Filomat 39:2 (2025), 451–468
https://doi.org/10.2298/FIL2502451H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Explicit inverses of some special matrices and their applications

Kadir Hilal[a], **Cahit Köme**[a,*]

[a]*Nevşehir Hacı Bektaş Veli University, Department of Mathematics, 50300, Nevşehir, Türkiye*

Abstract. As powerful mathematical structures, matrices have widespread use in real–world applications in various disciplines such as engineering, physics, computer science, and economics. In recent years, the topic of matrix inverses has started to attract the attention of many researchers. In this study, we investigate the inverse of some lower triangular polynomial matrices, which are formed by conditional polynomial sequences, with the help of some analytical techniques. We derive some correlations between conditional polynomial matrices and the $\mathcal{T}-$nomial matrices of the first and of the second kind. In addition, we get factorizations of the conditional polynomial matrices via $\mathcal{T}-$nomial matrices. Moreover, we obtain several combinatorial identities and provide more generalized results. Finally, we provide some numerical results that explain our method is faster and more efficient than MATHEMATICA's Inverse method while computing the inverses of non–singular conditional polynomial matrices.

1. Introduction

The topic of matrix inverses has been widely used in order to facilitate performance-requiring computations, especially in engineering problems, in recent years. Until now, many researchers have studied the inverses and factorizations of matrices which are associated with special number sequences [1, 2, 4–8, 10–14, 16]. For example, Zhang and Zhang obtained the Lucas matrix via the Pascal matrix of the first and second kinds. The authors also gave some interesting identities involving the Lucas numbers [2]. Falcon defined the $k-$Fibonacci matrix as an extension of the classical Fibonacci matrix [7]. He presented some combinatorial formulas involving the $k-$Fibonacci numbers by using the $k-$Fibonacci matrix and Pascal matrix. Miladinovic and Stanimirovic proposed the pseudoinverse of the generalized Fibonacci matrix and they gave the correlations between the generalized Fibonacci matrix and Pascal matrices [5]. Shen and He examined the Moore–Penrose inverse of the matrix whose nonzero entries are the classical Horadam numbers [11]. As a generalization of the studies [5] and [11], Shen et al. studied the Moore–Penrose inverse of the strictly lower triangular Toeplitz matrix and they derived a convolution formula containing the Horadam numbers. Also, the authors obtained several combinatorial identities by using this formula [12]. Radičić studied the Moore–Penrose inverse of the $k-$circulant matrices whose elements are the binomial coefficients [10]. Shen et al. studied the determinants and inverses of circulant matrices whose entries are

2020 *Mathematics Subject Classification*. Primary 05A10, 11B39, 15A09.
Keywords. Matrix Inverse, $\mathcal{T}-$nomial matrix, Combinatorial identities, Conditional sequences.
Received: 28 January 2024; Revised: 30 June 2024; Accepted: 23 October 2024
Communicated by Dijana Mosić
* Corresponding author: Cahit Köme
Email addresses: kadirhilal@hotmail.com (Kadir Hilal), cahit@nevsehir.edu.tr (Cahit Köme)
ORCID iDs: https://orcid.org/0000-0002-7850-7819 (Kadir Hilal), https://orcid.org/0000-0002-6488-9035 (Cahit Köme)

Fibonacci and Lucas numbers [4]. Yazlık and Taşkara illustrated a new generalization to compute determinants and inverses of the circulant matrix whose entries are the generalized $k-$Horadam numbers [13]. The authors also stated that the results in their work are the most general statements to obtain the inverses and determinants in such matrices having the elements of all second order sequences. Irmak and Köme studied the factorizations of the Lucas and inverse Lucas matrices [14]. The authors also investigated the Cholesky factorizations and obtained the upper and lower bounds for the eigenvalues of the symmetric Lucas matrix by using majorization techniques. Köme and Yazlık investigated the Moore–Penrose inverse of the conditional matrices via the convolution product formula [15]. They also expressed the Moore–Penrose inverse of the conditional matrices in the form of block matrices. Shen and Liu considered the lower Hessenberg matrix whose nonzero elements are the Horadam numbers. The authors computed the value of the determinant of this matrix with only the Horadam numbers and they derived the inverse of this matrix [16]. Köme obtained the explicit Moore–Penrose inverse of the singular conditional matrices whose elements are the generalized conditional sequences by using some analytical techniques [17]. Also, he investigated the correlations between singular conditional matrices and some combinatorial matrices such as Pascal matrices of the first and of the second kind.

The binomial coefficient, which is also known as a combination or combinatorial number, is the number of ways of picking r unordered outcomes from n possibilities. Towards the end of the twentieth century, several researchers have studied the applications and generalizations of binomial coefficients [26–31]. For example, Fontené studied the generalizations of binomial coefficients replacing the natural numbers by the terms of an arbitrary sequence $\{A_n\}_{n=0}^{\infty}$ of real or complex numbers [29]. Krot investigated the main theorems and definitions of the Fibonomial calculus which is a special case of $\psi-$extended Rota's finite operator calculus [27]. Gould gave a sequence of seven main theorems, generalizing all of the corresponding results previously found for ordinary and $q-$binomial coefficients to the most general situation for Fontené–Ward generalized binomial coefficients [25]. Seibert and Trojovský derived new identities, which are related to the generating function of the $k-$th powers of the Fibonacci numbers, for the Fibonomial coefficients [28]. Dziemianczuk developed the theory of $\mathcal{T}-$nomial coefficients by means of generating functions [31]. The author investigated a new combinatorial interpretation of $\mathcal{T}-$nomial coefficients and compared with the Konvalina way of objects' selections from weighted boxes.

In recent years, there has been a huge interest in modern science in the application of the Golden Section and Fibonacci numbers. The Fibonacci numbers, $\{F_n\}_{n=0}^{\infty}$, are the terms of the sequence $\{0, 1, 1, 2, 3, 5, \dots\}$ wherein each term is the sum of two consecutive terms, starting with the initial conditions $F_0 = 0$ and $F_1 = 1$. As $n \to \infty$, the ratio between successive Fibonacci numbers is called as golden ratio, $\tau = \frac{1+\sqrt{5}}{2} = 1.618\dots$, which plays an important role in arts, architecture, engineering, geometry, music, electrostatics, poetry, stock market trading and trigonometry [18]. Up until now, many researchers have studied the applications, generalizations and relations with other disciplines of the Fibonacci sequence [19–24]. In particular, Edson and Yayenie presented a notable generalization of the Fibonacci numbers, $\{q_n\}_{n=0}^{\infty}$, which is called as biperiodic Fibonacci sequence, and then the authors obtained the extended Binet's formula, generating function of this sequence [19]. Later, in [19], Bilgici studied the biperiodic Lucas sequence, $\{l_n\}_{n=0}^{\infty}$ and modified generalized Lucas sequence, $\{Q_n\}_{n=0}^{\infty}$, and he gave the Binet formulas, generating functions and some special identities for these sequences. Tan and Leung studied the generalized biperiodic Horadam sequence and investigated some congruence properties of the generalized Horadam sequence [21]. Motivated by the works [19, 20] and [21], Verma and Bala defined the generalized bivariate biperiodic Fibonacci polynomials, which we call as conditional polynomials sequence throughout this paper [22]. The authors also derived the Binet's formula, generating function, Catalan's identity, Cassini's identity, d'Ocagne's identity and Gelin Cesaro identity for the generalized bivariate biperiodic Fibonacci polynomials.

Up until now, several studies have been done on the inverses and factorization of matrices whose elements are classical number sequences [1–3, 5, 8]. Hereby, some natural questions are that: If we construct a lower triangular matrix with conditional polynomial sequences, can we find the inverse of this matrix explicitly? If yes, can we provide better performance in terms of the computational costs of the inverse of these conditional polynomial matrices compared to other methods? In this paper, we are going to answer these questions. Therefore, drawing inspiration from previous works on the inversion of matrices, we focus on the following outcomes:

- Constructing a conditional matrix with conditional entries,
- Establishing a new Binet formula for the conditional polynomial sequences,
- Proving auxiliary identities in order to obtain the conditional polynomial matrix inverse more effectively,
- Investigating the correlations between conditional polynomial matrices and $\mathcal{T}-$nomial matrices of the first and second kinds,
- Deriving some combinatorial identities involving conditional polynomial sequences,
- Providing comparative numerical results in order to explain the effectiveness of our method.

2. Preliminaries and Main Results

In this section, we introduce some definitions and preliminary facts which are used in this paper. Next, we obtain the Binet formula of the conditional polynomial sequence by virtue of the induction method. Later, we define the conditional polynomial matrix and its inverse with the help of the Binet formula and some analytical techniques.

Definition 2.1. *[31] A natural numbers' valued sequence $\mathcal{T} \equiv \{n_{\mathcal{T}}\}_{n\geq 1}$ constituted by n-th coefficient of the generating function $\mathcal{T}_{p,q}(x)$ expansion, i.e., $n_{\mathcal{T}} = [x^n]\mathcal{T}_{p,q}(x)$, where*

$$\mathcal{T}_{p,q}(x) = 1_{\mathcal{T}} \cdot \frac{x}{(1-px)(1-qx)}$$

while $p, q \in \mathbb{R}$ and $1_{\mathcal{T}} \in \mathbb{R}$ is called tileable and denoted by $\mathcal{T}_{p,q}$.

For $n \geqslant 1$, Dziemianczuk defined an explicit formula for $n-$th term of $\mathcal{T}$ as:

$$n_{\mathcal{T}} = \begin{cases} \frac{q^n-p^n}{q-p}, & q \neq p \\ nq^{n-1}, & q = p \end{cases}. \tag{1}$$

The explicit formula of $\mathcal{T}-$nomial coefficients while the sequence $\mathcal{T} = \mathcal{T}_{p,q}$ is

$$\left(\begin{array}{c} n \\ k \end{array}\right)_{\mathcal{T}} = \begin{cases} \prod_{i=1}^{k} \frac{\left(p^{n-i+1}-q^{n-i+1}\right)}{\left(p^i-q^i\right)}, & p \neq q \\ \left(\begin{array}{c} n \\ k \end{array}\right) p^{k(n-k)}, & q = p \end{cases}. \tag{2}$$

The $\mathcal{T}-$nomial coefficients satisfy binomial–like recurrence relation, i.e., for any $n, k \in \mathbb{N}$ we have

$$\left(\begin{array}{c} n \\ k \end{array}\right)_{\mathcal{T}} = p^{n-k}\left(\begin{array}{c} n-1 \\ k-1 \end{array}\right)_{\mathcal{T}} + q^k\left(\begin{array}{c} n-1 \\ k \end{array}\right)_{\mathcal{T}}$$

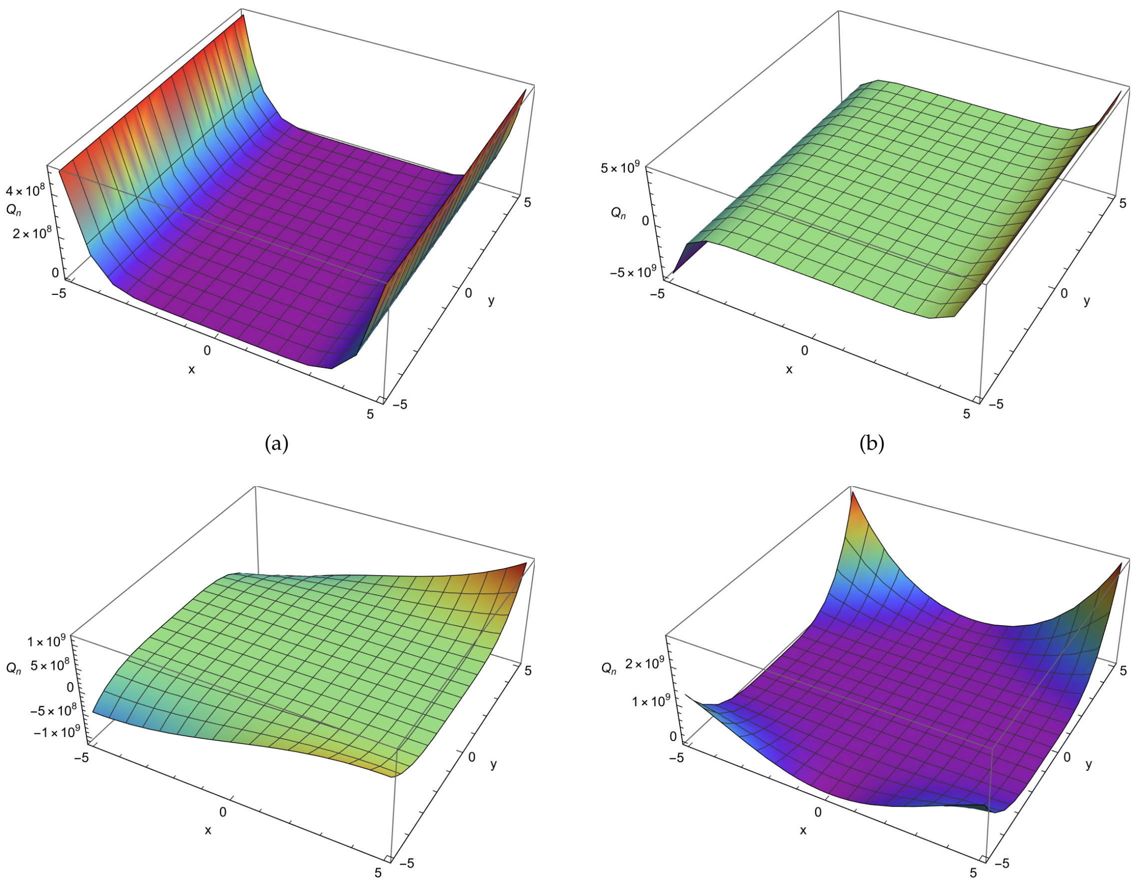
with initial values $\left(\begin{array}{c} n \\ 0 \end{array}\right)_{\mathcal{T}} = \left(\begin{array}{c} n \\ n \end{array}\right)_{\mathcal{T}} = 1$.

Definition 2.2. *[22] For any nonzero real numbers a, b and c, the conditional polynomial sequence is defined by the recurrence relation*

$$Q_n(x,y) = \begin{cases} axQ_{n-1}(x,y) + cyQ_{n-2}(x,y), & \text{if } n \text{ is even} \\ bxQ_{n-1}(x,y) + cyQ_{n-2}(x,y), & \text{if } n \text{ is odd} \end{cases}, \tag{3}$$

where $Q_0(x,y)$ and $Q_1(x,y) \neq 0$ are the arbitrary initial conditions.

Note that the conditional polynomial sequence can be reduced into infinite number sequences for the special cases of $a, b, c, x, y, Q_0(x,y)$ and $Q_1(x,y)$.



(a) (b) (c) (d)

Figure 1: The conditional polynomial sequence for different values of a, b, c, Q_0 and Q_1 on the intervals $x, y \in [-5, 5]$. **a)** $Q_0 = 0, Q_1 = 1, a = 2, b = 3, c = 0.1$ and $n = 10$ **b)** $Q_0 = 0, Q_1 = 1, a = 2, b = 3, c = 0.1$ and $n = 9$ **c)** $Q_0 = 2, Q_1 = x, a = 0.1, b = 3, c = 10$ and $n = 9$ **d)** $Q_0 = x, Q_1 = y, a = 0.1, b = 3, c = 10$ and $n = 9$.

Figure 1 shows that although conditional polynomial sequences have smaller values on the interval $[-3, 3]$, these polynomial sequences suddenly and sharply increase or decrease in larger ranges. This gives information about the difficulties that will be encountered in computations with conditional polynomial sequences.

The French mathematician Jacques–Marie Binet found an explicit formula of the Fibonacci sequence in 1843 and it was called as Binet's formula after this discovery.

The next definition explains generalized Binet formula of the sequence $\{Q_n(x, y)\}_{n=0}^{\infty}$. Moreover, $\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ is the parity function, i.e., $\xi(n) = 0$ when n is even and $\xi(n) = 1$ when n is odd. Let assume $\Delta(x, y) = a^2b^2x^2 + 4abcy \neq 0$. Also we have $\alpha(x, y) + \beta(x, y) = abx, \alpha(x, y) - \beta(x, y) = \sqrt{a^2b^2x^2 + 4abcy}$ and $\alpha(x, y)\beta(x, y) = -abcy$.

Theorem 2.3. *The Binet formula of the conditional polynomial sequence is*

$$Q_n(x, y) = \frac{a^{\xi(n+1)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \left(A\alpha^n(x, y) - B\beta^n(x, y) \right), \tag{4}$$

where $A = \frac{Q_1(x,y) - \frac{\beta(x,y)}{a} Q_0(x,y)}{\alpha(x,y) - \beta(x,y)}$, $B = \frac{Q_1(x,y) - \frac{\alpha(x,y)}{a} Q_0(x,y)}{\alpha(x,y) - \beta(x,y)}$, *$\alpha(x, y)$ and $\beta(x, y)$ are the roots of the polynomial* $\lambda^2 - \lambda abx - abcy$, *that is,* $\alpha(x,y) = \frac{abx+\sqrt{a^2b^2x^2+4abcy}}{2}$ *and* $\beta(x,y) = \frac{abx-\sqrt{a^2b^2x^2+4abcy}}{2}$.

Proof. We will prove the theorem by induction on n. The theorem clearly holds for $n = 0$ and $n = 1$. Now we suppose that the result is true for all positive integers less than or equal to n. Taking into account (3) and the hypothesis of induction, we need to show that the case also holds for $n + 1$. Hence, by using the identities $\alpha(x,y)x + cy = \frac{\alpha^2(x,y)}{ab}$ and $\beta(x,y)x + cy = \frac{\beta^2(x,y)}{ab}$, we have

$$
\begin{aligned}
Q_{n+1}(x,y) &= a^{1-\xi(n+1)}b^{\xi(n+1)}xQ_n(x,y) + cyQ_{n-1}(x,y) \\
&= \frac{a^{1-\xi(n+1)}(ab)^{\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} x\left(A\alpha^n(x,y) - B\beta^n(x,y)\right) \\
&\quad + \frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n-1}{2} \rfloor}} cy\left(A\alpha^{n-1}(x,y) - B\beta^{n-1}(x,y)\right) \\
&= a^{1-\xi(n+1)}A\alpha^{n-1}(x,y)\left(\frac{\alpha^2(x,y)}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}}\right) \\
&\quad - a^{1-\xi(n+1)}B\beta^{n-1}(x,y)\left(\frac{\beta^2(x,y)}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}}\right) \\
&= \frac{a^{\xi(n+2)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}}\left(A\alpha^{n+1}(x,y) - B\beta^{n+1}(x,y)\right),
\end{aligned}
$$

which completes the proof. □

Definition 2.4. *The conditional polynomial matrix of type s is defined by*

$$
\mathbb{Q}_n^{(s)}[x,y] = \left[q_{ij}^{(s)}\right] = \begin{cases} \left(\frac{b}{a}\right)^{\frac{\xi(i-j)}{2}} Q_{i-j+1}(x,y) & , i-j+s \geqslant 0 \\ 0 & , i-j+s < 0 \end{cases}, \tag{5}
$$

where $\{Q_n(x,y)\}_{n=0}^{\infty}$ *is the conditional polynomial sequence and the integer s stands for the shift of non–zero elements with respect to main diagonal.*

For the case $s < 0$, we observe that $|s|$ diagonals below the main diagonal in $\mathbb{Q}_n^{(s)}[x,y]$ are zeros and $rank\left(\mathbb{Q}_n^{(s)}[x,y]\right) = n - |s| < n$. Therefore, the matrix $\mathbb{Q}_n^{(s)}[x,y]$ is singular for $s < 0$. So, throughout this paper, we get $s = 0$ in order to remove singularity and to get a triangular matrix form. In addition, for the sake of simplicity, we take into account $\mathbb{Q}_n[x,y] = \left[q_{ij}\right]$ instead of $\mathbb{Q}_n^{(0)}[x,y]$.

Example 2.5. *The* 3×3 *conditional polynomial matrix of type* 0 *is equal to*

$$
\mathbb{Q}_3[x,y] = \begin{pmatrix} Q_1(x,y) & 0 & 0 \\ \frac{\sqrt{b}\left(aQ_1(x,y)x + cQ_0(x,y)y\right)}{\sqrt{a}} & Q_1(x,y) & 0 \\ Q_1(x,y)\left(abx^2 + cy\right) + bcQ_0(x,y)xy & \frac{\sqrt{b}\left(aQ_1(x,y)x + cQ_0(x,y)y\right)}{\sqrt{a}} & Q_1(x,y) \end{pmatrix}
$$

Lemma 2.6. *The following identity is valid for the conditional polynomial sequence* $\{Q_n(x,y)\}_{n=0}^{\infty}$ *satisfying* $Q_1(x,y) \neq 0$ *and for arbitrary integers i, j satisfying* $i \geqslant j + 2$*:*

$$
\begin{aligned}
&\Theta(x,y)\sum_{k=j+2}^{i}\left(\frac{b}{a}\right)^{\frac{\xi(i-k)}{2}} Q_{i-k+1}(x,y)(-1)^{k-j}\left(\frac{b}{a}\right)^{\frac{k-j}{2}} \frac{Q_0^{k-j-2}(x,y)\,(cy)^{k-j-1}}{Q_1^{k-j+1}(x,y)} \\
&= \frac{bcyQ_0(x,y)}{aQ_1^2(x,y)}\left(\frac{a}{b}\right)^{\frac{\xi(i-j)}{2}} Q_{i-j}(x,y) - \frac{cy}{Q_1(x,y)}\left(\frac{b}{a}\right)^{\frac{\xi(i-j)}{2}} Q_{i-j-1}(x,y)
\end{aligned} \tag{6}
$$

where $\Theta(x,y) = \frac{abxQ_0(x,y)Q_1(x,y) - aQ_1^2(x,y) + bcyQ_0^2(x,y)}{b}$.

Proof. Let us denote

$$\gamma(x,y)=\sum_{k=j+2}^{i}\left(\frac{b}{a}\right)^{\frac{\xi(i-k)}{2}}Q_{i-k+1}(x,y)(-1)^{k-j}\left(\frac{b}{a}\right)^{\frac{k-j}{2}}\frac{Q_0^{k-j-2}(x,y)\,(cy)^{k-j-1}}{Q_1^{k-j+1}(x,y)}.$$

By utilizing the Binet formula (4) and doing some algebraic operations, we get

$$\begin{aligned}
\gamma(x,y) &= \sum_{k=j+2}^{i}\left(\frac{b}{a}\right)^{\frac{\xi(i-k)}{2}}(-1)^{k-j}\left(\frac{b}{a}\right)^{\frac{k-j}{2}}\frac{Q_0^{k-j-2}(x,y)\,(cy)^{k-j-1}}{Q_1^{k-j+1}(x,y)}\left(\frac{a^{\xi(i-k+2)}}{(ab)^{\lfloor\frac{i-k+1}{2}\rfloor}}\left(A\alpha^{i-k+1}(x,y)-B\beta^{i-k+1}(x,y)\right)\right)\\
&= \sum_{k=j+2}^{i}\frac{1}{(ab)^{\frac{i-j}{2}}}\left(A\alpha^{i-k+1}(x,y)-B\beta^{i-k+1}(x,y)\right)(-b)^{k-j}\left(\frac{b}{a}\right)^{\frac{k-j}{2}}\frac{Q_0^{k-j-2}(x,y)\,(cy)^{k-j-1}}{Q_1^{k-j+1}(x,y)}\\
&= \frac{b^2cy}{Q_1^3(x,y)\,(ab)^{\frac{i-j}{2}}}\sum_{k=j+2}^{i}\left[\begin{array}{l}A\alpha^{i-j-1}(x,y)\left(-\frac{bcyQ_0(x,y)}{\alpha(x,y)Q_1(x,y)}\right)^{k-j-2}\\ \qquad -B\beta^{i-j-1}(x,y)\left(-\frac{bcyQ_0(x,y)}{\beta(x,y)Q_1(x,y)}\right)^{k-j-2}\end{array}\right]\\
&= \frac{b^2cy}{Q_1^3(x,y)\,(ab)^{\frac{i-j}{2}}}\left[\begin{array}{l}A\alpha^{i-j-1}(x,y)\dfrac{1-\left(-\frac{bcyQ_0(x,y)}{\alpha(x,y)Q_1(x,y)}\right)^{i-j-1}}{1+\frac{bcyQ_0(x,y)}{\alpha(x,y)Q_1(x,y)}}\\ \qquad -B\beta^{i-j-1}(x,y)\dfrac{1-\left(-\frac{bcyQ_0(x,y)}{\beta(x,y)Q_1(x,y)}\right)^{i-j-1}}{1+\frac{bcyQ_0(x,y)}{\beta(x,y)Q_1(x,y)}}\end{array}\right]\\
&= \frac{b^2cy}{Q_1^3(x,y)\,(ab)^{\frac{i-j}{2}}}\left[\begin{array}{l}A\dfrac{\alpha^{i-j}(x,y)-\alpha(x,y)\left(-\frac{bcyQ_0(x,y)}{Q_1}\right)^{i-j-1}}{\alpha(x,y)+\frac{bcyQ_0(x,y)}{Q_1(x,y)}}\\ \qquad -B\dfrac{\beta^{i-j}(x,y)-\beta(x,y)\left(-\frac{bcyQ_0(x,y)}{Q_1(x,y)}\right)^{i-j-1}}{\beta(x,y)+\frac{bcyQ_0(x,y)}{Q_1(x,y)}}\end{array}\right]\\
&= \frac{b^2cy}{Q_1^3(x,y)\,(ab)^{\frac{i-j}{2}}}\left[\frac{\begin{array}{l}A\left(\beta(x,y)+F(x,y)\right)\left(\alpha^{i-j}(x,y)-\alpha(x,y)\left(-F(x,y)\right)^{i-j-1}\right)\\ \qquad -B\left(\alpha(x,y)+F(x,y)\right)\left(\beta^{i-j}(x,y)-\beta(x,y)\left(-F(x,y)\right)^{i-j-1}\right)\end{array}}{\left(\alpha(x,y)+F(x,y)\right)\left(\beta(x,y)+F(x,y)\right)}\right],
\end{aligned}$$

where $F(x,y)=\frac{bcyQ_0(x,y)}{Q_1(x,y)}$. By virtue of the identities $\alpha(x,y)\beta(x,y)=-abcy$ and $\alpha(x,y)+\beta(x,y)=abx$, we obtain

$$\begin{aligned}
D(x,y) &= \left(\alpha(x,y)+\frac{bcyQ_0(x,y)}{Q_1(x,y)}\right)\left(\beta(x,y)+\frac{bcyQ_0(x,y)}{Q_1(x,y)}\right)\\
&= \frac{bcy\left(abxQ_0(x,y)Q_1(x,y)-aQ_1^2(x,y)+bcyQ_0^2(x,y)\right)}{Q_1^2(x,y)}
\end{aligned} \tag{7}$$

Furthermore, with the help of the equations (4) and (7), we have

$$\begin{aligned}
\gamma(x,y) &= \frac{b^2cy}{Q_1^3(x,y)\,(ab)^{\frac{i-j}{2}}\,D(x,y)}\begin{pmatrix} A\Big(\beta(x,y)\alpha(x,y)^{i-j}-\alpha(x,y)\beta(x,y)\left(-F(x,y)\right)^{i-j-1} \\ +\alpha(x,y)^{i-j}\left(F(x,y)\right)+\alpha(x,y)\left(-F(x,y)\right)^{i-j}\Big) \\ -B\Big(\alpha(x,y)\beta(x,y)^{i-j}-\beta(x,y)\alpha(x,y)\left(-F(x,y)\right)^{i-j-1} \\ +\beta(x,y)^{i-j}\left(F(x,y)\right)+\beta(x,y)\left(-F(x,y)\right)^{i-j}\Big)\end{pmatrix} \\
&= \frac{b^3Q_0(x,y)c^2y^2}{aQ_1^4(x,y)D(x,y)}\left(\frac{a}{(ab)^{\frac{i-j}{2}}}\left(A\alpha(x,y)^{i-j}-B\beta(x,y)^{i-j}\right)\right) \\
&\quad+\frac{b^2cy}{Q_1^3(x,y)\,(ab)^{\frac{i-j}{2}}\,D(x,y)}\left(-F(x,y)\right)^{i-j}Q_1(x,y) \\
&\quad-\frac{b^2c^2y^2}{Q_1^3(x,y)D(x,y)}\left(\frac{\sqrt{ab}}{(ab)^{\frac{i-j-1}{2}}}\left(A\alpha(x,y)^{i-j-1}-B\beta(x,y)^{i-j-1}\right)\right) \\
&\quad+\frac{b^3c^2y^2}{Q_1^3(x,y)(ab)^{\frac{i-j}{2}}D(x,y)}\left(-F(x,y)\right)^{i-j-1}Q_0(x,y) \\
&= \frac{b^3Q_0(x,y)c^2y^2}{aQ_1^4(x,y)D(x,y)}\left(\frac{a}{b}\right)^{\frac{\xi(i-j)}{2}}Q_{i-j}(x,y)-\frac{b^2c^2y^2}{Q_1^3D(x,y)}\left(\frac{b}{a}\right)^{\frac{\xi(i-j)}{2}}Q_{i-j-1}(x,y).
\end{aligned}$$

Therefore, we obtain

$$\Theta(x,y)\gamma(x,y)=\frac{bcyQ_0(x,y)}{aQ_1^2(x,y)}\left(\frac{a}{b}\right)^{\frac{\xi(i-j)}{2}}Q_{i-j}(x,y)-\frac{cy}{Q_1(x,y)}\left(\frac{b}{a}\right)^{\frac{\xi(i-j)}{2}}Q_{i-j-1}(x,y),$$

which completes the proof. □

In the partial case $a = b = A$, $c = B$, $Q_0(x,y) = a$, $Q_1(x,y) = b$ and $x = y = 1$, we obtain the following result for the second order non–degenerated recurrent sequence, $U_n^{(a,b)}$.

Corollary 2.7. *[1] The following identity is valid for the second order non-degenerated recurrent sequence $U_n^{(a,b)}$ satisfying $b \neq 0$ and for two arbitrary integers i, j satisfying $i \geq j+2$:*

$$\left(a^2B+abA-b^2\right)\sum_{k=j+2}^{i}(-1)^{k-j}\frac{a^{k-j-2}B^{k-j-1}}{b^{k-j+1}}U_{i-k+1}^{(a,b)}=\frac{aB}{b^2}U_{i-j}^{(a,b)}-\frac{B}{b}U_{i-j-1}^{(a,b)}.$$

In the case $a = b = 1$, $c = 1$, $Q_0(x,y) = a$, $Q_1(x,y) = b$ and $x = y = 1$, we obtain the following result for the generalized Fibonacci numbers, $F_n^{(a,b)}$.

Corollary 2.8. *For the generalized Fibonacci numbers $F_n^{(a,b)}, b \neq 0$ and for two arbitrary integers i, j satisfying $i \geqslant j+2$ the following is valid:*

$$\left(a^2+ab-b^2\right)\sum_{k=j+2}^{i}(-1)^{k-j}\frac{a^{k-j-2}}{b^{k-j+1}}F_{i-k+1}^{(a,b)}=\frac{a}{b^2}F_{i-j}^{(a,b)}-\frac{1}{b}F_{i-j-1}^{(a,b)}.$$

In the case $a = b = 1$, $c = 1$, $Q_0(x,y) = 2$, $Q_1(x,y) = 1$ and $x = y = 1$, we obtain the following result for the Lucas numbers, L_n.

Corollary 2.9. *[2] For the Lucas numbers and each $i \geq j+2$ the following is valid:*

$$5\sum_{k=j+2}^{i}(-1)^{k-j}2^{k-j-2}L_{i-k+1} = 2L_{i-j} - L_{i-j-1}.$$

Theorem 2.10. *For $Q_1(x,y) \neq 0$, the inverse of the conditional polynomial matrix, $\mathbb{Q}_n^{-1}[x,y] = \left[q'_{ij}\right]$, is equal to*

$$q'_{ij} = \begin{cases} (-1)^{i-j}\left(\frac{b}{a}\right)^{\frac{i-j}{2}}\Theta(x,y)\frac{Q_0^{i-j-2}(x,y)(cy)^{i-j-1}}{Q_1^{i-j+1}(x,y)} & ,i \geqslant j+2 \\ -\frac{\sqrt{b}(aQ_1(x,y)x+cQ_0(x,y)y)}{\sqrt{a}Q_1^2(x,y)} & ,i = j+1 \\ \frac{1}{Q_1(x,y)} & ,i=j \\ 0 & ,i<j \end{cases}. \tag{8}$$

Proof. Let $\sum_{k=1}^{n} q_{i,k}q'_{k,j} = \chi_{i,j}$. It is clear from the Theorem 2.10 that $\chi_{i,j} = 0$ for $i < j$. For the case $i = j$, we have

$$\chi_{i,i} = q_{i,i}q'_{i,i} = Q_1(x,y)\frac{1}{Q_1(x,y)} = 1.$$

For the case $i = j+1$, we obtain

$$\begin{aligned}\chi_{j+1,j} &= q_{j+1,j}q'_{j,j} + q_{j+1,j+1}q'_{j+1,j} \\ &= \frac{\sqrt{b}\,(aQ_1(x,y)x + cQ_0(x,y)y)}{\sqrt{a}}\frac{1}{Q_1(x,y)} + Q_1(x,y)\left(-\frac{\sqrt{b}\,(aQ_1(x,y)x + cQ_0(x,y)y)}{\sqrt{a}Q_1^2(x,y)}\right) \\ &= 0.\end{aligned}$$

For the last case $i \geqslant j+2$, we obtain

$$\begin{aligned}\chi_{i,j} &= \sum_{k=1}^{n} q_{i,k}q'_{k,j} \\ &= q_{i,j}q'_{j,j} + q_{i,j+1}q'_{j+1,j} + \sum_{k=j+2}^{i} q_{i,k}q'_{k,j} \\ &= \frac{1}{Q_1(x,y)}\left(\frac{b}{a}\right)^{\frac{\xi(i-j)}{2}}Q_{i-j+1}(x,y) - \frac{\sqrt{b}\,(aQ_1(x,y)x + cQ_0y)}{\sqrt{a}Q_1^2(x,y)}\left(\frac{b}{a}\right)^{\frac{\xi(i-j-1)}{2}}Q_{i-j}(x,y) \\ &\quad + \Theta(x,y)\sum_{k=j+2}^{i}\left(\frac{b}{a}\right)^{\frac{\xi(i-k)}{2}}Q_{i-k+1}(x,y)(-1)^{k-j}\left(\frac{b}{a}\right)^{\frac{k-j}{2}}\frac{Q_0^{k-j-2}(x,y)\,(cy)^{k-j-1}}{Q_1^{k-j+1}(x,y)} \\ &= \frac{1}{Q_1(x,y)}\left(\frac{b}{a}\right)^{\frac{\xi(i-j)}{2}}Q_{i-j+1}(x,y) - \left(\frac{b}{a}\right)\left(\frac{aQ_1(x,y)x}{Q_1^2(x,y)}\right)\left(\frac{a}{b}\right)^{\frac{\xi(i-j)}{2}}Q_{i-j}(x,y) \\ &\quad - \left(\frac{b}{a}\right)\left(\frac{Q_0(x,y)cy}{Q_1^2(x,y)}\right)\left(\frac{a}{b}\right)^{\frac{\xi(i-j)}{2}}Q_{i-j}(x,y) + \frac{bcyQ_0(x,y)}{aQ_1^2(x,y)}\left(\frac{a}{b}\right)^{\frac{\xi(i-j)}{2}}Q_{i-j}(x,y) \\ &\quad - \frac{cy}{Q_1(x,y)}\left(\frac{b}{a}\right)^{\frac{\xi(i-j)}{2}}Q_{i-j-1}(x,y) \\ &= \frac{1}{Q_1(x,y)}\left(\frac{b}{a}\right)^{\frac{\xi(i-j)}{2}}Q_{i-j+1}(x,y) - \frac{bx}{Q_1(x,y)}\left(\frac{a}{b}\right)^{\frac{\xi(i-j)}{2}}Q_{i-j}(x,y)\end{aligned}$$

$$
\begin{aligned}
&- \frac{cy}{Q_1(x,y)} \left(\frac{b}{a}\right)^{\frac{\xi(i-j)}{2}} Q_{i-j-1}(x,y) \\
&= \frac{\left(\frac{b}{a}\right)^{\frac{\xi(i-j)}{2}}}{Q_1(x,y)} \left(\underbrace{Q_{i-j+1}(x,y) - bx\left(\frac{a}{b}\right)^{\xi(i-j)} Q_{i-j}(x,y) - cyQ_{i-j-1}(x,y)}_{0} \right) \\
&= 0.
\end{aligned}
$$

Therefore, we can verify $\mathbb{Q}_n[x,y]\mathbb{Q}_n^{-1}[x,y] = \mathbb{I}_n$, where $\mathbb{I}_n$ is an $n \times n$ identity matrix. Analogously, one can verify $\mathbb{Q}_n^{-1}[x,y]\mathbb{Q}_n[x,y] = \mathbb{I}_n$. Thus, the proof is completed. □

Example 2.11. *The inverse of the matrix $\mathbb{Q}_4[x,y]$ is equal to*

$$
\begin{pmatrix}
\frac{1}{Q_1(x,y)} & 0 & 0 & 0 \\
-\frac{\sqrt{b}\left(aQ_1(x,y)x+cQ_0(x,y)y\right)}{\sqrt{a}Q_1(x,y)^2} & \frac{1}{Q_1(x,y)} & 0 & 0 \\
\frac{cy\left(abQ_1(x,y)Q_0(x,y)x-aQ_1(x,y)^2+bcQ_0(x,y)^2y\right)}{aQ_1(x,y)^3} & -\frac{\sqrt{b}\left(aQ_1(x,y)x+cQ_0(x,y)y\right)}{\sqrt{a}Q_1(x,y)^2} & \frac{1}{Q_1(x,y)} & 0 \\
-\frac{\sqrt{b}c^2Q_0(x,y)y^2\left(abQ_1(x,y)Q_0(x,y)x-aQ_1(x,y)^2+bcQ_0(x,y)^2y\right)}{a^{3/2}Q_1(x,y)^4} & \frac{cy\left(abQ_1(x,y)Q_0(x,y)x-aQ_1(x,y)^2+bcQ_0(x,y)^2y\right)}{aQ_1(x,y)^3} & -\frac{\sqrt{b}\left(aQ_1(x,y)x+cQ_0(x,y)y\right)}{\sqrt{a}Q_1(x,y)^2} & \frac{1}{Q_1(x,y)}
\end{pmatrix}.
$$

For $a = b = A$, $c = B$, $Q_0 = a$, $Q_1 = b$ and $x = y = 1$, we obtain the following result for the second order non–degenerated recurrent sequence, $U_n^{(a,b)}$.

Corollary 2.12. *[1] The inverse $\mathcal{U}_n^{-1(a,b,0)} = \left[u_{i,j}^{(a,b,0)}\right]$ of the matrix $\mathcal{U}_n^{(a,b,0)} = \left[u_{i,j}^{(a,b,0)}\right]$ $(b \neq 0)$ is equal to*

$$
u_{i,j}^{\prime(a,b,0)} = \begin{cases} (-1)^{i-j} \cdot \frac{a^2B+abA-b^2}{b^{i-j+1}} a^{i-j-2}B^{i-j-1}, & i \geq j+2, \\ -\frac{aB+bA}{b^2}, & i = j+1 \\ \frac{1}{b}, & i = j, \\ 0, & i < j \end{cases}. \tag{9}
$$

3. Conditional Polynomial Matrices and $\mathcal{T}$−nomial Matrices

In this section, we investigate the correlations between conditional polynomial matrices and $\mathcal{T}$−nomial matrices. For this purpose, firstly we start with the following definition.

Definition 3.1. *The $\mathcal{T}$−nomial matrix of the first kind is defined by*

$$
\mathbb{T}_n[t] = \left[\tau_{i,j}(t)\right] = \begin{cases} t^{i-j} \begin{bmatrix} i-1 \\ j-1 \end{bmatrix}_{\mathcal{T}}, & i-j \geqslant 0 \\ 0, & i-j < 0 \end{cases}. \tag{10}
$$

Note that, for $k > n$ we adopt two conditions, $0^0 = 1$ and $\binom{n}{k} = 0$, even in the case $k = 0$ throughout the paper. So, the following theorem investigates some correlations between the matrices $\mathbb{Q}_n[x,y]$ and $\mathbb{T}_n[t]$.

Theorem 3.2. *Let matrix $\mathbb{W}_n[t,x,y] = \left[\omega_{i,j}(t,x,y)\right]_{n\times n}$, whose entries are defined by*

$$
\omega_{i,j}(t,x,y) = t^{-j} \left[\frac{t^i}{Q_1(x,y)} \begin{bmatrix} i-1 \\ j-1 \end{bmatrix}_{\mathcal{T}} - \frac{t^{i-1}\left(\sqrt{b}\left(aQ_1(x,y)x + cQ_0(x,y)y\right)\right)}{\sqrt{a}Q_1^2(x,y)} \begin{bmatrix} i-2 \\ j-1 \end{bmatrix}_{\mathcal{T}} \right.
$$

$$+\Theta(x,y)\sum_{k=j}^{i-2}\frac{t^k\left(\frac{b}{a}\right)^{\frac{i-k}{2}}\left((-1)^{i-k}Q_0^{i-k-2}(x,y)(cy)^{i-k-1}\right)}{Q_1^{i-k+1}(x,y)}\begin{bmatrix}k-1\\j-1\end{bmatrix}_{\mathcal{T}}\Bigg], \tag{11}$$

where $\Theta(x,y)=\frac{abxQ_0(x,y)Q_1(x,y)-aQ_1^2(x,y)+bcyQ_0^2(x,y)}{b}$. *Then the* $\mathcal{T}-$*nomial matrix of the first kind can be factorized as*

$$\mathbb{T}_n[t]=\mathbb{Q}_n[x,y]\mathbb{W}_n[t,x,y].$$

Proof. Since the matrix $\mathbb{Q}_n[x,y]$ is invertible, it is sufficient to prove that $\mathbb{Q}_n^{-1}[x,y]\mathbb{T}_n[t]=\mathbb{W}_n[t,x,y]$. It is clear that $\omega_{i,j}(t,x,y)=0$ for $i<j$. So, we need to verify another cases. For the case $i=j$, we obtain

$$\omega_{j,j}(t,x,y)=q'_{j,j}\tau_{j,j}(t)=\frac{1}{Q_1(x,y)}=t^{-j}\frac{1}{Q_1(x,y)}t^j\begin{bmatrix}j-1\\j-1\end{bmatrix}_{\mathcal{T}}.$$

For the case $i=j+1$, we obtain

$$\begin{aligned}\omega_{j+1,j}(t,x,y)&=q'_{j+1,j}\tau_{j,j}(t)+q'_{j+1,j+1}\tau_{j+1,j}(t)\\&=-\frac{\sqrt{b}\,(aQ_1(x,y)x+cQ_0(x,y)y)}{\sqrt{a}Q_1^2(x,y)}+\frac{t\,[j]_{\mathcal{T}}}{Q_1(x,y)}\\&=t^{-j}\left[\frac{1}{Q_1(x,y)}t^{j+1}\begin{bmatrix}j\\j-1\end{bmatrix}_{\mathcal{T}}-\frac{\sqrt{b}\,(aQ_1(x,y)x+cQ_0(x,y)y)}{\sqrt{a}Q_1^2(x,y)}t^j\begin{bmatrix}j-1\\j-1\end{bmatrix}_{\mathcal{T}}\right].\end{aligned}$$

For the last case $i\geqslant j+2$, by virtue of Theorem 2.10, we get

$$\begin{aligned}\omega_{i,j}(t,x,y)&=q'_{i,i}\tau_{i,j}(x)+q'_{i,i-1}\tau_{i-1,j}(t)+\sum_{k=j}^{i-2}q'_{i,k}\tau_{k,j}(t)\\&=\Bigg[\frac{t^{i-j}}{Q_1(x,y)}\begin{bmatrix}i-1\\j-1\end{bmatrix}_{\mathcal{T}}-\frac{t^{i-j-1}\left(\sqrt{b}\,(aQ_1(x,y)x+cQ_0(x,y)y)\right)}{\sqrt{a}Q_1^2(x,y)}\begin{bmatrix}i-2\\j-1\end{bmatrix}_{\mathcal{T}}\\&\quad+\Theta(x,y)\sum_{k=j}^{i-2}\frac{t^{k-j}\left(\frac{b}{a}\right)^{\frac{i-k}{2}}\left((-1)^{i-k}Q_0^{i-k-2}(x,y)(cy)^{i-k-1}\right)}{Q_1^{i-k+1}(x,y)}\begin{bmatrix}k-1\\j-1\end{bmatrix}_{\mathcal{T}}\Bigg],\end{aligned}$$

which completes the proof. □

For $a=b=A$, $c=B$, $Q_0=a$, $Q_1=b$, $x=y=1$ in (3) and $p=q=1$ in (10), we obtain the following result for $\mathcal{U}_n^{(a,b,0)}$.

Corollary 3.3. *[1] The matrix* $\mathcal{G}_n[t;a,b](t\neq 0,b\neq 0)$, *whose entries are defined by*

$$\begin{aligned}g_{i,j}(t;a,b)=&t^{-j}\Bigg[\frac{1}{b}t^i\begin{pmatrix}i-1\\j-1\end{pmatrix}-\frac{aB+bA}{b^2}t^{i-1}\begin{pmatrix}i-2\\j-1\end{pmatrix}\\&+\sum_{k=j}^{i-2}(-1)^{i-k}\frac{a^2B+abA-b^2}{b^{i-k+1}}a^{i-k-2}B^{i-k-1}t^k\begin{pmatrix}k-1\\j-1\end{pmatrix}\Bigg]\end{aligned}$$

satisfies

$$\mathcal{P}_n[t]=\mathcal{U}_n^{(a,b,0)}\mathcal{G}_n[t;a,b].$$

For $a=b=A$, $c=B$, $Q_0=0$, $Q_1=1$, $x=y=1$ in (3) and $p=q=t=1$ in (10), we obtain the following result for the generalized Fibonacci matrix.

Corollary 3.4. *[3] The matrix $\mathcal{M}_n$, whose entries are defined by*

$$m_{ij} = \binom{i-1}{j-1} - \binom{i-2}{j-1} - \binom{i-3}{j-1},$$

satisfies $\mathcal{P}_n = \mathcal{F}_n \mathcal{M}_n$, where $\mathcal{P}_n$ is the Pascal matrix and $\mathcal{F}_n$ is the Fibonacci matrix.

For $a = b = A$, $c = B$, $Q_0 = 2$, $Q_1 = 1$, $x = y = 1$ in (3) and $p = q = 1$ in (10), we obtain the following result for $\mathcal{L}_n$.

Corollary 3.5. *[2] The generalized Pascal matrix of the first kind and the Lucas matrix satisfy $\mathcal{P}_n[t] = \mathcal{L}_n \mathcal{G}_n[t; 2, 1]$, where*

$$g_{i,j}(t;2,1) = t^{-j}\left[t^{i}\binom{i-1}{j-1} - 3t^{i-1}\binom{i-2}{j-1} + 5(-1)^{i}2^{i-2}\sum_{k=j}^{i-2}(-1)^{k}\binom{k-1}{j-1}\left(\frac{t}{2}\right)^{k}\right].$$

Corollary 3.6. *For $Q_1(x, y) \neq 0$, The matrix $\mathbb{W}_n\left[-\sqrt{\frac{b}{a}}\frac{cQ_0(x,y)y}{Q_1(x,y)}, x, y\right]$ is defined by*

$$\begin{aligned}\omega_{i,j}\left(-\sqrt{\frac{b}{a}}\frac{cQ_0(x,y)y}{Q_1(x,y)}, x, y\right) = \frac{\left(-\sqrt{b}cQ_0(x,y)y\right)^{i-j-2}}{\left(\sqrt{a}Q_1(x,y)\right)^{i-j+1}}\sqrt{a}bcy\Bigg[&cQ_0(x,y)^2 y\begin{bmatrix} i-1 \\ j-1 \end{bmatrix}_{\mathcal{T}} \\ &+Q_0(x,y)\left(aQ_1(x,y)x + cQ_0(x,y)y\right)\begin{bmatrix} i-2 \\ j-1 \end{bmatrix}_{\mathcal{T}} \\ &+\Theta(x,y)\sum_{k=j}^{i-2}\begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_{\mathcal{T}}\Bigg]\end{aligned}$$

satisfies

$$\mathbb{T}_n\left[-\sqrt{\frac{b}{a}}\frac{cQ_0(x,y)y}{Q_1(x,y)}\right] = \mathbb{Q}_n[x,y]\mathbb{W}_n\left[-\sqrt{\frac{b}{a}}\frac{cQ_0(x,y)y}{Q_1(x,y)}, x, y\right].$$

Proof. The proof is clear by taking $t = -\sqrt{\frac{b}{a}}\frac{cQ_0(x,y)y}{Q_1(x,y)}$ in Theorem 3.2. □

The following theorem investigates the matrix $\mathbb{H}_n[x] = \left[h_{i,j}\right], (i, j = 1, \ldots, n)$ which gives a similar correlation between the matrix $\mathbb{Q}_n[x, y]$ and the $\mathcal{T}-$nomial matrix of the first kind:

Theorem 3.7. *Let matrix $\mathbb{H}_n[t, x, y] = \left[h_{i,j}(t, x, y)\right]_{n\times n}$, whose entries are defined by*

$$\begin{aligned}h_{i,j}(t,x,y) = t^{i}\Bigg[&\frac{t^{-j}}{Q_1(x,y)}\begin{bmatrix} i-1 \\ j-1 \end{bmatrix}_{\mathcal{T}} - \frac{t^{-j-1}\left(\sqrt{b}\left(aQ_1(x,y)x + cQ_0(x,y)y\right)\right)}{\sqrt{a}Q_1^2(x,y)}\begin{bmatrix} i-1 \\ j \end{bmatrix}_{\mathcal{T}} \\ &+\Theta(x,y)\sum_{k=j+2}^{i}\frac{t^{-k}\left(\frac{b}{a}\right)^{\frac{k-j}{2}}\left((-1)^{k-j}Q_0^{k-j-2}(x,y)(cy)^{k-j-1}\right)}{Q_1^{k-j+1}(x,y)}\begin{bmatrix} i-1 \\ k-1 \end{bmatrix}_{\mathcal{T}}\Bigg],\end{aligned} \tag{12}$$

where $\Theta(x,y) = \frac{abxQ_0(x,y)Q_1(x,y) - aQ_1^2(x,y) + bcyQ_0^2(x,y)}{b}$. Then the $\mathcal{T}-$nomial matrix of the first kind can be factorized as

$$\mathbb{T}_n[t] = \mathbb{H}_n[t,x,y]\mathbb{Q}_n[x,y].$$

Proof. The proof of the theorem can be done similar to Theorem 3.2. So we omit it. □

For $a = b = A$, $c = B$, $Q_0 = a$, $Q_1 = b$, $x = y = 1$ in (3) and $p = q = 1$ in (10), we obtain the following result for $\mathcal{U}_n^{(a,b,0)}$.

Corollary 3.8. *[1] The matrix $\mathcal{H}_n[t; a, b]$, $(b \neq 0)$, defined by*

$$h_{i,j}(t; a, b) = t^i \left[\frac{1}{b} t^{-j} \binom{i-1}{j-1} - \frac{aB + bA}{b^2} t^{-j-1} \binom{i-1}{j} + \sum_{k=j+2}^{i} (-1)^{k-j} \frac{a^2B + abA - b^2}{b^{k-j+1}} a^{k-j-2} B^{k-j-1} t^{-k} \binom{i-1}{k-1} \right]$$

satisfies

$$\mathcal{P}_n[t] = \mathcal{H}_n[t; a, b] \mathcal{U}_n^{(a,b,0)}.$$

For $a = b = A$, $c = B$, $Q_0 = 2$, $Q_1 = 1$, $x = y = 1$ in (3) and $p = q = 1$ in (10), we obtain the following result for $\mathcal{L}_n$.

Corollary 3.9. *[2] The Lucas matrix satisfies $\mathcal{P}_n[t] = \mathcal{H}_n[t; 2, 1] \mathcal{L}_n$, where*

$$h_{i,j}(t; 2, 1) = t^{i-j-1} \left[t \binom{i-1}{j-1} - 3 \binom{i-1}{j} + (-1)^j \frac{5t^{j+1}}{2^{j+2}} \sum_{k=j+2}^{i} (-1)^k \binom{i-1}{k-1} 2^k t^{-k} \right].$$

Definition 3.10. *The $\mathcal{T}-$nomial matrix of the second kind is defined by*

$$\mathbb{S}_n[t] = \left[s_{i,j}(t) \right] = \begin{cases} t^{i+j-2} \begin{bmatrix} i-1 \\ j-1 \end{bmatrix}_{\mathcal{T}}, & i - j \geqslant 0 \\ 0, & i - j < 0 \end{cases}. \tag{13}$$

The following theorem investigates some correlations between the matrices $\mathbb{Q}_n[x, y]$ and $\mathbb{S}_n[t]$.

Theorem 3.11. *Let matrix $\mathbb{K}_n[t, x, y] = \left[\kappa_{i,j}(t, x, y) \right]_{n \times n}$, whose entries are defined by*

$$\kappa_{i,j}(t, x, y) = t^j \left[\frac{t^{i-2}}{Q_1(x, y)} \begin{bmatrix} i-1 \\ j-1 \end{bmatrix}_{\mathcal{T}} - \frac{t^{i-3} \left(\sqrt{b} \left(aQ_1(x, y)x + cQ_0(x, y)y \right) \right)}{\sqrt{a} Q_1^2(x, y)} \begin{bmatrix} i-2 \\ j-1 \end{bmatrix}_{\mathcal{T}} + \Theta(x, y) \sum_{k=j}^{i-2} \frac{t^{k-2} \left(\frac{b}{a} \right)^{\frac{i-k}{2}} \left((-1)^{i-k} Q_0^{i-k-2}(x, y)(cy)^{i-k-1} \right)}{Q_1^{i-k+1}(x, y)} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_{\mathcal{T}} \right], \tag{14}$$

where $\Theta(x, y) = \frac{abxQ_0(x,y)Q_1(x,y) - aQ_1^2(x,y) + bcyQ_0^2(x,y)}{b}$. Then the $\mathcal{T}-$nomial matrix of the second kind can be factorized as

$$\mathbb{S}_n[t] = \mathbb{Q}_n[x, y] \mathbb{K}_n[t, x, y].$$

Proof. In order to prove the theorem, it is sufficient to prove that $\mathbb{Q}_n^{-1}[x, y] \mathbb{S}_n[t] = \mathbb{K}_n[t, x, y]$. It is clear that $\kappa_{i,j}(t, x, y) = 0$ for $i < j$. So, we need to verify another cases. For the case $i = j$, we obtain

$$\kappa_{j,j}(t, x, y) = q'_{j,j} s_{j,j}(t) = \frac{t^{2j-2}}{Q_1(x, y)} = t^j \frac{1}{Q_1(x, y)} t^{j-2} \begin{bmatrix} j-1 \\ j-1 \end{bmatrix}_{\mathcal{T}}.$$

For the case $i = j + 1$, we obtain

$$\kappa_{j+1,j}(t, x, y) = q'_{j+1,j} s_{j,j}(t) + q'_{j+1,j+1} s_{j+1,j}(t)$$

$$= -\frac{\sqrt{b}\,(aQ_1(x,y)x + cQ_0(x,y)y)}{\sqrt{a}Q_1^2(x,y)}t^{2j-2} + \frac{t^{2j-1}\,[j]_{\mathcal{T}}}{Q_1(x,y)}$$

$$= t^j\left[\frac{1}{Q_1(x,y)}t^{j-1}\begin{bmatrix} j \\ j-1 \end{bmatrix}_{\mathcal{T}} - \frac{\sqrt{b}\,(aQ_1(x,y)x + cQ_0(x,y)y)}{\sqrt{a}Q_1^2(x,y)}t^{j-2}\begin{bmatrix} j-1 \\ j-1 \end{bmatrix}_{\mathcal{T}}\right].$$

For the last case $i \geqslant j+2$, by virtue of Theorem 2.10, we get

$$\begin{aligned}
\kappa_{i,j}(t,x,y) &= q'_{i,i}s_{i,j}(t) + q'_{i,i-1}s_{i-1,j}(t) + \sum_{k=j}^{i-2} q'_{i,k}s_{k,j}(t) \\
&= \left[\frac{t^{i+j-2}}{Q_1(x,y)}\begin{bmatrix} i-1 \\ j-1 \end{bmatrix}_{\mathcal{T}} - \frac{t^{i+j-3}\left(\sqrt{b}\,(aQ_1(x,y)x + cQ_0(x,y)y)\right)}{\sqrt{a}Q_1^2(x,y)}\begin{bmatrix} i-2 \\ j-1 \end{bmatrix}_{\mathcal{T}}\right. \\
&\quad \left. +\Theta(x,y)\sum_{k=j}^{i-2}\frac{t^{k+j-2}\left(\frac{b}{a}\right)^{\frac{i-k}{2}}\left((-1)^{i-k}Q_0^{i-k-2}(x,y)(cy)^{i-k-1}\right)}{Q_1^{i-k+1}(x,y)}\begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_{\mathcal{T}}\right],
\end{aligned}$$

which completes the proof. □

Corollary 3.12. *For $Q_1(x,y) \neq 0$, the matrix $\mathbb{K}_n\left[-\sqrt{\frac{b}{a}}\frac{cQ_0(x,y)y}{Q_1(x,y)}, x, y\right]$ is defined by*

$$\begin{aligned}
\kappa_{i,j}\left(-\sqrt{\frac{b}{a}}\frac{cQ_0(x,y)y}{Q_1(x,y)}, x, y\right) &= \frac{\left(-\sqrt{b}cQ_0(x,y)y\right)^{i+j-4}}{\left(\sqrt{a}Q_1(x,y)\right)^{i+j-1}}\sqrt{a}bcy\left[cQ_0(x,y)^2y\begin{bmatrix} i-1 \\ j-1 \end{bmatrix}_{\mathcal{T}}\right. \\
&\quad +Q_0(x,y)\,(aQ_1(x,y)x + cQ_0(x,y)y)\begin{bmatrix} i-2 \\ j-1 \end{bmatrix}_{\mathcal{T}} \\
&\quad \left. +\Theta(x,y)\sum_{k=j}^{i-2}\begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_{\mathcal{T}}\right]
\end{aligned}$$

satisfies

$$\mathbb{S}_n\left[-\sqrt{\frac{b}{a}}\frac{cQ_0(x,y)y}{Q_1(x,y)}\right] = \mathbb{Q}_n[x,y]\mathbb{K}_n\left[-\sqrt{\frac{b}{a}}\frac{cQ_0(x,y)y}{Q_1(x,y)}, x, y\right].$$

Proof. *The proof is clear by taking $t = -\sqrt{\frac{b}{a}}\frac{cQ_0(x,y)y}{Q_1(x,y)}$ in Theorem 3.11.* □

Theorem 3.13. *Let matrix $\mathbb{L}_n[t,x,y] = \left[\ell_{i,j}(t,x,y)\right]_{n\times n}$, whose entries are defined by*

$$\begin{aligned}
\ell_{i,j}(t,x,y) &= t^i\left[\frac{t^{j-2}}{Q_1}\begin{bmatrix} i-1 \\ j-1 \end{bmatrix}_{\mathcal{T}} - \frac{t^{j-1}\left(\sqrt{b}\,(aQ_1(x,y)x + cQ_0(x,y)y)\right)}{\sqrt{a}Q_1^2(x,y)}\begin{bmatrix} i-1 \\ j \end{bmatrix}_{\mathcal{T}}\right. \\
&\quad \left. +\Theta(x,y)\sum_{k=j+2}^{i}\frac{t^{k-2}\left(\frac{b}{a}\right)^{\frac{k-j}{2}}\left((-1)^{k-j}Q_0^{k-j-2}(x,y)(cy)^{k-j-1}\right)}{Q_1^{k-j+1}(x,y)}\begin{bmatrix} i-1 \\ k-1 \end{bmatrix}_{\mathcal{T}}\right].
\end{aligned} \tag{15}$$

Then the $\mathcal{T}$−nomial matrix of the second kind can be factorized as $\mathbb{S}_n[t] = \mathbb{L}_n[t,x,y]\mathbb{Q}_n[x,y]$.

Proof. The proof of the theorem can be done similar to Theorem 3.11. So we omit it. □

For $a = b = A$, $c = B$, $Q_0 = a$, $Q_1 = b$, $x = y = 1$ in (3) and $p = q = 1$ in (13), we obtain the following result for $\mathcal{U}_n^{(a,b,0)}$.

Corollary 3.14. *[1] The matrices* $\mathcal{S}_n[t;a,b] = \left[s_{i,j}(t;a,b)\right]$ *and* $\mathcal{T}_n[t;a,b] = \left[t_{i,j}(t;a,b)\right]$, $i, j = 1, \ldots, n$, $(b \neq 0)$ *whose entries are defined by*

$$s_{i,j}(t;a,b) = t^j\left[\frac{1}{b}t^{i-2}\binom{i-1}{j-1} - \frac{aB+bA}{b^2}t^{i-3}\binom{i-2}{j-1} + \sum_{k=j}^{i-2}(-1)^{i-k}\frac{a^2B+abA-b^2}{b^{i-k+1}}a^{i-k-2}B^{i-k-1}t^{k-2}\binom{k-1}{j-1}\right], \quad (16)$$

$$t_{i,j}(t;a,b) = t^i\left[\frac{1}{b}t^{j-2}\binom{i-1}{j-1} - \frac{aB+bA}{b^2}t^{j-1}\binom{i-1}{j} + \sum_{k=j+2}^{i}(-1)^{k-j}\frac{a^2B+abA-b^2}{b^{k-j+1}}a^{k-j-2}B^{k-j-1}t^{k-2}\binom{i-1}{k-1}\right] \quad (17)$$

which satisfy

$$\mathcal{Q}_n[t] = \mathcal{U}_n^{(a,b,0)}\mathcal{S}_n[t;a,b] \quad (18)$$

and

$$\mathcal{Q}_n[t] = \mathcal{T}_n[t;a,b]\mathcal{U}_n^{(a,b,0)}. \quad (19)$$

4. Some Combinatorial Identities

In this section we investigate some combinatorial identities involving the conditional polynomials.

Theorem 4.1. *For any positive integers i and j satisfying* $i \geqslant j+2$, *and* $Q_1(x,y) \neq 0$, *we have*

$$\begin{aligned}
&\left(-\sqrt{\frac{b}{a}}\frac{cyQ_0(x,y)}{Q_1(x,y)}\right)^{i-j}\begin{bmatrix} i-1 \\ j-1 \end{bmatrix}_{\mathcal{T}} \\
&= \frac{Q_{i-j+1}(x,y)}{Q_1(x,y)} - Q_{i-j}(x,y)\frac{\sqrt{b}\left(Q_1(x,y)ax + Q_0(x,y)cy(p+q+1)\right)}{\sqrt{a}Q_1^2(x,y)} \\
&\quad + \sum_{k=j+2}^{i} Q_{i-k+1}(x,y)\frac{\left(-Q_0(x,y)\sqrt{b}cy\right)^{k-j-2}}{\left(Q_1(x,y)\sqrt{a}\right)^{k-j+1}}\sqrt{a}bcy\left[cQ_0^2(x,y)y\begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_{\mathcal{T}}\right. \\
&\qquad \left. + Q_0(x,y)\left(aQ_1(x,y)x + cQ_0(x,y)y\right)\begin{bmatrix} k-2 \\ j-1 \end{bmatrix}_{\mathcal{T}} + \Theta(x,y)\sum_{m=j}^{k-2}\begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_{\mathcal{T}}\right].
\end{aligned}$$

Proof. By virtue of Corollary 3.6, we get the following equalities:

$$\begin{aligned}
\omega_{j,j}\left(-\sqrt{\frac{b}{a}}\frac{cyQ_0(x,y)}{Q_1(x,y)}, x, y\right) &= \frac{1}{Q_1(x,y)}, \\
\omega_{j+1,j}\left(-\sqrt{\frac{b}{a}}\frac{cyQ_0(x,y)}{Q_1(x,y)}, x, y\right) &= -\frac{\sqrt{b}\left(Q_1(x,y)ax + Q_0(x,y)cy(p+q+1)\right)}{\sqrt{a}Q_1^2(x,y)}.
\end{aligned}$$

In addition, for $i \geqslant j+2$, the proof can be derived by applying identities

$$\left(-\sqrt{\frac{b}{a}}\frac{cyQ_0(x,y)}{Q_1(x,y)}\right)^{i-j}\begin{bmatrix} i-1 \\ j-1 \end{bmatrix}_{\mathcal{T}}$$

$$= q_{i,j}\omega_{j,j}\left(-\sqrt{\frac{b}{a}}\frac{cyQ_0(x,y)}{Q_1(x,y)},x,y\right)+q_{i,j+1}\omega_{j+1,j}\left(-\sqrt{\frac{b}{a}}\frac{cyQ_0(x,y)}{Q_1(x,y)},x,y\right)$$
$$+\sum_{k=j}^{i-2}q_{i,k}\omega_{k,j}\left(-\sqrt{\frac{b}{a}}\frac{cyQ_0(x,y)}{Q_1(x,y)},x,y\right).$$

□

Theorem 4.2. *For any positive integers i and j satisfying $i \geqslant j+2$, and $Q_1(x,y) \neq 0$, we have*

$$\left(-\sqrt{\frac{b}{a}}\frac{cyQ_0(x,y)}{Q_1(x,y)}\right)^{i+j-2}\begin{bmatrix} i-1 \\ j-1 \end{bmatrix}_{\mathcal{T}}$$
$$= Q_{i-j+1}(x,y)\frac{bc^2Q_0^2(x,y)y^2}{aQ_1^3(x,y)} - Q_{i-j}(x,y)\frac{b^{3/2}(cyQ_0(x,y))^2\left(aQ_1(x,y)x+cQ_0(x,y)y(p+q+1)\right)}{a^{3/2}Q_1^4(x,y)}$$
$$+\sum_{k=j+2}^{i}Q_{i-k+1}(x,y)\frac{\left(-Q_0(x,y)\sqrt{b}cy\right)^{k+j-4}}{\left(Q_1(x,y)\sqrt{a}\right)^{k+j-1}}\sqrt{a}bcy\left[cQ_0^2(x,y)y\begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_{\mathcal{T}}\right.$$
$$\left.+Q_0(x,y)\left(aQ_1(x,y)x+cQ_0(x,y)y\right)\begin{bmatrix} k-2 \\ j-1 \end{bmatrix}_{\mathcal{T}}+\Theta(x,y)\sum_{m=j}^{k-2}\begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_{\mathcal{T}}\right].$$

Proof. By virtue of Corollary 3.12, we get the following equalities:

$$\kappa_{j,j}\left(-\sqrt{\frac{b}{a}}\frac{cyQ_0(x,y)}{Q_1(x,y)},x,y\right)=\frac{bc^2Q_0^2(x,y)y^2}{aQ_1^3(x,y)},$$
$$\kappa_{j+1,j}\left(-\sqrt{\frac{b}{a}}\frac{cyQ_0(x,y)}{Q_1(x,y)},x,y\right)=-\frac{b^{3/2}(cyQ_0(x,y))^2\left(aQ_1(x,y)x+cQ_0(x,y)y(p+q+1)\right)}{a^{3/2}Q_1^4(x,y)}.$$

In addition, for $i \geqslant j+2$, the proof can be derived by applying identities

$$\left(-\sqrt{\frac{b}{a}}\frac{cyQ_0(x,y)}{Q_1(x,y)}\right)^{i-j+2}\begin{bmatrix} i-1 \\ j-1 \end{bmatrix}_{\mathcal{T}}$$
$$= q_{i,j}\kappa_{j,j}\left(-\sqrt{\frac{b}{a}}\frac{cyQ_0(x,y)}{Q_1(x,y)},x,y\right)+q_{i,j+1}\kappa_{j+1,j}\left(-\sqrt{\frac{b}{a}}\frac{cyQ_0(x,y)}{Q_1(x,y)},x,y\right)+\sum_{k=j}^{i-2}q_{i,k}\kappa_{k,j}\left(-\sqrt{\frac{b}{a}}\frac{cyQ_0(x,y)}{Q_1(x,y)},x,y\right).$$

□

Theorem 4.3. *For $1 \leqslant r \leqslant n$, and $Q_1(x,y) \neq 0$, we have*

$$\begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_{\mathcal{T}} = \sum_{l=r}^{n}Q_{n-l+1}(x,y)\left[\frac{1}{Q_1(x,y)}\begin{bmatrix} l-1 \\ r-1 \end{bmatrix}_{\mathcal{T}} - \frac{\left(\sqrt{b}\left(aQ_1(x,y)x+cQ_0(x,y)y\right)\right)}{\sqrt{a}Q_1^2(x,y)}\begin{bmatrix} l-2 \\ r-1 \end{bmatrix}_{\mathcal{T}}\right.$$
$$\left.+\Theta(x,y)\sum_{k=r}^{l-2}\frac{\left(\frac{b}{a}\right)^{\frac{l-k}{2}}\left((-1)^{l-k}Q_0^{l-k-2}(x,y)(cy)^{l-k-1}\right)}{Q_1^{l-k+1}(x,y)}\begin{bmatrix} k-1 \\ r-1 \end{bmatrix}_{\mathcal{T}}\right].$$

Proof. In the partial case $t=1$ in Corollary 3.2, we obtain

$$\omega_{i,j}(1,x,y)=\left[\frac{1}{Q_1(x,y)}\begin{bmatrix} i-1 \\ j-1 \end{bmatrix}_{\mathcal{T}} - \frac{\left(\sqrt{b}\left(aQ_1(x,y)x+cQ_0(x,y)y\right)\right)}{\sqrt{a}Q_1^2(x,y)}\begin{bmatrix} i-2 \\ j-1 \end{bmatrix}_{\mathcal{T}}\right.$$

$$+\Theta(x,y)\sum_{k=j}^{i-2}\frac{\left(\frac{b}{a}\right)^{\frac{i-k}{2}}\left((-1)^{i-k}Q_0^{i-k-2}(x,y)(cy)^{i-k-1}\right)}{Q_1^{i-k+1}(x,y)}\begin{bmatrix}k-1\\j-1\end{bmatrix}_{\mathcal{T}}\Bigg].$$

Now, the proof follows from

$$\begin{bmatrix}n-1\\r-1\end{bmatrix}_{\mathcal{T}}=t_{n,r}=\sum_{l=r}^{n}Q_{n-l+1}(x,y)\omega_{l,r}(1,x,y).$$

□

5. Numerical Tests

The inverse of a square matrix can be easily calculated with the 'Inverse' method in `MATHEMATICA`. Although this method is robust in many calculations, it performs slower calculations when the matrix becomes larger. In this section, we illustrate some numerical results for inverses of conditional polynomial matrices. We compare the effectiveness of the results obtained in Theorem 2.10 and the Inverse method. We performed all computations using the following hardware specifications:

- Intel(R) Core(TM) i5-4590 CPU @ 3.30GHz Processor
- 8 GB DDR4 RAM
- 64–bit Windows 11 Education operating system.

Now, we give the following results which are obtained by using Theorem 2.10 and `MATHEMATICA`'s 'Inverse" method.

n	`Inverse (MATHEMATICA)`	Theorem 2.10	Error
2	0.000548	0.000076	0.000472
3	0.0011367	0.0001158	0.0010209
4	0.0028468	0.0003337	0.0025131
5	0.0088339	0.0003432	0.0084907
6	0.0162643	0.0004823	0.015782
7	0.0276735	0.0006898	0.0269837
8	0.0547237	0.0008647	0.053859
9	0.0817499	0.0011106	0.0806393
10	0.130648	0.0013663	0.129282
11	0.19043	0.0016814	0.188749
12	0.784246	0.0020061	0.78224
13	1.12259	0.0028654	1.11972
14	1.65156	0.0043366	1.64723
15	2.51047	0.0032862	2.50718
16	4.16315	0.0039752	4.15918
17	6.97054	0.0054974	6.96504
18	13.1865	0.0056553	13.1809
19	24.8668	0.0065218	24.8602
20	50.5861	0.0073745	50.5787

Table 1: Computational times (in seconds) for the inverse of the matrix $\mathbb{Q}_n[x,y]$ via MATHEMATICA's Inverse method, Theorem 2.10 and Error amounts for different values of n.

In Table 1, the inverses of matrices from 2×2 to 20×20 are computed with both MATHEMATICA's Inverse method and Theorem 2.10, and the computation times are presented as numerical values. When the numerical results are observed, while the computation time increases sharply in computations made with MATHEMATICA's Inverse method, the computation time increases more stable in computations made with Theorem 2.10. Thus, it seems that Theorem 2.10 is more effective than the Inverse method for inverting higher dimensional matrices. Especially, in Figure 2 (a), we see that the computation time is sharply increasing after $n = 10$. In particular, for $n = 20$, we compute the inverse of the matrix $\mathbb{Q}_{20}[x, y]$ in 50.5861 seconds via Inverse method. On the other hand, in Figure 2 (b) and (c), we give the results via Theorem 2.10 for $n = 20$ and $n = 100$, respectively. In particular, for $n = 20$, we obtain the inverse of the matrix $\mathbb{Q}_{20}[x, y]$ in 0.0073745 seconds via Theorem 2.10. So, for this case, it is observed that the computations made with the Theorem 2.10 are approximately 6.8596×10^3 faster compared to the `MATHEMATICA`'s Inverse method. As we can see from the numerical results and figures, all computations made with Theorem 2.10 work more effectively than MATHEMATICA's Inverse method. Thus, the results we have presented have largely eliminated the difficulties in the computation of the conditional polynomial matrices.

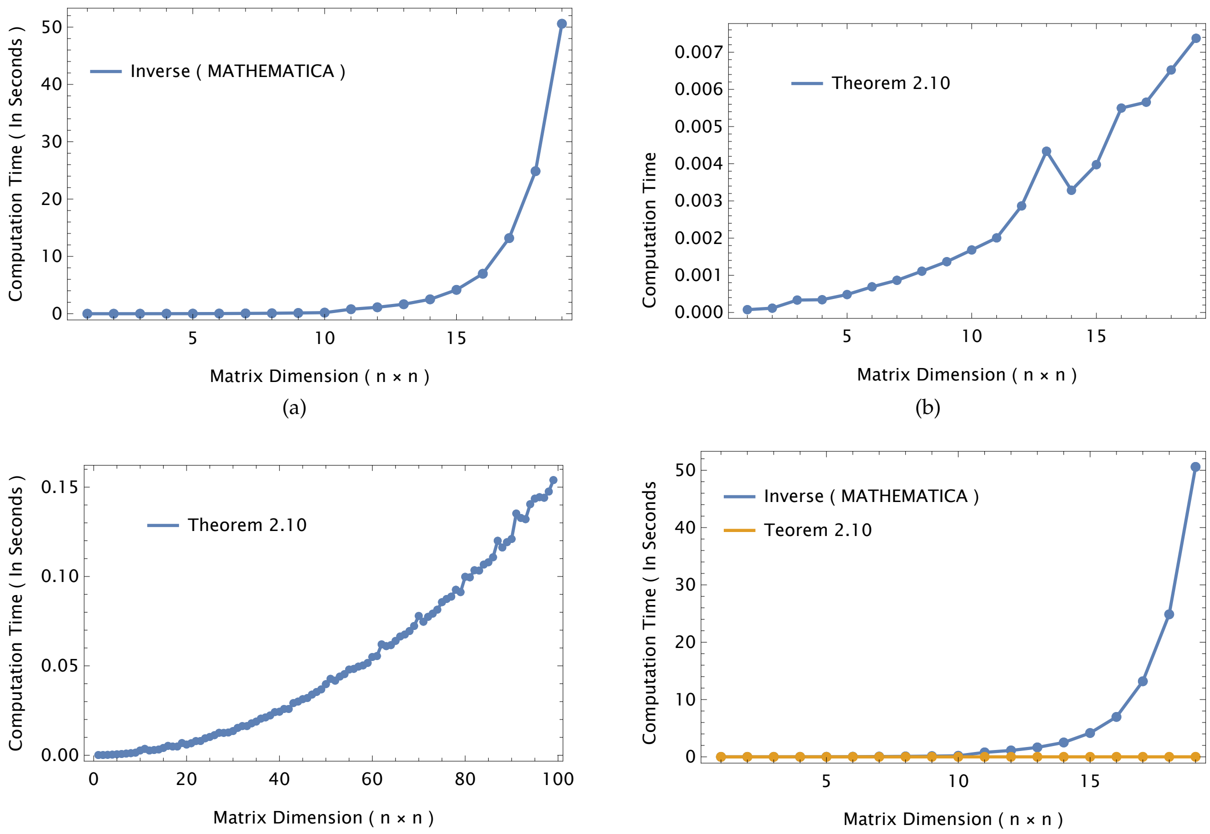


Figure 2: Computation performances for the inverse of the matrix $\mathbb{Q}_n[x, y]$ via Theorem 2.10 and Inverse method in `MATHEMATICA`.

Acknowledgment

This study is a part of the first author's Master thesis.

Declarations

Conflict of Interest The authors have no conflicts of interest to declare that are relevant to the content of this article.

References

[1] Stanimirović, P., Nikolov, J. & Stanimirović, I. A generalization of Fibonacci and Lucas matrices. *Discrete Applied Mathematics*. **156**, 2606-2619 (2008)
[2] Zhang, Z. & Zhang, Y. The Lucas matrix and some combinatorial identities. *Indian Journal of Pure and Applied Mathematics*. **38**, 457 (2007)
[3] Lee, G., Kim, J. & Cho, S. Some combinatorial identities via Fibonacci numbers. *Discrete Applied Mathematics*. **130**, 527-534 (2003)
[4] Shen, S., Cen, J. & Hao, Y. On the determinants and inverses of circulant matrices with Fibonacci and Lucas numbers. *Applied Mathematics and Computation*. **217**, 9790-9797 (2011)
[5] Miladinovic, M. & Stanimirovic, P. Singular case of generalized Fibonacci and Lucas matrices. *Journal of the Korean Mathematical Society*. **48**, 33-48 (2011)
[6] Mosić, D., & Stanimirović, P. S. Composite outer inverses for rectangular matrices. Quaestiones Mathematicae, **44(1)**, 45-72 (2021).
[7] Falcon, S. The $k-$Fibonacci matrix and the Pascal matrix. *Open Mathematics*. **9**, 1403-1410 (2011)
[8] Stanimirović, P. & Miladinović, M. Inversion of the generalized Fibonacci matrix by convolution. *International Journal of Computer Mathematics*. **88**, 1519-1532 (2011)
[9] Lee, G., Kim, J. & Lee, S. Factorizations and eigenvalues of Fibonacci and symmetric Fibonacci matrices. *Fibonacci Quarterly*. **40**, 203-211 (2002)
[10] Radičić, B. The inverse and the Moore–Penrose inverse of a $k-$circulant matrix with binomial coefficients. *Bulletin of the Belgian Mathematical Society–Simon Stevin*. **27**, 29-42 (2020)
[11] Shen, S. & He, J. Moore–Penrose inverse of generalized Fibonacci matrix and its applications. *International Journal of Computer Mathematics*. **93**, 1756-1770 (2016)
[12] Shen, S., Liu, W. & Feng, L. Inverse and Moore–Penrose inverse of Toeplitz matrices with classical Horadam numbers. *Operator and Matrices*. **11**, 929-939 (2017)
[13] Yazlik, Y. & Taskara, N. On the inverse of circulant matrix via generalized $k-$Horadam numbers. *Applied Mathematics and Computation*. **223** pp. 191-196 (2013)
[14] Irmak, N. & Köme, C. Linear algebra of the Lucas matrix. *Hacettepe Journal of Mathematics and Statistics*. **50**, 549-558 (2021)
[15] Köme, C. & Yazlik Y. Inverse and Moore–Penrose inverse of conditional matrices via convolution. *Journal of Applied Mathematics and Computing*. **70**(1), 417–433 (2024)
[16] Shen, S. & Liu, W. Inversion of lower Hessenberg matrix involving classical Horadam numbers. *International Journal of Algebra*. **10**, 343-350 (2016)
[17] Köme, C. Moore–Penrose inverse of the singular conditional matrices and its applications. *Indian Journal of Pure and Applied Mathematics* (2024), 55-138, (2024)
[18] Koshy, T. Fibonacci and Lucas Numbers With Applications. *John Wiley & Sons*. (2001)
[19] Edson, M. & Yayenie, O. A New Generalization of Fibonacci Sequence & Extended Binet's Formula. *Integers*. **9**, 639-654 (2009)
[20] Bilgici, G. Two generalizations of Lucas sequence. *Applied Mathematics and Computation*. **245** pp. 526-538 (2014)
[21] Elif, T. & Leung, H. A note on congruence properties of the generalized bi–periodic Horadam sequence. *Hacettepe Journal of Mathematics and Statistics*. **49**, 2084-2093 (2020)
[22] Verma, V. & Bala, A. On properties of generalized bi–variate bi–periodic Fibonacci polynomials. *International Journal of Advanced Science and Technology*. **29**, 8065-8072 (2020)
[23] Ait–Amrane, Rosa N. & Belbachir, H. Bi–periodic $r-$Fibonacci sequence and bi–periodic $r-$Lucas sequence of type s. *Hacettepe Journal of Mathematics and Statistics*. **51**(3), 680–699 (2022)
[24] Catarino, P. & Spreafico, E. A Note on Bi–Periodic Leonardo Sequence. *Armenian Journal of Mathematics*, **16**(5), 1–17, 2024
[25] Gould, H. The bracket function and Fontené–Ward generalized binomial coefficients with application to Fibonomial coefficients. *Fibonacci Quart*. **7** pp. 23-40 (1969)
[26] Hoggatt Jr, V. Fibonacci numbers and generalized binomial coefficients. *Fibonacci Quart*. **5** pp. 383–400 (1967)
[27] Krot, E. An introduction to finite Fibonomial calculus. *Open Mathematics*. **2**, 754-766 (2004)
[28] Seibert, J. & Trojovsky, P. On some identities for the Fibonomial coefficients. *Mathematica Slovaca*. **55**, 9–19 (2005)
[29] Fontené, G. Generalisation d'une formule connue. *Nouvelles Annales De Mathématiques: Journal Des Candidats Aux Écoles Polytechnique Et Normale*. **15** pp. 112-112 (1915)
[30] Kwasniewski, A. Information on combinatorial interpretation of Fibonomial coefficients. *ArXiv Preprint Math/0402291*. (2004)
[31] Dziemianczuk, M. Generalization of Fibonomial coefficients. *ArXiv Preprint ArXiv:0908.3248*. (2009)