e_{tu} : t \leq u\}$ forms an $\mathfrak{R}$-linear basis of $I(S,\mathfrak{R})$ and is known as standard basis.

Proposition 2.1. *Let A_1, A_2 be two $\mathfrak{R}$-algebras. Then A_1 and A_2 have no improper mixed triple derivations if and only if $A_1 \bigoplus A_2$ has no improper mixed triple derivation.*

Proof. Assume that A_1 and A_2 have no improper mixed triple derivations. We show that $A_1 \bigoplus A_2$ has no improper mixed triple derivation. Let $\mathfrak{T}$ be a mixed triple derivation on $A_1 \bigoplus A_2$ and $\mathfrak{T}(a,b) = (\mathfrak{T}_1(a), \mathfrak{T}_2(b))$, where $\mathfrak{T}_i : A_i \to A_i$ $(i = 1,2)$ are linear maps. Also, it can be easily verified that $\mathfrak{T}_i$ $(i = 1,2)$ is a mixed triple derivation on A_i $(i = 1,2)$. Therefore, $\mathfrak{T}_i = \Psi_i + \phi_i$ $(i = 1,2)$, where $\Psi_i : A_i \to A_i (i = 1,2)$ are derivations and $\phi_i : A_i \to A_i$ $(i = 1,2)$ are central valued maps on A_i. Consequently, $\Psi : A_1 \bigoplus A_2 \to A_1 \bigoplus A_2$ given by $\Psi(a,b) = (\Psi_1(a), \Psi_2(b))$ is a derivation and $\Phi : A_1 \bigoplus A_2 \to A_1 \bigoplus A_2$ given by $\Phi(a,b) = (\phi_1(a), \phi_2(b))$ is a central valued map. Therefore $\mathfrak{T} = \Psi + \Phi$ and hence $A_1 \bigoplus A_2$ has no improper mixed triple derivation.

Conversely, we assume that $A_1 \bigoplus A_2$ has no improper mixed triple derivation. We prove that A_1 (similarly A_2) has no improper mixed triple derivation also. Let $\mathfrak{T}_1$ be a mixed triple derivation of A_1. Define a map on $A_1 \bigoplus A_2$ by $\mathfrak{T}(a,b) = (\mathfrak{T}_1(a), 0)$. Clearly $\mathfrak{T}$ is a mixed triple derivation on $A_1 \bigoplus A_2$. So $\mathfrak{T}$ is proper, i.e., $\mathfrak{T} = \Psi + \Phi$, where $\Psi : A_1 \bigoplus A_2 \to A_1 \bigoplus A_2$ is a derivation and $\Phi : A_1 \bigoplus A_2 \to \mathcal{Z}(A_1 \bigoplus A_2)$ is a central valued map. Hence

$$\begin{aligned} \mathfrak{T}(a,0) &= (\Psi + \Phi)(a,0) \\ (\mathfrak{T}_1(a),0) &= \Psi(a,0) + \Phi(a,0) \\ (\mathfrak{T}_1(a),0) &= (\Psi_1(a),0) + (\phi_1(a),0) \\ (\mathfrak{T}_1(a),0) &= (\Psi_1(a) + \phi_1(a),0). \end{aligned}$$

Hence $\mathfrak{T}_1(a) = (\Psi_1 + \phi_1)(a)$, which gives us that $\mathfrak{T}_1$ is proper on A_1. □

The main result of our paper follows as:

Theorem 2.2. *Let $\mathfrak{R}$ be a 2-torsion free commutative unital ring and $\mathfrak{T}$ be a mixed triple derivation of $I(S,\mathfrak{R})$, where S is finite. Then $\mathfrak{T}$ is proper.*

Let $S = \cup S_i$ be the decomposition of S into distinct connected and finite components. Also, let $\delta_i = \sum_{t\in S_i} e_{tt}$. Therefore from [16, Theorem 1.3.13], $\{\delta_i\}$ forms a complete set of central primitive idempotents of $I(S,\mathfrak{R})$, i.e., $I(S,\mathfrak{R}) = \bigoplus \delta_i I(S,\mathfrak{R})$. Clearly $\delta_i I(S,\mathfrak{R}) \cong I(S_i,\mathfrak{R})$ for each i. Therefore, using Proposition 2.1, we prove our Theorem 2.2, when S is connected.

Lemma 2.3. [21, Theorem 2.2] *Let $\mathfrak{T} : I(S,\mathfrak{R}) \to I(S,\mathfrak{R})$ be an $\mathfrak{R}$-linear operator. Then $\mathfrak{T}$ is a derivation if and only if $\mathfrak{T}$ satisfies*

$$\mathfrak{T}(e_{ij}) = \sum_{t\in L_i} B^{ii}_{ti} e_{tj} + B^{ij}_{ij} e_{ij} + \sum_{u\in R_j} B^{jj}_{ju} e_{iu}$$

for all $e_{ij} \in \mathfrak{B}$ and B^{ij}_{tu} satisfies

$$B^{ii}_{ij} + B^{jj}_{ij} = 0, \quad \text{if } i \leq j;$$

$$B^{ij}_{ij} + B^{jk}_{jk} = B^{ik}_{ik}, \quad \text{if } i \leq j \leq k.$$

Lemma 2.4. *Let S be connected and $\mathfrak{T}$ be a mixed triple derivation of $I(S,\mathfrak{R})$. Then*

$$\mathfrak{T}(e_{ii}) = \sum_{t\in L_i} B^{ii}_{ti} e_{ti} + \sum_{i\neq t\in S} B^{ii}_{tt} e_{tt} + \sum_{u\in R_i} B^{ii}_{iu} e_{iu} \tag{1}$$

$$\mathfrak{T}(e_{ij}) = \sum_{t\in L_i} B^{ii}_{ti} e_{tj} + B^{ij}_{ij} e_{ij} + \sum_{u\in R_j} B^{jj}_{ju} e_{iu}, \quad \text{if } i \neq j. \tag{2}$$

Proof. Assume first that $|S| = 1$, let i be the unique element of S, so the result holds trivially.

Now assume that $|S| \geq 2$ and let $i \in$ S . Since S is connected, there is an element $j \neq i$ comparable with i. Assume we choose a path with the starting vertex j and end vertex i, i.e., for $e_{ij} \in \mathfrak{B}$ with $i < j$.

For the end vertex i, with $i \neq t \in S$, we have $[e_{ii} \circ e_{tt},\ e_{tt}] = 0$. Therefore,

$$\mathfrak{T}([e_{ii} \circ e_{tt},\ e_{tt}]) = 0$$
$$\mathfrak{T}(e_{ii})e_{tt} - e_{tt}\mathfrak{T}(e_{ii}) + e_{ii}\mathfrak{T}(e_{tt})e_{tt} - e_{tt}\mathfrak{T}(e_{tt})e_{ii} = 0.$$

Operating in previous relation by e_{tt} from left and by e_{uu} from right, we get

$$e_{tt}\mathfrak{T}(e_{ii})e_{tt}e_{uu} - e_{tt}\mathfrak{T}(e_{ii})e_{uu} + e_{tt}e_{ii}\mathfrak{T}(e_{tt})e_{tt}e_{uu} - e_{tt}\mathfrak{T}(e_{tt})e_{ii}e_{uu} = 0.$$

Hence

$$\begin{aligned} e_{tt}\mathfrak{T}(e_{ii})e_{uu} &= 0, \text{ if } i \neq t < u \neq i \\ e_{tt}\sum_{t\leq u} B^{ii}_{tu}e_{tu}e_{uu} &= 0, \text{ if } \ i \neq t < u \neq i \\ \sum_{t\leq u} B^{ii}_{tu}e_{tu} &= 0, \text{ if } \ i \neq t < u \neq i. \\ B^{ii}_{tu} &= 0, \text{ if } i \neq t < u \neq i. \end{aligned} \tag{3}$$

Also, we have $[e_{ii} \circ e_{ii},\ e_{ij}] = 2e_{ij}$. Therefore

$$\mathfrak{T}([e_{ii} \circ e_{ii},\ e_{ij}]) = 2\mathfrak{T}(e_{ij})$$
$$2\{\mathfrak{T}(e_{ii})e_{ij} + e_{ii}\mathfrak{T}(e_{ii})e_{ij} - e_{ij}\mathfrak{T}(e_{ii})e_{ii} + e_{ii}\mathfrak{T}(e_{ij}) - \mathfrak{T}(e_{ij})e_{ii}\} = 2\mathfrak{T}(e_{ij}).$$

Multiplying the above relation by e_{ii} from left and by e_{jj} from right, we get

$$2\{e_{ii}\mathfrak{T}(e_{ii})e_{ij} + e_{ii}\mathfrak{T}(e_{ii})e_{ij} + e_{ii}\mathfrak{T}(e_{ij})e_{jj}\} = 2e_{ii}\mathfrak{T}(e_{ij})e_{jj}.$$

Now using 2-torsion freeness of $\mathfrak{R}$, we have

$$\begin{aligned} 4e_{ii}\mathfrak{T}(e_{ii})e_{ij} &= 0 \\ 4e_{ii}\sum_{t\leq u} B^{ii}_{tu}e_{tu}e_{ij} &= 0 \\ B^{ii}_{ii} &= 0. \end{aligned} \tag{4}$$

Now utilizing the relations (3) and (4), we have

$$\begin{aligned} \mathfrak{T}(e_{ii}) &= \sum_{t\leq u} B^{ii}_{tu}e_{tu} \\ &= \sum_{t\in S} B^{ii}_{tt}e_{tt} + \sum_{t<u} B^{ii}_{tu}e_{tu} \\ &= \sum_{i\neq t\in S} B^{ii}_{tt}e_{tt} + \sum_{i=t<u} B^{ii}_{iu}e_{iu} + \sum_{i\neq t<u} B^{ii}_{tu}e_{tu} \\ &= \sum_{i\neq t\in S} B^{ii}_{tt}e_{tt} + \sum_{i<u} B^{ii}_{iu}e_{iu} + \sum_{i\neq t<u\neq i} B^{ii}_{tu}e_{tu} + \sum_{i\neq t<u=i} B^{ii}_{ti}e_{ti} \\ &= \sum_{i\neq t\in S} B^{ii}_{tt}e_{tt} + \sum_{i<u} B^{ii}_{iu}e_{iu} + \sum_{i\neq t<u\neq i} B^{ii}_{tu}e_{tu} + \sum_{i\neq t<i} B^{ii}_{ti}e_{ti} \\ &= \sum_{i\neq t\in S} B^{ii}_{tt}e_{tt} + \sum_{i<u} B^{ii}_{iu}e_{iu} + \sum_{t<i} B^{ii}_{ti}e_{ti}. \end{aligned}$$

Therefore,

$$\mathfrak{T}(e_{ii}) = \sum_{i\neq t\in S} B^{ii}_{tt}e_{tt} + \sum_{t<i} B^{ii}_{ti}e_{ti} + \sum_{u>i} B^{ii}_{iu}e_{iu}. \tag{5}$$

Let us now consider the starting vertex j. We have $[e_{ij} \circ e_{jj},\ e_{jj}] = e_{ij}$. Therefore

$$\begin{aligned} \mathfrak{T}([e_{ij} \circ e_{jj},\ e_{jj}]) &= \mathfrak{T}(e_{ij}) \\ \mathfrak{T}(e_{ij})e_{jj} - e_{jj}\mathfrak{T}(e_{ij}) + e_{ij}\mathfrak{T}(e_{jj})e_{jj} - e_{jj}\mathfrak{T}(e_{jj})e_{ij} + e_{ij}\mathfrak{T}(e_{jj}) &= \mathfrak{T}(e_{ij}). \end{aligned}$$

Multiplying the above relation by e_{ii} from left and by e_{jj} from right, we get

$$e_{ii}\mathfrak{T}(e_{ij})e_{jj} + e_{ij}\mathfrak{T}(e_{jj})e_{jj} + e_{ij}\mathfrak{T}(e_{jj})e_{jj} = e_{ii}\mathfrak{T}(e_{ij})e_{jj}.$$

Now using 2-torsion freeness of $\mathfrak{R}$, we get

$$\begin{aligned} 2e_{ij}\mathfrak{T}(e_{jj})e_{jj} &= 0 \\ 2e_{ij}\sum_{t\leq u} B^{jj}_{tu}e_{tu}e_{jj} &= 0 \\ B^{jj}_{jj} &= 0. \end{aligned} \tag{6}$$

Now for any $j \neq u \in S$, we have $[e_{uu} \circ e_{uu},\ e_{jj}] = 0$. Therefore

$$\begin{aligned} \mathfrak{T}([e_{uu} \circ e_{uu},\ e_{jj}]) &= 0 \\ 2\{e_{uu}\mathfrak{T}(e_{uu})e_{jj} - e_{jj}\mathfrak{T}(e_{uu})e_{uu} + e_{uu}\mathfrak{T}(e_{jj}) - \mathfrak{T}(e_{jj})e_{uu}\} &= 0. \end{aligned}$$

Applying e_{tt} on left and e_{uu} on right in the above relation and using 2-torsion freeness, we get

$$2\{e_{tt}e_{uu}\mathfrak{T}(e_{uu})e_{jj}e_{uu} - e_{tt}e_{jj}\mathfrak{T}(e_{uu})e_{uu} + e_{tt}e_{uu}\mathfrak{T}(e_{jj})e_{uu} - e_{tt}\mathfrak{T}(e_{jj})e_{uu}\} = 0.$$

$$\begin{aligned} 2e_{tt}\mathfrak{T}(e_{jj})e_{uu} &= 0, \text{ if } j\neq t<u\neq j \\ e_{tt}\sum_{t\leq u} B^{jj}_{tu}e_{tu}e_{uu} &= 0, \text{ if } j\neq t<u\neq j \\ e_{tt}\sum_{t\leq u} B^{jj}_{tu}e_{tu} &= 0, \text{ if } j\neq t<u\neq j \\ B^{jj}_{tu} &= 0, \quad \text{if} \quad j\neq t<u\neq j. \end{aligned} \tag{7}$$

By doing a similar computation as in the above case, we get

$$\mathfrak{T}(e_{jj}) = \sum_{j\neq u\in S} B^{jj}_{uu}e_{uu} + \sum_{t<j} B^{jj}_{tj}e_{tj} + \sum_{u>j} B^{jj}_{ju}e_{ju}. \tag{8}$$

Since by given hypothesis S is connected, each element $t \in S$ must be either a starting vertex or an end vertex of a path, therefore from the above relations (5) and (8), we get the desired form of $\mathfrak{T}(e_{tt})$ for any $t \in S$.

Next we describe the form of $\mathfrak{T}(e_{ij})$ where $i \neq j$. Now, we have

$$\begin{aligned} e_{ij} &= [e_{ii} \circ e_{ij},\ e_{jj}] \\ \mathfrak{T}(e_{ij}) &= \mathfrak{T}([e_{ii} \circ e_{ij},\ e_{jj}]) \\ \mathfrak{T}(e_{ij}) &= \mathfrak{T}(e_{ii})e_{ij} + e_{ij}\mathfrak{T}(e_{ii})e_{jj} - e_{jj}\mathfrak{T}(e_{ii})e_{ij} + e_{ii}\mathfrak{T}(e_{ij})e_{jj} - e_{jj}\mathfrak{T}(e_{ij})e_{ii} + e_{ij}\mathfrak{T}(e_{jj}) - \mathfrak{T}(e_{jj})e_{ij}. \end{aligned}$$

Utilizing (1) in the previous relation, we get

$$\mathfrak{T}(e_{ij}) = \sum_{t<i} B^{ii}_{ti}e_{tj} + \sum_{u>j} B^{jj}_{ju}e_{iu} + (B^{ij}_{ij} + B^{ii}_{jj} - B^{jj}_{ii})e_{ij} - (B^{jj}_{ji} + B^{ii}_{ji})e_{jj} - B^{ij}_{ji}e_{ji}. \tag{9}$$

Also, we have $-e_{ij} = [e_{ij} \circ e_{jj}, e_{ii}]$. Therefore

$$\begin{aligned} -\mathfrak{T}(e_{ij}) &= \mathfrak{T}([e_{ij} \circ e_{jj}, e_{ii}]) \\ \mathfrak{T}(e_{ij}) &= \mathfrak{T}(e_{ii})e_{ij} - e_{ij}\mathfrak{T}(e_{ii}) + e_{ii}\mathfrak{T}(e_{ij})e_{jj} - e_{jj}\mathfrak{T}(e_{ij})e_{ii} + e_{ij}\mathfrak{T}(e_{jj}) + e_{ii}\mathfrak{T}(e_{jj})e_{ij} - e_{ij}\mathfrak{T}(e_{jj})e_{ii}. \end{aligned}$$

Utilizing (1) in previous relation, we obtain

$$\mathfrak{T}(e_{ij}) = \sum_{t<i} B^{ii}_{ti} e_{tj} + \sum_{u>j} B^{jj}_{ju} e_{iu} + (B^{ij}_{ij} - B^{ii}_{jj} + B^{jj}_{ii})e_{ij} - (B^{jj}_{ji} + B^{ii}_{ji})e_{ii} - B^{ij}_{ji} e_{ji}. \tag{10}$$

From (9) and (10), we have

$$2(B^{ii}_{jj} - B^{jj}_{ii})e_{ij} - (B^{jj}_{ji} + B^{ii}_{ji})e_{jj} + (B^{jj}_{ji} + B^{ii}_{ji})e_{ii} = 0. \tag{11}$$

Operating e_{jj} on left in the last relation, we get $(B^{jj}_{ji} + B^{ii}_{ji}) = 0$. Similarly, operating left side by e_{ii} and right side by e_{jj} in (11) and using 2-torsion freeness of $\mathfrak{R}$, we arrive at $(B^{ii}_{jj} - B^{jj}_{ii}) = 0$. Indeed, we also have $[e_{ii} \circ e_{ii},\ e_{ij}] = 2e_{ij}$. So

$$\begin{aligned} &2\mathfrak{T}(e_{ij}) = \mathfrak{T}([e_{ii} \circ e_{ii},\ e_{ij}]) \\ &2\mathfrak{T}(e_{ij}) = 2\{\mathfrak{T}(e_{ii})e_{ij} + e_{ii}\mathfrak{T}(e_{ii})e_{ij} - e_{ij}\mathfrak{T}(e_{ii})e_{ii} + e_{ii}\mathfrak{T}(e_{ij}) - \mathfrak{T}(e_{ij})e_{ii}\}. \end{aligned} \tag{12}$$

Operating e_{jj} on left and e_{ii} on right in the previous relation, we get $B^{ij}_{ji} = 0$. Using these relations in either (9) or (10), we conclude that for $i \neq j$, $\mathfrak{T}(e_{ij})$ has the form

$$\mathfrak{T}(e_{ij}) = \sum_{t<i} B^{ii}_{ti} e_{tj} + \sum_{u>j} B^{jj}_{ju} e_{iu} + B^{ij}_{ij} e_{ij}. \tag{13}$$

This completes the proof. □

Lemma 2.5. *Let S be finite and connected. If $\mathfrak{T}$ is a mixed triple derivation on $I(S, \mathfrak{R})$, then the coefficients B^{ij}_{tu} are subject to the following relations:*

$$B^{ii}_{ij} + B^{jj}_{ij} = 0, \quad \text{if} \quad i < j; \tag{14}$$

$$B^{ii}_{kk} + B^{ii}_{ll} = 0, \ \text{if} \ k < l,\ l \neq i \neq k; \tag{15}$$

$$B^{pp}_{ii} = B^{pp}_{ll}, \ \text{if} \ i < l, \ l \neq p \neq i; \tag{16}$$

$$B^{il}_{il} + B^{lq}_{lq} = B^{iq}_{iq}, \ \text{if} \ i < l < q, \ i \neq q; \tag{17}$$

$$B^{il}_{il} + B^{li}_{li} = 0, \ \text{if} \ i \neq l; \tag{18}$$

$$B^{ij}_{ij} + B^{jl}_{jl} + B^{li}_{li} = 0, \ \text{if} \ i < j < l < i; \tag{19}$$

$$B^{ij}_{ij} + B^{jl}_{jl} + B^{lq}_{lq} = B^{iq}_{iq}, \ \text{if} \ i < j < l < q, \ i \neq q. \tag{20}$$

Proof. We have

$$[e_{ij} \circ e_{kl}, e_{pq}] = \delta_{jk}(\delta_{lp} e_{iq} - \delta_{qi} e_{pl}) + \delta_{li}(\delta_{jp} e_{kq} - \delta_{qk} e_{pj}). \tag{21}$$

Applying mixed triple derivation $\mathfrak{T}$ on above identity and by Lemma 2.4, we have the following cases:

1. $i = j,\ k = l,\ p = q$;
2. $i = j,\ k = l,\ p \neq q$;
3. $i = j,\ k \neq l,\ p = q$;

4. $i = j,\ k \neq l,\ p \neq q$;
5. $i \neq j,\ k = l,\ p = q$;
6. $i \neq j,\ k = l,\ p \neq q$;
7. $i \neq j,\ k \neq l,\ p = q$;
8. $i \neq j,\ k \neq l,\ p \neq q$.

We will deal with the Cases (1), (3), (4), (8) and rest other cases are symmetric to these four cases.
Case (1). $i = j,\ k = l,\ p = q$. From (21), we have

$$[e_{ii} \circ e_{kk},\ e_{pp}] = \delta_{ik}(\delta_{kp}e_{ip} - \delta_{pi}e_{pk}) + \delta_{ki}(\delta_{ip}e_{kp} - \delta_{pk}e_{pi}). \tag{22}$$

Now there are two subcases:
Subcase 1.1. Assuming $k = i$ in (22), we have

$$\begin{aligned} 2(\delta_{ip}e_{ip} - \delta_{pi}e_{pi}) &= [e_{ii} \circ e_{ii}, e_{pp}] \\ 2\{\mathfrak{T}(\delta_{ip}e_{ip} - \delta_{pi}e_{pi})\} &= \mathfrak{T}([e_{ii} \circ e_{ii}, e_{pp}]) \\ 2\{\mathfrak{T}(\delta_{ip}e_{ip} - \delta_{pi}e_{pi})\} &= [\mathfrak{T}(e_{ii}) \circ e_{ii}, e_{pp}] + [e_{ii} \circ \mathfrak{T}(e_{ii}), e_{pp}] + [e_{ii} \circ e_{ii}, \mathfrak{T}(e_{pp})] \\ 2\{\mathfrak{T}(\delta_{ip}e_{ip}) - \mathfrak{T}(\delta_{pi}e_{pi})\} &= 2\{\mathfrak{T}(e_{ii})e_{ii}e_{pp} + e_{ii}\mathfrak{T}(e_{ii})e_{pp} - e_{pp}\mathfrak{T}(e_{ii})e_{ii} - e_{pp}e_{ii}\mathfrak{T}(e_{ii}) + e_{ii}\mathfrak{T}(e_{pp}) - \mathfrak{T}(e_{pp})e_{ii}\}. \end{aligned}$$

Now, there are two subcases:
Subcase 1.1.1. $i = p$. Then the last relation holds trivially.
Subcase 1.1.2. $i \neq p$. In this situation, the last relation reduces to

$$e_{ii}\mathfrak{T}(e_{ii})e_{pp} - e_{pp}\mathfrak{T}(e_{ii})e_{ii} + e_{ii}\mathfrak{T}(e_{pp}) - \mathfrak{T}(e_{pp})e_{ii} = 0.$$

Using (1) in the above relation, we conclude that

$$B^{ii}_{ip} + B^{pp}_{ip} = 0, \text{ if } i < p,$$

$$B^{pp}_{pi} + B^{ii}_{pi} = 0, \text{ if } p < i.$$

This proves (14).
Subcase 1.2. Assuming $k \neq i$ in (22), we get $[e_{ii} \circ e_{kk},\ e_{pp}] = 0$. Thus

$$\begin{aligned} 0 &= \mathfrak{T}([e_{ii} \circ e_{kk},\ e_{pp}]) \\ 0 &= \mathfrak{T}(e_{ii})\delta_{kp}e_{kp} + e_{kk}\mathfrak{T}(e_{ii})e_{pp} - e_{pp}\mathfrak{T}(e_{ii})e_{kk} - \delta_{pk}e_{pk}\mathfrak{T}(e_{ii}) + e_{ii}\mathfrak{T}(e_{kk})e_{pp} + \mathfrak{T}(e_{kk})\delta_{ip}e_{ip} \\ &\quad - \delta_{pi}e_{pi}\mathfrak{T}(e_{kk}) - e_{pp}\mathfrak{T}(e_{kk})e_{ii}. \end{aligned} \tag{23}$$

Now we have three further subcases:
Subcase 1.2.1. $k = p,\ i \neq p$, we obtain

$$\mathfrak{T}(e_{ii})e_{kk} - e_{kk}\mathfrak{T}(e_{ii}) + e_{ii}\mathfrak{T}(e_{kk})e_{kk} - e_{kk}\mathfrak{T}(e_{kk})e_{ii} = 0.$$

Using (1) in the previous relation, we conclude that $C^{ii}_{ik} + C^{kk}_{ik} = 0$, if $i < k$ or $C^{kk}_{ki} + C^{ii}_{ki} = 0$, if $k < i$.
Subcase 1.2.2. $k \neq p,\ i = p$. This case is symmetric to Subcase 1.2.1.
Subcase 1.2.3. $k \neq p,\ i \neq p$. Then the relation (23) holds trivially.
Case (3). Assuming $i = j,\ k \neq l,\ p = q$ in (21), we arrive at

$$[e_{ii} \circ e_{kl}, e_{pp}] = \delta_{ik}(\delta_{lp}e_{ip} - \delta_{pi}e_{pl}) + \delta_{li}(\delta_{ip}e_{kp} - \delta_{pk}e_{pi}). \tag{24}$$

Now there are three subcases:
Subcase 3.1. $i \neq l$ and $i \neq k$. We have $[e_{ii} \circ e_{kl},\ e_{pp}] = 0$. Thus

$$\begin{aligned} 0 &= \mathfrak{T}([e_{ii} \circ e_{kl},\ e_{pp}]) \\ &= \mathfrak{T}(e_{ii})\delta_{lp}e_{kp} + e_{kl}\mathfrak{T}(e_{ii})e_{pp} - e_{pp}\mathfrak{T}(e_{ii})e_{kl} - \delta_{pk}e_{pl}\mathfrak{T}(e_{ii}) + e_{ii}\mathfrak{T}(e_{kl})e_{pp} + \mathfrak{T}(e_{kl})\delta_{ip}e_{ip} \\ &\quad - \delta_{pi}e_{pi}\mathfrak{T}(e_{kl}) - e_{pp}\mathfrak{T}(e_{kl})e_{ii}. \end{aligned} \tag{25}$$

Now we have three subcases:
Subcase 3.1.1. $l \neq p, p \neq k$. From (25), we have

$$e_{kl}\mathfrak{T}(e_{ii})e_{pp} - e_{pp}\mathfrak{T}(e_{ii})e_{kl} + e_{ii}\mathfrak{T}(e_{kl})e_{pp} + \mathfrak{T}(e_{kl})\delta_{ip}e_{ip} - \delta_{pi}e_{pi}\mathfrak{T}(e_{kl}) - e_{pp}\mathfrak{T}(e_{kl})e_{ii} = 0.$$

Now if $i \neq p$, then above relation holds trivially and if $i = p$, then we have $B^{ll}_{li}+B^{ii}_{li} = 0$ for $l < i$ and $B^{ii}_{ik}+B^{kk}_{ik} = 0$ for $i < k$.
Subcase 3.1.2. $l = p$ and $p \neq k$. From (25), we obtain

$$\mathfrak{T}(e_{ii})e_{kl} + e_{kl}\mathfrak{T}(e_{ii})e_{ll} - e_{ll}\mathfrak{T}(e_{ii})e_{kl} + e_{ii}\mathfrak{T}(e_{kl})e_{ll} - e_{ll}\mathfrak{T}(e_{kl})e_{ii} = 0.$$

Applying (1) and (2), we have $B^{ii}_{ik} + B^{kk}_{ik} = 0$ for $i < k$ and $B^{ii}_{kk} + B^{ii}_{ll} = 0$ for $k < l,\ l \neq i \neq k$, which proves (15).
Subcase 3.1.3. $l \neq p$ and $p = k$. This case is symmetric to Subcase 3.1.2.
Subcase 3.2. $i = k$ and $i \neq l$. From (24), we deduce that

$$\begin{aligned}
\delta_{lp}e_{ip} - \delta_{pi}e_{pl} &= [e_{ii} \circ e_{il},\ e_{pp}]\\
\mathfrak{T}(\delta_{lp}e_{ip}) - \mathfrak{T}(\delta_{pi}e_{pl}) &= \mathfrak{T}([e_{ii} \circ e_{il},\ e_{pp}])\\
\mathfrak{T}(\delta_{lp}e_{ip}) - \mathfrak{T}(\delta_{pi}e_{pl}) &= \mathfrak{T}(e_{ii})\delta_{lp}e_{ip} + e_{il}\mathfrak{T}(e_{ii})e_{pp} - e_{pp}\mathfrak{T}(e_{ii})e_{il} - \delta_{pi}e_{pl}\mathfrak{T}(e_{ii}) + e_{ii}\mathfrak{T}(e_{il})e_{pp} + \mathfrak{T}(e_{il})\delta_{ip}e_{ip}\\
&\quad - \delta_{pi}e_{pi}\mathfrak{T}(e_{il}) - e_{pp}\mathfrak{T}(e_{il})e_{ii} + e_{il}\mathfrak{T}(e_{pp}) - \mathfrak{T}(e_{pp})e_{il}.
\end{aligned} \tag{26}$$

Now we have three subcases:
Subcase 3.2.1. $l \neq p, p \neq i$. From (26), we have

$$0 = e_{il}\mathfrak{T}(e_{ii})e_{pp} - e_{pp}\mathfrak{T}(e_{ii})e_{il} + e_{ii}\mathfrak{T}(e_{il})e_{pp} - e_{pp}\mathfrak{T}(e_{il})e_{ii} + e_{il}\mathfrak{T}(e_{pp}) - \mathfrak{T}(e_{pp})e_{il}.$$

Using (1), (2) and (14), we arrive at $C^{pp}_{ii} = C^{pp}_{ll}$, if $i < l$ and $l \neq p \neq i$, which gives (16).
Subcase 3.2.2. $l = p,\ p \neq i$. From (26), we have

$$\mathfrak{T}(e_{il}) = \mathfrak{T}(e_{ii})e_{il} + e_{il}\mathfrak{T}(e_{ii})e_{ll} - e_{ll}\mathfrak{T}(e_{ii})e_{il} + e_{ii}\mathfrak{T}(e_{il})e_{ll} - e_{ll}\mathfrak{T}(e_{il})e_{ii} + e_{il}\mathfrak{T}(e_{ll}) - \mathfrak{T}(e_{ll})e_{il}.$$

Using (1) and (2), we get $C^{ll}_{li} + C^{ii}_{li} = 0$, if $l < i$.
Subcase 3.2.3. $l \neq p,\ p = i$. This case is symmetric to Subcase 3.2.2.
Subcase 3.3. $i \neq k,\ i = l$. This case is symmetric to Subcase 3.2.
Case (4). $i = j,\ k \neq l,\ p \neq q$. From (21), we have

$$[e_{ii} \circ e_{kl}, e_{pq}] = \delta_{ik}(\delta_{lp}e_{iq} - \delta_{qi}e_{pl}) + \delta_{li}(\delta_{ip}e_{kq} - \delta_{qk}e_{pi}). \tag{27}$$

We have three subcases:
Subcase 4.1. $k = i,\ i \neq l$. From (27), we have

$$[e_{ii} \circ e_{il}, e_{pq}] = \delta_{lp}e_{iq} - \delta_{qi}e_{pl}.$$

Now we further have four subcases:
Subcase 4.1.1. $l \neq p,\ q \neq i$, we have $[e_{ii} \circ e_{il}, e_{pq}] = 0$. Hence

$$\begin{aligned}
0 &= \mathfrak{T}([e_{ii} \circ e_{il}, e_{pq}])\\
&= e_{il}\mathfrak{T}(e_{ii})e_{pq} - e_{pq}\mathfrak{T}(e_{ii})e_{il} + e_{ii}\mathfrak{T}(e_{il})e_{pq} + \mathfrak{T}(e_{il})\delta_{ip}e_{iq} - e_{pq}\mathfrak{T}(e_{il})e_{ii} + e_{il}\mathfrak{T}(e_{pq}) - \mathfrak{T}(e_{pq})e_{il}.
\end{aligned}$$

Now if $i = p$, then, we have $C^{qq}_{qi} + C^{ii}_{qi} = 0$, if $q < i$ and $C^{ll}_{li} + C^{ii}_{li} = 0$, if $l < i$. If $i \neq p$, then we get $C^{qq}_{qi} + C^{ii}_{qi} = 0$, if $q < i$ and $C^{ll}_{lp} + C^{pp}_{lp} = 0$, if $l < p$.
Subcase 4.1.2. $l = p,\ i \neq q$, we have $e_{iq} = [e_{ii} \circ e_{il}, e_{lq}]$. This shows that

$$\begin{aligned}
\mathfrak{T}(e_{iq}) &= \mathfrak{T}([e_{ii} \circ e_{il}, e_{lq}]),\\
\mathfrak{T}(e_{iq}) &= \mathfrak{T}(e_{ii})e_{iq} + e_{il}\mathfrak{T}(e_{ii})e_{lq} - e_{lq}\mathfrak{T}(e_{ii})e_{il} + e_{ii}\mathfrak{T}(e_{il})e_{lq} - e_{lq}\mathfrak{T}(e_{il})e_{ii} + e_{il}\mathfrak{T}(e_{lq}) - \mathfrak{T}(e_{lq})e_{il}.
\end{aligned}$$

Now using (1) and (2), we conclude that

$$B^{iq}_{iq}e_{iq} = B^{ii}_{ll}e_{iq} - B^{ii}_{qi}e_{ll} + B^{il}_{il}e_{iq} + B^{lq}_{lq}e_{iq} - B^{qq}_{qi}e_{ll}.$$

Now using (15) and (16), we get $B^{il}_{il} + B^{lq}_{lq} = B^{iq}_{iq}$, for $i < l < q$ and $i \neq q$, which proves (17).
Subcase 4.1.3. $l = p,\ i = q$, we have $e_{ii} - e_{ll} = [e_{ii} \circ e_{il},\ e_{li}]$. Thus

$$\begin{aligned}\mathfrak{T}(e_{ii} - e_{ll}) &= \mathfrak{T}([e_{ii} \circ e_{il},\ e_{li}])\\ \mathfrak{T}(e_{ii}) - \mathfrak{T}(e_{ll}) &= \mathfrak{T}(e_{ii})e_{ii} + e_{il}\mathfrak{T}(e_{ii})e_{li} - e_{li}\mathfrak{T}(e_{ii})e_{il} - e_{ll}\mathfrak{T}(e_{ii}) + e_{ii}\mathfrak{T}(e_{il})e_{li} - e_{li}\mathfrak{T}(e_{il}) - e_{li}\mathfrak{T}(e_{il})e_{ii}\\ &\quad + e_{il}\mathfrak{T}(e_{li}) - \mathfrak{T}(e_{li})e_{il}.\end{aligned}$$

Now utilizing (1) and (2), we get

$$\sum_{i\neq t\in S} B^{ii}_{tt}e_{tt} - \sum_{l\neq u\in S} B^{ll}_{uu}e_{uu} = (B^{ii}_{ll} + B^{il}_{il} + B^{li}_{li})e_{ii} - (B^{ll}_{li} + B^{ii}_{li})e_{li} - (B^{ii}_{ll} + B^{il}_{il} + B^{li}_{li})e_{ll}.$$

Hence, utilizing (15) and (16), we deduce that $B^{il}_{il} + B^{li}_{li} = 0$, if $i \neq l$, which proves (18).
Subcase 4.1.4. $l \neq p,\ i = q$. This case is symmetric to Subcase 4.1.2.
Subcase 4.2. $k \neq i,\ i = l$. This case is symmetric to Subcase 4.1.
Subcase 4.3. $k \neq i,\ i \neq l$. From (27), we have $[e_{ii} \circ e_{kl}, e_{pq}] = 0$. Thus

$$\begin{aligned}0 &= \mathfrak{T}([e_{ii} \circ e_{kl}, e_{pq}])\\ &= \mathfrak{T}(e_{ii})\delta_{lp}e_{kq} + e_{kl}\mathfrak{T}(e_{ii})e_{pq} - e_{pq}\mathfrak{T}(e_{ii})e_{kl} - \delta_{qk}e_{pl}\mathfrak{T}(e_{ii}) + e_{ii}\mathfrak{T}(e_{kl})e_{pq} + \mathfrak{T}(e_{kl})\delta_{ip}e_{iq}\\ &\quad - \delta_{qi}e_{pi}\mathfrak{T}(e_{kl}) - e_{pq}\mathfrak{T}(e_{kl})e_{ii}.\end{aligned} \tag{28}$$

Now we consider three subcases:
Subcase 4.3.1. $l \neq p,\ q \neq k,\ i \neq p,\ i \neq q$, we have

$$e_{kl}\mathfrak{T}(e_{ii})e_{pq} - e_{pq}\mathfrak{T}(e_{ii})e_{kl} + e_{ii}\mathfrak{T}(e_{kl})e_{pq} - e_{pq}\mathfrak{T}(e_{kl})e_{ii} = 0.$$

Using (1) and (2), the last relation holds trivially.
Subcase 4.3.2. $l \neq p,\ q \neq k,\ i \neq p,\ i = q$, we have

$$e_{kl}\mathfrak{T}(e_{ii})e_{pi} - e_{pi}\mathfrak{T}(e_{ii})e_{kl} + e_{ii}\mathfrak{T}(e_{kl})e_{pi} - e_{pi}\mathfrak{T}(e_{kl}) - e_{pi}\mathfrak{T}(e_{kl})e_{ii} = 0.$$

Now using (1) and (2), we deduce that $C^{ii}_{ik} + C^{kk}_{ik} = 0$, if $i < k$.
Subcase 4.3.3. $l = p,\ q \neq k,\ i \neq p,\ i = q$, we have

$$\mathfrak{T}(e_{ii})e_{ki} + e_{kl}\mathfrak{T}(e_{ii})e_{li} - e_{li}\mathfrak{T}(e_{ii})e_{kl} + e_{ii}\mathfrak{T}(e_{kl})e_{li} - e_{li}\mathfrak{T}(e_{kl}) - e_{li}\mathfrak{T}(e_{kl})e_{ii} = 0.$$

Now using (1) and (2), we conclude that $C^{ii}_{ik} + C^{kk}_{ik} = 0$, if $i < k$ or $C^{ii}_{kk} + C^{ii}_{ll} = 0$, if $k < i,\ k \neq i \neq l$.
Case (8). $i \neq j,\ k \neq l,\ p \neq q$. From (21), we have

$$[e_{ij} \circ e_{kl},\ e_{pq}] = \delta_{jk}(\delta_{lp}e_{iq} - \delta_{qi}e_{pl}) + \delta_{li}(\delta_{jp}e_{kq} - \delta_{qk}e_{pj}). \tag{29}$$

Now we consider two subcases:
Subcase 8.1. $j = k,\ l \neq i$, we have

$$[e_{ij} \circ e_{jl}, e_{pq}] = (\delta_{lp}e_{iq} - \delta_{qi}e_{pl}). \tag{30}$$

Now we further have the following four subcases:
Subcase 8.1.1. $l = p,\ q = i$, we deduce that

$$\begin{aligned}e_{ii} - e_{ll} &= [e_{ij} \circ e_{jl}, e_{li}]\\ \mathfrak{T}(e_{ii} - e_{ll}) &= \mathfrak{T}([e_{ij} \circ e_{jl}, e_{li}])\\ \mathfrak{T}(e_{ii}) - \mathfrak{T}(e_{ll}) &= [\mathfrak{T}(e_{ij}) \circ e_{jl}, e_{li}] + [e_{ij} \circ \mathfrak{T}(e_{jl}),\ e_{li}] + [e_{ij} \circ e_{jl},\ \mathfrak{T}(e_{li})]\\ \mathfrak{T}(e_{ii}) - \mathfrak{T}(e_{ll}) &= \mathfrak{T}(e_{ij})e_{ji} + e_{jl}\mathfrak{T}(e_{ij})e_{li} - e_{li}\mathfrak{T}(e_{ij})e_{jl} + e_{ij}\mathfrak{T}(e_{jl})e_{li} - e_{lj}\mathfrak{T}(e_{jl}) - e_{li}\mathfrak{T}(e_{jl})e_{ij}\\ &\quad + e_{il}\mathfrak{T}(e_{li}) - \mathfrak{T}(e_{li})e_{il}.\end{aligned}$$

Now using (1), (2), (15) and (16), we get

$$B^{ij}_{ij} + B^{jl}_{jl} + B^{li}_{li} = 0, \text{ if } i < j < l < i,$$

which is the relation (19).
Subcase 8.1.2. $l = p,\ q \neq i$. From (30), we obtain

$$\begin{aligned} e_{iq} &= [e_{ij} \circ e_{jl}, e_{lq}] \\ \mathfrak{T}(e_{iq}) &= \mathfrak{T}([e_{ij} \circ e_{jl}, e_{lq}]) \\ \mathfrak{T}(e_{iq}) &= [\mathfrak{T}(e_{ij}) \circ e_{jl}, e_{lq}] + [e_{ij} \circ \mathfrak{T}(e_{jl}), e_{lq}] + [e_{ij} \circ e_{jl}, \mathfrak{T}(e_{lq})] \\ \mathfrak{T}(e_{iq}) &= \mathfrak{T}(e_{ij})e_{jq} + e_{jl}\mathfrak{T}(e_{ij})e_{lq} - e_{lq}\mathfrak{T}(e_{ij})e_{jl} + e_{ij}\mathfrak{T}(e_{jl})e_{lq} - e_{lq}\mathfrak{T}(e_{jl})e_{ij} + e_{il}\mathfrak{T}(e_{lq}) - \mathfrak{T}(e_{lq})e_{il}. \end{aligned}$$

Now using (1) and (2), we get

$$B^{ij}_{ij} + B^{jl}_{jl} + B^{lq}_{lq} = B^{iq}_{iq}, \text{ if } i < j < l < q,\ i \neq q,$$

which is the relation (20).
Subcase 8.1.3. $l \neq p,\ q = i$. This case is symmetric to Subcase 8.1.2.
Subcase 8.1.4. $l \neq p,\ q \neq i$. From (30), we have $[e_{ij} \circ e_{jl}, e_{pq}] = 0$. Thus

$$\mathfrak{T}([e_{ij} \circ e_{jl}, e_{pq}]) = 0.$$

Using (1) and (2), we get previously proved relations.
Subcase 8.2. $j \neq k,\ l = i$. It is symmetric to Subcase 8.1. □

Now, we are ready to prove Theorem 2.2 as follows:

Proof. We prove this theorem by using cardinality of S. If $|S| = 1$, then $I(S, \mathfrak{R}) \cong \mathfrak{R}$ and the result holds trivially. Now assume that $|S| \geq 2$. Let $\mathfrak{T} : I(S, \mathfrak{R}) \to I(S, \mathfrak{R})$ be a mixed triple derivation. We have to show that $\mathfrak{T}$ is proper. Now as $\mathfrak{T}$ is a mixed triple derivation, so by Lemmas 2.4 and 2.5, it has the form

$$\mathfrak{T}(e_{ii}) = \sum_{t \in L_i} B^{ii}_{ti} e_{ti} + \sum_{i \neq t \in S} B^{ii}_{tt} e_{tt} + \sum_{u \in R_i} B^{ii}_{iu} e_{iu} \tag{31}$$

$$\mathfrak{T}(e_{ij}) = \sum_{t \in L_i} B^{ii}_{ti} e_{tj} + B^{ij}_{ij} e_{ij} + \sum_{u \in R_j} B^{jj}_{ju} e_{iu}, \text{ if } i \neq j, \tag{32}$$

where the coefficients B^{ij}_{tu} satisfy the relations (14) - (20).
Now we define an $\mathfrak{R}$-linear operator Ψ on $I(S, \mathfrak{R})$ as

$$\Psi(e_{ij}) = \sum_{t \in L_i} B^{ii}_{ti} e_{tj} + B^{ij}_{ij} e_{ij} + \sum_{u \in R_j} B^{jj}_{ju} e_{iu}, \text{ if } i \leq j, \tag{33}$$

for all $e_{ij} \in \mathfrak{B}$. Using Lemma 2.5, B^{ij}_{tu} satisfies $B^{ii}_{il} + B^{ll}_{il} = 0,\ i < l$ and $B^{il}_{il} + B^{lq}_{lq} = B^{iq}_{iq}$, for $i \leq l \leq q$. By [21, Theorem 2.2], Ψ is a derivation. Now consider the $\mathfrak{R}$-linear map $\phi : I(S, \mathfrak{R}) \to I(S, \mathfrak{R})$ by $\phi(e_{ij}) = (\mathfrak{T} - \Psi)(e_{ij})$, for all $e_{ij} \in \mathfrak{B}$. So now we only have to show that ϕ is a central valued map and we will get our result. By $\mathfrak{R}$-linearity of ϕ, we have

$$\begin{aligned} \phi(e_{ij}) &= (\mathfrak{T} - \Psi)(e_{ij}), \\ \phi(e_{ij}) &= \mathfrak{T}(e_{ij}) - \Psi(e_{ij}). \end{aligned} \tag{34}$$

Using relations (31) - (33) in (34), we conclude that ϕ is a central valued map of $I(S, \mathfrak{R})$. Hence $\mathfrak{T}$ is proper. □

3. The general case

Here in this section, we examine the mixed triple derivation on $I(S,\mathfrak{R})$, where S is a locally finite preordered set. Let $\tilde{I}(S,\mathfrak{R})$ be the subalgebra of $I(S,\mathfrak{R})$, which is an $\mathfrak{R}$-linear subspace of $I(S,\mathfrak{R})$ generated by $\{e_{xy} : x \leq y\}$. This implies that $\tilde{I}(S,\mathfrak{R})$ represents the set of functions $f \in I(S,\mathfrak{R})$, which are nonzero only at a finite number of $(x,\ y)$. Moreover $\tilde{I}(S,\mathfrak{R})$=$I(S,\mathfrak{R})$ if and only if S is finite.

An $\mathfrak{R}$-linear map $\mathfrak{T} : \tilde{I}(S,\mathfrak{R}) \to I(S,\mathfrak{R})$ is called a mixed triple derivation if

$$\mathfrak{T}([f \circ g,\ h]) = [\mathfrak{T}(f) \circ g,\ h] + [f \circ \mathfrak{T}(g),\ h] + [f \circ g,\ \mathfrak{T}(h)],$$

for all $f, g, h \in \tilde{I}(S,\mathfrak{R})$. Clearly, Lemmas 2.4 and 2.5 remain still valid when we replace a finite set S by a locally finite set S and $I(S,\mathfrak{R})$ by $\tilde{I}(S,\mathfrak{R})$. Although the sums $\mathfrak{T}(e_{ij}) = \sum_{t\leq u} B^{ij}_{tu} e_{tu}$ are now infinite, multiplying by e_{pq} either on left or on right works as in the finite case. We also keep in mind the following important observation

$$e_{xx} f e_{yy} = f(x,y) e_{xy} \tag{35}$$

for all $f \in I(S,\mathfrak{R})$ and $x \leq y$, which will be used frequently.

Definition 3.1. *Let $f \in I(S,\mathfrak{R})$ and $x \leq y$. Then the restriction of f to $\{z \in S \mid x \leq z \leq y\}$ is denoted by $f|_{[x,y]}$ and is defined as*

$$f|_{[x,y]} = \sum_{x\leq u\leq v\leq y} f(u,v) e_{uv}.$$

Obviously, the above sum is finite, so $f|_{[x,y]} \in \tilde{I}(S,\mathfrak{R})$.

Definition 3.2. *Let $f \in I(S,\mathfrak{R})$. The diagonal function f_d is defined as*

$$f_d(x,y) = \begin{cases} f(x,y), & x = y \\ 0, & x \neq y. \end{cases}$$

Note that a diagonal function is constant on a set S if for any $x, y \in S$, we have $f(x,x) = f(y,y)$.

Lemma 3.3. *[22, Lemma 3.1] Let $\mathfrak{A} : I(S,\mathfrak{R}) \to \tilde{I}(S,\mathfrak{R})$ be a map defined by $\mathfrak{A}(f) = f|_{[x,y]}$. Then $\mathfrak{A}$ is an algebra homomorphism.*

Lemma 3.4. *Let $\mathfrak{T}$ be a mixed triple derivation of $I(S,\mathfrak{R})$ and $x < y$. Then*

$$\mathfrak{T}(f)(x,y) = \mathfrak{T}(f|_{[x,y]})(x,y). \tag{36}$$

Moreover, if $\mathfrak{T}$ is a derivation, then (36) holds for $x = y$ also.

Proof. *From [22, Lemma 3.4], we just verify (36) only when $x < y$. From (35), we have*

$$\begin{aligned}\mathfrak{T}(f)(x,y) &= [e_{xx} \circ \mathfrak{T}(f),\ e_{yy}](x,y)\\ &= \{\mathfrak{T}([e_{xx} \circ f,\ e_{yy}]) + [e_{yy},\ \mathfrak{T}(e_{xx}) \circ f] + [\mathfrak{T}(e_{yy}),\ e_{xx} \circ f]\}(x,y)\\ &= \mathfrak{T}([e_{xx} \circ f,\ e_{yy}])(x,y) + [e_{yy},\ \mathfrak{T}(e_{xx}) \circ f](x,y) + [\mathfrak{T}(e_{yy}),\ e_{xx} \circ f](x,y)\\ &= f(x,y)\mathfrak{T}(e_{xy})(x,y) - f(y,x)\mathfrak{T}(e_{yx})(x,y) + (\mathfrak{T}(e_{xx})f)(x,x) + (f\mathfrak{T}(e_{xx}))(x,x)\\ &\quad - (f\mathfrak{T}(e_{xx}))(x,y) - (\mathfrak{T}(e_{xx})f)(x,y) + \mathfrak{T}(e_{yy})(x,x)f(x,y) + (\mathfrak{T}(e_{yy})f)(y,y)\\ &\quad - (f\mathfrak{T}(e_{yy}))(x,y) - f(x,x)\mathfrak{T}(e_{yy})(x,y).\end{aligned} \tag{37}$$

Now in particular, replacing f by $f|_{[x,y]}$ in the last relation, we have

$$\begin{aligned}\mathfrak{T}(f|_{[x,y]})(x,y) &= f|_{[x,y]}(x,y)\mathfrak{T}(e_{xy})(x,y) - f|_{[x,y]}(y,x)\mathfrak{T}(e_{yx})(x,y) + (\mathfrak{T}(e_{xx})f|_{[x,y]})(x,x)\\ &\quad + (f|_{[x,y]}\mathfrak{T}(e_{xx}))(x,x) - (f|_{[x,y]}\mathfrak{T}(e_{xx}))(x,y) - (\mathfrak{T}(e_{xx})f|_{[x,y]})(x,y)\\ &\quad + \mathfrak{T}(e_{yy})(x,x)f|_{[x,y]}(x,y) + (\mathfrak{T}(e_{yy})f|_{[x,y]})(y,y) - (f|_{[x,y]}\mathfrak{T}(e_{yy}))(x,y)\\ &\quad - f|_{[x,y]}(x,x)\mathfrak{T}(e_{yy})(x,y).\end{aligned} \tag{38}$$

Obviously, we have $f(x,y) = f|_{[x,y]}(x,y)$ and $f(x,x) = f|_{[x,y]}(x,x)$. Also applying [22, Lemma 3.2], the third, fourth, fifth, sixth, eighth and ninth terms of (38) coincide with (37). Hence, we only need to show that the second term of (38) coincides with (37) also. Now, if $y \nleq x$, then obviously both equal to 0 and we get our result and if $y \leq x$, then $x \leq y \leq z \iff z \leq y \leq x$, which implies that $f(y,x) = f|_{[y,x]}(y,x) = f|_{[x,y]}(y,x)$. Hence, we get (36). □

Proposition 3.5. *Suppose $\Psi : \tilde{I}(S,\mathfrak{R}) \to I(S,\mathfrak{R})$ is a derivation. Then Ψ can be extended uniquely to a derivation $\tilde{\Psi}$ of $I(S,\mathfrak{R})$.*

Proof. By given hypothesis, $\Psi : \tilde{I}(S,\mathfrak{R}) \to I(S,\mathfrak{R})$ is a derivation. Now we define a map $\tilde{\Psi}$ on $I(S,\mathfrak{R})$ as $\tilde{\Psi}(f)(x,y) = \Psi(f|_{[x,y]})(x,y)$, for all $f \in I(S,\mathfrak{R}), x \leq y$. Then $\tilde{\Psi}$ is a linear extension of Ψ and is a derivation of $I(S,\mathfrak{R})$ by [22, Remark 3.7]. Now we show $\tilde{\Psi}$ is unique. Let Γ be another derivation of $I(S,\mathfrak{R})$ such that $\Gamma(h) = \Psi(h)$ for all $h \in \tilde{I}(S,\mathfrak{R})$. From Lemma 3.3, we have

$$\Gamma(f)(x,y) = \Gamma(f|_{[x,y]})(x,y) = \Psi(f|_{[x,y]})(x,y) = \tilde{\Psi}(f|_{[x,y]})(x,y),$$

for all $f \in I(S,\mathfrak{R})$ and $x \leq y$. Hence $\Gamma = \Psi$. □

Lemma 3.6. *Let $\mathfrak{T}$ be a mixed triple derivation on $I(S,\mathfrak{R})$, where S is connected. Then $\mathfrak{T}(f)(x,x) = \mathfrak{T}(f)(y,y)$, for all $x,y \in S$.*

Proof. By given hypothesis S is connected, without loss of generality, we assume $x < y$. Now, we have

$$\begin{aligned}\mathfrak{T}([e_{xy},f])(x,y) &= \mathfrak{T}([e_{xx}\circ e_{xy},\ f])(x,y)\\ &= \{[\mathfrak{T}(e_{xx})\circ e_{xy},\ f] + [e_{xx}\circ \mathfrak{T}(e_{xy}),\ f] + [e_{xy},\mathfrak{T}(f)]\}(x,y)\\ &= ([\mathfrak{T}(e_{xx})\circ e_{xy},\ f])(x,y) + ([e_{xx}\circ \mathfrak{T}(e_{xy}),\ f])(x,y) + ([e_{xy},\mathfrak{T}(f)])(x,y)\\ &= \mathfrak{T}(e_{xx})(x,x)f(y,y) + (\mathfrak{T}(e_{xx})f)(y,y) - 2(f\mathfrak{T}(e_{xx}))(x,x) - f(x,x)\mathfrak{T}(e_{xx})(y,y)\\ &\quad +(\mathfrak{T}(e_{xy})f)(x,y) + \mathfrak{T}(e_{xy})(x,x)f(x,y) - f(x,x)\mathfrak{T}(e_{xy})(x,y)\\ &\quad +\mathfrak{T}(f)(y,y) - \mathfrak{T}(f)(x,x).\end{aligned} \tag{39}$$

Replacing f by $f|_{[x,y]}$ in the previous relation, we get

$$\begin{aligned}\mathfrak{T}([e_{xy},f|_{[x,y]}])(x,y) &= \mathfrak{T}(e_{xx})(x,x)f|_{[x,y]}(y,y) + (\mathfrak{T}(e_{xx})f|_{[x,y]})(y,y) - 2(f|_{[x,y]}\mathfrak{T}(e_{xx}))(x,x)\\ &\quad -f|_{[x,y]}(x,x)\mathfrak{T}(e_{xx})(y,y) + (\mathfrak{T}(e_{xy})f|_{[x,y]})(x,y) + \mathfrak{T}(e_{xy})(x,x)f|_{[x,y]}(x,y)\\ &\quad -f|_{[x,y]}(x,x)\mathfrak{T}(e_{xy})(x,y) + \mathfrak{T}(f|_{[x,y]})(y,y) - \mathfrak{T}(f|_{[x,y]})(x,x).\end{aligned} \tag{40}$$

From Lemmas 3.3 and 3.4, we have

$$\mathfrak{T}([e_{xy},f])(x,y) = \mathfrak{T}([e_{xy},f]|_{[x,y]})(x,y) = \mathfrak{T}([e_{xy}|_{[x,y]},f|_{[x,y]}])(x,y) = \mathfrak{T}([e_{xy},f|_{[x,y]}])(x,y).$$

Comparing (39) and (40), we have $f(x,y) = f|_{[x,y]}(x,y)$, $f(x,x) = f|_{[x,y]}(x,x)$ and $f(y,y) = f|_{[x,y]}(y,y)$. So first, fourth, sixth and seventh terms of (40) coincide with corresponding terms of right hand side of (39). Also, from [22, Lemma 3.2], we have $(\mathfrak{T}(e_{xy})f)(x,y) = (\mathfrak{T}(e_{xy})f|_{[x,y]})(x,y)$. Also, from [22, Lemma 3.2], we have

$$(\mathfrak{T}(e_{xx})f|_{[x,y]})(y,y) = (\mathfrak{T}(e_{xx})(f|_{[x,y]})|_{[y,y]})(y,y) = (\mathfrak{T}(e_{xx})f|_{[y,y]})(y,y) = (\mathfrak{T}(e_{xy})f)(y,y).$$

Therefore, we have

$$\mathfrak{T}(f)(y,y) - \mathfrak{T}(f)(x,x) = \mathfrak{T}(f|_{[x,y]})(y,y) - \mathfrak{T}(f|_{[x,y]})(x,x).$$

The right hand side of previous relation is zero by Lemma 2.5. Hence, we get our result. □

Theorem 3.7. *Let S be connected and $\Re$ be a 2-torsion free commutative ring with unity. If $\mathfrak{T}$ is a mixed triple derivation of $I(S,\Re)$, then $\mathfrak{T} = \Psi + \phi$, where $\Psi : I(S,\Re) \to I(S,\Re)$ is a derivation and $\phi : I(S,\Re) \to \mathcal{Z}(I(S,\Re))$ is an $\Re$-linear map.*

Proof. Let $\mathfrak{T}$ be a mixed triple derivation of $I(S,\Re)$. Let ϕ, $\Psi : I(S,\Re) \to I(S,\Re)$ be maps defined as $\phi(f) = \mathfrak{T}(f)_d$ and $\Psi(f) = \mathfrak{T}(f) - \phi(f)$. From Lemma 3.6, ϕ is a central valued map on $I(S,\Re)$. Now, we only have to show that Ψ is a derivation on $I(S,\Re)$. Let's restrict Ψ on $\tilde{I}(S,\Re)$, by Theorem 2.2, $\Psi : \tilde{I}(S,\Re) \to I(S,\Re)$ is a derivation. Also from Proposition 3.5, Ψ can be extended to a derivation $\tilde{\Psi}$ of $I(S,\Re)$. Clearly we have

$$\tilde{\Psi}(f)(x,y) = \Psi(f|_{[x,y]})(x,y) = \mathfrak{T}(f|_{[x,y]})(x,y) - \phi(f|_{[x,y]})_d(x,y). \tag{41}$$

Now if $x < y$, then the relation (41) reduces to $\tilde{\Psi}(f)(x,y) = \mathfrak{T}(f|_{[x,y]})(x,y)$. Now using Lemma 3.4, we get $\tilde{\Psi}(f)(x,y) = \mathfrak{T}(f)(x,y) = \Psi(f)(x,y)$, which is the required result for this case. Now if $x = y$, then the right hand side of (41) is zero. On the other hand, we have $\Psi(f)(x,x) = \mathfrak{T}(f)(x,x) - \phi(f)_d(x,x) = 0$. Hence, we deduce that $\tilde{\Psi} = \Psi$ and thus Ψ is a derivation of $I(S,\Re)$. □

Finally, we have constructed an example which shows that the condition of 2-torsion freeness of $\Re$ is essential in Theorem 3.7.

Example 3.8. *Let $\Re = \mathbb{Z}_2$ and $S = \{x_1, x_2\}$ with the relation $x_i \leq x_j$ if $i \leq j$ and $i, j \in \{1,2\}$. Then*

$$I(S,\Re) \cong \mathcal{T}_2(\Re),$$

the algebra of all 2×2 upper triangular matrices over $\Re$. We define an $\Re$-linear map $\mathfrak{T} : \mathcal{T}_2(\Re) \to \mathcal{T}_2(\Re)$ as $\mathfrak{T}\left\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right\} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$. Then $\mathfrak{T}$ is a mixed triple derivation on $\mathcal{T}_2(\Re)$. If $\mathfrak{T}$ is proper, then $\mathfrak{T} = \Psi + \phi$, where $\Psi : \mathcal{T}_2(\Re) \to \mathcal{T}_2(\Re)$ is a derivation and $\phi : \mathcal{T}_2(\Re) \to \mathcal{Z}(\mathcal{T}_2(\Re))$ is an $\Re$-linear map. Thus $\phi(A) = \lambda_A I$, where $\lambda_A \in \Re$ and I is the 2×2 identity matrix. Hence, $\mathfrak{T} - \phi = \Psi$ is a derivation. But

$$(\mathfrak{T} - \phi)\left\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right\} = \begin{bmatrix} a - \lambda_A I & b \\ 0 & c - \lambda_A I \end{bmatrix},$$

which is a contradiction. Thus $\mathfrak{T}$ is improper.

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