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Mappings preserving sum of products $\{a,b\}_* + b^*a$ or preserving $\{a,b\}_* + a^*b$ on generalized matrix $*$-rings

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Abstract. Let A be a generalized matrix $*$-ring and let B be an unital $*$-ring. For $a, b \in A$, we define $\{a,b\}_* = ab + ba^*$. In this paper, we prove that, under some additional conditions, a bijective map $\varphi : A \to B$ that satisfies $\varphi(\{a,b\}_* + b^*a) = \{\varphi(a), \varphi(b)\}_* + \varphi(b)^*\varphi(a)$ for all $a, b \in A$ or satisfies $\varphi(\{a,b\}_* + a^*b) = \{\varphi(a), \varphi(b)\}_* + \varphi(a)^*\varphi(b)$ for all $a, b \in A$ is a $*$-ring isomorphism.

1. Introduction

Two algebraic structures A and B are considered essentially the same if they are isomorphic, meaning that there is a bijection $\varphi : A \to B$ between them that preserves their operations. However, we often just have a bijection $\varphi : A \to B$ that preserves only a part of their structures. For example, we may have two rings A and B and a bijection $\varphi : A \to B$ that preserves multiplication, but we still do not know if it preserves addition.

Therefore, it is interesting to find some additional conditions under which a bijective map preserving a part of their structures is necessarily an isomorphism. As a main example, Martindale III in [5] found that for associative rings A and B such that A is prime and contains a nontrivial idempotent, every multiplicative bijective map $\varphi : A \to B$ is additive.

Motivated by this, many authors paid more attention to generalizations or bijective mappings on rings preserving other important operations. For example, Li and Xiao in [8] extended the result of Martindale III from prime associative rings to a larger class of associative rings called generalized matrix rings and they also studied the Jordan product $a \circ b = ab + ba$ and the triple Jordan product $\{abc\} = abc + cba$.

Other authors also considered rings with involution, because of their applications to functional analysis, such as in C^*-algebras and bounded linear operators in Hilbert spaces, and they considered operations using

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also the involution. For instance, Cui, Li, Fang, and Lu examined the characterization of specific mappings preserving the operation $\{a,b\}_* = ab + ba^*$ or preserving $[a,b]_* = ab - ba_*$ in a factor von Neumann algebra [2, 3]. Also, Taghavi et al. in [6] investigated mappings preserving the operation $\{a,b\}_* + b^*a$ in the context of associative prime complex $*$-algebras.

Recently, many authors have studied the other mappings not necessarily preserving a certain operation, but satisfying interesting identities. For example, Ferreira and Marietto in [4] and other authors in references therein have explored and analyzed rings with involution A, an n-ary operation $p(x_1,\ldots,x_n)$ in A, such as the triple Lie products $p(a,b,c) = [[a,b],c]$ and its variants with involutions, and mappings $\varphi : A \to A$ that resemble derivations concerning p, that is, they satisfy:

$$\varphi\left(p(x_1,\ldots,x_n)\right) = \sum_{i=1}^{n} p(x_1,\ldots,x_{i-1},\varphi(x_i),x_{i+1},\ldots,x_n)$$

for all $x_1,\ldots,x_n \in A$. They proved that, if A is an unital prime complex $*$-algebra containing a nontrivial symmetric idempotent and $\varphi : A \to A$ is a derivation concerning p, then φ is a $*$-derivation, that is, φ preserves involution and is a derivation concerning the original multiplication.

For another example, Cheung in [1], investigated a class of rings called triangular algebras and, for such a ring A, he investigated what we call commuting mappings, which are mappings $\varphi : A \to A$ satisfying $[\varphi(a),a] = 0$ for all $a \in A$. He proved that, for a triangular algebra A satisfying some additional conditions, every commuting mapping $\varphi : A \to A$ is proper, that is, there is an element x in the center $Z(A)$ of A and a linear mapping h with image in $Z(A)$ such that $\varphi(a) = ax + h(a)$ for all $a \in A$.

Most of these results were obtained for prime associative rings or other particular cases of generalized matrix rings. Therefore, it would be interesting to see whether these results can be extended to generalized matrix rings. For example, Xiao and Wei in [7] extended the result of Cheung for generalized matrix rings.

In Section 3, we will extend the result of [6] about mappings preserving $\{a,b\}_* + b^*a$ to generalized matrix rings. In Section 4, we will obtain the same result for mappings preserving a similar operation $\{a,b\}_* + a^*b$. But before all this, in Section 2 of this paper, we will introduce some notations and terminology to state more clearly our results and show by some examples that these two operations are not directly equivalent.

2. Notation and terminology

In this paper, by a ring, we mean an *associative* ring. By a *unital* ring we mean a ring with an identity element. Let A be a ring.

- A is *prime* if, for all ideals I and J of A such that $IJ = 0$, we have $I = 0$ or $J = 0$. Because we are assuming that A is associative, it is equivalent to say that for any $x, y \in A$, if for all $a \in A$ we have $xay = 0$, then $x = 0$ or $y = 0$.
- A is *semiprime* if, for any ideal I of A such that $I^2 = 0$, we have $I = 0$. Because we are assuming that A is associative, it is equivalent to say that for any $x \in A$, if for all $a \in A$ we have $xax = 0$, then $x = 0$.

An *involution* in a ring A is a function $a \mapsto a^*$ from A to A such that for any $a, b \in A$:

- $(a+b)^* = a^* + b^*$,
- $(ab)^* = b^*a^*$,
- $a^{**} = a$.

A *$*$-ring* is a ring A endowed with an involution. In this case, for every element $x \in A$, we say that x is *symmetric* if $x^* = x$, and x is a *projection* if it is symmetric and idempotent.

A *complex involution* in a $\mathbb{C}$-algebra A is an involution $a \mapsto a^*$ such that for any $\lambda \in \mathbb{C}$ and $x \in A$ we have $(\lambda x)^* = \bar{\lambda}x^*$. A *complex $*$-algebra* is a $\mathbb{C}$-algebra with a complex involution.

Let A and B be two $*$-rings and $\varphi\colon A \to B$ a map. We have the following definitions:

- φ is *additive* if $\varphi(a+b)=\varphi(a)+\varphi(b)$ for any $a,b\in G$.
- φ is *ring isomorphism* if φ is an additive bijection that satisfies $\varphi(ab)=\varphi(a)\varphi(b)$ for all $a,b\in G$.
- φ *preserves involution* if $\varphi(a^*)=\varphi(a)^*$ for all elements $a\in G$.
- φ is a $*$*-ring isomorphism* if φ is an isomorphism that preserves involution.

The following theorem is the main result of the article [6].

Theorem 2.1. *Let A and B be two prime unital complex $*$-algebras such that A has a projection e satisfying $e\notin\{0,1\}$. Suppose that $\varphi: A\to B$ is a bijective mapping satisfying* ***at least one of*** *the following:*

- $\varphi(\{a,b\}_*+b^*a)=\{\varphi(a),\varphi(b)\}_*+\varphi(b)^*\varphi(a)$ *for all $a,b\in A$,*
- $\varphi(\{a^*,b\}_*+ab^*)=\{\varphi(a)^*,\varphi(b)\}_*+\varphi(a)\varphi(b)^*$ *for al $a,b\in A$.*

Then φ is a $*$-ring isomorphism.

We will extend this result to a larger class of rings. Thus, we will briefly present the important concepts about generalized matrix rings. Let R and S be unital rings, let M be an (R,S)-bimodule and let N be an (S,R)-bimodule. Let us also consider two bimodule homomorphisms $\Phi: M\otimes_S N\to R$ and $\Psi: N\otimes_R M\to S$ satisfying the following associativity conditions: $(mn)m'=m(nm')$ and $(nm)n'=n(mn')$ for all $m,m'\in M$ and $n,n'\in N$, where we put $mn=\Phi(m\otimes n)$ and $nm=\Psi(n\otimes m)$. Let:

$$A=\mathrm{Mat}(R,S,M,N)=\left\{\begin{pmatrix} r & m\\ n & s\end{pmatrix}: r\in R, s\in S, m\in M, n\in N\right\}$$

be the set of all 2×2 matrices. Observe that, with the obvious matrix operations of addition and multiplication, A is a ring, and we call it *generalized matrix ring*. Set:

$$e_1=\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

and:

$$e_2=\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

For $i,j\in\{1,2\}$ let $A_{ij}=e_iAe_j$. Then we can write $A=A_{11}\oplus A_{12}\oplus A_{21}\oplus A_{22}$. Also, for every $a\in A$ and $i,j\in\{1,2\}$, let $a_{ij}=e_iae_j$, so that the element a_{ij} belongs to A_{ij}. By a direct calculation, for any $a,b\in A$, we have $a_{ij}b_{kl}=0$ if $j\neq k$, where $i,j,k,l\in\{1,2\}$.

In Section 2.1 of [7], Xiao and Wei highlight a simpler approach to generalized matrix algebras. Namely, we can have a correspondence between generalized matrix rings and pairs (A,e) where A is an unital ring and e is an idempotent element of A, as in the following proposition.

Proposition 2.2. *Let A be an unital ring and $e\in A$ be an idempotent element. Then the function:*

$$a\mapsto\begin{pmatrix} eae & ea(1-e)\\ (1-e)ae & (1-e)a(1-e)\end{pmatrix}$$

is an isomorphism:

$$A\to\begin{pmatrix} eAe & eA(1-e)\\ (1-e)Ae & (1-e)A(1-e)\end{pmatrix}.$$

We say that a generalized matrix algebra $A = \mathrm{Mat}(R, S, M, N)$ is *secondary-faithful* if M is faithful as a left R-module and faithful as a right S-module and N is faithful as a left S-module and faithful as a right R-module. With the above correspondence, is it equivalent to say that A_{12} is faithful as a left A_{11}-module and faithful as a right A_{22}-module and A_{21} is faithful as a left A_{22}-module and faithful as a right A_{11}-module.

Example 2.3. *Let A be a **prime** unital ring with an idempotent $e \notin \{0, 1\}$. Then A is a secondary-faithful generalized matrix ring.*

Indeed, for $a_{11} \in A_{11}$, if for every $x_{12} \in A_{12}$ we have $a_{11}x_{12} = 0$, then for every $x \in A$ we have $e_1xe_2 \in A_{12}$, so $0 = a_{11}e_1xe_2 = a_{11}xe_2$, but $e_2 \neq 0$ and A is prime, so $a_{11} = 0$. Thus A_{12} if a faithful left A_{11}-module.

Moreover, for $a_{22} \in A_{22}$, if for every $x_{12} \in A_{12}$ we have $x_{12}a_{22} = 0$, then for every $x \in A$ we have $e_1xe_2 \in A_{12}$, so $0 = e_1xe_2a_{22} = e_1xa_{22}$, but $e_1 \neq 0$ and A is prime, so $a_{22} = 0$. Thus A_{12} is a faithful right A_{22}-module.

The converse of Example 2.3 is not true. There are generalized matrix rings that are secondary-faithful but are not even semiprime.

Example 2.4. *We consider the following set:*

$$A = \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ \mathbb{C} & \mathbb{C} \end{pmatrix}.$$

Addition in A is coordinatewise, but multiplication in A is given by:

$$\begin{pmatrix} a_1 & m_1 \\ n_1 & b_1 \end{pmatrix} * \begin{pmatrix} a_2 & m_2 \\ n_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & a_1m_2 + m_1b_2 \\ n_1a_2 + b_1n_2 & b_1b_2 \end{pmatrix}.$$

To show that A is not semiprime, consider the element:

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that $a * x * a = 0$ for every $x \in A$.

A *generalized matrix $*$-ring* is a generalized matrix ring A with an involution such that $e_1^* = e_1$. Notice that it implies that $e_2 = 1 - e_1$ satisfies $e_2^* = e_2$ as well. It is also easy to see that, because of the properties of involution, for $A = \mathrm{Mat}(R, S, M, N)$ to be secondary-faithful, it suffices for M to be a faithful left R-module and a faithful right S-module.

Example 2.5. *The ring in the Example 2.4 with the usual conjugate-transposition is a generalized matrix $*$-ring that is secondary-faithful but not semiprime.*

Let G be an abelian group and n be a positive integer. We say G is *n-torsion free* if $nx = 0$ implies $x = 0$ for every $x \in G$. We say G is *n-divisible* if for every $x \in G$ there is exactly one $y \in G$, denoted by $\frac{x}{n}$, such that $ny = x$. If X, Y and Z are abelian groups, we say that a function $\varphi : X \times Y \to Z$ is *biadditive* if:

- $\varphi(x + y, z) = \varphi(x, z) + \varphi(y, z)$,
- $\varphi(x, y + z) = \varphi(x, y) + \varphi(x, z)$.

With this picture in mind, in this paper, we will discuss when a bijective mapping that preserves sums of products is a $*$-ring isomorphism in the case of the generalized matrix $*$-rings that are 2-divisible, 3-torsion free and secondary-faithful.

In the Section 3, we will prove the following theorem.

Theorem 2.6. *Let A be generalized matrix $*$-ring that is 2-divisible, 3-torsion free and secondary-faithful. Let B be a unital $*$-ring. Let $\varphi : A \to B$ be a bijective mapping satisfying **at least one of** the following:*

- $\varphi(\{a, b\}_* + b^*a) = \{\varphi(a), \varphi(b)\}_* + \varphi(b)^*\varphi(a)$ *for all* $a, b \in A$,
- $\varphi(\{a^*, b\}_* + ab^*) = \{\varphi(a)^*, \varphi(b)\}_* + \varphi(a)\varphi(b)^*$ *for all* $a, b \in A$.

Then φ *is a* $*$*-ring isomorphism.*

Indeed it is sufficient to consider just one case, because of the following proposition, as indicated in Lemma 1.1 of [6]:

Proposition 2.7. *Let* A *and* B *be two* $*$*-rings and* $\varphi : A \to B$ *a mapping. Then, the following statements are equivalent:*

- φ *satisfies* $\varphi(\{a, b\}_* + b^*a) = \{\varphi(a), \varphi(b)\}_* + \varphi(b)^*\varphi(a)$ *for* $a, b \in A$,
- φ *satisfies* $\varphi(\{a^*, b\}_* + ab^*) = \{\varphi(a)^*, \varphi(b)\}_* + \varphi(a)\varphi(b)^*$ *for* $a, b \in A$.

Also, in Section 4, we will prove the following theorem.

Theorem 2.8. *Let* A *be generalized matrix* $*$*-ring that is* 2*-divisible,* 3*-torsion free and secondary-faithful. Let* B *be a unital* $*$*-ring. Let* $\varphi : A \to B$ *be a bijective mapping satisfying* ***at least one of*** *the following:*

- $\varphi(\{a, b\}_* + a^*b) = \{\varphi(a), \varphi(b)\}_* + \varphi(a)^*\varphi(b)$,
- $\varphi(a^* \circ b + ab) = \varphi(a)^* \circ \varphi(b) + \varphi(a)\varphi(b)$.

Then φ *is a* $*$*-ring isomorphism.*

Indeed it is sufficient to consider just one case, because of the following proposition.

Proposition 2.9. *Let* A *and* B *be two* $*$*-rings and* $\varphi : A \to B$ *a mapping. The following properties are equivalent:*

- φ *satisfies* $\varphi(\{a, b\}_* + a^*b) = \{\varphi(a), \varphi(b)\}_* + \varphi(a)^*\varphi(b)$ *for* $a, b \in A$,
- φ *satisfies* $\varphi(a^* \circ b + ab) = \varphi(a)^* \circ \varphi(b) + \varphi(a)\varphi(b)$ *for* $a, b \in A$.

The following examples show that the properties of the Propositions 2.7 and 2.9 are *not* equivalent so that the study of these new mappings is not just a trivial rearrangement of the previous study by Taghavi.

Example 2.10. *Let* $A = \mathbb{Z}_2 \times \mathbb{Z}_2$ *be the direct product of two copies of the ring* $\mathbb{Z}_2$ *of integers modulo* 2*, let* $(x, y)^* = (y, x)$ *and let* $\varphi(x, y) = (x + y, y)$*. Then* φ *satisfies:*

$$\varphi(\{a, b\}_* + b^*a) = \{\varphi(a), \varphi(b)\}_* + \varphi(b)^*\varphi(a)$$

for $a, b \in A$*, but:*

$$\varphi(\{a_0, b_0\}_* + a_0^*b_0) \neq \{\varphi(a_0), \varphi(b_0)\}_* + \varphi(a_0)^*\varphi(b_0)$$

for $a_0 = (1, 1)$ *and* $b_0 = (0, 1)$.

Example 2.11. *Let* $A = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *be the direct product of three copies of the ring* $\mathbb{Z}_2$ *of integers modulo* 2*, let* $(x, y, z)^* = (x, z, y)$ *and let* $\varphi(x, y, z) = (y, x, z)$*. Then* φ *satisfies:*

$$\varphi(\{a, b\}_* + a^*b) = \{\varphi(a), \varphi(b)\}_* + \varphi(a)^*\varphi(b)$$

for $a, b \in A$*, but:*

$$\varphi(\{a_0, b_0\}_* + b_0^*a_0) \neq \{\varphi(a_0), \varphi(b_0)\}_* + \varphi(b_0)^*\varphi(a_0)$$

for $a_0 = (1, 1, 1)$ *and* $b_0 = (0, 1, 1)$.

The following example shows that the statements of Theorems 2.6 and 2.8 are no longer true if we replace "secondary-faithful" by "semiprime with $e \notin \{0, 1\}$".

Example 2.12. *Consider the ring $A = \mathbb{Z}_{35}$ of integers modulo 35. We define $x^* = x$, $e = 15$ and the following function $\varphi\colon A \to A$ given by $\varphi(x) = 9x^3$. Then A is a generalized matrix $*$-ring that is 2-divisible, 3-torsion free and semiprime[1)] with $e \notin \{0, 1\}$. Also, φ is bijective and for all $a, b \in A$ we have:*

$$\varphi(\{a, b\}_* + b^* a) = \{\varphi(a), \varphi(b)\}_* + \varphi(b)^* \varphi(a),$$

$$\varphi(\{a, b\}_* + a^* b) = \{\varphi(a), \varphi(b)\}_* + \varphi(a)^* \varphi(b),$$

but $\varphi(1 + 1) \neq \varphi(1) + \varphi(1)$.

Before proving the main results, we will also present auxiliary claims, some of which have the same proof as in [6].

Claim 2.13. *Let A be a generalized matrix $*$-ring. Then $A_{ij}^* \subset A_{ji}$, for $i, j \in \{1, 2\}$.*

Proof. If $a_{ij} \in A_{ij}$ then:

$$a_{ij}^* = (e_i a_{ij} e_j)^* = e_j^* a_{ij}^* e_i^* = e_j a_{ij}^* e_i \in A_{ji}.$$

Therefore $a_{ij}^* \in A_{ji}$. □

It is easy to check the following result.

Claim 2.14. *Let A and B be abelian groups. Let $P : A \times A \to A$ and $Q : B \times B \to B$ be bilinear functions. Let $\varphi : A \to B$ be a function satisfying: $\varphi(P(a, b)) = Q(\varphi(a), \varphi(b))$ for any $a, b \in A$. Let $h, x_1, \ldots, x_n \in A$ such that:*

$$\varphi(h) = \varphi(x_1) + \cdots + \varphi(x_n).$$

Then for every $t \in A$ we have:

- $\varphi(P(t, h)) = \varphi(P(t, x_1)) + \cdots + \varphi(P(t, x_n))$,
- $\varphi(P(h, t)) = \varphi(P(x_1, t)) + \cdots + \varphi(P(x_n, t))$.

The proof is the same as that of Lemma 2.1 of [6].

Claim 2.15. *Let A and B be abelian groups. Let $P : A \times A \to A$ and $Q : B \times B \to B$ be bilinear functions. Let $\varphi : A \to B$ be a **surjective** function satisfying $\varphi(P(a, b)) = Q(\varphi(a), \varphi(b))$ for all $a, b \in A$. Then $\varphi(0) = 0$.*

The proof is the same as that of Lemma 2.2 of [6].

Claim 2.16. *Let A be a $*$-ring that is 3-torsion free and let $x \in A$. If $2x + x^* = 0$, then $x = 0$.*

Proof. If $2x + x^* = 0$, then $4x + 2x^* = 0$ and also:

$$0 = 0^* = (2x + x^*)^* = x + 2x^*.$$

Subtracting it from the previous equality, we obtain $3x = 0$, so $x = 0$. □

1) Indeed, for every integer $n > 1$, then $\mathbb{Z}_n$ is semiprime if and only if $n = p_1 \cdots p_r$ where $p_1, \ldots, p_r$ are distinct primes.

3. First main theorem

In this section, we will prove Theorem 2.6 through the following two Theorems 3.1 and 3.9. The first consists of proving that φ is additive.

Theorem 3.1. *Let A be a generalized matrix $*$-ring that is 3-torsion free and secondary-faithful. Let B be an abelian group. Let $Q : B \times B \to B$ be a biadditive function. Let $\varphi : A \to B$ be a bijective function which satisfies:*

$$\varphi(\{a, b\}_* + b^* a) = Q(\varphi(a), \varphi(b))$$

for all $a, b \in A$. Then φ is additive.

We will prove it by several lemmas, whose statements have the same hypotheses as the Theorem 3.1. Some of them have the same proof as in [6]. Also, we will use the following abbreviation:

$$P(a, b) = \{a, b\}_* + b^* a$$

for every $a, b \in G$. It is easy to see that P is a biadditive function, therefore we can apply Claims 2.13 to 2.16.

We have the following helpful formulas, that hold for the idempotent elements e_i and for arbitrary $x \in A$ and $q_{ij} \in A_{ij}$, where $i \neq j$:

i) $P(e_i, x) = (2x_{ii} + x_{ii}^*) + x_{ij} + (x_{ji} + x_{ij}^*)$,

ii) $P(x, e_i) = (2x_{ii} + x_{ii}^*) + (x_{ij} + x_{ji}^*) + x_{ji}$,

iii) $P(q_{ij}, x) = (q_{ij}x_{ji} + x_{ij}q_{ij}^*) + (x_{ii}^* q_{ij} + q_{ij}x_{jj}) + x_{jj}q_{ij}^* + x_{ij}^* q_{ij}$,

iv) $P(x, q_{ij}) = q_{ij}x_{ij}^* + (x_{ii}q_{ij} + q_{ij}x_{jj}^*) + q_{ij}^* x_{ii} + (x_{ji}q_{ij} + q_{ij}^* x_{ij})$.

Lemma 3.2. *If $i \neq j$, then:*

a) $\varphi(a_{ii} + b_{ij}) = \varphi(a_{ii}) + \varphi(b_{ij})$,

b) $\varphi(a_{ii} + c_{ji}) = \varphi(a_{ii}) + \varphi(c_{ji})$.

The proof is the same as that of Property 2.1 of [6].

Lemma 3.3. *If $i \neq j$, then $\varphi(a_{ii} + b_{ij} + d_{jj}) = \varphi(a_{ii}) + \varphi(b_{ij}) + \varphi(d_{jj})$.*

The proof is the same as that Property 2.2 of [6].

Lemma 3.4. *If $i \neq j$, then $\varphi(a_{ij} + b_{ij}) = \varphi(a_{ij}) + \varphi(b_{ij})$.*

The proof is the same as that of Property 2.3 of [6].

Lemma 3.5. $\varphi(a_{ii} + b_{ii}) = \varphi(a_{ii}) + \varphi(b_{ii})$.

Proof. Let $f \in A$ such that $\varphi(f) = \varphi(a_{ii}) + \varphi(b_{ii})$. By Claim 2.14, for every $t \in A$:

$$\varphi(P(f, t)) = \varphi(P(a_{ii}, t)) + \varphi(P(b_{ii}, t)). \tag{1}$$

By (1) with $t = e_j$, (ii) and Claim 2.15:

$$\varphi((2f_{jj} + f_{jj}^*) + (f_{ji} + f_{ij}^*) + f_{ij}) = \varphi(0) + \varphi(0) = \varphi(0).$$

Cancelling φ and using Claim 2.16, we get $f_{jj} = f_{ji} = f_{ij} = 0$, so $f = f_{ii}$. Now, let $q_{ji} \in A_{ji}$. By (1) with $t = q_{ji}$ and Lemma 3.4:

$$\varphi(q_{ji}f_{ii}^*) = \varphi(q_{ji}a_{ii}^*) + \varphi(q_{ji}b_{ii}^*) = \varphi(q_{ji}a_{ii}^* + q_{ji}b_{ii}^*).$$

Cancelling φ and using the fact that A_{ji} is a faithful right A_{ii}-module, we conclude that $f_{ii} = a_{ii} + b_{ii}$. □

Lemma 3.6. *If $i \neq j$, then $\varphi(a_{ii} + b_{ij} + c_{ji}) = \varphi(a_{ii}) + \varphi(b_{ij} + c_{ji})$.*

Proof. Let $f \in A$ be such that $\varphi(f) = \varphi(a_{ii}) + \varphi(b_{ij} + c_{ji})$. By Claim 2.14:

$$\varphi(P(f, e_j)) = \varphi(P(a_{ii}, e_j)) + \varphi(P(b_{ij} + c_{ji}, e_j)).$$

By Claim 2.15:

$$\varphi((2f_{jj} + f_{jj}^*) + (f_{ji} + f_{ij}^*) + f_{ij}) = \varphi(0) + \varphi((c_{ji} + b_{ij}^*) + b_{ij}) = \varphi((c_{ji} + b_{ij}^*) + b_{ij}).$$

Cancelling φ and using Claim 2.16, we have $f_{jj} = 0$, $f_{ji} = c_{ji}$ and $f_{ij} = b_{ij}$, so $f = f_{ii} + b_{ij} + c_{ji}$. Now, let $q_{ji} \in A_{ji}$. By Claim 2.14:

$$\begin{aligned}&\varphi(P(P(P(f, q_{ji}), e_j), e_i))\\ &= \varphi(P(P(P(a_{ii}, q_{ji}), e_j), e_i) + \varphi(P(P(P(b_{ij} + c_{ji}, q_{ji}), e_j), e_i)).\end{aligned}$$

By Claim 2.15:

$$\varphi(f_{ii}q_{ji}^* + q_{ji}f_{ii}^*) = \varphi(a_{ii}q_{ji}^* + q_{ji}a_{ii}^*) + \varphi(0) = \varphi(a_{ii}q_{ji}^* + q_{ji}a_{ii}^*).$$

Thus $f_{ii}q_{ji}^* + q_{ji}f_{ii}^* = a_{ii}q_{ji}^* + q_{ji}a_{ii}^*$. Therefore $f_{ii}q_{ji}^* = a_{ii}q_{ji}^*$ and $q_{ji}f_{ii}^* = q_{ji}a_{ii}^*$. Because A_{ji} is a faithful right A_{ii}-module, then $f_{ii} = a_{ii}$, so $f = a_{ii} + b_{ij} + c_{ji}$. □

Lemma 3.7. *If $i \neq j$ then $\varphi(b_{ij} + c_{ji}) = \varphi(b_{ij}) + \varphi(c_{ji})$.*

Proof. By Lemmas 3.2, 3.5 and 3.6:

$$\begin{aligned}&\varphi(b_{ij} + c_{ji}) + \varphi(3e_j + c_{ji}b_{ij})\\ &= \varphi(3e_j + b_{ij} + c_{ji} + c_{ji}b_{ij})\\ &= \varphi(P(e_j + c_{ji}, e_j + b_{ij}))\\ &= Q(\varphi(e_j + b_{ij}), \varphi(e_j + c_{ji}))\\ &= Q(\varphi(e_j) + \varphi(b_{ij}), \varphi(e_j) + \varphi(c_{ji}))\\ &= Q(\varphi(e_j), \varphi(e_j)) + Q(\varphi(b_{ij}), \varphi(e_j)) + Q(\varphi(e_j), \varphi(c_{ji})) + Q(\varphi(b_{ij}), \varphi(c_{ji}))\\ &= \varphi(P(e_j, e_j)) + \varphi(P(e_j, b_{ij})) + \varphi(P(c_{ji}, e_j)) + \varphi(P(c_{ji}, b_{ij}))\\ &= \varphi(3e_j) + \varphi(b_{ij}) + \varphi(c_{ji}) + \varphi(c_{ji}b_{ij})\\ &= \varphi(b_{ij}) + \varphi(c_{ji}) + \varphi(3e_j + c_{ji}b_{ij}).\end{aligned}$$

Therefore $\varphi(b_{ij} + c_{ji}) = \varphi(b_{ij}) + \varphi(c_{ji})$. □

Lemma 3.8. $\varphi(a_{11} + b_{12} + c_{21} + d_{22}) = \varphi(a_{11}) + \varphi(b_{12}) + \varphi(c_{21}) + \varphi(d_{22})$.

The proof is the same as that of Property 2.6 of [6].

Proof of Theorem 3.1. Now, using Lemmas 3.4, 3.5 and 3.8, it is easy to see that φ is additive. □

Now we focus our attention to the second theorem of this section, that consists of proving that φ is a $*$-ring isomorphism.

Theorem 3.9. *Let A be a unital $*$-ring that is 2-divisible and 3-torsion free. Let B be a unital $*$-ring. Let $\varphi : A \to B$ be an* ***additive*** *bijective function such that:*

$$\varphi(\{x, y\}_* + y^*x) = \{\varphi(x), \varphi(y)\}_* + \varphi(y)^*\varphi(x). \tag{2}$$

for any $x, y \in A$. Then φ is a $*$-ring isomorphism.

Because $\varphi : A \to B$ is an additive bijective function and A is 2-divisible and 3-torsion free, then B is 2-divisible and 3-torsion free too.

The following lemmas have the same hypotheses as the Theorem 2.8 and we use them to prove the Theorem 2.8.

Lemma 3.10. $\varphi(1) = 1$.

Proof. We will prove that, if $a = \varphi^{-1}(1)$, then $2a - 2a^* = a^2 - a^{*2}$, that is Equation (2.3) of [6]. The remainder of the proof is the same as in the proof of Lemma 2.4 of [6]. For every $x \in A$, using (2) with $y = a$ we have:

$$\varphi(\{x, a\}_* + a^* x) = \{\varphi(x), 1\}_* + 1^* \cdot \varphi(x) = 2\varphi(x) + \varphi(x)^*. \tag{3}$$

In particular, using (3) with $x = 1$, we have:

$$\varphi(2a + a^*) = 2\varphi(1) + \varphi(1)^*. \tag{4}$$

Let $b = 2a + a^* - 1$. Then $\varphi(b) = \varphi(b)^*$, so by (3) with $x = b$, we have:

$$\varphi(\{b, a\}_* + a^* b) = 2\varphi(b) + \varphi(b)^* = 3\varphi(b) = \varphi(3b),$$

thus $ba + ab^* + a^*b = 3b$. Because $b = 2a + a^* - 1$, we have:

$$8a + 4a^* - 3 = 3a^2 + a^{*2} + 2aa^* + 3a^*a,$$

and taking involution:

$$4a + 8a^* - 3 = a^2 + 3a^{*2} + 2aa^* + 3a^*a,$$

thus $4a - 4a^* = 2a^2 - 2a^{*2}$. Because A is 2-divisible, we conclude that $2a - 2a^* = a^2 - a^{*2}$. □

Lemma 3.11. *For all $a \in A$ we have $\varphi(a^*) = \varphi(a)^*$.*

The proof is the same as that of Lemma 2.5 of [6].

Lemma 3.12. *For all $a, b \in A$ we have $\varphi(ab) = \varphi(a)\varphi(b)$.*

Proof. We first divide into some steps.

a) Let $a^* = a$ and $b^* = b$. Using (2) with $(x, y) = (b, a)$ and with $(x, y) = (a, b)$ and using Lemma 3.11, we obtain:

$$2\varphi(ab) + \varphi(ba) = 2\varphi(a)\varphi(b) + \varphi(b)\varphi(a),$$

$$\varphi(ab) + 2\varphi(ba) = \varphi(a)\varphi(b) + 2\varphi(b)\varphi(a).$$

Because B is 3-torsion free, we have $\varphi(ab) = \varphi(a)\varphi(b)$.

b) Let $a^* = -a$ and $b^* = b$. Using (2) with $(x, y) = (a, b)$ and using Lemma 3.11, we obtain $\varphi(ab) = \varphi(a)\varphi(b)$.

c) Let $a^* = a$ e $b^* = -b$. Using (2) with $(x, y) = (a, b)$ and using Lemma 3.11, we obtain $\varphi(ab) = \varphi(a)\varphi(b)$.

d) Let $a^* = -a$ and $b^* = -b$. Using (2) with $(x, y) = (b, a)$ and with $(x, y) = (a, b)$ and using Lemma 3.11, we obtain:

$$-2\varphi(ab) + \varphi(ba) = -2\varphi(a)\varphi(b) + \varphi(b)\varphi(a),$$

$$\varphi(ab) - 2\varphi(ba) = \varphi(a)\varphi(b) - 2\varphi(b)\varphi(a).$$

Because B is 3-torsion free, we have $\varphi(ab) = \varphi(a)\varphi(b)$.

Now, let $a, b \in A$. Because A is 2-divisible, we can consider $a_1 = \frac{a+a^*}{2}$, $a_2 = \frac{a-a^*}{2}$, $b_1 = \frac{b+b^*}{2}$ and $b_2 = \frac{b-b^*}{2}$. Then $a = a_1 + a_2$, $b = b_1 + b_2$, $a_1^* = a_1$, $a_2^* = -a_2$, $b_1^* = b_1$ and $b_2^* = -b_2$. Therefore:

$$\begin{aligned}
\varphi(ab) &= \varphi((a_1 + a_2)(b_1 + b_2)) \\
&= \varphi(a_1b_1 + a_2b_1 + a_1b_2 + a_2b_2) \\
&= \varphi(a_1b_1) + \varphi(a_2b_1) + \varphi(a_1b_2) + \varphi(a_2b_2) \\
&= \varphi(a_1)\varphi(b_1) + \varphi(a_2)\varphi(b_1) + \varphi(a_1)\varphi(b_2) + \varphi(a_2)\varphi(b_2) \\
&= (\varphi(a_1) + \varphi(a_2))(\varphi(b_1) + \varphi(b_2)) \\
&= \varphi(a_1 + a_2)\varphi(b_1 + b_2) \\
&= \varphi(a)\varphi(b),
\end{aligned}$$

concluding the proof. □

Proof of Theorem 3.9. Now, using Lemmas 3.11 and 3.12, it is easy to see that φ is a $*$-ring isomorphism. □

4. Second main theorem

In this section, we will prove Theorem 2.8 through the following two Theorems 4.1 and 4.9. The first consists of proving that φ is additive.

Theorem 4.1. *Let A be a generalized matrix $*$-ring that is 3-torsion free and secondary-faithful. Let B be an abelian group. Let $Q : B \times B \to B$ be a biadditive function. Let $\varphi : A \to B$ be a bijective function which satisfies $\varphi(\{a, b\}_* + a^*b) = Q(\varphi(a), \varphi(b))$ for all $a, b \in A$. Then φ is additive.*

We will prove it by several lemmas, whose statements have the same hypotheses as the Theorem 4.1. Also we will use the following abbreviation $P(a, b) = \{a, b\}_* + a^*b$ for every $a, b \in A$. It is easy to see that P is a biadditive function. Therefore, we can apply the Claims 2.13 to 2.16.

We have the following helpful formulas, that hold for the idempotent elements e_i and for arbitrary $x \in A$ and $q_{ij} \in A_{ij}$, where $i \neq j$:

i) $P(e_i, x) = 3x_{ii} + 2x_{ij} + x_{ji}$,

ii) $P(x, e_i) = (x_{ii} + 2x_{ii}^*) + x_{ji}^* + (x_{ji} + x_{ij}^*)$,

iii) $P(q_{ij}, x) = (q_{ij}x_{ji} + x_{ij}q_{ij}^*) + q_{ij}x_{jj} + (x_{jj}q_{ij}^* + q_{ij}^*x_{ii}) + q_{ij}^*x_{ij}$,

iv) $P(x, q_{ij}) = q_{ij}x_{ij}^* + (x_{ii}q_{ij} + q_{ij}x_{jj}^* + x_{ii}^*q_{ij}) + (x_{ji}q_{ij} + x_{ij}^*q_{ij})$.

Lemma 4.2. *If $i \neq j$, then:*

a) $\varphi(a_{ii} + b_{ij}) = \varphi(a_{ii}) + \varphi(b_{ij})$,

b) $\varphi(a_{ii} + c_{ji}) = \varphi(a_{ii}) + \varphi(c_{ji})$.

Proof. **a)** Let $t \in A$ such that $\varphi(t) = \varphi(a_{ii}) + \varphi(b_{ij})$. Using Claim 2.14:

$$\varphi(P(t, e_j)) = \varphi(P(a_{ii}, e_j)) + \varphi(P(b_{ij}, e_j)).$$

By Claim 2.15 we have:

$$\varphi((t_{jj} + 2t_{jj}^*) + t_{ij}^* + (t_{ij} + t_{ji}^*)) = \varphi(0) + \varphi(b_{ij}^* + b_{ij}) = \varphi(b_{ij}^* + b_{ij}).$$

Cancelling φ and using Claim 2.16 we obtain $t_{ij} = b_{ij}$, $t_{ji} = 0$ and $t_{jj} = 0$. Now let $q_{ji} \in A_{ji}$. Using Claim 2.14:

$$\varphi(P(P(t, q_{ji}), e_j)) = \varphi(P(P(a_{ii}, q_{ji}), e_j)) + \varphi(P(P(b_{ij}, q_{ji}), e_j)).$$

By Claim 2.15:

$$\varphi(t_{ii}q^*_{ji}) = \varphi(a_{ii}q^*_{ji}) + \varphi(0) = \varphi(a_{ii}q^*_{ji}).$$

This shows that $t_{ii}q^*_{ji} = a_{ii}q^*_{ji}$, so $q_{ji}t^*_{ii} = q_{ji}a^*_{ii}$. Therefore, because A_{ji} is a faithful right A_{ii}-module, we get $t_{ii} = a_{ii}$.

b) Let $s \in A$ such that $\varphi(s) = \varphi(a_{ii}) + \varphi(c_{ji})$. Using Claim 2.14:

$$\varphi(P(s, e_j)) = \varphi(P(a_{ii}, e_j)) + \varphi(P(c_{ji}, e_j)).$$

Thus by Claim 2.15 we have:

$$\varphi((s_{jj} + 2s^*_{jj}) + s^*_{ij} + (s_{ij} + s^*_{ji})) = \varphi(0) + \varphi(c_{ji}) = \varphi(c_{ji}).$$

Cancelling φ and using Claim 2.16, we obtain $s_{ij} = 0$, $s_{ji} = c_{ji}$ and $s_{jj} = 0$. Now let $q_{ij} \in A_{ij}$. Using Claim 2.14:

$$\varphi(P(P(q_{ij}, s), e_j)) = \varphi(P(P(q_{ij}, a_{ii}), e_j)) + \varphi(P(P(q_{ij}, c_{ji}), e_j)).$$

By Claim 2.15:

$$\varphi(q^*_{ij}s_{ii}) = \varphi(q^*_{ij}a_{ii}) + \varphi(0) = \varphi(q^*_{ij}a_{ii}).$$

This shows that $q^*_{ij}s_{ii} = q^*_{ij}a_{ii}$, so $s^*_{ii}q_{ij} = a^*_{ii}q_{ij}$. Therefore, because A_{ij} is a faithful left A_{ii}-module, we get $s_{ii} = a_{ii}$. □

Lemma 4.3. *If $i \neq j$, then $\varphi(a_{ii} + b_{ij} + c_{ji}) = \varphi(a_{ii}) + \varphi(b_{ij} + c_{ji})$.*

Proof. Let $t \in A$ such that $\varphi(t) = \varphi(a_{ii}) + \varphi(b_{ij} + c_{ji})$. Using Claim 2.14:

$$\varphi(P(t, e_j)) = \varphi(P(a_{ii}, e_j)) + \varphi(P(b_{ij} + c_{ji}, e_j)).$$

By Claim 2.15 we have:

$$\varphi((t_{jj} + 2t^*_{jj}) + t^*_{ij} + (t_{ij} + t^*_{ji})) = \varphi(0) + \varphi(b^*_{ij} + (b_{ij} + c^*_{ji})) = \varphi(b^*_{ij} + (b_{ij} + c^*_{ji})).$$

Cancelling φ and using Claim 2.16, we obtain $t_{ij} = b_{ij}$, $t_{ji} = c_{ji}$ and $t_{jj} = 0$. Now let $q_{ij} \in A_{ij}$. By Claim 2.14:

$$\begin{aligned}&\varphi(P(P(P(q_{ij}, t), e_j), e_i))\\ &= \varphi(P(P(P(q_{ij}, a_{ii}), e_j), e_i)) + \varphi(P(P(P(q_{ij}, b_{ij} + c_{ji}), e_j), e_i)).\end{aligned}$$

Using (ii) and (iii) and Claim 2.15:

$$\varphi(q'_{ij}t_{ii}) = \varphi(q'_{ij}a_{ii}) + \varphi(0) = \varphi(q'_{ij}a_{ii}),$$

so that $q'_{ij}t_{ii} = q'_{ij}a_{ii}$, thus $t'_{ii}q_{ij} = a'_{ii}q_{ij}$, therefore, because but A_{ij} is a faithful left A_{ii}-module, we obtain $t_{ii} = a_{ii}$. □

Lemma 4.4. *If $i \neq j$, then $\varphi(a_{ii} + b_{ij} + c_{jj}) = \varphi(a_{ii}) + \varphi(b_{ij}) + \varphi(c_{jj})$.*

Proof. Let $t \in A$ such that $\varphi(t) = \varphi(a_{ii}) + \varphi(b_{ij}) + \varphi(c_{jj})$. Using Claim 2.14, for every $x \in A$:

$$\varphi(P(t, x)) = \varphi(P(a_{ii}, x)) + \varphi(P(b_{ij}, x)) + \varphi(P(c_{jj}, x)). \tag{5}$$

Using (5) with $x = e_i$, Lemma 3.2 and Claim 2.15:

$$\varphi((t_{ii} + 2t^*_{ii}) + t^*_{ji} + (t_{ji} + t^*_{ij})) = \varphi(a_{ii} + 2a^*_{ii}) + \varphi(b^*_{ij}) + \varphi(0) = \varphi((a_{ii} + 2a^*_{ii}) + b^*_{ij}).$$

By Claim 2.16, this implies $t_{ii} = a_{ii}$, $t_{ij} = b_{ij}$ and $t_{ji} = 0$. Now, using (5) with $x = e_j$, Lemma 3.3 and Claim 2.15:

$$\varphi((t_{jj} + 2t^*_{jj}) + b^*_{ij} + b_{ij}) = \varphi(0) + \varphi(b^*_{ij} + b_{ij}) + \varphi(c_{jj} + 2c^*_{jj}) = \varphi((c_{jj} + 2c^*_{jj}) + b^*_{ij} + b_{ij}).$$

By Claim 2.16, we infer that $t_{jj} = c_{jj}$. □

Lemma 4.5. *If $i \neq j$, then $\varphi(a_{ij} + b_{ij}) = \varphi(a_{ij}) + \varphi(b_{ij})$.*

Proof. By Lemma 4.2 and Claim 2.15:

$$\begin{aligned}
\varphi(a_{ij} + b_{ij}) &= \varphi(P(P(e_i + a_{ij}, e_i + b_{ij}^*), e_j)) \\
&= Q(Q(\varphi(e_i + a_{ij}), \varphi(e_i + b_{ij}^*)), \varphi(e_j)) \\
&= Q(Q(\varphi(e_i) + \varphi(a_{ij}), \varphi(e_i) + \varphi(b_{ij}^*)), \varphi(e_j)) \\
&= Q(Q(\varphi(a_{ij}), \varphi(b_{ij}^*)), \varphi(e_j)) + Q(Q(\varphi(a_{ij}), \varphi(e_i)), \varphi(e_j)) \\
&+ Q(Q(\varphi(e_i), \varphi(b_{ij}^*)), \varphi(e_j)) + Q(Q(\varphi(e_i), \varphi(e_i)), \varphi(e_j)) \\
&= \varphi(P(P(a_{ij}, b_{ij}^*), e_j)) + \varphi(P(P(a_{ij}, e_i), e_j)) \\
&+ \varphi(P(P(e_i, b_{ij}^*), e_j)) + \varphi(P(P(e_i, e_i), e_j)) \\
&= \varphi(0) + \varphi(a_{ij}) + \varphi(b_{ij}) + \varphi(0) \\
&= \varphi(a_{ij}) + \varphi(b_{ij}).
\end{aligned}$$

Therefore $\varphi(a_{ij} + b_{ij}) = \varphi(a_{ij}) + \varphi(b_{ij})$. □

Lemma 4.6. *$\varphi(a_{ii} + b_{ii}) = \varphi(a_{ii}) + \varphi(b_{ii})$.*

Proof. Let $t \in A$ such that $\varphi(t) = \varphi(a_{11}) + \varphi(b_{11})$. By Claim 2.14:

$$\varphi(P(t, e_j)) = \varphi(P(a_{ii}, e_j)) + \varphi(P(b_{ii}, e_j)).$$

By Claim 2.15:

$$\varphi((t_{jj} + 2t_{jj}^*) + t_{ij}^* + (t_{ij} + t_{ji}^*)) = \varphi(0) + \varphi(0) = \varphi(0).$$

By Claim 2.16, we have $t_{ij} = 0$, $t_{ji} = 0$ and $t_{jj} = 0$. Now let $q_{12} \in A_{12}$. By Claim 2.14:

$$\varphi(P(q_{ij}, t)) = \varphi(P(q_{ij}, a_{ii})) + \varphi(P(q_{ij}, b_{ii})).$$

By Lemma 4.5:

$$\varphi(q_{ij}^* x_{ii}) = \varphi(q_{ij}^* a_{ii}) + \varphi(q_{ij}^* b_{ii}) = \varphi(q_{ij}^* a_{ii} + q_{ij}^* b_{ii}).$$

Because A_{ij} is a faithful left A_{ii}-module, then $t_{ii} = a_{ii} + b_{ii}$. □

Lemma 4.7. *$\varphi(a_{11} + b_{12} + c_{21} + d_{22}) = \varphi(a_{11} + d_{22}) + \varphi(b_{12} + c_{21})$.*

Proof. Let $t \in A$ be such that $\varphi(t) = \varphi(a_{11} + d_{22}) + \varphi(b_{12} + c_{21})$. By Claim 2.14:

$$\varphi(P(t, e_1)) = \varphi(P(a_{11} + d_{22}, e_1)) + \varphi(P(b_{12} + c_{21}, e_1)).$$

By Lemma 3.3:

$$\begin{aligned}
&\varphi((t_{11} + 2t'_{11}) + t'_{21} + (t_{21} + t'_{12})) = \varphi(a_{11} + 2a'_{11}) + \varphi(c'_{21} + (c_{21} + b'_{12})) \\
&= \varphi((a_{11} + 2a'_{11}) + c'_{21} + (c_{21} + b'_{12})).
\end{aligned}$$

By Claim 2.16, we have $t_{11} = a_{11}$, $t_{12} = b_{12}$ and $t_{21} = c_{21}$. Analogously we have $t_{22} = d_{22}$. □

Lemma 4.8. *If $i \neq j$, then $\varphi(a_{ij} + b_{ji}) = \varphi(a_{ij}) + \varphi(b_{ji})$.*

Proof. By Lemmas 4.2 and 4.7 and Claim 2.15:

$$\begin{aligned}
&\varphi(a_{ij}b_{ji} + b_{ji}a_{ij}) + \varphi(a_{ij} + b_{ji})\\
&= \varphi(a_{ij}b_{ji} + b_{ji}a_{ij} + a_{ij} + b_{ji})\\
&= \varphi(P(e_i + a^*_{ij}, e_j + b_{ji}))\\
&= Q(\varphi(e_i + a^*_{ij}), \varphi(e_j + b_{ji}))\\
&= Q(\varphi(e_i) + \varphi(a^*_{ij}), \varphi(e_j) + \varphi(b_{ji}))\\
&= Q(\varphi(a^*_{ij}), \varphi(b_{ji})) + Q(\varphi(a^*_{ij}), \varphi(e_j)) + Q(\varphi(e_i), \varphi(b_{ji})) + Q(\varphi(e_i), \varphi(e_j))\\
&= \varphi(P(a^*_{ij}, b_{ji})) + \varphi(P(a^*_{ij}, e_j)) + \varphi(P(e_i, b_{ji})) + \varphi(P(e_i, e_j))\\
&= \varphi(a_{ij}b_{ji} + b_{ji}a_{ij}) + \varphi(a_{ij}) + \varphi(b_{ji}) + \varphi(0)\\
&= \varphi(a_{ij}b_{ji} + b_{ji}a_{ij}) + \varphi(a_{ij}) + \varphi(b_{ji}).
\end{aligned}$$

Therefore $\varphi(a_{ij} + b_{ji}) = \varphi(a_{ij}) + \varphi(b_{ji})$. □

Proof of Theorem 4.1. Now using the Lemmas 4.4 to 4.8 and the Claim 2.15, it is easy to see that φ is additive. □

Now we focus our attention to the second theorem of this section, that consists of proving that φ is a $*$-ring isomorphism.

Theorem 4.9. *Let A be a unital $*$-ring that is 2-divisible and 3-torsion free. Let B be a unital $*$-ring. Let $\varphi : A \to B$ be an **additive** bijective function which satisfies*

$$\varphi(\{x, y\}_* + x^*y) = \{\varphi(x), \varphi(y)\}_* + \varphi(x)^*\varphi(y) \tag{6}$$

for all $x, y \in A$. Then φ is a $*$-ring isomorphism.

Because $\varphi : A \to B$ is an additive bijective function and A is 2-divisible and 3-torsion free, then B is 2-divisible and 3-torsion free too.

In order to prove that we will prove some more lemmas. They have the same hypotheses as the Theorem 4.9.

Lemma 4.10. $\varphi(1) = 1$.

Proof. Let $a = \varphi^{-1}(1)$. By (6) with $(x, y) = (a, 1)$, we have $\varphi(a + 2a^*) = 3\varphi(1) = \varphi(3)$. Therefore, by Claim 2.16, we infer that $a = 1$. □

Lemma 4.11. $\varphi(a^*) = \varphi(a)^*$ *for all* $a \in A$.

Proof. Let $a \in A$. By (6) with $(x, y) = (a, 1)$, we have $\varphi(a) + 2\varphi(a^*) = \varphi(a) + 2\varphi(a)^*$, hence $2\varphi(a^*) = 2\varphi(a)^*$, but B is 2-divisible, so we have $\varphi(a^*) = \varphi(a)^*$. □

Lemma 4.12. $\varphi(ab) = \varphi(a)\varphi(b)$ *for all* $a, b \in A$.

Proof. We first divide into some steps.

a) Let $a^* = a$. Using (6) with $(x, y) = (a, b)$ and Lemma 4.11, we infer:

$$2\varphi(ab) + \varphi(ba) = 2\varphi(a)\varphi(b) + \varphi(b)\varphi(a).$$

b) Let $b^* = -b$. Using (6) with $(x, y) = (b, a)$ and Lemma 4.11, we infer $\varphi(ab) = \varphi(a)\varphi(b)$.

c) Let $a^* = a$ and $b^* = b$. By (a), we have:

$$2\varphi(ab) + \varphi(ba) = 2\varphi(a)\varphi(b) + \varphi(b)\varphi(a),$$

$$\varphi(ab) + 2\varphi(ba) = \varphi(a)\varphi(b) + 2\varphi(b)\varphi(a).$$

Because B is 3-torsion free, we have $\varphi(ab) = \varphi(a)\varphi(b)$

d) Let $a^* = -a$ and $b^* = b$. By (a) and (b) we have:

$$\varphi(ab) + 2\varphi(ba) = \varphi(a)\varphi(b) + 2\varphi(b)\varphi(a),$$

$$\varphi(ba) = \varphi(b)\varphi(a).$$

Thus $\varphi(ab) = \varphi(a)\varphi(b)$.

The remainder of the proof is analogous to that of Lemma 3.12. □

Proof of Theorem 4.9. Just use the Lemmas 4.11 and 4.12. □

5. Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

6. Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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