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Weighted statistical soft convergence in soft topologies

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Dedicated to Professor Mahmut Ergüt in honor of 52 years of contributions to education and research.

Abstract. Soft sets and statistical convergence are both mathematical tools with which some generalizations can be made. In this study, we defined the weighted statistical convergence of soft point sequences in soft topological spaces and examined it from some aspects. Furthermore, statistical convergence was attained using g-density, a more versatile density function than asymptotic density. Regarding weighted soft statistical convergence concept, we also establish some relationships between soft topology and the classical topology induced by it within the framework of the statistical convergence concept and some clusters associated with statistical convergence were defined.

1. Introduction

There are two important concepts that will form the basis of this study. These are soft topology and statistical convergence. Both of these concepts are mathematical tools defined to make some generalizations. In order to overcome various types of uncertainties seen in various fields, Molodtsov [24] defined soft sets and also established the fundamental results of the new theory, which can be seen as a new mathematical tool. As a generalization of the classic convergence of real sequences, statistical convergence was first introduced independently by Steinhaus [34] and Fast [15].

Uncertain data are involved in a lot of real-world issues in disciplines like economics, engineering, environmental science, medicine, and social sciences. Because these problems contain a variety of uncertainties, we are unable to solve them using traditional methods. Although there are many theories in the literature that address uncertainties, each has its own drawbacks. The notion of soft sets was developed to represent ambiguity and uncertainty. Since its definition, soft set theory has also been studied in a variety of contexts. Maji et al. [22] defined and examined a number of the fundamental concepts of soft set theory. Soft groups were defined and their fundamental characteristics were deduced by Aktaş and Çağman [1]. Soft topological spaces and their fundamental characteristics were characterized by Shabir and Naz [32] and Çağman et al. [9] and that paper also defines some soft separation axioms. The convergence of sequence in soft topological spaces was examined by Varol and Aygün [35]. The neighborhood characteristics of a soft element were examined by Nazmul and Samanta [28].

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Likewise, convergence plays a crucial role in various fields such as physics, economics, and engineering, particularly in analyzing trends and predicting future behaviors. Many convergence concepts have emerged after the concept of topologically convergence in classical analysis. One of the most intensively studied types of convergence, which is a generalization of topological convergence, is statistical convergence. This concept has been examined for many mathematical structures and by making generalizations. Hence it has wide applications in many fields such as summability theory, number theory, measure theory, trigonometric series. Among the dense literature, foundational works on statistical convergence are considered Schoenberg [33], Maddox [23], Connor [12], Fridy [16, 17] and Šalát [31].

Firstly, while the convergence of soft point sequences according to soft topology is given by Demir and Ozbakır [14] and Di Maio and Kočinac [21] extended the topological convergence to statistical convergence in topological spaces as usual convergence is extended to statistical convergence with the help of natural density. In the following years, many studies were carried out on soft sets and statistical convergence. However, although the concepts of both soft topology and statistical convergence have been shown to be useful in a wide variety of fields, to the best of our knowledge, no study has yet been conducted on statistical convergence in soft topological spaces. For this reason, the aim of this paper is to introduce the concept of statistical convergence of the sequences of soft points in soft topological spaces in order to partially fill this gap in the literature and to provide direction for future studies.

The paper is organized into three sections. In the second part, which is divided into two subsections, an overview of soft topology and statistical convergence is given, respectively. In the third section, which is given in three parts, firstly, the statistical convergence of soft point sequences is defined and some of its properties are examined. In the second part, the relations between soft topology and reduced topology are discussed in terms of statistical convergence. In the last part, some point sets related to weighted statistical convergence are given.

2. Preliminaries

As a preliminary, for the convenience of the reader, we first provide some definitions and well-known results in soft set theory and statistical convergence, including properties and set-theoretic operations.

2.1. Soft topology

In the rest of this study, we use the capital letters X, E, and $P(X)$ to denote the universe, the set of parameters and power set of X, respectively.

Definition 2.1. ([24]) A pair of (F, A) is called a soft set over X where F is a mapping given by $F : A \subseteq E \to P(X)$.

In other words, the soft set is a parametrized family of subsets of the set X. The set of all soft sets over X is denoted by $SS(X, E)$. For $e \in E$, every set $F(e)$ may be considered the set of e-approximate elements of the soft set (F, E). Considering $F(e) = \varnothing$ for every $e \in E \setminus A$, (F, E) can be written instead of (F, A).

According to the definition given by [24], some fundamental operations in soft set theory used in this study are defined below.

Definition 2.2. ([22]) Let (F, E),$(G, E) \in SS(X, E)$.

(1) If for every $e \in E$, $F(e) = \varnothing$, then (F, E) is said to be a null soft set, denoted by $\varnothing_s$ (see [22]).

(2) If for every $e \in E$, $F(e) = X$, then (F, E) is said to be an absolute soft set, denoted by X_s (see [22]).

(3) (F, E) is a soft subset of (G, E) (denoted by $(F, E) \subset (G, E)$) if $F(e) \subset G(e)$ for $e \in E$. Hence, (F, E) is equal to (G, E) (denoted by $(F, E) = (G, E)$) if $F(e) = G(e)$ for $e \in E$ (see [29]).

(4) The soft complement of (F, E) is the soft set $(F, E)^c = (F^c, E)$, where $F^c : E \to P(X)$, $F^c(e) = X \setminus F(e)$ for every $e \in E$ (see [32]).

(5) The union $(F,E)\cup(G,E)$ is a soft set $(H,E)\in SS(X,E)$ such that $H:E\to P(X)$, $H(e)=F(e)\cup G(e)$ for every $e\in E$ (see [22]).

(6) The intersection $(F,E)\cap(G,E)$ is a soft set $(I,E)\in SS(X,E)$ such that $I:E\to P(X)$, $I(e)=F(e)\cap G(e)$ for every $e\in E$ (see [22]).

While the concept of topology is defined for the same parameter set in [32], it is defined using different parameter sets in [9].

Definition 2.3. ([32]) For the parameter set E, $\tau_s\subseteq SS(X,E)$ is said to be soft topology on X_s and the triple (X_s,τ_s,E) is called soft topological space provided that the following assertions hold:

(1) $\varnothing_s, X_s\in\tau_s$.

(2) τ_s is closed under the union of any collection of soft sets in τ_s.

(3) τ_s is closed under the intersection of a finite number of soft sets in τ_s.

The members of τ_s are called soft open sets, and a soft set $(F,E)\in SS(X,E)$ is called a soft closed set if $(F,E)^c\in\tau_s$. Unless otherwise stated, since E will always be taken as the parameter set, we will use the notation (X,τ) instead of (X_s,τ_s,E) for the sake of brevity.

Definition 2.4. ([9, 32])
Let (X,τ) be a soft topological space.

(1) A subcollection $\mathfrak{B}$ of τ is called a soft base for τ if every member of τ can be expressed as the union of some members of $\mathfrak{B}$.

(2) The soft closure of $(F,E)\in SS(X,E)$ is the soft set $\overline{(F,E)}=\bigcap\{(G,E):(F,E)\subseteq(G,E)\text{ and }(G,E)^c\in\tau\}$. Then, $\overline{(F,E)}$ is the smallest soft closed set over X which contains (F,E) and (F,E) is closed if and only if $(F,E)=\overline{(F,E)}$.

(3) For $P_e^x\in SP(X)$, $P_e^x\in\overline{(F,E)}$ if and only if $(F,E)\cap(U,E)\neq\varnothing_s$ for every $(U,E)\in\mathcal{N}_\tau(P_e^x)$.

In the literature, the concept of soft points was given in [5], [36], [28], and [13]. However, there are differences between these definitions. In this study, the soft set definition will be as follows.

Definition 2.5. ([13, 28]) A soft set (F,E) over X is said to be a soft point, if there is $e\in E$ such that $F(e)=\{x\}$ for some $x\in X$ and $F(e')=\varnothing$ for all $e'\in E\setminus\{e\}$. The soft point defined in this way is denoted by P_e^x. Throughout this study, $SP(X,E)$ denotes the set of all soft points over X.

Definition 2.6. ([13, 28]) Let $(F,E),(G,E),(H,E)\in SS(X,E)$ and $P_e^x\in SP(X,E)$.

(1) $P_{e_1}^{x_1}, P_{e_2}^{x_2}\in SP(X,E)$ are said to be equal if $e_1=e_2$ and $x_1=x_2$. Thus, these points are said to be not equal if $e_1\neq e_2$ or $x_1\neq x_2$.

(2) The soft point P_e^x is said to belong to the soft set (F,E), denoted by $P_e^x\in(F,E)$, if $x\in F(e)$. Then, $P_e^x\notin(F,E)$ if and only if $x\notin F(e)$.

(3) (F,E) is called soft neighborhood of the soft point P_e^x if there exists a soft open set (G,E) such that $P_e^x\in(G,E)\subseteq(F,E)$. The set of all soft neighborhoods of P_e^x is denoted by $\mathcal{N}_\tau(P_e^x)$.

Proposition 2.7. ([13, 28]) *Let $(F,E),(G,E)\in SS(X,E)$ and $P_e^x\in SP(X,E)$. Then we have:*

(1) $P_e^x\in(F,E)\iff P_e^x\notin(F,E)^c$.

(2) $P_e^x\in[(F,E)\cup(G,E)]\iff P_e^x\in(F,E)$ *or* $P_e^x\in(G,E)$.

(3) $P_e^x \in [(F,E) \cap (G,E)] \iff P_e^x \in (F,E)$ *and* $P_e^x \in (G,E)$.

(4) $(F,E) \subseteq (G,E) \iff P_e^x \in (F,E)$ *implies* $P_e^x \in (G,E)$.

Definition 2.8. ([14]) A soft topological space (X,τ) is called soft Hausdorff space if for every distinct two soft points $P_{e_1}^{x_1}, P_{e_2}^{x_2} \in SP(X,E)$ there exist soft open sets (F,E) and (G,E) such that $P_{e_1}^{x_1} \in (F,E)$, $P_{e_2}^{x_2} \in (G,E)$ and $(F,E) \cap (G,E) = \emptyset_s$.

Definition 2.9. ([7, 30]) Let (X,τ) be a soft topological space and $P_e^x \in SP(X,E)$.

(1) The subfamily $B_\tau(P_e^x)$ of $\mathcal{N}_\tau(P_e^x)$ is called a soft neighborhoods base of P_e^x if, for each $(F,E) \in \mathcal{N}_\tau(P_e^x)$, there exists $(U,E) \in B_\tau(P_e^x)$ such that $P_e^x \in (U,E) \subseteq (F,E)$.

(2) (X,τ) is called a soft first-countable space if there exists a countable soft neighborhood base for each soft point P_e^x.

Proposition 2.10. ([7]) *Let (X,τ) be a soft topological space and $P_e^x \in SP(X,E)$. If (X,τ) is soft first countable space then there exists a countable soft neighborhood base $B_\tau(P_e^x) = \{(U_n,E) : n \in \mathbb{N}\}$ such that $(U_{n+1},E) \subset (U_n,E)$ for each $n \in \mathbb{N}$.*

2.2. Density and statistical convergence

The statistical convergence depends on density of subsets of $\mathbb{N}$. The natural density of $K \subset \mathbb{N}$, which is the main tool for this convergence is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \le n : k \in K\}|$$

where $|\{k \le n : k \in K\}|$ denotes the number of elements of K that do not exceed n (see [16]).

Remark 2.11. A sequence x=(x_k) of complex numbers is said to be statistically convergent to some number L if $\delta(\{k \le n : |x_k - L| \ge \varepsilon\}) = 0$ for every positive number ε (see [15]). In the event that a sequence is statistically convergent, an infinite number of its terms may continue to exist outside the statistical limit for each $\varepsilon > 0$, as long as the set made up of these indices of terms has zero natural density. This is the essential feature that distinguishes statistical convergence from topological convergence. All topological convergent sequences are statistically convergent because a finite set has zero natural density.

Di Maio and Kočinac [21] extended the topological convergence to statistical convergence in topological spaces, just as usual convergence is extended to statistical convergence with the help of natural density.

Definition 2.12. ([21]) A sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space X is said to converge statistically (or shortly, s-converge) to $x \in X$, if for every neighborhood U of x, $\delta(\{n \in N : x_n \notin U\}) = 0$.

Different generalizations of natural density have been defined in the literature, such as α-density, weighted density, f-density, where f is an unlimited modular function. Thus, some generalizations of statistical convergence have been obtained with these density definitions. For example, further studies on statistical convergence in the context of these different density functions can be found in [8, 25–27], [2, 3, 19].

A modified notion of natural density, given by Balcerzak et al. [6], is defined by considering a weight function. The function $g : \mathbb{N} \to [0,\infty)$ is said to be a weight function whenever it satisfies the following properties:

$$\lim_{n\to\infty} g(n) = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{n}{g(n)} \ne 0.$$

Henceforth, the set of all weight functions that satisfy properties given above is denoted by $\mathcal{G}$.

Definition 2.13. ([6]) The density of a set $A \subseteq \mathbb{N}^+$ with respect to a weight function $g \in \mathcal{G}$ is defined by the following limit

$$\delta_g(A) := \lim_{n\to\infty} \frac{1}{g(n)} |\{k \in A : k \leq n\}|$$

whenever it exists. This type of density is abbreviated as g-weight density.

Clearly, the g-weight density coincides with the natural density when $g(n) = n$. Similarly, if we consider the weight function g as $g(n) = n^\alpha$ for $\alpha \in [0, 1]$ then the g-weight density reduces to the α-density which is defined in [11]. In [4], weighted statistical rough convergence in normed spaces was introduced and investigated from different perspectives by using the g-weight density.

Therefore, it is evident that g-weight density is a generalized form of the natural density and satisfies similar properties.For sets $A, B \subset \mathbb{N}$, the following properties hold: $\delta_g(A) = 0$ whenever A is a finite set, $\delta_g(\mathbb{N} \setminus A) = \delta_g(\mathbb{N}) - \delta_g(A)$, and $A \subseteq B$ implies $\delta_g(A) \leq \delta_g(B)$.

Using g-density, the following definition generalize the concept of statistical convergence in topological spaces, as presented by Di Maio and Kočinac [21].

Definition 2.14. A sequence (x_n) in a topological space X is said to weighted statistically g-convergent (or shortly, gs-converge) to $x \in X$, if for every neighborhood U of x, $\delta_g(\{n \in N : x_n \notin U\}) = 0$.

The convergence of soft point sequences according to soft topology was given in [14] as follows:

Definition 2.15. ([14]) Let (X, τ) be a soft topological space and $\left(P_{e_n}^{x_n}\right) \subset SP(X, E)$. The sequence $\left(P_{e_n}^{x_n}\right)$ convergent to $P_e^x \in SP(X, E)$ (denoted by $P_{e_n}^{x_n} \xrightarrow{\tau} P_e^x$) if for every $(F, A) \in \mathcal{N}_\tau(P_e^x)$ there exist an $n_0 \in \mathbb{N}$ such that $P_{e_n}^{x_n} \in (F, A)$ for all $n \geq n_0$.

Remark 2.16. Accordingly, many results in [21] and [14] can be reproduced using g-density. However our goal is the statistical soft convergence of the sequence of soft points via g-density.

3. Main results

The results obtained in this section will be presented in three parts.

3.1. Properties of weighted g-statistical soft convergence

In this section, we introduce the concept of weighted g-statistical convergence of sequence of soft points in soft topological spaces. Our primary goal is to investigate the fundamental properties of this type of convergence.

Definition 3.1. Let (X, τ) be a soft topological space. The sequence $\left(P_{e_n}^{x_n}\right) \subset SP(X, E)$ is said to be weighted g-statistical soft convergent *(briefly gss-convergent)* to $P_e^x \in SP(X, E)$ provided that for every $(F, A) \in \mathcal{N}(P_e^x)$

$$\delta_g\left(\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \notin (F, A)\right\}\right) = 0.$$

In this case, we write $P_{e_n}^{x_n} \xrightarrow{gss} P_e^x$.

The following simple proposition states that statistical convergence is a generalization of topological convergence.

Proposition 3.2. *If the soft point sequence $\left(P_{e_n}^{x_n}\right)$ is τ-convergence to $P_e^x \in SP(X, E)$ then it is gss-convergent to P_e^x for any $g \in G$.*

Proof. Suppose that $P_{e_n}^{x_n} \xrightarrow{\tau} P_e^x$ holds. Then, for every $(F,A) \in \mathcal{N}_\tau(P_e^x)$ there exists an $n_0 \in \mathbb{N}$ such that $P_{e_n}^{x_n} \in (F,A)$ for all $n > n_0$. Hence, we have the following inclusion

$$\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \notin (F,A)\right\} \subseteq \{1,2,3,...,n_0\}.$$

Therefore, the monotonicity property of g-density gives that the following inequality

$$\delta_g\left(\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \notin (F,A)\right\}\right) = \delta_g(\{1,2,3,...,n_0\}) = 0.$$

This implies that $P_{e_n}^{x_n} \xrightarrow{gss} P_e^x$. □

However, the converse of the above proposition may not generally be true.

Example 3.3. Consider the set $E = \{e\}$, the family $\mathfrak{B} = \{(F_\lambda, E) = \{e = \{\lambda\}\} \mid \lambda \in \mathbb{N}\}$ is a base of the soft topology $\tau = \left\{\bigcup_{\lambda \in J}(F_\lambda E)\right\}$ it generates over $\mathbb{N}$. If the sequence of soft points $(P_e^{x_n})$ is convergence to $P_e^x \in SP(X,E)$ then there exists an $n_0 \in \mathbb{N}$ and the sequence should be of the form

$$\left(P_e^{x_1}, P_e^{x_2}, ..., P_e^{x_{n_0}}, P_e^x, P_e^x, P_e^x, ...\right)$$

(see Example 3.25 in [14]). Define the sequence $\left(P_{e_n}^{x_n}\right) \subset SP(X,E)$ as follows:

$$P_e^{x_n} = \begin{cases} P_e^0, & n = k^2, k \in \mathbb{N} \\ P_e^1, & n \neq k^2, k \in \mathbb{N} \end{cases}.$$

Obviously, this sequence is not τ-convergent. Choose the weight function $g(n) = 2n$. Hence, since $\left|\left\{1,4,9,16,...,n^2,...\right\}\right| \leq \sqrt{n}$ holds, for all $(F,A) \in \mathcal{N}_\tau\left(P_e^1\right)$ we have

$$\delta_g\left(\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \notin (F,A)\right\}\right) = \lim_{n\to\infty} \frac{\left|\left\{1,4,9,16,..,n^2,...\right\}\right|}{g(n)} \leq \lim_{n\to\infty} \frac{\sqrt{n}}{g(n)} = 0$$

which means $P_{e_n}^{x_n} \xrightarrow{gss} P_e^1$.

On the other hand, statistical soft limit points do not need to be unique.

Example 3.4. For the parameter set $E = \{e\}$, $\tau = \{(F_a, \{e\}) : F_a(e) = [a,\infty), a \in \mathbb{R}\}$ is a soft topology on $\mathbb{R}$. Considering the sequence $\left(P_{e_n}^{x_n}\right) = \left\{P_e^1, P_e^2, ..., P_e^n, P_e^{n+1}, ...\right\}$, the statement $P_{e_n}^{x_n} \xrightarrow{gss} P_e^x$ holds for each $x \in \mathbb{R}$ where $g(n) = n$.

Theorem 3.5. *In soft Hausdorff spaces (X,τ), the limit of a gss-convergent sequence is uniquely determined.*

Proof. Let (X,τ) be a soft Hausdorff topological space, $\left(P_{e_n}^{x_n}\right) \subset SP(X,E)$. Suppose that $P_{e_n}^{x_n} \xrightarrow{gss} P_e^x$ and $P_{e_n}^{x_n} \xrightarrow{gss} P_f^y$. Since (X,τ) is a soft Hausdorff topological space, there exist a $(F,E) \in \mathcal{N}_\tau(P_e^x)$ and a $(G,E) \in \mathcal{N}_\tau\left(P_f^y\right)$ provided that $(F,E) \cap (G,E) = \emptyset_s$. Then, $\delta_g\left(\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \notin (F,E)\right\}\right) = 0$ and $\delta_g\left(\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \notin (G,E)\right\}\right) = 0$ hold. On the other hand,

$$\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \notin (F,E) \cap (G,E)\right\} = \left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \notin (F,E)\right\} \cup \left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \notin (G,E)\right\}$$

implies

$$\begin{aligned} 1 &= \delta_g(\mathbb{N}) = \delta_g\left(\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \notin (F,E)\right\} \cup \left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \notin (G,E)\right\}\right) \\ &\leq \delta_g\left(\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \notin (F,E)\right\}\right) + \delta_g\left(\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \notin (G,E)\right\}\right) = 0 \end{aligned}$$

which leads to a contradiction. □

Choose an arbitrary sequence $\left(P_{e_n}^{x_n}\right) \subset SP(X,E)$. For the natural number sequence $(n_k)_{k\in\mathbb{N}}$ such that $n_k \leq n_{k+1}$ for each $k \in \mathbb{N}$, $\left(P_{e_{n_k}}^{x_{n_k}}\right)_{k\in\mathbb{N}}$ is called a subsequence of $\left(P_{e_n}^{x_n}\right)$. Unlike the case in classical topological convergence, a subsequence of a statistically convergent sequence of soft points does not need to be statistically convergent.

Example 3.6. For the parameter set $E = \{e\}$, the collection $B = \left\{\left(F_{(a,b)}, E\right) = \{e = (a,b)\} \mid a, b \in \mathbb{R}\right\}$ is a base of the soft topology $\tau = \left\{\bigcup \left(F_{(a,b)}, E\right)\right\}$ on $\mathbb{R}$. The sequence $\left(P_{e_n}^{x_n}\right)$ where $P_e^{x_n} = \begin{cases} P_e^n, & if\ \ n = k^2, k \in \mathbb{N} \\ P_e^{\frac{1}{n}}, & otherwise \end{cases}$ is *gss*-converge to P_e^0. However, the subsequence $P_e^{x_{n_k}} = \left\{P_e^{x_{n_k}} \mid n_k = k^2, k \in \mathbb{N}\right\}$ of $\left(P_e^{x_{n_k}}\right)$ does not statistically convergent.

However, a type of subsequence in which statistical convergence is preserved can be given as follows. Remind that a subset A of $\mathbb{N}$ such that $\delta(A) = \delta_g(\mathbb{N})$ is called g-statistically dense.

Definition 3.7. A subsequence $\left(P_{e_{n_k}}^{x_{n_k}}\right)_{n_k\in A}$ of the sequence $\left(P_{e_n}^{x_n}\right) \subset SP(X,E)$ is g-statistical dense if there exist a dense set $A \subset \mathbb{N}$ of indices n_k such that $\{n_1 < n_2 < n_3 < ...\}$ and $\delta_g(A) = \delta_g(\mathbb{N})$.

Theorem 3.8. *A sequence $\left(P_{e_n}^{x_n}\right)$ is gss-convergent to P_e^x if and only if each g-statistical dense subsequence is gss-convergent to P_e^x.*

Proof. Suppose that $P_{e_n}^{x_n} \xrightarrow{gss} P_e^x$ and $\left(P_{e_{n_k}}^{x_{n_k}}\right)_{n_k\in A}$ is a statistical dense subsequence of $\left(P_{e_n}^{x_n}\right)$ which is not *gss*-convergent to P_e^x. In that case there is a neighborhood (F,A) of P_e^x such that $\delta_g\left(\left\{k \in \mathbb{N} \mid P_{e_{n_k}}^{x_{n_k}} \notin (F,A)\right\}\right) > 0$. As $\delta_g(A) = \delta_g(\mathbb{N})$ we have

$$0 < \delta_g\left(\left\{k \in \mathbb{N} \mid P_{e_{n_k}}^{x_{n_k}} \notin (F,A)\right\}\right) \leq \delta_g\left(\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \notin (F,A)\right\}\right).$$

However this is a contradiction to $P_{e_n}^{x_n} \xrightarrow{gss} P_e^x$. Conversely, since every sequence is a dense subsequence of itself it clearly is that $P_{e_n}^{x_n} \xrightarrow{gss} P_e^x$. □

Remark 3.9. It should be noted that in the last theorem, it is possible for a sequence to be statistically convergent if all dense sequences are statistically convergent to the same point. However, if there are dense sequences that statistically converge to different points, the sequence may not be statistically convergent. Also, the existence of a *gss*-convergent dense subsequence is not sufficient for the sequence to be statistical convergent.

Its well known, the statistical convergence of real sequences is equivalent to the existence of a topologically convergent dense subsequence (see [31]). However, in topological spaces, topological (statistical) convergence of any subsequence do not imply topological (statistical) convergence of the sequence. For this reason, Di Maio and Kočinac [21] gave a new definition called s^*-convergence. Also, we give a parallel definition for soft spaces in terms g-density as follows.

Definition 3.10. The sequence $\left(P_{e_n}^{x_n}\right) \subset SP(X,E)$ is said to be gss^*-convergent to $P_e^x \in X$ if there is $A \subseteq \mathbb{N}$ with $\delta_g(A) = \delta_g(\mathbb{N})$ such that $\left\{P_{e_m}^{x_m}\right\}_{m\in A} \xrightarrow{\tau} P_e^x$. In this case we write $P_{e_n}^{x_n} \xrightarrow{gss^*} P_e^x$.

Theorem 3.11. *The convergence $P_{e_n}^{x_n} \xrightarrow{gss^*} P_e^x$ implies $P_{e_n}^{x_n} \xrightarrow{gss} P_e^x$.*

Proof. Let $(F,E) \in \mathcal{N}(P_e^x)$. Since $P_{e_n}^{x_n} \xrightarrow{gss^*} P_e^x$, there are $A \subseteq \mathbb{N}$ with $\delta_g(A) = \delta_g(\mathbb{N})$ and $n_0 \in \mathbb{N}$ such that $P_{e_n}^{x_n} \in (F,E)$ for $n \geq n_0$ and $n \in A$. Then, we have

$$\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \notin (F,E)\right\} \subseteq \{1, 2, .., n_0\} \cup (\mathbb{N} \setminus A)$$

and thus

$$\begin{aligned}\delta_g(\{1,2,..,n_0\}\cup(\mathbb{N}\setminus A)) &= \delta_g(\{1,2,..,n_0\})+\delta_g((\mathbb{N}\setminus A))\\ &= 0+\left(\delta_g(\mathbb{N})-\delta_g(A)\right)\\ &= 0\end{aligned}$$

it follows $P_{e_n}^{x_n} \xrightarrow{gss} P_e^x$. □

The following example indicates that the converse of the theorem above does not hold in general.

Example 3.12. If $X = M\cup\{0,1\}$ where M is an uncountable set and $E = \{e\}$ then $\tau = \{(F,E)\subset X \mid (F,E)^c \text{ is finite}\}\cup\{\emptyset_s\}$ defines a soft topology on X. It is easy to see that the sequence $P_e^{x_n} = \begin{cases} P_e^1, & \text{for infinitely many } n\\ P_e^0, & \text{otherwise}\end{cases}$ is gss-convergent to P_e^0, but the sequence is not gss^*-convergent.

Theorem 3.13. *Let (X,τ) be a first countable soft topological space and $\left(P_{e_n}^{x_n}\right)\subset SP(X,E)$. If $\left(P_{e_n}^{x_n}\right)$ is gss-convergent to $P_e^x\in SP(X,E)$ then $P_{e_n}^{x_n}\xrightarrow{gss^*}P_e^x$.*

Proof. Suppose that $\left(P_{e_n}^{x_n}\right)$ is gss-convergent to $P_e^x \in SP(X,E)$. Since (X,τ) is a first countable soft topological space, there exists a countable decreasing soft open neighborhood base (U_n,E) at P_e^x. Let us define the following sets for each $j\in\mathbb{N}$:

$$K_j = \{n\in\mathbb{N} : x_n \in (U_n,E)\} \text{ and } K_j(n) = \left\{k\in K_j : k\le n\right\}.$$

Since the convergence $P_{e_n}^{x_n}\xrightarrow{gss}P_e^x$ holds, we have $\delta_g\left(K_j\right)=\delta_g(\mathbb{N})$ and $K_1\supset K_2\supset\cdots\supset K_j\supset K_{j+1}\supset\cdots$ for every $j\in\mathbb{N}$. Choose a strictly increasing sequence (t_j) consisting of positive real numbers such that $\lim_{j\to\infty} t_j = \delta_g(\mathbb{N})$. By the definition of K_1, we can pick any element $k_1\in K_1$ such that $\frac{|K_1(n)|}{g(n)} > t_1$ holds for all $n>k_1$. Also, we can pick an element $k_2\in K_2$ such that $k_2>k_1$ and $\frac{|K_2(n)|}{g(n)}>t_2$ holds for all $n\ge k_2$. Continuing this iterative procedure leads us to a sequence (k_j) of natural numbers such that $k_j\in K_j$ for all $j=1,2,\cdots$, and also we have $\frac{|K_j(n)|}{g(n)}>t_j$ for all $n\ge k_j$.

Let us construct the set $K\subseteq\mathbb{N}$ as follows: every natural number within $[1,k_1]$ is included in K, and every natural number within $\left[k_j,k_{j+1}\right]\cap K_j$ for each $j=1,2,\cdots$ is in the set K. Let $K=\{n_1<n_2<...\}$. Consequently, $\frac{|K(n)|}{g(n)}\ge\frac{|K_j(n)|}{g(n)}>t_j$ holds for all $k_j\le n<k_{j+1}$. When we take the limit as n approaches infinity in the last inequality, we attain the result that $\delta_g(K)=\delta_g(\mathbb{N})$. Let (U,E) be a neighborhood of P_e^x. Then, there exists a natural number $j_0\in\mathbb{N}$ such that $\left(U_{j_0},E\right)\subset(U,E)$. For each $n\in K\cap\left[k_j,\infty\right)$, Then, there exists a unique $j\ge j_0$ such that $b_j\le n<b_{j+1}$ and so, by the definition of K, we obtain $n\in K_j$. Thus, we have $P_{e_n}^{x_n}\in\left(U_j,E\right)\subset\left(U_{j_0},E\right)\subset(U,E)$ for each $n\in K\cap\left[k_j,\infty\right)$. This gives that $\left(P_{e_n}^{x_n}\right)$ is τ-convergent to P_e^x on the g-dense set K, i.e. $\left(P_{e_n}^{x_n}\right)$ is gss^*-convergent to P_e^x. □

Theorem 3.14. *Let (X,τ) be a soft topological space, $\left(P_{e_n}^{x_n}\right)\subset SP(X,E)$, and $g,h\in\mathcal{G}$. Then, the following assertions hold.*

(1) *Suppose that there exists $M>0$ and $k_0\in\mathbb{N}$ such that $\frac{g(n)}{h(n)}\le M$ holds for all $n\ge k_0$. In that case,*

$$P_{e_n}^{x_n}\xrightarrow{gss}P_e^x \quad\Longrightarrow\quad P_{e_n}^{x_n}\xrightarrow{hss}P_e^x.$$

(2) *Suppose that there exists $m>0$ and $k_0\in\mathbb{N}$ such that $m\le\frac{g(n)}{h(n)}$ holds for all $n\ge k_0$. In that case,*

$$P_{e_n}^{x_n}\xrightarrow{hss}P_e^x \quad\Longrightarrow\quad P_{e_n}^{x_n}\xrightarrow{gss}P_e^x.$$

(3) *Suppose that there exists $m, M > 0$ and $k_0 \in \mathbb{N}$ such that $m \leq \frac{g(n)}{h(n)} \leq M$ holds for all $n \geq k_0$. In that case,*

$$P_{e_n}^{x_n} \xrightarrow{gss} P_e^x \iff P_{e_n}^{x_n} \xrightarrow{hss} P_e^x.$$

Proof. If $\left(P_{e_n}^{x_n}\right)$ is gss-convergent to P_e^x, then for every $(F, A) \in \mathcal{N}(P_e^x)$

$$\delta_g\left(\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \notin (F, A)\right\}\right) = 0 \tag{1}$$

holds. On the other hand, from the existence of a number $M > 0$ and $k_0 \in \mathbb{N}$ such that $\frac{g(n)}{h(n)} \leq M$ holds for all $n \geq k_0$, we have

$$\frac{1}{h(n)}\left|\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \notin (F, A)\right\}\right| \leq M \frac{1}{g(n)}\left(\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \notin (F, A)\right\}\right).$$

Thus, by taking the limit as $n \to \infty$, the equality (1) implies that $\left(P_{e_n}^{x_n}\right)$ is hss-convergent to P_e^x. The proof of (2) is similar and (3) is a consequence of (1) and (2). □

It is natural to ask whether the concept of statistical convergence is preserved under continuous functions. Kharal and Ahmed introduced the concept of soft mapping using a subset of the parameter set [20]. The continuity of soft mappings defined on the parameter set E was provided by Hazra et al. [18]. Initially, let us recall the continuous functions that are defined between soft topological spaces.

Definition 3.15. ([18, 20]) Let (X, τ_1, E_1) and (Y, τ_2, E_2) be two soft topological spaces, and let $f : X \to Y$ and $p : E_1 \to E_2$ be two mappings. Then, the mapping is defined as

$$\varphi_{(f,p)} : SS(X, E_1) \to SS(Y, E_2), \varphi_{(f,p)}(F, E_1) = (F^{\varphi}, E_2)$$

where for each $e' \in E_2$,

$$F^{\varphi}(e') = \begin{cases} f\left(\bigcup\limits_{e \in p^{-1}(e')} F(e)\right), & p^{-1}\{e'\} \neq \varnothing \\ \varnothing, & p^{-1}\{e'\} = \varnothing \end{cases}$$

is called a soft mapping. In that case, the inverse of $\varphi_{(f,p)}$ is also a soft mapping defined by

$$\varphi_{(f,p)}^{-1} : SS(Y, E_2) \to SS(X, E_1), \varphi_{(f,p)}^{-1}(F, E_2) = \left(F^{\varphi^{-1}}, E_1\right)$$

where for each $e \in E_1$,

$$F^{\varphi^{-1}}(e) = f^{-1}\{F(p(e))\}.$$

Moreover, $\varphi_{(f,p)}$ is injective (or surjective) if f and p are both injective (or surjective) and $\varphi_{(f,p)} : SS(X, E_1) \to SS(Y, E_2)$ is said to be soft continuous if $\varphi_{(f,p)}^{-1}(F, E_2) \in \tau_1$ for every $(F, E_2) \in \tau_2$.

Lemma 3.16. *Let (X, τ_1, E_1) and (Y, τ_2, E_2) be two soft topological spaces, $f : X \to Y$ and $p : E_1 \to E_2$ be two mappings and let $P_e^x \in SP(X, E_1)$. If p injective then $\varphi_{(f,p)}(P_e^x) = P_{p(e)}^{f(x)} \in SP(Y, E_2)$.*

Proof. If p is injective then there exists only an element $e' \in E_2$ such that $p(e) = e'$. Then, from the definition of soft mapping, we have

$$F^{\varphi}(e') = \begin{cases} f(F(e)), & p^{-1}\{e'\} = e \\ \varnothing, & p^{-1}\{e'\} \neq e \end{cases} = \begin{cases} \{f(x)\}, & p^{-1}\{e'\} = e \\ \varnothing, & p^{-1}\{e'\} \neq e \end{cases}.$$

This implies that $(F^{\varphi}, E_2) = P_{p(e)}^{f(x)} \in SP(Y, E_2)$. □

Next theorem states that *gss*-convergence is preserved under an injective soft continuous mapping.

Theorem 3.17. *Let (X,τ_1,E_1) and (Y,τ_2,E_2) be two soft topological spaces, $\varphi_{(f,p)}:(X,\tau_1)\to(Y,\tau_2)$ an injective soft continuous mapping. If a sequence $\left(P_{e_n}^{x_n}\right)\in SP(X,E_1)$ is gss-convergent to P_e^x, thus $\left(\varphi_{(f,p)}\left(P_{e_n}^{x_n}\right)\right)$ is gss-convergent to $\varphi_{(f,p)}\left(P_e^x\right)\in SP(Y,E_2)$.*

Proof. Consider $(F,E_2)\in N_{\tau_2}\left(\varphi_{(f,p)}\left(P_e^x\right)\right)$. Soft continuity of $\varphi_{(f,p)}$ implies $\varphi_{(f,p)}^{-1}\{(F,E_2)\}\in\tau_1$. Hence, since $P_e^x\in\varphi_{(f,p)}^{-1}\{(F,E_2)\}\in\mathcal{N}_{\tau_1}\left(P_e^x\right)$, we have

$$\delta_g\left(\left\{n\in\mathbb{N}\mid P_{e_n}^{x_n}\notin\varphi_{(f,p)}^{-1}\{(F,E_2)\}\right\}\right)=0.$$

Recall that the inclusion $(F,E_2)\subseteq\varphi_{(f,p)}^{-1}\left(\varphi_{(f,p)}\{(F,E_2)\}\right)$ holds for every (F,E_2) (see [18]). Then, from the equality

$$\left\{n\in\mathbb{N}\mid\varphi_{(f,p)}\left(P_{e_n}^{x_n}\right)\notin(F,E_2)\right\}\subseteq\left\{n\in\mathbb{N}\mid P_{e_n}^{x_n}\notin\varphi_{(f,p)}^{-1}\{(F,E_2)\}\right\}$$

and the monotonicity of g-density, we obtain

$$\delta_g\left(\left\{n\in\mathbb{N}\mid\varphi_{(f,p)}\left(P_{e_n}^{x_n}\right)\notin(F,E_2)\right\}\right)=0.$$

Consequently, since (F,E_2) is choosen arbitrarily, it follows that $\varphi_{(f,p)}\left(P_{e_n}^{x_n}\right)\xrightarrow{gss}\varphi_{(f,p)}\left(P_e^x\right)$ in (Y,τ_2,E_2). □

3.2. *On the topologies induced from soft topology*

As shown in [32], each soft topology defines a topology in the classical sense over the universe set for each parameter. To be more clear, if (X,τ) is a soft topological space then, the collection

$$\tau_e=\{F(e):(F,E)\in\tau\}$$

for every $e\in E$, defines a topology on X (see proposition 5 in [32]). However, the converse of this fact does not hold in general (see Example 2 in [32]).

In this section, the connection between τ-convergence and τ_e-convergence is initially covered. Subsequently, a similar analysis of this relationship is provided within the framework of statistical convergence.

Theorem 3.18. *Let P_e^x be arbitrary element in $SP(X,E)$ and the condition $F(e_n)\subseteq F(e)$ holds for every $(F,E)\in\mathcal{N}_\tau\left(P_e^x\right)$ and every $n\in\mathbb{N}$. In that case $P_{e_n}^{x_n}\xrightarrow{\tau}P_e^x$ implies $x_n\xrightarrow{\tau_e}x$.*

Proof. Choose a $F(e)\in\mathcal{N}_{\tau_e}(x)$. Suppose that $P_{e_n}^{x_n}\xrightarrow{\tau}P_e^x$ holds. Then, for every $(F,E)\in\mathcal{N}_\tau\left(P_e^x\right)$ there is $n_0\in\mathbb{N}$ such that $P_{e_n}^{x_n}\in(F,E)$ for every $n\geq n_0$. It means $x_n\in F(e_n)$ for every $n\geq n_0$. From the assumption $F(e_n)\subseteq F(e)$, we have $x_n\in F(e)$ for every $n\geq n_0$. That is $x_n\xrightarrow{\tau_e}x$. □

The following example illustrates that the theorem's assumption must be satisfied.

Example 3.19. For $X=\{x_1,x_2,x_3\}$ and $E=\{e_1,e_2,e_3\}$, $\tau=\{X_s,\varnothing_s,(F_1,E),(F_2,E)\}$ is a soft topology where

$$(F_1,E)=\{F_1(e_1)=\varnothing,F_1(e_2)=\{x_2\},F_1(e_3)=\{x_3\}\}$$

and

$$(F_2,E)=\{F_2(e_1)=\{x_1\},F_2(e_2)=\{x_2,x_3\},F_2(e_3)=X\}.$$

Define the sequences $(x_n) = \{x_1, x_2, x_3, x_2, x_3, ...\}$ and $\left(P_{e_n}^{x_n}\right) = \left\{P_{e_1}^{x_1}, P_{e_2}^{x_2}, P_{e_3}^{x_3}, P_{e_2}^{x_2}, P_{e_3}^{x_3}, ...\right\}$ where

$$P_{e_n}^{x_n} = \begin{cases} P_{e_1}^{x_1}, & n = 1 \\ P_{e_2}^{x_2}, & n = 2k \\ P_{e_3}^{x_3}, & n = 2k+1 \end{cases} \quad \text{and} \quad x_n = \begin{cases} x_1, & n = 1 \\ x_2, & n = 2k \\ x_3, & n = 2k+1 \end{cases}.$$

However, the sequence (x_n) is not convergent to x for the topology τ_{e_3} whereas the sequence $\left(P_{e_n}^{x_n}\right)_{n\in\mathbb{N}}$ converges to $P_{e_3}^{x}$.

Theorem 3.20. *Let P_e^x be an arbitrary element in $SP(X,E)$ and the condition $F(e) \subseteq F(e_n)$ holds for every $(F,E) \in \mathcal{N}_\tau(P_e^x)$ and for every $n \in \mathbb{N}$. In that case $x_n \xrightarrow{\tau_e} x$ implies $P_{e_n}^{x_n} \xrightarrow{\tau} P_e^x$.*

Proof. Choose a $(F,E) \in \mathcal{N}_\tau(P_e^x)$. Suppose that $x_n \xrightarrow{\tau_e} x$. Then for every neighborhood $F(e)$ of x there is $n_0 \in \mathbb{N}$ such that $x_n \in F(e)$ for every $n \geq n_0$. Based on the assumption $F(e) \subseteq F(e_n)$, it follows that $P_{e_n}^{x_n} \in (F,E)$ for every $n \geq n_0$, indicating that $P_{e_n}^{x_n} \xrightarrow{\tau} P_e^x$. □

Similarly, the following example illustrates that the theorem's assumption must be satisfied.

Example 3.21. For $X = \{x_1, x_2, x_3\}$ and $E = \{e_1, e_2, e_3\}$, $\tau = \{X_s, \varnothing_s, (F_1,E), (F_2,E), (F_3,E), (F_4,E)\}$ is a soft topological space on X where

$$\begin{aligned} (F_1,E) &= \{F_1(e_1) = X, F_1(e_2) = \{x_3\}, F_1(e_3) = X\}, \\ (F_2,E) &= \{F_2(e_1) = \varnothing, F_2(e_2) = \{x_2,x_3\}, F_2(e_3) = X\}, \\ (F_3,E) &= \{F_3(e_1) = \varnothing, F_3(e_2) = \{x_3\}, F_3(e_3) = X\}, \\ (F_4,E) &= \{F_4(e_1) = X, F_4(e_2) = \{x_2,x_3\}, F_4(e_3) = X\}. \end{aligned}$$

Consider the sequences $(x_n) = \{x_1, x_2, x_3, x_2, x_3, ...\}$ and $\left(P_{e_n}^{x_n}\right) = \left\{P_{e_1}^{x_1}, P_{e_2}^{x_2}, P_{e_3}^{x_3}, P_{e_2}^{x_2}, P_{e_3}^{x_3}, ...\right\}$ as described in the previous example. In that case, although the sequence (x_n) converges to x_1 for the topology τ_{e_1}, the sequence $\left(P_{e_n}^{x_n}\right)_{n\in\mathbb{N}}$ is not convergent to $P_{e_1}^{x_1}$.

As a consequence, the following result is clear.

Corollary 3.22. *For any element $P_e^x \in SP(X,E)$, if the condition $F(e_n) = F(e)$ holds for every $(F,E) \in \mathcal{N}_\tau(P_e^x)$ and for every $n \in \mathbb{N}$ then*

$$x_n \xrightarrow{\tau_e} x \iff P_{e_n}^{x_n} \xrightarrow{\tau} P_e^x.$$

Proof. Suppose that for at least one $n \in \mathbb{N}, F(e_n) \neq F(e)$. It is a contradiction from Example 3.19 or Example 3.21. Converse direction is clear from Theorem 3.18 and Theorem 3.20. □

Theorem 3.23. *Let (X,τ) be first countable soft space. If for every $(F,E) \in \mathcal{N}_\tau(P_e^x)$, the condition $F(e) \subseteq F(e_n)$ holds for every $n \in \mathbb{N}$ then $x_n \xrightarrow{gs-\tau_e} x$ implies $P_{e_n}^{x_n} \xrightarrow{gss-\tau} P_e^x$.*

Proof. Suppose that $x_n \xrightarrow{gs-\tau_e} x$ holds. From Theorem 3.13, (x_n) is gs^*-*converges* to x on τ_e, i.e. there is $A \subseteq \mathbb{N}$ with $\delta_g(A) = 1$ such that $(x_{n_k})_{n_k\in A} \xrightarrow{\tau_e} x$. Thus, by using Theorem 3.20, we obtain that $\left(P_{e_{n_k}}^{x_{n_k}}\right)$ is τ-convergent to P_e^x which means that $\left(P_{e_n}^{x_n}\right)$ is gss^*-convergent to P_e^x. Thus, by Theorem 3.11, we obtain that $\left(P_{e_n}^{x_n}\right)$ is gss-convergent to P_e^x. □

Theorem 3.24. *Let (X,τ) be first countable soft space. If for every $(F,E) \in \mathcal{N}_\tau(P_e^x)$, the condition $F(e_n) \subseteq F(e)$ holds for every $n \in \mathbb{N}$ then $P_{e_n}^{x_n} \xrightarrow{gss-\tau} P_e^x$ implies $x_n \xrightarrow{gs-\tau_e} x$.*

Proof. Suppose that $P_{e_n}^{x_n} \xrightarrow{gss-\tau} P_e^x$ holds. From Theorem 3.13, $\left(P_{e_n}^{x_n}\right)$ is gss^*-*converges* to P_e^x on τ, i.e. there is $A \subseteq \mathbb{N}$ with $\delta_g(A) = 1$ such that $\left(P_{e_{n_k}}^{x_{n_k}}\right)_{n_k\in A} \xrightarrow{\tau_e} P_e^x$. Thus, by using Theorem 3.18, we obtain that $(x_{n_k})_{n_k\in A}$ is τ_e-convergent to x which means that (x_n) is gs^*-convergent to x. Again, by Theorem 3.11, we obtain that (x_n) is gs-convergent to x. □

3.3. Some point sets related weighted statistical convergence

This section provides definitions for the g-statistical limit point and g-statistical cluster point of the sequence of soft points related to statistical convergence together with some of the most fundamental results. We will begin by defining the term upper weighted g-density which will be used in these definitions.

The upper weighted g-density of the set $A \subseteq \mathbb{N}$ is defined by

$$\overline{\delta_g}(A) = \lim_{n\to\infty} \frac{|\{k \in A \mid k \leq n\}|}{g(n)}$$

whenever the limit exists.

Definition 3.25. A subsequence $\left(P_{e_{n_k}}^{x_{n_k}}\right)_{n_k \in A}$ of the sequence $\left(P_{e_n}^{x_n}\right) \subset SP(X,E)$ is g-thin if there exists a dense set $A \subset \mathbb{N}$ of indices n_k such that $\{n_1 < n_2 < n_3 < \dots\}$ and $\delta_g(A) = 0$.

Similarly, a subsequence $\left(P_{e_{n_k}}^{x_{n_k}}\right)_{n_k \in A}$ of the sequence $\left(P_{e_n}^{x_n}\right) \subset SP(X,E)$ is g-nonthin if there exists a set $A \subset \mathbb{N}$ of indices n_k such that $\{n_1 < n_2 < n_3 < ...\}$ and $\delta_g(A) \neq 0$ which means that $\delta_g(A) > 0$ or $\delta_g(A)$ does not exist. Also, $\delta_g(A) \neq 0$ is equivalent to upper weighted g-density $\overline{\delta_g}(A)$ is positive.

Definition 3.26. Let $\left(P_{e_n}^{x_n}\right)$ be a sequence of soft points in the soft topological space (X,τ) and $P_e^x \in SP(X,E)$.

(1) $P_e^x \in SP(X,E)$ is said to be a g-statistical limit point of $\left(P_{e_n}^{x_n}\right)$ if there is a nonthin subsequence $\left(P_{e_{n_k}}^{x_{n_k}}\right)_{n_k \in A}$ such that $P_{e_{n_k}}^{x_{n_k}} \xrightarrow{\tau} P_e^x$, i.e. The set of all g-statistical limit points of $\left(P_{e_n}^{x_n}\right)$ is denoted by $\Lambda_g\left(P_{e_n}^{x_n}\right)$.

(2) $P_e^x \in SP(X,E)$ is said to be a g-statistical cluster point of $\left(P_{e_n}^{x_n}\right)$ if, for each $(F,E) \in \mathcal{N}_\tau(P_e^x)$, $\delta_g\left(\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \in (F,E)\right\}\right) \neq 0$ or equivalently the assertion

$$\overline{\delta_g}\left(\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \in (F,E)\right\}\right) > 0$$

holds. The set of all g-statistical cluster points of $\left(P_{e_n}^{x_n}\right)$ is denoted by $\Theta_g\left(P_{e_n}^{x_n}\right)$.

Theorem 3.27. *For any sequence $\left(P_{e_n}^{x_n}\right) \subset SP(X,E)$, the inclusion $\Lambda_g\left(P_{e_n}^{x_n}\right) \subset \Theta_g\left(P_{e_n}^{x_n}\right)$ holds.*

Proof. Suppose that $P_e^x \in \Lambda_g\left(P_{e_n}^{x_n}\right)$ and $\left(P_{e_{n_k}}^{x_{n_k}}\right)$ is a nonthin subsequence of $\left(P_{e_n}^{x_n}\right)$ such that $P_{e_{n_k}}^{x_{n_k}} \xrightarrow{\tau} P_e^x$ and $\overline{\delta}_g(\{n_k \mid k \in \mathbb{N}\}) = \alpha > 0$. Then, for $(F,E) \in \mathcal{N}_\tau(P_e^x)$, there exists $k_{i_0} \in \mathbb{N}$ such that $P_{e_{n_k}}^{x_{n_k}} \in (F,E)$ for every $k > k_{i_0}$. Thus, we have

$$\{n_k \mid k \in \mathbb{N}\} \setminus \left\{n_{k_1}, n_{k_2}, ..., n_{k_{i_0}}\right\} \subseteq \left\{n \in\mid P_{e_n}^{x_n} \in (F,E)\right\}.$$

Accordingly, the inequalities

$$\begin{aligned}\overline{\delta}_g\left(\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \in (F,E)\right\}\right) &\geq \overline{\delta}_g(\{n_k \mid k \in \mathbb{N}\}) - \overline{\delta}_g(\{k_1, k_2, ..., k_{i_0}\}) \\ &\geq \alpha - 0 \\ &> 0\end{aligned}$$

hold. This indicates $P_e^x \in \Theta\left(P_{e_n}^{x_n}\right)$. □

The union of any collection of soft points can be considered a soft set and every soft set can be expressed as the union of all soft points belonging to it (see Proposition 3.5 in [13]). Therefore, $\Theta\left(P_{e_n}^{x_n}\right)$ is a soft set if considered the form $\Theta\left(P_{e_n}^{x_n}\right) = \bigcup\left\{P_e^x \mid P_e^x \in \Theta\left(P_{e_n}^{x_n}\right)\right\}$.

Theorem 3.28. *For any sequence* $\left(P_{e_n}^{x_n}\right) \subset SP(X,E)$*, the set* $\Theta_g\left(P_{e_n}^{x_n}\right)$ *is* τ*-closed.*

Proof. Choose an arbitrary element $P_e^x \in \overline{\Theta_g\left(P_{e_n}^{x_n}\right)}$ and $(F,E) \in \mathcal{N}_\tau\left(P_e^x\right)$. Then, there is an element $P_a^y \in SP(X,E)$ in $(F,E) \cap \Theta_g\left(P_{e_n}^{x_n}\right)$. Considering $(G,E) \in \mathcal{N}_\tau\left(P_a^y\right)$ such that $(G,E) \subset (F,E)$, since $P_a^y \in (G,E) \cap \Theta_g\left(P_{e_n}^{x_n}\right)$ we have $\overline{\delta_g}\left(\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \in (G,E)\right\}\right) > 0$. Hence, from the inclusion

$$\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \in (G,E)\right\} \subseteq \left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \in (F,E)\right\}$$

and the monotonicity of upper g-density we obtain

$$0 < \overline{\delta_g}\left(\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \in (G,E)\right\}\right) \leq \overline{\delta_g}\left(\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \in (F,E)\right\}\right).$$

It follows that $P_e^x \in \Theta_g\left(P_{e_n}^{x_n}\right)$. □

Theorem 3.29. *Let* $\left(P_{e_n}^{x_n}\right), \left(P_{e_n}^{y_n}\right) \subset SP(X,E)$ *be two sequences such that* $\delta_g\left(\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \neq P_{e_n}^{y_n}\right\}\right) = 0$*. In that case, the equalities*

$$\Theta_g\left(P_{e_n}^{x_n}\right) = \Theta_g\left(P_{e_n}^{y_n}\right) \quad \text{and} \quad \Lambda_g\left(P_{e_n}^{x_n}\right) = \Lambda_g\left(P_{e_n}^{y_n}\right)$$

hold.

Proof. Assume that $P_e^z \in \Theta_g\left(P_{e_n}^{x_n}\right)$ and $(F,E) \in \mathcal{N}_\tau\left(P_e^z\right)$. Then, $\overline{\delta_g}\left(\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \in (F,E)\right\}\right) > 0$ holds. On the other hand, the inclusion

$$\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \in (F,E)\right\} \setminus \left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \neq P_{e_n}^{y_n}\right\} \subseteq \left\{n \in \mathbb{N} \mid P_{e_n}^{y_n} \in (F,E)\right\}$$

holds and so

$$0 < \overline{\delta_g}\left(\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \in (F,E)\right\}\right) - \overline{\delta_g}\left(\left\{n \in \mathbb{N} \mid P_{e_n}^{x_n} \neq P_{e_n}^{y_n}\right\}\right) \leq \overline{\delta_g}\left(\left\{n \in \mathbb{N} \mid P_{e_n}^{y_n} \in (F,E)\right\}\right).$$

Consequently, we obtain $\overline{\delta_g}\left(\left\{n \in \mathbb{N} \mid P_{e_n}^{y_n} \in (F,E)\right\}\right) > 0$ which means $P_e^z \in \Theta_g\left(P_{e_n}^{y_n}\right)$. By symmetry, we can also prove $\Theta_g\left(P_{e_n}^{y_n}\right) \subseteq \Theta_g\left(P_{e_n}^{x_n}\right)$.

Now, choose $P_e^z \in \Lambda_g\left(P_{e_n}^{x_n}\right)$. Then, there exists a set $A = \{n_1 < n_2 < ... < n_k < ...\} \subseteq \mathbb{N}$ such that $\overline{\delta_g}(A) > 0$ and $P_{e_{n_k}}^{x_{n_k}} \to P_e^z$. Hence, for each $(F,E) \in N\left(P_e^z\right)$ there exists $k_{i_0} \in \mathbb{N}$ such that $P_{e_{n_k}}^{x_{n_k}} \in (F,E)$ for every $k > k_0$. Similarly, from the inclusion

$$\left\{n_k \in \mathbb{N} \mid P_{e_{n_k}}^{x_{n_k}} \in (F,E)\right\} \setminus \left\{n_k \in \mathbb{N} \mid P_{e_{n_k}}^{x_{n_k}} \neq P_{e_{n_k}}^{y_{n_k}}\right\} \subseteq \left\{n_k \in \mathbb{N} \mid P_{e_{n_k}}^{y_{n_k}} \in (F,E)\right\}$$

and properties of the upper g-density, we have

$$0 \neq \overline{\delta_g}\left(\left\{k \in \mathbb{N} \mid P_{e_{n_k}}^{x_{n_k}} \in (F,E)\right\}\right) - \overline{\delta_g}\left(\left\{k \in \mathbb{N} \mid P_{e_{n_k}}^{x_{n_k}} \neq P_{e_{n_k}}^{y_{n_k}}\right\}\right) \leq \overline{\delta_g}\left(\left\{k \in \mathbb{N} \mid P_{e_{n_k}}^{y_{n_k}} \in (F,E)\right\}\right).$$

It follows that $P_e^z \in \Lambda_g\left(P_{e_n}^{y_n}\right)$. Again, by symmetry, we can also prove $\Lambda_g\left(P_{e_n}^{y_n}\right) \subseteq \Lambda_g\left(P_{e_n}^{x_n}\right)$. □

4. Conclusion

This study establishes a connection between the concepts of soft sets and statistical convergence, and defines the statistical convergence of soft point sequences, that correspond to classical point sequences. In this definition, a more general concept of weighted density is used instead of the natural density. The relationship with topological convergence is demonstrated, and some fundamental results concerning convergence have been presented. As is well known, soft topology induces a classical topology. In this regard, the relationships between soft topology and the classical topology derived from it are examined in the context of statistical convergence. Additionally, some point sets related to convergence are defined, and their essential characteristics are outlined.

This work contributes to the literature by introducing a new form of convergence in soft topological spaces, that have not been previously considered, and lays the foundation for further investigation into the application of statistical convergence in various disciplines, such as economics, engineering, and environmental science, where soft sets are already used. Future research may extend the current framework using alternative density functions and evaluate the applicability of these newly defined convergence concepts to practical problems characterized by uncertainty.

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