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Weighted generalized group inverse in Banach *-algebras

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Abstract. In this paper, we introduce the notion of weighted generalized group inverse in a Banach algebra with proper involution. This is a natural generalization of weighted weak group inverse for a complex matrix and Hilbert space operator. We present several characterizations and representations of this generalized inverse. In addition, a new partial order on elements in a Banach *-algebra is investigated by using the weighted generalized group inverse and some known results are thus generalized.

1. Introduction

Let $A \in \mathbb{C}^{n\times n}$ be a complex matrix. The group inverse of A is defined as the matrix $X \in \mathbb{C}^{n\times n}$ satisfies the equations:

$$XA^2 = A, AX^2 = X, AX = XA.$$

Such X is unique if exists, denoted by $A^{\#}$. As is well known, a square complex matrix A has group inverse if and only if $rank(A) = rank(A^2)$.

In [20], Wang and Chen introduced and studied a weak group inverse for square complex matrices. A square complex matrix A has weak group inverse X if it satisfies the equations:

$$AX^2 = X, AX = A^{\textcircled{\dagger}}A.$$

Here, $A^{\textcircled{\dagger}}$ is the core-EP inverse of A (see [7, 8, 11, 13, 15]). Weak group inverse was also generalized to a rectangular matrix and Hilbert space operator (see [5, 17]). We refer the reader to [18, 21–24] for more results on weak group inverse.

Let $\mathcal{B}(X, Y)$ be the set of all bounded linear operators from X to Y, where X and Y are infinite-dimensional complex Hilbert spaces. Very recently, Mosić and D. Zheng introduced and studied weighted weak group inverse for Hilbert space operators. Let $A \in \mathcal{B}(X, Y)$ and $W \in \mathcal{B}(Y, X) \setminus \{0\}$. The W-weighted weak group inverse of A is defined as

$$A^{\otimes,W} = (A^{\textcircled{d},W}W)^2A.$$

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Here, $A^{\circledast,W}$ is the weighted core-EP inverse of A (see [12, 14, 17]).

A Banach algebra is called a Banach *-algebra if there exists an involution $* : x \to x^*$ satisfying $(x+y)^* = x^* + y^*, (\lambda x)^* = \overline{\lambda} x^*, (xy)^* = y^* x^*, (x^*)^* = x$. The involution $*$ is proper if $x^*x = 0 \Longrightarrow x = 0$ for any $x \in \mathcal{A}$. The algebra $\mathbb{C}^{n\times n}$ of all $n \times n$ complex matrices is a Banach algebra with conjugate transpose $*$ as its proper involution. If X is a Hilbert space then the algebra $\mathcal{B}(X)$ of all bounded linear operators from on X, with the usual operations and norm, is a Banach algebra with the adjoint operation as its proper involution. Every C^*-algebra is a Banach *-algebra that satisfies an additional condition known as the C^*-identity. The goal of this paper is to generalize (weighted) weak group inverse for complex matrices and Hilbert space operators to elements in a Banach algebra with proper involution. Some known results are thus generalized to wider cases.

Let $\mathcal{A}$ be a Banach algebra with proper involution $*$. An element $a \in \mathcal{A}$ has g-Drazin inverse (i.e., generalized Drazin inverse) if there exists $x \in \mathcal{A}$ such that $ax^2 = x, ax = xa, a - a^2x \in \mathcal{A}^{qnil}$. Such x is unique, if exists, and denote it by a^d. Here, $\mathcal{A}^{qnil} = \{z \in \mathcal{A} \mid 1 + \lambda z \in \mathcal{A}^{-1} \text{ for any } \lambda \in \mathbb{C}\}$. Evidently, $a \in \mathcal{A}^{qnil} \Leftrightarrow \lim\limits_{n\to\infty} \| a^n \|^{\frac{1}{n}} = 0$. As a generalization of weak group inverse mentioned above, the author introduced and studied generalized group inverse (see [2]). An element $a \in \mathcal{A}$ has generalized group inverse if there exists $x \in \mathcal{A}$ such that

$$x = ax^2, (a^*a^2x)^* = a^*a^2x, \lim_{n\to\infty} \|a^n - xa^{n+1}\|^{\frac{1}{n}} = 0.$$

Such x is unique, if exists, and denote it by $a^{®}$. Here, we list several characterizations of generalized group inverse.

Theorem 1.1. *(see [2, Theorem 2.2, Theorem 4.1 and Theorem 5.1]) Let $\mathcal{A}$ be a Banach *-algebra, and let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{®}$.

(2) *There exist $x, y \in \mathcal{A}$ such that*

$$a = x + y, x^*y = yx = 0, x \in \mathcal{A}^{\#}, y \in \mathcal{A}^{qnil}.$$

(3) $a \in \mathcal{A}^d$ and there exists $x \in \mathcal{A}$ such that

$$x = ax^2, (a^d)^*a^2x = (a^d)^*a, \lim_{n\to\infty} \|a^n - xa^{n+1}\|^{\frac{1}{n}} = 0.$$

(4) *There exists an idempotent $p \in \mathcal{A}$ such that*

$$a + p \in \mathcal{A}^{-1}, (a^*ap)^* = a^*ap \text{ and } pa = pap \in \mathcal{A}^{qnil}.$$

(5) $a \in \mathcal{A}^d$ *and*

$$a^d\mathcal{A} = q\mathcal{A} \text{ and } a^*aq = q^*a^*a$$

for an idempotent $q \in \mathcal{A}$.

In Section 2, we extend the definition of weighted weak group inverse for a Hilbert space operator to an elements in a Banach algebra. We obtain some characterizations of weighted generalized group inverse, in particular, the representations of weighted generalized group inverse in terms of generalized group inverses. Recall that an element $a \in \mathcal{A}$ has generalized w-core-EP inverse if there exist $x \in \mathcal{A}$ such that

$$a(wx)^2 = x, (wawx)^* = wawx, \lim_{n\to\infty} \|(aw)^n - (xw)(aw)^{n+1}\|^{\frac{1}{n}} = 0.$$

The preceding x is unique if exists, and we denote it by $a^{\circledast,w}$ (see [3]). An element $a \in \mathcal{A}$ has generalized core-EP inverse if the w mentioned above is 1, and denote its generalized core-EP inverse by $a^{\circledast}$ (see [1]). In

Section 3, we establish the representations of weighted generalized group inverse as a subclass of weighted generalized core-EP inverses. In Section 4, we characterize the weighted generalized group inverse in terms of involved images and kernels. A new property of weighted group inverse is presented by using three systems of equations. Finally, in the last Section, we investigate constrained binary relations and some properties of weighted generalized group orders are derived by using weighted generalized group inverses.

Throughout the paper, all Banach algebras are complex with a proper involution $*$. We use $\mathcal{A}^{-1}, \mathcal{A}^{\#}, \mathcal{A}^{d}, \mathcal{A}^{\textcircled{w}}$, $\mathcal{A}^{\textcircled{d}}$ and $\mathcal{A}^{\textcircled{g}}$ to denote the sets of all invertible, group invertible, g-Drazin invertible, weak group invertible, generalized core-EP invertible and generalized group invertible elements in $\mathcal{A}$, respectively. An element $a \in \mathcal{A}$ has $(1,3)$-inverse x if it satisfies the equations $axa = a$ and $(ax)^* = ax$ and denote x by $a^{(1,3)}$.

2. Weighted generalized group inverse

The purpose of this section is to introduce a new generalized inverse which is a natural generalization of group inverse in a *-Banach algebra. Our starting points is the following.

Theorem 2.1. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) There exists $x \in \mathcal{A}$ such that

$$x = a(wx)^2, [(wa)^*(wa)^2wx]^* = (wa)^*(wa)^2wx,$$
$$\lim_{n\to\infty} \|(aw)^n - (xw)(aw)^{n+1}\|^{\frac{1}{n}} = 0.$$

(2) $wa \in \mathcal{A}^{\textcircled{g}}$.

In this case, $x = a[(wa)^{\textcircled{g}}]^2$.

Proof. (1) ⇒ (2) By hypothesis, we can find $x \in \mathcal{A}$ such that

$$a(wx)^2 = x, [(wa)^*(wa)^2wx]^* = (wa)^*(wa)^2wx, \lim_{n\to\infty} \|(aw)^{n-1} - (xw)(aw)^n\|^{\frac{1}{n-1}} = 0.$$

Then

$$(wa)(wx)^2 = wx, [(wa)^*(wa)^2wx]^* = (wa)^*(wa)^2wx.$$

Furthermore, we have

$$\begin{array}{cl} & \|(wa)^n - (wx)(wa)^{n+1}\|^{\frac{1}{n}} \\ = & \|w(aw)^{n-1}a - wxw(aw)^n a\|^{\frac{1}{n}} \\ = & \|w[(aw)^{n-1} - xw(aw)^n]a\|^{\frac{1}{n}} \\ \leq & \|w\|^{\frac{1}{n}}[\|(aw)^{n-1} - xw(aw)^n\|^{\frac{1}{n-1}}]^{\frac{n-1}{n}}\|a\|^{\frac{1}{n}}. \end{array}$$

Therefore

$$\lim_{n\to\infty} \|(wa)^n - (wx)(wa)^{n+1}\|^{\frac{1}{n}} = 0.$$

Hence,

$$wa \in \mathcal{A}^{\textcircled{g}} \text{ and } (wa)^{\textcircled{g}} = wx.$$

Accordingly,

$$x = a(wx)^2 = a[(wa)^{\textcircled{g}}]^2,$$

as desired.

(2) ⇒ (1) Let $x = a[(wa)^{\textcircled{g}}]^2$. Then we verify that

$$\begin{array}{rcl} a(wx)^2 & = & awa[(wa)^{\textcircled{g}}]^2wa[(wa)^{\textcircled{g}}]^2 \\ & = & a[(wa)^{\textcircled{g}}]^2 = x, \\ (wa)^*(wa)^2wx & = & (wa)^*(wa)^2wa[(wa)^{\textcircled{g}}]^2 \\ & = & (wa)^*(wa)^2(wa)^{\textcircled{g}}, \\ ((wa)^*(wa)^2wx)^* & = & [(wa)^*(wa)^2(wa)^{\textcircled{g}}]^* = (wa)^*(wa)^2(wa)^{\textcircled{g}} \\ & = & (wa)^*(wa)^2wx. \end{array}$$

We easily check that

$$\begin{array}{rcl}(xw)(aw)^{n+1} & = & a[(wa)^{®}]^2w(aw)^{n+1}\\ & = & (aw)^n - a[(wa)^{n-1} - (wa)^{®}(wa)^n]w\\ & - & a(wa)^{®}[(wa)^n - (wa)^{®}(wa)^{n+1}]w\end{array}$$

Hence, we have

$$\begin{array}{rl} & \|(aw)^n - (xw)(aw)^{n+1}\|^{\frac{1}{n}}\\ \leq & \|a\|^{\frac{1}{n}}\|(wa)^{n-1} - (wa)^{®}(wa)^n\|^{\frac{1}{n}}\|w\|^{\frac{1}{n}}\\ + & \|a(wa)^{®}\|^{\frac{1}{n}}\|(wa)^n - (wa)^{®}(wa)^{n+1}\|^{\frac{1}{n}}\|w\|^{\frac{1}{n}}.\end{array}$$

Therefore

$$\lim_{n\to\infty}\|(aw)^n - (xw)(aw)^{n+1}\|^{\frac{1}{n}} = 0,$$

the result follows. □

Corollary 2.2. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) The system of conditions

$$x = a(wx)^2, [(wa)^*(wa)^2wx]^* = (wa)^*(wa)^2wx,$$
$$\lim_{n\to\infty}\|(aw)^n - (xw)(aw)^{n+1}\|^{\frac{1}{n}} = 0.$$

is consistent and it has the unique solution given by $x = a[(wa)^{®}]^2$.

(2) $wa \in \mathcal{A}^{®}$.

Proof. (1) ⇒ (2) This is obvious by Theorem 2.1.

(2) ⇒ (1) Since $wa \in \mathcal{A}^{®}$, by the argument above, $a[(wa)^{®}]^2$ satisfies the preceding equations. If x satisfies the system of conditions mentioned above, then $wx = (wa)^{®}$. Therefore $x = a(wx)^2 = a[(wa)^{®}]^2$, as asserted. □

The preceding unique solution x is called the generalized w-group inverse of a, and denote it by $a^{®,w}$. That is, $a^{®,w} = a[(wa)^{®}]^2$. We use $\mathcal{A}^{®,w}$ to denote the set of all generalized w-group invertible elements in $\mathcal{A}$. By the argument above, we now derive

Corollary 2.3. *Let $a, w \in \mathcal{A}$. Then*

(1) $a^{®,w} = x$.

(2) $wa \in \mathcal{A}^{®}$ and $(wa)^{®} = wx$.

An element $a \in \mathcal{A}$ has generalized w-Drazin inverse if there exist $x \in \mathcal{A}$ such that

$$awx = xwa, xwawx = x, a - awxwa \in \mathcal{A}^{qnil}.$$

Such x is unique, if exists, and denote it by $a^{d,w}$. Evidently, $a^{d,w} = (aw)^d a(wa)^d$ (see [4]). Let $\mathcal{A}^{d,w}$ be the set of all generalized w-Drazin invertible elements in $\mathcal{A}$.

Theorem 2.4. *Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}^{®,w}$ if and only if*

(1) $a \in \mathcal{A}^{d,w}$;

(2) there exists $x \in \mathcal{A}$ such that

$$x = a(wx)^2, [(wa)^d]^*(wa)^2wx = [(wa)^d]^*wa,$$
$$\lim_{n\to\infty}\|(aw)^n - (xw)(aw)^{n+1}\|^{\frac{1}{n}} = 0.$$

Proof. $\Longrightarrow$ In view of Theorem 2.1, $wa \in \mathcal{A}^{®}$. By virtue of [2, Theorem 2.2], $wa \in \mathcal{A}^{d}$. Hence $a \in \mathcal{A}^{d,w}$. Set $x = (wa)^{®}$. Then

$$x = a(wx)^2, \lim_{n\to\infty} \|(aw)^n - (xw)(aw)^{n+1}\|^{\frac{1}{n}} = 0.$$

Since $(wa)^{®} = wx$, it follows by [2, Theorem 2.2] that $[(wa)^d]^*(wa)^2wx = [(wa)^d]^*wa$, as required.

$\Longleftarrow$ Since $a \in \mathcal{A}^{d,w}$, $wa \in \mathcal{A}^d$. By hypothesis, there exists $x \in \mathcal{A}$ such that

$$x = a(wx)^2, [(wa)^d]^*(wa)^2wx = [(wa)^d]^*wa,$$
$$\lim_{n\to\infty} \|(aw)^{n-1} - (xw)(aw)^n\|^{\frac{1}{n-1}} = 0.$$

Then $xw = wa(wx)^2$. Moreover, we see that

$$\begin{array}{rl} & \|(wa)^n - (wx)(wa)^{n+1}\|^{\frac{1}{n}} \\ \leq & \|w\|^{\frac{1}{n}}\Big(\|(ax)^{n-1} - (xw)(aw)^n\|^{\frac{1}{n-1}}\Big)^{1-\frac{1}{n}}\|a\|^{\frac{1}{n}}. \end{array}$$

Thus, we have

$$\lim_{n\to\infty} \|(wa)^n - wx(wa)^{n+1}\|^{\frac{1}{n}} = 0.$$

In light of [2, Theorem 2.2], $wa \in \mathcal{A}^{®}$. Therefore we complete the proof by Theorem 2.1. □

Corollary 2.5. *Let $a \in \mathcal{A}$ and $k \in \mathbb{N}$. Then $a \in \mathcal{A}^{®,w}$ if and only if*

(1) $a \in \mathcal{A}^{d,w}$;
(2) there exists $x \in \mathcal{A}$ such that

$$x = a(wx)^2, [(wa)^k]^*(wa)^2wx = [(wa)^k]^*wa,$$
$$\lim_{n\to\infty} \|(aw)^n - (xw)(aw)^{n+1}\|^{\frac{1}{n}} = 0.$$

Proof. $\Longrightarrow$ Let $x = a^{®,w}$. In view of Theorem 2.1, we have

$$x = a(wx)^2, (wa)^*(wa)^2wx = (wa)^*wa,$$
$$\lim_{n\to\infty} \|(aw)^n - (xw)(aw)^{n+1}\|^{\frac{1}{n}} = 0.$$

We have shown that $[(wa)^{k-1}]^*(wa)^*(wa)^2wx = [(wa)^{k-1}]^*(wa)^*wa$. That is, $[(wa)^k]^*(wa)^2wx = [(wa)^k]^*wa$, as desired.

$\Longleftarrow$ By hypothesis, there exists $x \in \mathcal{A}$ such that

$$x = a(wx)^2, [(wa)^k]^*(wa)^2wx = [(wa)^k]^*wa,$$
$$\lim_{n\to\infty} \|(aw)^n - (xw)(aw)^{n+1}\|^{\frac{1}{n}} = 0.$$

Then

$$[((wa)^d)^{k+1}]^*[(wa)^k]^*(wa)^2wx = [((wa)^d)^{k+1}]^*[(wa)^k]^*wa.$$

Therefore $[(wa)^d]^*(wa)^2wx = [(wa)^d]^*wa$, This completes the proof by Theorem 2.4. □

Corollary 2.6. *Let $a \in \mathcal{A}$ and $k \in \mathbb{N}$. Then $a \in \mathcal{A}^{®}$ if and only if*

(1) $a \in \mathcal{A}^d$;
(2) there exists $x \in \mathcal{A}$ such that

$$x = ax^2, (a^k)^*a^2x = (a^k)^*, \lim_{n\to\infty} \|a^n - xa^{n+1}\|^{\frac{1}{n}} = 0.$$

Proof. This is obvious by choosing $w = 1$ in Corollary 2.5. □

3. Representations by weighted generalized core-EP inverses

Every square complex matrix has core-EP inverse, and so has weighted generalized core-EP inverse (see [19]). Every weighted g-Drazin invertible bounded linear Hilbert operator has weighted generalized core-EP inverse (see [17]). In view of [2, Theorem 6.1], $\mathcal{A}^{\textcircled{d},w} \subseteq \mathcal{A}^{\textcircled{g},w}$. Thus, weighted generalized core-EP invertible elements form a rich subclass of weighted generalized group inverse. Let $a \in \mathcal{A}^{\textcircled{d},w}$. The aim of this section is to present the representations of weighted generalized group inverse $a^{\textcircled{g},w}$ as a subclass of weighted generalized core-EP inverse.

Theorem 3.1. *Let $a \in \mathcal{A}^{\textcircled{d},w}$. Then*

$$a^{\textcircled{g},w} = a((wa)^{\textcircled{d}})^3wa = (a^{\textcircled{d},w}w)^2a.$$

Proof. In view of [3, Corollary 2.2], we have $a^{\textcircled{d},w} = a[(wa)^{\textcircled{d}}]^2$. By virtue of Corollary 2.2, $a^{\textcircled{g},w} = a[(wa)^{\textcircled{g}}]^2$. Applying [2, Theorem 6.1], we see that $(wa)^{\textcircled{g}} = [(wa)^{\textcircled{d}}]^2wa$. Thus,

$$\begin{array}{rcl} a^{\textcircled{g},w} & = & a[(wa)^{\textcircled{d}}]^2wa[(wa)^{\textcircled{d}}]^2wa \\ & = & a[(wa)^{\textcircled{d}}]^2(wa)^{\textcircled{d}}wa \\ & = & a[(wa)^{\textcircled{d}}]^3wa. \end{array}$$

On the other hand, we verify that

$$\begin{array}{rcl} (a^{\textcircled{d},w}w)^2a & = & [a((wa)^{\textcircled{d}})^2w]^2a \\ & = & [a((wa)^{\textcircled{d}})^2w][a((wa)^{\textcircled{d}})^2w]a \\ & = & a((wa)^{\textcircled{d}})^3wa. \end{array}$$

Therefore $a^{\textcircled{g},w} = (a^{\textcircled{d},w}w)^2a$, as asserted. □

In [17], Mosić and Zhang introduced and studied the weighted weak group inverse for Hilbert space operators. Evidently, weighted weak group inverse and weighted generalized group inverse coincide with each other for a Hilbert space operator as the following shows.

Corollary 3.2. *Let X be a Hilbert space, $W \in \mathcal{B}(X) \setminus \{0\}$ and $A \in \mathcal{B}(X)^{d,W}$. Then*

$$A^{\textcircled{g},W} = (A^{\textcircled{d},W}W)^2A.$$

Proof. This is obvious by Theorem 3.1. □

We are ready to prove:

Theorem 3.3. *Let $a \in \mathcal{A}^{\textcircled{d},w}$. Then $a^{\textcircled{g},w} = x$ if and only if*

$$a(wx)^2 = x, awx = a^{\textcircled{d},w}wa.$$

Proof. $\Longrightarrow$ In view of Theorem 3.1, $x = a((wa)^{\textcircled{d}})^3wa$. Since $wa((wa)^{\textcircled{d}})^2 = (wa)^{\textcircled{d}}$ and $a^{\textcircled{d},w} = a((wa)^{\textcircled{d}})^2$, we check that

$$\begin{array}{rcl} a(wx)^2 & = & awa((wa)^{\textcircled{d}})^3wawa((wa)^{\textcircled{d}})^3wa \\ & = & a((wa)^{\textcircled{d}})^2(wa)^{\textcircled{d}}wa \\ & = & a((wa)^{\textcircled{d}})^3wa = x, \\ awx & = & awa((wa)^{\textcircled{d}})^3wa \\ & = & a((wa)^{\textcircled{d}})^2wa = a^{\textcircled{d},w}wa, \end{array}$$

as desired.

$\Longleftarrow$ By hypothesis, we have

$$awxwx = x, awx = a^{\textcircled{d},w}wa.$$

Then we see that

$$\begin{array}{rcl} x & = & a(wx)^2 = (awx)wx = (a^{\textcircled{d},w}wa)wx \\ & = & a^{\textcircled{d},w}w(awx) = a^{\textcircled{d},w}w[a^{\textcircled{d},w}wa] \\ & = & (a^{\textcircled{d},w}w)^2a \end{array}$$

In light of Theorem 3.1, $x = a^{\textcircled{g},w}$, as asserted. □

Corollary 3.4. *Let $a \in \mathcal{A}^{\textcircled{d}}$. Then $a^{\textcircled{g}} = x$ if and only if*

$$ax^2 = x, ax = a^{\textcircled{d}}a.$$

Proof. This is obvious by choosing $w = 1$ in Theorem 3.3. □

In [20], Wang and Chen introduced and studied the weighted weak group inverse for complex matrices. As an immediate consequence of corollary 3.4, the weak group inverse and generalized group inverse coincide with each other for a square complex matrix.

Corollary 3.5. *Let $A \in \mathbb{C}^{n\times n}$. Then $X = A^{\textcircled{g}}$ if and only if X satisfies the system of equations*

$$AX^2 = X, AX = A^{\textcircled{\dagger}}A.$$

We are ready to prove:

Theorem 3.6. *Let $a \in \mathcal{A}^{\textcircled{d},w}$. Then the following are equivalent:*

(1) $a^{\textcircled{g},w} = x$.
(2) $a^{\textcircled{d},w}wawx = x, awx = a^{\textcircled{d},w}wa$.
(3) $xwawx = x, awx = a^{\textcircled{d},w}wa, xwa^{\textcircled{d},w} = a^{\textcircled{d},w}wa^{\textcircled{d},w}$.

Proof. (1) ⇒ (3) In view of Theorem 3.3,

$$a(wx)^2 = x, awx = a^{\textcircled{d},w}wa.$$

By virtue of [2, Theorem 3.1], $a^{\textcircled{d},w}wawa^{\textcircled{d},w} = a^{\textcircled{d},w}$. Applying Theorem 3.1, we have

$$\begin{array}{rcl} xwa^{\textcircled{d},w} & = & [(a^{\textcircled{d},w}w)^2a]wa^{\textcircled{d},w} \\ & = & a^{\textcircled{d},w}w[a^{\textcircled{d},w}wawa^{\textcircled{d},w}] \\ & = & a^{\textcircled{d},w}wa^{\textcircled{d},w}. \end{array}$$

Then

$$xwawx = xw(awx) = (xwa^{\textcircled{d},w})wa = (a^{\textcircled{d},w}wa^{\textcircled{d},w})wa = [a^{\textcircled{d},w}w]^2a.$$

In light of Theorem 3.1, we have $xwawx = x$, as desired.

(3) ⇒ (2) By hypothesis, we have

$$\begin{array}{rcl} a^{\textcircled{d},w}wawx & = & a^{\textcircled{d},w}w(awx) \\ & = & a^{\textcircled{d},w}w(a^{\textcircled{d},w}wa) \\ & = & (a^{\textcircled{d},w}wa^{\textcircled{d},w})wa \\ & = & (xwa^{\textcircled{d},w})wa \\ & = & xw(a^{\textcircled{d},w}wa) \\ & = & xw(awx) \\ & = & x, \end{array}$$

as required.

(2) ⇒ (1) By hypothesis, $a^{\textcircled{d},w}wawx = x, awx = a^{\textcircled{d},w}wa$. Then $a(wx)^2 = (awx)wx = (a^{\textcircled{d},w}wa)wx = x$. In view of Theorem 3.3, $a^{\textcircled{g},w} = x$; hence the result. □

Corollary 3.7. *Let $a \in \mathcal{A}^{\textcircled{d}}$. Then the following are equivalent:*

(1) $a^{\textcircled{g}} = x$.
(2) $a^{\textcircled{d}}ax = x, ax = a^{\textcircled{d}}a$.
(3) $xax = x, ax = a^{\textcircled{d}}a, xa^{\textcircled{d}} = a^{\textcircled{d}}wa^{\textcircled{d}}$.

Proof. This is obvious by choosing $w = 1$ in Theorem 3.6. □

4. Characterizations involving images and kernels

Let $im(a) = \{ar \mid r \in \mathcal{A}\}$ and $ker(b) = \{r \in \mathcal{A} \mid br = 0\}$. The notation $p_{im(a),ker(b)}$ denotes the idempotent $p \in \mathcal{A}$ such that $im(p) = im(a)$ and $ker(p) = ker(b)$. We easily check that the preceding idempotent p is uniquely determined by $a, b \in \mathcal{A}$. The goal of this section is to characterize the weighted generalized group inverse by using involved images and kernels.

Theorem 4.1. *Let $a \in \mathcal{A}^{\textcircled{d},w}$. Then the following are equivalent:*

(1) $a^{\textcircled{g},w} = x$.
(2) $wawx = p_{im(wa^{d,w}),ker(a^{\textcircled{d},w}wa)}, im(x) \subseteq im(a^{d,w})$.

Proof. (1) ⇒ (2) In view of Theorem 3.1, $a^{\textcircled{g},w} = a((wa)^{\textcircled{d}})^3 wa$. Then

$$wawx = waw[a((wa)^{\textcircled{d}})^3 wa] = (wa)^{\textcircled{d}} wa.$$

Let $p = wawx$. Then $p^2 = (wa)^{\textcircled{d}} wa(wa)^{\textcircled{d}} wa = (wa)^{\textcircled{d}} wa = p$. Since $a^{d,w} = (aw)^d a(wa)^d$, we have

$$\begin{aligned} p &= (wa)^{\textcircled{d}} wa = w[(aw)^d]^2 awa = w(aw)^d a \\ &= w(aw)^d aw[(aw)^d]^2 awa = w(aw)^d a(wa)^d wa; \end{aligned}$$

hence, $im(p) \subseteq im(wa^{d,w})$. On the other hand, we have

$$\begin{aligned} wa^{d,w} &= w(aw)^d a(wa)^d = wa[(wa)^d]^2 wa(wa)^d \\ &= (wa)^d wa(wa)^d = p(wa)^d; \end{aligned}$$

hence, $im(wa^{d,w}) \subseteq im(p)$. Thus $im(p) = im(wa^{d,w})$. Likewise, $ker(p) = ker(a^{\textcircled{d},w}wa)$. Moreover, we verify that

$$\begin{aligned} x &= a((wa)^{\textcircled{d}})^3 wa \\ &= a[(wa)^{\textcircled{d}}(wa(wa)^{\textcircled{d}})^{(1,3)}]^3 wa \\ &= a[(wa)^d]^3 [wa(wa)^{\textcircled{d}}]^{(1,3)} \\ &= aw[(aw)^d]^2 a[(wa)^d]^2 [wa(wa)^{\textcircled{d}}]^{(1,3)} \\ &= [(aw)^d a(wa)^d](wa)[wa(wa)^{\textcircled{d}}]^{(1,3)} \\ &= a^{d,w}(wa)[wa(wa)^{\textcircled{d}}]^{(1,3)}. \end{aligned}$$

This implies that $im(x) \subseteq im(a^{d,w})$, as required.

(2) ⇒ (1) By the preceding discussion, $x = a^{\textcircled{g},w}$ satisfies

$$wawx = p_{im(wa^{d,w}),ker(a^{\textcircled{d},w}wa)}, im(x) \subseteq im(a^{d,w}).$$

Assume that $wawy = p_{im(wa^{d,w}),ker(a^{\textcircled{d},w}wa)}, im(y) \subseteq im(a^{d,w})$. Then $waw(x-y) = 0$, and so $x - y \in ker(waw) \subseteq ker(a^{d,w}waw)$. On the other hand, $x - y \in im(a^{d,w}) \subseteq im(a^{d,w}waw)$. Therefore $x - y \in im(a^{d,w}waw) \bigcap ker(a^{d,w}waw) = 0$, and so $y = x$, thus yielding the result. □

Corollary 4.2. *Let $a \in \mathcal{A}^{\textcircled{d}}$. Then the following are equivalent:*

(1) $a^{\textcircled{g}} = x$.
(2) $ax = p_{im(a^d),ker(a^{\textcircled{d}}a)}, im(x) \subseteq im(a^d)$.

Proof. We easily obtain the result by choosing $w = 1$ in Theorem 4.1. □

Theorem 4.3. *Let $a \in \mathcal{A}^{\textcircled{d},w}$. Then*

(1) $a^{\textcircled{g},w} = a^{\textcircled{d},w} p_{im(wa^{d,w}, ker(a^{\textcircled{d},w}wa)}$.
(2) $a^{\textcircled{g},w} = a^{d,w} p_{im(wa^{d,w}, ker(a^{\textcircled{d},w}wa)}$.

(3) $(aw)^3(aw)^d$ is group invertible and

$$a^{\textcircled{g},w} = [(aw)^3(aw)^d]^{\#}awa^{\textcircled{d},w}wa.$$

Proof. (1) In view of Theorem 3.6, we have $a^{\textcircled{g},w} = a^{\textcircled{d},w}wawa^{\textcircled{g},w}$. Set $q = wawa^{\textcircled{g},w}$. By using Theorem 4.1, $q = p_{im(wa^{d,w},ker(a^{\textcircled{d},w}wa)}$, as required.

(2) By virtue of Theorem 3.1, we have

$$\begin{array}{rcl} a^{\textcircled{g},w} & = & (a^{\textcircled{d},w}w)^2a \\ & = & a^{d,w}waw(a^{\textcircled{d},w}w)^2a \\ & = & a^{d,w}[wa^{\textcircled{d},w}wa]. \end{array}$$

In light of Theorem 3.6, $awx = a^{\textcircled{d},w}wa$; whence, $wawx = wa^{\textcircled{d},w}wa$. It follows by Theorem 4.1 that $wawx = p_{im(wa^{d,w})}$. Therefore $a^{\textcircled{g},w} = a^{d,w}p_{im(wa^{d,w}),ker(a^{\textcircled{d},w})wa}$.

(3) Clearly, $[(aw)^3(aw)^d]^{\#} = [(aw)^d]^2$. By virtue of Theorem 3.3., we derive

$$\begin{array}{rcl} [(aw)^3(aw)^d]^{\#}awa^{\textcircled{d},w}wa & = & [(aw)^d]^2awa^{\textcircled{d},w}wa \\ & = & (aw)^d[a^{\textcircled{d},w}wa] \\ & = & (aw)^d(aw)a^{\textcircled{g},w} \\ & = & a^{\textcircled{g},w}, \end{array}$$

and so the result is proved. □

As an immediate consequence, we derive

Corollary 4.4. *Let $a \in \mathcal{A}^{\textcircled{d}}$. Then*

(1) $a^{\textcircled{g},w} = a^{\textcircled{d}}p_{im(wa^d,ker(a^{\textcircled{d}}a)}$.
(2) $a^{\textcircled{g},w} = a^dp_{im(a^d,ker(a^{\textcircled{d}}a)}$.
(3) a^3a^d is group invertible and

$$a^{\textcircled{g},w} = [a^3a^d]^{\#}aa^da.$$

Let $A, B \in M_2(\mathcal{A})$. We say that A is simply equivalent to B if A is equivalent to B by a column and a row transformations, i.e.,

$$\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} A \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} = B.$$

We denote it by $A \dot{\sim} B$.

Theorem 4.5. *Let $a \in \mathcal{A}^{\textcircled{d},w}$. Then*

(1) there exists a unique $x \in \mathcal{A}$ such that

$$ker(x) = im(wa)^d, x^2 = x, [(wa)^d]^*wax = 0;$$

(2) there exists a unique $y \in \mathcal{A}$ such that

$$ker(y) = im(aw)^d, y^2 = y, [(wa)^d]^*(wa)^2wy = 0;$$

(3) there exists a unique $z \in \mathcal{A}$ such that

$$\begin{pmatrix} waw & 1-x \\ 1-y & z \end{pmatrix} \dot{\sim} \begin{pmatrix} waw & 0 \\ 0 & 0 \end{pmatrix}.$$

In this case, $a^{\textcircled{g},w} = z$.

Proof. Let $x = 1 - wawa^{\textcircled{g},w}$, $y = 1 - a^{\textcircled{g},w}waw$ and $z = a^{\textcircled{g},w}$. One directly checks that equalities in (1) and (2) hold. In view of Theorem 3.6, $z = zaz = a^{\textcircled{g},w}wawa^{\textcircled{g},w}$. Then we verify that

$$\begin{pmatrix} 1 & 0 \\ -a^{\textcircled{g},w} & 1 \end{pmatrix}\begin{pmatrix} waw & 1-x \\ 1-y & z \end{pmatrix}\begin{pmatrix} 1 & -a^{\textcircled{g},w} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} waw & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} waw & 1-x \\ 1-y & z \end{pmatrix} \dot{\sim} \begin{pmatrix} waw & 0 \\ 0 & 0 \end{pmatrix}.$$

Claim 1. Assume that there exists $x' \in \mathcal{A}$ such that $ker(x') = im(wa)^d, (x')^2 = x', [(wa)^d]^*wax' = 0$. Then $im(1-x') = ker(x') = im(wa)^d$. By using [2, Theorem 3.4], we have $im(wa)^d = im(wa^{\textcircled{g}})$. Then $im(wa)^d = im(wa^{\textcircled{g},w})$ by Corollary 2.3. Since $wa^{\textcircled{g},w} = wa^{\textcircled{g},w}wawa^{\textcircled{g},w}$, we have $im(wa^{\textcircled{g},w}) = im(wa^{\textcircled{g},w}wa)$. Thus,

$$im(wa)^d = im(wa^{\textcircled{g},w}) = im(wa^{\textcircled{g},w}wa).$$

Since $1 - x'$ and $wa^{\textcircled{g},w}wa$ are idempotents, we have

$$1 - x' = wa^{\textcircled{g},w}wa(1 - x').$$

In view of Theorem 2.1 and [2, Theorem 6.1], $a^{\textcircled{g},w} = a[(wa)^{\textcircled{g}}]^2 = a[((wa)^{\textcircled{d}})^2(wa)]^2$, and so

$$\begin{array}{rcl} wa^{\textcircled{g},w}wa & = & (wa)[((wa)^{\textcircled{d}})^2(wa)]^2 \\ & = & (wa)^{\textcircled{d}}(wa)[(wa)^{\textcircled{d}}]^2(wa) \\ & = & [(wa)^{\textcircled{d}}]^2(wa) \\ & = & (wa)^{\textcircled{d}}[(wa)^d]^2((wa)^d)^{(1,3)}(wa) \\ & = & (wa)^{\textcircled{d}}(wa)^d[(wa)^d((wa)^d)^{(1,3)}](wa) \\ & = & (wa)^{\textcircled{d}}(wa)^d[(wa)^d((wa)^d)^{(1,3)}]^*(wa) \\ & = & (wa)^{\textcircled{d}}(wa)^d[((wa)^d)^{(1,3)}]^*[(wa)^d]^*(wa). \end{array}$$

Hence,

$$wa^{\textcircled{g},w}wax' = (wa)^{\textcircled{d}}(wa)^d[((wa)^d)^{(1,3)}]^*[(wa)^d]^*(wa)x' = 0.$$

Therefore $1 - x' = wa^{\textcircled{g},w}wa = 1 - x$, and so $x' = x$. The uniqueness of x is proved.

Claim 2. Assume that there exists $y' \in \mathcal{A}$ such that $ker(y') = im(aw)^d, (y')^2 = y', [(wa)^d]^*(wa)^2wy' = 0$. Then $im(1-y') = ker(y') = im(aw)^d$. Since $a^{\textcircled{g},w}w = a^{\textcircled{g},w}wawa^{\textcircled{g},w}w$, we see that $im(a^{\textcircled{g},w}w) = im(a^{\textcircled{g},w}waw)$. In view of [2, Theorem 3.4], we have $im(a^{\textcircled{g},w}w) \subseteq im(aw)^d$. By using Theorem 3.1 and [1, Theorem 1.2], we directly check that

$$\begin{array}{rcl} a^{\textcircled{g},w}wa(wa)^dw & = & [a((wa)^{\textcircled{d}})^3wa]wa(wa)^dw \\ & = & a[((wa)^d)^2((wa)^d)^{(1,3)}]^3(wa)^d(wa)^2w \\ & = & a[(wa)^d]^2w \\ & = & (aw)^d. \end{array}$$

and so $im(aw)^d \subseteq im(a^{\textcircled{g},w}w)$. Therefore

$$im(aw)^d = im(a^{\textcircled{g},w}w) = im(a^{\textcircled{g},w}waw).$$

Since $1 - y'$ and $a^{\textcircled{g},w}waw$ are idempotents, we get

$$1 - y' = a^{\textcircled{g},w}waw(1 - y').$$

As in the argument above, we see that

$$\begin{array}{rcl} a^{®,w}waw & = & a[((wa)^{\textcircled{d}})^2(wa)]^2waw \\ & = & a[(wa)^{\textcircled{d}}]^2(wa)^{\textcircled{d}}(wa)^2w \\ & = & a[(wa)^{\textcircled{d}}]^2[(wa)^d]^2(wa)^d)^{(1,3)}(wa)^2w \\ & = & a[(wa)^{\textcircled{d}}]^2(wa)^d[(wa)^d((wa)^d)^{(1,3)}](wa)^2w \\ & = & a[(wa)^{\textcircled{d}}]^2(wa)^d[(wa)^d((wa)^d)^{(1,3)}]^*(wa)^2w \\ & = & a[(wa)^{\textcircled{d}}]^2(wa)^d[((wa)^d)^{(1,3)}]^*[(wa)^d]^*(wa)^2w. \end{array}$$

This implies that

$$a^{®,w}wawy' = a[(wa)^{\textcircled{d}}]^2(wa)^d[((wa)^d)^{(1,3)}]^*[(wa)^d]^*(wa)^2wy' = 0.$$

Accordingly, $1 - y' = a^{®,w}waw = 1 - y$; hence, $y' = y$. The uniqueness of y is proved.

Claim 3. Assume that there exists $z' \in \mathcal{A}$ such that

$$\begin{pmatrix} waw & 1-x' \\ 1-y' & z' \end{pmatrix} \dot{\sim} \begin{pmatrix} waw & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}\begin{pmatrix} waw & 1-x' \\ 1-y' & z' \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} waw & 0 \\ 0 & 0 \end{pmatrix}.$$

By the argument above, $x' = 1 - wawa^{®,w}$ and $y' = 1 - a^{®,w}waw$. Then

$$\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}\begin{pmatrix} waw & wawa^{®,w} \\ a^{®,w}waw & z' \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} waw & 0 \\ 0 & 0 \end{pmatrix}.$$

Obviously, we have

$$\begin{array}{rl} & \begin{pmatrix} waw & wawa^{®,w} \\ a^{®,w}waw & z' \end{pmatrix} \\ = & \begin{pmatrix} 1 & 0 \\ a^{®,w} & 1 \end{pmatrix}\begin{pmatrix} waw & 0 \\ 0 & z' - a^{®,w}wawa^{®,w} \end{pmatrix}\begin{pmatrix} 1 & a^{®,w} \\ 0 & 1 \end{pmatrix}. \end{array}$$

Then

$$\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}\begin{pmatrix} waw & 0 \\ 0 & z' - a^{®,w}wawa^{®,w} \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} waw & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, we see that

$$\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}\begin{pmatrix} waw & 0 \\ 0 & z' - a^{®,w}wawa^{®,w} \end{pmatrix} = \begin{pmatrix} waw & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

Accordingly, $z' - a^{®,w}wawa^{®,w} = 0$, and then $z' = a^{®,w}$, thus yielding the result. □

Corollary 4.6. *Let $a \in \mathcal{A}^{\textcircled{d}}$. Then*

(1) there exists a unique $x \in \mathcal{A}$ such that

$$ker(x) = im(a^d), x^2 = x, (a^d)^*ax = 0;$$

(2) there exists a unique $y \in \mathcal{A}$ such that

$$ker(y) = im(a^d), y^2 = y, (a^d)^*a^2y = 0;$$

(3) there exists a unique $z \in \mathcal{A}$ such that

$$\begin{pmatrix} a & 1-x \\ 1-y & z \end{pmatrix} \dot{\sim} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

In this case, $a^{®} = z$.

Proof. This is an immediate consequence of Theorem 4.5. □

Let $N(X)$ and $R(X)$ represent the null space and range space of a complex matrix X, respectively. As an immediate consequence, we improve [6, Theorem 4.2] as follows.

Corollary 4.7. *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$. Then*

(1) there exists a unique matrix X such that

$$N(X) = R(WA)^D, X^2 = X, [(WA)^D]^*WAX = 0;$$

(2) there exists a unique matrix Y such that

$$N(Y) = R(AW)^D, Y^2 = Y, [(WA)^D]^*(WA)^2WY = 0;$$

(3) there exists a unique matrix Z such that

$$\begin{pmatrix} WAW & I-X \\ I-Y & Z \end{pmatrix} \dot{\sim} \begin{pmatrix} WAW & 0 \\ 0 & 0 \end{pmatrix}.$$

In this case, $A^{®,W} = Z$.

Proof. Since very rectangular matrix can be regarded as a subblock of a square matrix by adding some zero entries, the result is true by Theorem 4.5. □

5. Weighted generalized group orders

Our main concern in this section is to describe the relations between two elements in a Banach *-algebra by means of weighted generalized group inverses. Let $a, b, w \in \mathcal{A}$ and $a \in \mathcal{A}^{®,w}$. Our starting point is the following:

Definition 5.1. (1) *$a \leq_r^{®,w} b$ if $(aw)a^{®,w} = (bw)a^{®,w}$.*
(2) *$a \leq_l^{®,w} b$ if $a^{®,w}(wa) = a^{®,w}(wb)$.*
(3) *$a \leq^{®,w} b$ if $a \leq_r^{®,w} b$ and $a \leq_l^{®,w} b$.*

Lemma 5.2. *Let $a, b, w \in \mathcal{A}$ and $a \in \mathcal{A}^{®,w}$. Then*

(1) $a \leq_r^{®,w} b$ if and only if $a(wa)^d = b(wa)^d$.
(2) $a \leq_l^{®,w} b$ if and only if $(wa)^{\circledast}(wa)^2 = (wa)^{\circledast}(wa)(wb)$.

Proof. (1) $\Longrightarrow$ Since $a \leq_r^{®,w} b$, we have $(aw)a^{®,w} = (bw)a^{®,w}$. In view of Theorem 2.1,

$$a^{®,w} = a[(wa)^{®}]^2.$$

Then we derive

$$(aw)a[(wa)^{®}]^2 = (bw)a[(wa)^{®}]^2.$$

Since $wa[(wa)^{\textcircled{g}}]^2 = (wa)^{\textcircled{g}}$, we obtain

$$a(wa)^{\textcircled{g}} = b(wa)^{\textcircled{g}}.$$

In light of [2, Theorem 6.1], $(wa)^{\textcircled{g}} = [(wa)^{\textcircled{\dagger}}]^2 wa$. Then we have

$$\begin{aligned} a[(wa)^{\textcircled{\dagger}}]^2 &= a[(wa)^{\textcircled{\dagger}}]^2 wa(wa)^{\textcircled{\dagger}} \\ &= b[(wa)^{\textcircled{\dagger}}]^2 wa(wa)^{\textcircled{\dagger}} \\ &= b[(wa)^{\textcircled{\dagger}}]^2. \end{aligned}$$

By virtue of [1, Theorem 1.2], $(wa)^{\textcircled{\dagger}} = [(wa)^d]^2((wa)^d)^{(1,3)}$, and then $[(wa)^{\textcircled{\dagger}}]^2 wa = [(wa)^d]^3$. Therefore

$$\begin{aligned} a(wa)^d &= a[(wa)^d]^3(wa)^2 \\ &= b[(wa)^d]^3(wa)^2 \\ &= b(wa)^d. \end{aligned}$$

$\Longleftarrow$ Since $a(wa)^d = b(wa)^d$, we have $a(wa)^d = b(wa)^d$. As $(wa)^{\textcircled{\dagger}} = [(wa)^d]^2((wa)^d)^{(1,3)}$, we get $a(wa)^{\textcircled{\dagger}} = b(wa)^{\textcircled{\dagger}}$. This implies that

$$\begin{aligned} awa^{\textcircled{g},w} &= awa[(wa)^{\textcircled{g}}]^2 \\ &= awa[(wa)^{\textcircled{\dagger}}]^2 wa(wa)^{\textcircled{g}} \\ &= a(wa)^{\textcircled{\dagger}} wa(wa)^{\textcircled{g}} \\ &= b(wa)^{\textcircled{\dagger}} wa(wa)^{\textcircled{g}} \\ &= bwa[(wa)^{\textcircled{\dagger}}]^2 wa(wa)^{\textcircled{g}} \\ &= bwa[(wa)^{\textcircled{g}}]^2 \\ &= bwa^{\textcircled{g},w}. \end{aligned}$$

Therefore $a \leq_r^{\textcircled{g},w} b$.

(2) Analogously to the preceding discussion, the result follows. □

Lemma 5.3. *Let $a, b \in \mathcal{A}$ and $a \in \mathcal{A}^{\textcircled{\dagger}}$. Then*

(1) $aa^{\textcircled{g}} = ba^{\textcircled{g}}$ if and only if $aa^d = ba^d$.
(2) $a^{\textcircled{g}}a = a^{\textcircled{g}}b$ if and only if $a^{\textcircled{\dagger}}a^2 = a^{\textcircled{\dagger}}ab$.

Proof. (1) Assume that $aa^{\textcircled{g}} = ba^{\textcircled{g}}$. As in the proof of Lemma 5.2, we have $aa^d = ba^d$. Conversely, assume that $aa^d = ba^d$. In view of [1, Theorem 1.2], we have $a^{\textcircled{g}} = [a^{\textcircled{\dagger}}]^2 a = (a^d)^2 (a^d)^{(1,3)} a^{\textcircled{\dagger}} a$. Therefore $aa^{\textcircled{g}} = (aa^d)[a^d(a^d)^{(1,3)}a^{\textcircled{\dagger}}a] = (ba^d)[a^d(a^d)^{(1,3)}a^{\textcircled{\dagger}}a] = ba^{\textcircled{g}}$, as required.

(2) This is proved in the similar way to the above. □

We are ready to prove:

Theorem 5.4. *Let $a, b, w \in \mathcal{A}$ and $a \in \mathcal{A}^{\textcircled{g},w}$. Then*

(1) $a \leq^{\textcircled{g},w} b$.
(2) $a(wa)^d = b(wa)^d$ and $(wa)^{\textcircled{\dagger}}(wa)^2 = (wa)^{\textcircled{\dagger}}(wa)(wb)$.
(3) $a(wa)^{\textcircled{g}} = b(wa)^{\textcircled{g}}$ and $(wa)^{\textcircled{g}}(wa) = (wa)^{\textcircled{g}}(wb)$.

Proof. (1) ⇒ (2) This is obvious by combining (1) and (2) in Lemma 5.2.

(2) ⇒ (3) Since $a(wa)^d = b(wa)^d$, we have $wa(wa)^d = wb(wa)^d$. The implication is obtained by Lemma 5.3.

(3) ⇒ (1) Since $a(wa)^{\textcircled{g}} = b(wa)^{\textcircled{g}}$, by virtue of Lemma 5.3, $a(wa)^d = b(wa)^d$. According to Lemma 5.2, $a \leq_r^{\textcircled{g},w} b$. On the other hand, $(wa)^{\textcircled{g}}(wa) = (wa)^{\textcircled{g}}(wb)$. Since $a^{\textcircled{g},w} = a[(wa)^{\textcircled{g}}]^2$, we have $a^{\textcircled{g},w}(wa) = a^{\textcircled{g},w}(wb)$. Then $a \leq_l^{\textcircled{g},w} b$. So the theorem is true. □

The relation $\leq^{\textcircled{g},w}$ is a pre-order as the following shows.

Corollary 5.5. *If $a \leq^{\textcircled{g},w} b$ and $b \leq^{\textcircled{g},w} c$, then $a \leq^{\textcircled{g},w} c$.*

Proof. In view of Theorem 5.4, we have

$$a(wa)^{\textcircled{g}} = b(wa)^{\textcircled{g}}, (wa)^{\textcircled{g}}(wa) = (wa)^{\textcircled{g}}(wb);$$
$$b(wb)^{\textcircled{g}} = c(wb)^{\textcircled{g}}, (wb)^{\textcircled{g}}(wb) = (wb)^{\textcircled{g}}(wc).$$

Then

$$\begin{array}{rl} & a(wa)^{\textcircled{g}} = b(wa)^{\textcircled{g}} = b[wa(wa)^{\textcircled{g}}](wa)^{\textcircled{g}} \\ = & b(wb)[(wa)^{\textcircled{g}}]^2 = bw[b(wa)^{\textcircled{g}}](wa)^{\textcircled{g}} \\ = & bw[a(wa)^{\textcircled{g}}](wa)^{\textcircled{g}} = bw[b(wb)[(wa)^{\textcircled{g}}]^2(wa)^{\textcircled{g}} \\ = & b(wb)^2[(wa)^{\textcircled{g}}]^3 = \cdots = b(wb)^n[(wa)^{\textcircled{g}}]^{n+1} \\ = & b[(wb)^n - (wb)^d(wb)^{n+1}][(wa)^{\textcircled{g}}]^{n+1} + b(wb)^d(wb)^{n+1}[(wa)^{\textcircled{g}}]^{n+1}; \\ & c(wa)^{\textcircled{g}} = c[wa(wa)^{\textcircled{g}}](wa)^{\textcircled{g}} = c[wb(wa)^{\textcircled{g}}](wa)^{\textcircled{g}} \\ = & cwb[(wa)^{\textcircled{g}}]^2 = \cdots = c(wb)^n[(wa)^{\textcircled{g}}]^{n+1} \\ = & c[(wb)^n - (wb)^d(wb)^{n+1}][(wa)^{\textcircled{g}}]^{n+1} + c(wb)^d(wb)^{n+1}[(wa)^{\textcircled{g}}]^{n+1}. \end{array}$$

Since $bc(wb)^d = c(wb)^d$ and

$$\lim_{n\to\infty} \|(wb)^n - (wb)^d(wb)^{n+1}\|^{\frac{1}{n}} = \lim_{n\to\infty} \|(wb - (wb)^d(wb)^2)^n\|^{\frac{1}{n}} = 0,$$

we see that

$$\lim_{n\to\infty} \|a(wa)^{\textcircled{g}} - c(wa)^{\textcircled{g}}\|^{\frac{1}{n}} = 0.$$

Therefore $a(wa)^{\textcircled{g}} = c(wa)^{\textcircled{g}}$. By a similar route, we check that $(wa)^{\textcircled{g}}(wa) = (wa)^{\textcircled{g}}(wc)$. Therefore $a \leq^{\textcircled{g},w} c$, the corollary is true. □

Let $aw, wa \in \mathcal{A}^{\textcircled{d}}$. Then $a \in \mathcal{A}^{\textcircled{g},w}$ and $a^{\textcircled{g},w} = (aw)^{\textcircled{g}}a(wa)^{\textcircled{g}}$. We are now ready to prove the main result of this section.

Theorem 5.6. *Let $aw, wa \in \mathcal{A}^{\textcircled{d}}$. Then the following statements are equivalent:*

(1) $a \leq_r^{\textcircled{g},w} b$ if and only if $wa \leq^{\textcircled{g}} wb$.
(2) $a \leq_l^{\textcircled{g},w} b$ if and only if $aw \leq^{\textcircled{g}} bw$.
(3) $a \leq^{\textcircled{g},w} b$ if and only if $wa \leq^{\textcircled{g}} wb$ and $aw \leq^{\textcircled{g}} bw$.

Proof. (1) ⇒ (2) Since $a \leq^{\textcircled{g},w} b$, we have

$$(aw)a^{\textcircled{g},w} = (bw)a^{\textcircled{g},w}, a^{\textcircled{g},w}(wa) = a^{\textcircled{g},w}(wb).$$

In view of Theorem 2.1, $a^{\textcircled{g},w} = a[(wa)^{\textcircled{g}}]^2$. Hence

$$\begin{array}{rcl} (wa)(wa)^{\textcircled{g}} & = & w(aw)a[(wa)^{\textcircled{g}}]^2 \\ & = & w(aw)a^{\textcircled{g},w} \\ & = & w(bw)a^{\textcircled{g},w} \\ & = & w(bw)a[(wa)^{\textcircled{g}}]^2 \\ & = & (wb)(wa)^{\textcircled{g}}. \end{array}$$

Furthermore, we check that

$$\begin{array}{rcl} (wa)^{\textcircled{g}}(wa) & = & wa[(wa)^{\textcircled{g}}]^2(wa) \\ & = & w[a((wa)^{\textcircled{g}})^2](wa) \\ & = & w[a^{\textcircled{g},w}(wb)] \\ & = & w[a[(wa)^{\textcircled{g}}]^2(wb)] \\ & = & [wa((wa)^{\textcircled{g}})^2](wb) \\ & = & (wa)^{\textcircled{g}}(wa). \end{array}$$

Therefore $wa \leq^{\textcircled{g}} wb$. Analogously, we show that $aw \leq^{\textcircled{g}} bw$, as required.

(2) ⇒ (1) Since $wa \leq^{\circledast} wb$ and $aw \leq^{\circledast} bw$, we have

$$(wa)(wa)^{\circledast} = (wb)(wa)^{\circledast}, (wa)^{\circledast}(wa) = (wa)^{\circledast}(wb);$$
$$(aw)(aw)^{\circledast} = (bw)(aw)^{\circledast}, (aw)^{\circledast}(aw) = (aw)^{\circledast}(bw).$$

In view of [2, Theorem 2.1],

$$\begin{array}{ll} & (aw)(aw)^d[(aw)(aw)^d]^{(1,3)} \\ = & (aw)(aw)^{\circledast} \\ = & (bw)(aw)^{\circledast} \\ = & (bw)(aw)^d[(aw)(aw)^d]^{(1,3)}, \end{array}$$

and so

$$\begin{array}{ll} & (aw)[(aw)^d]^2(aw)(aw)^d[(aw)(aw)^d]^{(1,3)}(aw)(aw)^d \\ = & (bw)[(aw)^d]^2(aw)(aw)^d[(aw)(aw)^d]^{(1,3)}(aw)(aw)^d. \end{array}$$

We infer that $aw[(aw)^d]^2(aw)(aw)^d = bw[(aw)^d]^2(aw)(aw)^d$, hence $aw(aw)^d = bw(aw)^d$. By using Cline's formula and [2, Theorem 2.1], we derive

$$\begin{array}{rcl} a(wa)^{\circledast} & = & a(wa)^d[(wa)(wa)^d]^{(1,3)} \\ & = & aw[(aw)^d]^2a[(wa)(wa)^d]^{(1,3)} \\ & = & [aw(aw)^d](aw)^da[(wa)(wa)^d]^{(1,3)} \\ & = & [bw(aw)^d](aw)^da[(wa)(wa)^d]^{(1,3)} \\ & = & bw[(aw)^d]^2a[(wa)(wa)^d]^{(1,3)} \\ & = & b(wa)^d[(wa)(wa)^d]^{(1,3)} \\ & = & b(wa)^{\circledast}. \end{array}$$

Since $a^{\circledast,w} = a[(wa)^{\circledast}]^2$, we verify that

$$\begin{array}{rcl} (aw)a^{\circledast,w} & = & (aw)a[(wa)^{\circledast}]^2 \\ & = & a[wa((wa)^{\circledast})^2] \\ & = & a(wa)^{\circledast} \\ & = & b(wa)^{\circledast} \\ & = & (bw)a[(wa)^{\circledast}]^2 \\ & = & (bw)a^{\circledast,w}. \end{array}$$

Likewise, we prove that $a^{\circledast,w}(wa) = a^{\circledast,w}(wb)$. Therefore $a \leq^{\circledast,w} b$, as asserted. □

Let $p, q \in \mathcal{A}$ be projections and $x \in \mathcal{A}$. Then $x = pxq + px(1-q) + (1-p)xq + (1-p)x(1-q)$. We write x as in the matrix form:

$$x = \begin{pmatrix} pxq & px(1-q) \\ (1-p)xq & (1-p)x(1-q) \end{pmatrix}_{p\times q}.$$

Theorem 5.7. *Let $aw, wa \in \mathcal{A}^{\circledcirc}$. Then the following statements are equivalent:*

(1) $a \leq_r^{\circledast,w} b$.
(2) a, w and b are represented as

$$a = \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix}_{p\times q}, w = \begin{pmatrix} w_1 & w_2 \\ 0 & w_3 \end{pmatrix}_{q\times p}, b = \begin{pmatrix} a_1 & b_2 \\ 0 & b_3 \end{pmatrix}_{p\times q},$$

where

$$p = (aw)(aw)^{\circledcirc}, q = (wa)(wa)^{\circledcirc},$$
$$a_1 \in (p\mathcal{A}q)^{-1}, w_1 \in (q\mathcal{A}p)^{-1},$$
$$a_3w_3 \in ((1-p)\mathcal{A}(1-p))^{qnil},$$
$$w_3a_3 \in ((1-q)\mathcal{A}(1-q))^{qnil},$$
$$(a_2 - b_2)w_3 + (a_1w_1)^{-1}(a_1w_2 + a_2w_3)(a_3 - b_3)w_3 = 0.$$

Proof. Let $p = (aw)(aw)^{ⓓ}$ and $q = (wa)(wa)^{ⓓ}$. Then we have

$$[1-(aw)(aw)^{ⓓ}](aw)^d = 0, [1-(wa)(wa)^{ⓓ}](wa)^d = 0.$$

Moreover, we verify that

$$\begin{array}{rcl} (1-p)aq & = & [1-(aw)(aw)^{ⓓ}]a(wa)(wa)^{ⓓ} \\ & = & [1-(aw)(aw)^{ⓓ}](aw)a(wa)(wa)^d(wa)^{ⓓ} \\ & = & [1-(aw)(aw)^{ⓓ}](aw)^3[(aw)^d]^2a(wa)^{ⓓ} \\ & = & [1-(aw)(aw)^{ⓓ}](aw)^d(aw)^2a(wa)^{ⓓ} \\ & = & 0; \\ (1-q)wp & = & [1-(wa)(wa)^{ⓓ}]w(aw)(aw)^{ⓓ} \\ & = & [1-(wa)(wa)^{ⓓ}](wa)^d(wa)^2a(aw)^{ⓓ} \\ & = & 0; \end{array}$$

$$\begin{array}{rcl} a_1w_1 & = & (aw)(aw)^{ⓓ}a(wa)(wa)^{ⓓ}w(aw)(aw)^{ⓓ} \\ & = & (aw)(aw)^{ⓓ}a(wa)(wa)^{ⓓ}w(aw)(aw)^{ⓓ} \\ & \in & (p\mathcal{A}p)^{-1}; \\ w_1a_1 & = & (wa)(wa)^{ⓓ}w(aw)(aw)^{ⓓ} \\ & \in & (q\mathcal{A}q)^{-1}; \\ a_2w_2 & = & [1-(aw)(aw)^{ⓓ}]a[1-(wa)(wa)^{ⓓ}]w[1-(aw)(aw)^{ⓓ}] \\ & \in & [(1-p)\mathcal{A}(1-p)]^{qnil}; \\ w_2a_2 & = & [1-(wa)(wa)^{ⓓ}]w[1-(aw)(aw)^{ⓓ}]a[1-(wa)(wa)^{ⓓ}] \\ & \in & [(1-q)\mathcal{A}(1-q)]^{qnil}. \end{array}$$

Write $b = \begin{pmatrix} b_1 & b_2 \\ b_4 & b_3 \end{pmatrix}_{p\times q}$. Then we have

$$aw = \begin{pmatrix} a_1w_1 & a_1w_2 + a_2w_3 \\ 0 & a_3w_3 \end{pmatrix}_{p\times p}.$$

Thus,

$$(aw)^{ⓖ} = \begin{pmatrix} (a_1w_1)^{-1} & (a_1w_1)^{-2}(a_1w_2 + a_2w_3) \\ 0 & 0 \end{pmatrix}_{p\times p},$$

and so

$$aw(aw)^{ⓖ} = \begin{pmatrix} 1 & (a_1w_1)^{-1}(a_1w_2 + a_2w_3) \\ 0 & 0 \end{pmatrix}_{p\times p}.$$

Furthermore, we have

$$bw = \begin{pmatrix} b_1w_1 & b_1w_2 + b_2w_3 \\ b_4w_1 & b_4w_2 + b_3w_3 \end{pmatrix}_{p\times p}.$$

Then

$$bw(aw)^{ⓖ} = \begin{pmatrix} b_1w_1(a_1w_1)^{-1} & b_1w_1(a_1w_1)^{-2}(a_1w_2 + a_2w_3) \\ b_4w_1(a_1w_1)^{-1} & b_4w_1(a_1w_1)^{-2}(a_1w_2 + a_2w_3) \end{pmatrix}_{p\times p}.$$

Moreover, we have

$$\begin{array}{rl} & (aw)^{ⓖ}aw \\ = & \begin{pmatrix} 1 & (a_1w_1)^{-1}(a_1w_2 + a_2w_3) + (a_1w_1)^{-2}(a_1w_2 + a_2w_3)a_3w_3 \\ 0 & 0 \end{pmatrix}, \\ & (aw)^{ⓖ}bw \\ = & \begin{pmatrix} 1 & (a_1w_1)^{-1}(b_1w_2 + b_2w_3) + (a_1w_1)^{-2}(a_1w_2 + a_2w_3)(b_4w_2 + b_3w_3) \\ 0 & 0 \end{pmatrix}. \end{array}$$

(1) ⇒ (2) By Lemma 5.2 and Lemma 5.3, we have $aw(aw)^{®} = bw(aw)^{®}$, hence, $b_1w_1(a_1w_1)^{-1} = 1$, and so $b_1w_1 = a_1w_1$. Since w_1 is invertible, we obtain $a_1 = b_1$. Since $b_4w_1(a_1w_1)^{-1} = 0$, we see that $b + 4w_1 = 0$, and so $b_4 = 0$. On the other hand, we have $(aw)^{®}aw = (aw)^{®}bw$. This implies that

$$\begin{array}{rl} & (a_1w_1)^{-1}(a_1w_2 + a_2w_3) + (a_1w_1)^{-2}(a_1w_2 + a_2w_3)a_3w_3 \\ = & (a_1w_1)^{-1}(b_1w_2 + b_2w_3) + (a_1w_1)^{-2}(a_1w_2 + a_2w_3)b_3w_3. \end{array}$$

Hence,

$$\begin{array}{rl} & (a_1w_2 + a_2w_3) + (a_1w_1)^{-1}(a_1w_2 + a_2w_3)a_3w_3 \\ = & (a_1w_2 + b_2w_3) + (a_1w_1)^{-1}(a_1w_2 + a_2w_3)b_3w_3. \end{array}$$

Accordingly, $(a_2 - b_2)w_3 + (a_1w_1)^{-1}(a_1w_2 + a_2w_3)(a_3 - b_3)w_3 = 0$.

(2) ⇒ (1) By direct computations, we have

$$(aw)(aw)^{®} = \begin{pmatrix} 1 & (a_1w_1)^{-1}(a_1w_2 + a_2w_3) \\ 0 & 0 \end{pmatrix}_{p\times p} = (bw)(aw)^{®}.$$

Likewise, we check that $(aw)^{®}(aw) = (bw)^{®}(aw)$. Therefore $a \leq^{®,w} b$, as asserted. □

Corollary 5.8. *Let $A, B \in \mathbb{C}^{n\times n}$. Then the following are equivalent:*

(1) $A \leq_r^{®,W} B$.
(2) A, W and B are represented as

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}_{P\times Q}, W = \begin{pmatrix} W_1 & W_2 \\ 0 & W_3 \end{pmatrix}_{Q\times P}, B = \begin{pmatrix} A_1 & B_2 \\ 0 & B_3 \end{pmatrix}_{P\times Q},$$

where

$$\begin{array}{c} P = (AW)(AW)^{\textcircled{d}}, Q = (WA)(WA)^{\textcircled{d}}, \\ A_1 \in (p\mathbb{C}^{n\times n}q)^{-1}, W_1 \in (q\mathbb{C}^{n\times n}p)^{-1}, \\ A_3W_3 \in N((I_n - P)\mathbb{C}^{n\times n}(I_n - P)), \\ W_3A_3 \in N((I_n - Q)\mathbb{C}^{n\times n}(I_n - Q)), \\ (A_2 - B_2)W_3 + (A_1W_1)^{-1}(A_1W_2 + A_2W_3)(A_3 - B_3)W_3 = 0. \end{array}$$

Proof. This is obvious by Theorem 5.7. □

Analogously, we now derive

Theorem 5.9. *Let $aw, wa \in \mathcal{A}^{\textcircled{d}}$. Then the following statements are equivalent:*

(1) $a \leq_l^{®,w} b$.
(2) a, w and b are represented as

$$a = \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix}_{p\times q}, w = \begin{pmatrix} w_1 & w_2 \\ 0 & w_3 \end{pmatrix}_{q\times p},$$
$$b = \begin{pmatrix} a_1 - w_1^{-1}w_2b_4 & b_2 \\ b_4 & b_3 \end{pmatrix}_{p\times q},$$

where

$$\begin{array}{c} p = (aw)(aw)^{\textcircled{d}}, q = (wa)(wa)^{\textcircled{d}}, \\ a_1 \in (p\mathcal{A}q)^{-1}, w_1 \in (q\mathcal{A}p)^{-1}, \\ a_3w_3 \in ((1-p)\mathcal{A}(1-p))^{qnil}, \\ w_3a_3 \in ((1-q)\mathcal{A}(1-q))^{qnil}, w_3b_4 = 0, \\ b_2 = a_2 + w_1^{-1}w_2(a_3 - b_3) + (w_1a_1w_1)^{-1}(w_1a_2 + w_2a_3)w_3(a_3 - b_3). \end{array}$$

References

[1] H. Chen and M. Sheibani, On generalized core-EP invertibility in a Banach algebra, arXiv:2309.09862 [math.FA]. https://doi.org/10.48550/arXiv.2309.09862.

[2] H. Chen and M. Sheibani, Generalized group inverse in a Banach *-algebra, preprint, 2023. https://doi.org/10.21203/rs.3.rs-3338906/v1.

[3] H. Chen and M. Sheibani, Weighted generalized core-EP inverse in Banach *-algebras, preprint, 2023. https://doi.org/10.21203/rs.3.rs-3332600/v1.

[4] A. Dajić and J.J. Koliha, The weighted g-Drazin inverse for operators, *J. Aust. Math. Soc.*, **82**(2007), 163–181.

[5] D.E. Ferreyra; V. Orquera and N. Thome, A weak group inverse for rectangular matrices, *Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat.*, **113**(2019), 3727–3740.

[6] D.E. Ferreyra; V. Orquera and N. Thome, Representation s of weighted WG inverse and a rank equation's solution, *Linear Multilinear Algebra*, **71**(2023), 226–241.

[7] Y. Gao and J. Chen, Pseudo core inverses in rings with involution, *Commun. Algebra*, **46**(2018), 38–50.

[8] Y. Gao and J. Chen, The pseudo core inverse of a lower triangular matrix, *Rev. R. Acad. Cienc. Exactas Fis.Nat., Ser. A Mat.*, **113**(2019), 423–434.

[9] J. Marovt and Mosić, On some orders in *-rings based on the core-EP decompositions, *J. Algebra Appl.*, **21**(2022), https://doi.org/10.1142/ S0219498822500104.

[10] D. Mosić, Weighted binary relations for operators on Banach spaces, *Aequat. Math.*, **90**(2016), 787–798.

[11] D. Mosić, Core-EP pre-order of Hilbert space operators, *Quaest. Math.*, **41**(2018), 585–600.

[12] D. Mosić, Weighted core-EP inverse of an operator between Hilbert spaces, *Linear Multilinear Algebra*, **67**(2019), 278–298.

[13] D. Mosić, Core-EP inverse in ring with involution, *Publ. Math. Debrecen*, **96**(2020), 427–443.

[14] D. Mosić, Weighted core-EP inverse and weighted core-EP pre-orders in a C^*-algebra, *J. Aust. Math. Soc.*, **111**(2021), 76–110.

[15] D. Mosić, Core-EP inverses in Banach algebras, *Linear Multilinear Algebra*, **69**(2021), 2976–2989.

[16] D. Mosić and D.S. Djordjevic, The gDMP inverse of Hilbert space operators, *J. Spectral Theory*, **8**(2018), 555–573.

[17] D. Mosić and D. Zhang, Weighted weak group inverse for Hilbert space operators, *Front. Math. China*, **15**(2020), 709–726.

[18] D. Mosić; P.S. Stanimirovic, Representations for the weak group inverse, *Appl. Math. Comput.*, **397**(2021), Article ID 125957, 19 p.

[19] H. Wang, Core-EP decomposition and its applications, *Linear Algebra Appl.*, **508**(2016), 289–300.

[20] H. Wang; J. Chen, Weak group inverse, *Open Math.*, **16**(2018), 1218–1232.

[21] H. Wang and X. Liu, The weak group matrix, *Aequat. Math.*, **93**(2019), 1261–1273.

[22] H. Yan; H. Wang; K. Zuo and Y. Chen, Further characterizations of the weak group inverse of matrices and the weak group matrix, *AIMS Math.*, **6**(2021), 9322–9341.

[23] M. Zhou; J. Chen and Y. Zhou, Weak group inverses in proper *-rings, *J. Algebra Appl.*, **19**(2020), https://doi.org/10.1142/S0219498820502382.

[24] M. Zhou; J. Chen; Y. Zhou and N. Thome, Weak group inverses and partial isometries in proper *-rings, *Linear Multilinear Algebra*, **70**(2021), 1–16.