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Application of best proximity point(pair) theorem and measure of noncompactness to a system of integro differential equations in Banach space

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Abstract. This article explores the existence of an optimal solution for our proposed system of integro differential equations in Banach space by generalizing the best proximity point (pair) theorem and utilizing a new contraction operator. With the aid of an appropriate example, the applicability of our findings has also been demonstrated.

1. Introduction

The measure of noncompactness (MNC) and the best proximity point theorem are widely used to solve various kinds of integral equations. The first best proximity point theorem was proven by Ky Fan. For a nonempty subset Y of a normed linear space (NLS) E, we defined a map $T : Y \to E$. If the distance between $d \in Y$ and $T(d)$ is as small as possible, then $d \in Y$ represents the best approximate point of T in E.

The paper will follow the following format: We begin by reviewing some basic terms and concepts related to best proximity theory. The best proximity point theorem for cyclic and noncyclic contractive operators is then established. Next, we illustrate their particular cases. Lastly, we utilize our results to explore the optimum solutions of a system of Integro differential equations.

Many researchers considered using the concept of MNC, which was first introduced by Kuratowski and then further developed by Hausdorff, to obtain significant extensions of the theory of compact operators. The key ability is to apply MNC to check if a mapping satisfied some significant inequalities. We thus include a brief history that can help the reader in understanding our problem and motivation. We revisit the fundamental fixed point problem in a Banach space Z, taking some regularity assumptions from Schauder [2].

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In [14] the authors considered an infinite system of three point boundary value problem of p-Laplacian operator for the existence of solution in a new sequence space related to the tempered sequence space l_p^α, $p \geq 1$ with the help of Hausdorff *MNC* and an example is provided to illustrate their new results in tempered sequence spaces. Using *MNC*, the authors of [16] first addressed the *BPP* results. Subsequently, they applied these results to investigate whether a system of second-order differential equations has optimal solutions. In [19], the aim of the authors is to formulate theoretical outcomes for the qualitative analysis of fractional-order integro differential equations with integral type conditions. The idea is derived using fixed point theory and fractional calculus. For application and numerical verification purposes, an example is also investigated. In [20] the authors using Meir-Keeler condensing operators to study the criteria under which an infinite system of integral equations in three variables has a solution in the Banach tempering sequence space C_0^β and l_1^β with the help of Hausdorff *MNC*. At last they provided an example to demonstrate the implications of their established condition.

Motivated by these studies, we developed a newly defined contraction operator utilizing *MNC* to construct a best proximity point theorem and investigated the existence of optimum solutions for a system of integro differential equations in Banach space.

Theorem 1.1. *[10] For a nonempty, bounded, closed and convex subset* B *of a Banach space Z, consider* $\mathtt{L} : \mathsf{B} \to \mathsf{B}$ *be continuous and compact, then* $\mathtt{L}$ *admits at least a fixed point.*

Clearly, it is the generalization of Brouwer fixed point theorem.

Consider a Banach space Z and a closed ball $\mathcal{S}(r,s) = \{k \in Z : \|k - r\| \leq s\}$ in Z. Suppose $\overline{H}$ (*for all nonempty set H*) denotes the closure of H and $\overline{conv}(H)$ denotes the closed and convex hull of the non empty set H which is the smallest convex and closed set containing H.

Also K_Z and S_Z represents the family of non empty bounded subsets of Z and the subfamily of Z consisting all relatively compact sets, respectively; $\mathbb{R} = (-\infty, \infty)$; and $\mathbb{R}_+ = [0, \infty)$.

A measure of noncompactness (MNC) is defined axiomatically as follows:

Definition 1.2. *[1] A map* $\mathcal{M} : K_Z \to \mathbb{R}_+$ *is a* MNC *(measure of noncompactness) in the Banach space Z, if the following conditions are holds for* $\mathcal{M}$*:*

1. $ker\mathcal{M} = \{\mathtt{X} \in K_Z : \mathcal{M}(\mathtt{X}) = 0\} \neq \phi$,
2. $\mathtt{X} \in ker\mathcal{M}$ *if and only if* $\mathtt{X}$ *is relatively compact,*
3. $\mathtt{X}_1 \subseteq \mathtt{X}_2 \Rightarrow \mathcal{M}(\mathtt{X}_1) \leq \mathcal{M}(\mathtt{X}_2)$,
4. $\mathcal{M}\left(\overline{\mathtt{X}}\right) = \mathcal{M}(\mathtt{X})$,
5. $\mathcal{M}(\overline{conv}(\mathtt{X})) = \mathcal{M}(\mathtt{X})$,
6. $\mathcal{M}(\zeta\mathtt{X}_1 + (1-\zeta)\mathtt{X}_2) \leq \zeta\mathcal{M}(\mathtt{X}_1) + (1-\zeta)\mathcal{M}(\mathtt{X}_2)$, *for* $\zeta \in [0,1]$,
7. $\max\{\mathcal{M}(\mathtt{X}_1), \mathcal{M}(\mathtt{X}_2)\} = \mathcal{M}(\mathtt{X}_1 \cup \mathtt{X}_2)$,
8. *The set* $\mathtt{X}_\infty = \cap_{n=1}^\infty \mathtt{X}_n$ *is compact and non empty, if* $(\mathtt{X}_n)$ *is a decreasing sequence of closed sets which are non empty in* K_Z *and* $\lim_{n\to\infty} \mathcal{M}(\mathtt{X}_n) = 0$.

In particular, the space $Z = \mathcal{C}(\mathtt{I})$, where $\mathtt{I}$ is the closed and bounded interval, is the set of real valued continuous functions on $\mathtt{I}$. Then Z is a Banach space with the norm
$\|\mathrm{D}\| = \sup\{|\mathrm{D}(w)| : w \in \mathtt{I}\}, \mathrm{D} \in Z$.
Assume $\mathrm{K}(\neq \phi) \subseteq Z$ be bounded. For $\mathrm{D} \in \mathrm{K}$ and $q > 0$, the modulus of continuity of D, represented by $\mathtt{H}(\mathrm{D}, q)$ i.e., $\mathtt{H}(\mathrm{D}, q) = \sup\{|\mathrm{D}(w_1) - \mathrm{D}(w_2)| : w_1, w_2 \in \mathtt{I}, |w_1 - w_2| \leq q\}$.
Furthermore, we define
$\mathtt{H}(\mathrm{K}, q) = \sup\{\mathtt{H}(\mathrm{D}, q) : \mathrm{D} \in \mathrm{K}\}$; $\mathtt{H}_0(\mathrm{K}) = \lim_{q\to 0} \mathtt{H}(\mathrm{K}, q)$.
A Housdorff MNC Σ is given by
$\Sigma(\mathrm{K}) = \frac{1}{2}\mathtt{H}_0(\mathrm{K})$(see [2]).
It is widely known that the map $\mathtt{H}_0$ is an MNC in Z.

2. Preliminaries

We collect some fundamental definitions and notations needed for the paper.

Definition 2.1. *[10] Consider Z be a Banach space. Then,*

1. *Z is a uniformly convex Banach space, if there exists a strictly increasing function* $\mathbb{B} : (0,2] \to [0,1]$ *such that*

$$\begin{cases} \|i-j\| \leq G, \\ \|k-j\| \leq G, \Longrightarrow \|\frac{i+k}{2} - j\| \leq \left(1 - \mathbb{B}\left(\frac{l}{G}\right)\right) G; \\ \|i-k\| \geq l, \end{cases}$$

for all $i,k,j \in Z$, $G > 0$ *and* $l \in [0,2G]$.

2. *Z is a strictly convex Banach space, if, for all* $i,k,j \in Z$ *and* $G > 0$, *the following conditions hold:*

$$\begin{cases} \|i-j\| \leq G, \\ \|k-j\| \leq G, \Longrightarrow \|\frac{i+k}{2} - j\| < G. \\ i \neq k. \end{cases}$$

Consider a normed linear space (NLS) E. For any two non-empty subset $\mathtt{D}_1, \mathtt{D}_2$ of E, the pair $(\mathtt{D}_1, \mathtt{D}_2)$ is closed $\Longleftrightarrow$ both $\mathtt{D}_1$, $\mathtt{D}_2$ are closed; $(\mathtt{D}_1, \mathtt{D}_2) \subseteq (\mathcal{A}_1, \mathcal{A}_2) \Longleftrightarrow \mathtt{D}_1 \subseteq \mathcal{A}_1, \mathtt{D}_2 \subseteq \mathcal{A}_2$.

In addition, we denote by $dist\,(\mathcal{A}_1, \mathcal{A}_2) = \inf\{\|\mu - \nu\| : (\mu, \nu) \in \mathcal{A}_1 \times \mathcal{A}_2\}$,
$\mathcal{A}_0 = \{\mu \in \mathcal{A}_1 : \textit{there exists } \nu_1 \in \mathcal{A}_2 \textit{ so that } \|\mu - \nu_1\| = dist\,(\mathcal{A}_1, \mathcal{A}_2)\}$,
$\mathcal{B}_0 = \{\nu \in \mathcal{A}_2 : \textit{there exists } \mu_1 \in \mathcal{A}_1 \textit{ so that } \|\mu_1 - \nu\| = dist\,(\mathcal{A}_1, \mathcal{A}_2)\}$.

Definition 2.2. *[10] Consider E as a* NLS. *A non-empty pair* $(\mathcal{A}_1, \mathcal{A}_2)$ *of E is proximinal if* $\mathcal{A}_1 = \mathcal{A}_0$ *and* $\mathcal{A}_2 = \mathcal{B}_0$. *For a reflexive Banach space D, if the pair* $(\mathcal{A}_1, \mathcal{A}_2)$ *be a closed, nonempty, convex and bounded in D, then* $(\mathcal{A}_0, \mathcal{B}_0)$ *is also a closed, nonempty, convex and bounded pair.*

Consider a function $T : \mathcal{A}_1 \cup \mathcal{A}_2 \to \mathcal{A}_1 \cup \mathcal{A}_2$. *We say that T is,*

1. *relatively nonexpansive, if* $\|T(\mu) - T(\nu)\| \leq \|\mu - \nu\|$ *for any* $(\mu, \nu) \in \mathcal{A}_1 \times \mathcal{A}_2$,
2. *cyclic, if* $T(\mathcal{A}_1) \subseteq \mathcal{A}_2$ *and* $T(\mathcal{A}_2) \subseteq \mathcal{A}_1$,
3. *non cyclic, if* $T(\mathcal{A}_1) \subseteq \mathcal{A}_1$ *and* $T(\mathcal{A}_2) \subseteq \mathcal{A}_2$,
4. *compact, if* $\left(\overline{T(\mathcal{A}_1)}, \overline{T(\mathcal{A}_2)}\right)$ *is compact.*

Definition 2.3. *[10] Consider* $(\mathcal{A}_1, \mathcal{A}_2)$ *as a nonempty pair in a Banach space Z and* $\mathrm{F} : \mathcal{A}_1 \cup \mathcal{A}_2 \to \mathcal{A}_1 \cup \mathcal{A}_2$ *be a cyclic function, then* $b \in \mathcal{A}_1 \cup \mathcal{A}_2$ *is called a* $\mathcal{BPP}$ *of* F *if* $\|b - \mathrm{F}(b)\| = dist\,(\mathcal{A}_1, \mathcal{A}_2)$.
If F *is noncyclic, then the pair* $(b, \nu) \in \mathcal{A}_1 \times \mathcal{A}_2$ *is a best proximity pair if* $\|b - \nu\| = dist\,(\mathcal{A}_1, \mathcal{A}_2)$, *where* $b = \mathrm{F}(b), \nu = \mathrm{F}(\nu)$.

Corollary 2.4. *[10] Suppose a Banach space Z and a nonempty, convex and compact pair* $(\mathcal{A}_1, \mathcal{A}_2)$ *in Z. Let we have a cyclic and relatively nonexpansive mapping* $T : \mathcal{A}_1 \cup \mathcal{A}_2 \to \mathcal{A}_1 \cup \mathcal{A}_2$. *Then T has a* $\mathcal{BPP}$.

Corollary 2.5. *[10] Suppose a strictly convex Banach space Z and a compact, nonempty and convex pair* $(\mathcal{A}_1, \mathcal{A}_1)$ *in Z. Let we have a relatively nonexpansive and noncyclic mapping* $T : \mathcal{A}_1 \cup \mathcal{A}_2 \to \mathcal{A}_1 \cup \mathcal{A}_2$. *Then T has a best proximity pair.*

The following theorems 2.6 and 2.7 are the extended form of corollaries 2.4 and 2.5.

Theorem 2.6. *[10] Suppose a reflexive Banach space Z and a convex, nonempty, closed and bounded pair* $(\mathcal{A}_1, \mathcal{A}_2)$ *in Z. Let we have a relatively nonexpansive and cyclic mapping* $T : \mathcal{A}_1 \cup \mathcal{A}_2 \to \mathcal{A}_1 \cup \mathcal{A}_2$. *Then T has a* $\mathcal{BPP}$, *if T is compact.*

Theorem 2.7. *[10] Suppose a reflexive, strictly convex Banach space Z and a convex, nonempty, closed and bounded pair* $(\mathcal{A}_1, \mathcal{A}_2)$ *in Z. Let we have a relatively nonexpansive and noncyclic mapping* $T : \mathcal{A}_1 \cup \mathcal{A}_2 \to \mathcal{A}_1 \cup \mathcal{A}_2$. *Then T has a best proximity pair, if T is compact.*

3. Main result

Definition 3.1. *[7] Let* $\mathtt{D}$ *be the set of all maps* $\mathbb{P} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *with*

(i) $\max\{\mu, \nu\} \le \mathbb{P}(\mu, \nu)\,;\ \mu, \nu \ge 0,$

(ii) $\mathbb{P}$ *is continuous,*

(iii) $\mathbb{P}(\mu_1 + \mu_2, \nu_1 + \nu_2) \le \mathbb{P}(\mu_1, \nu_1) + \mathbb{P}(\mu_2, \nu_2)\,;\ \mu_1, \mu_2, \nu_1, \nu_2 \ge 0.$

As an example, we can consider $\mathbb{P}(\mu, \nu) = \mu + \nu$ for all $\mu, \nu > 0$.

$\mathcal{A}_1$ and $\mathcal{A}_2$ will be nonempty convex subsets of a Banach space Z in this section.

Definition 3.2. *Consider* $(\mathcal{A}_1, \mathcal{A}_2)$ *as a pair of convex and nonempty subsets of a Banach space* Z *equipped with a* MNC $\mathcal{M}$*, and* $\mathbb{U}$, $\mathbb{V}$ *are nondecreasing continuous functions. A mapping* $T : \mathcal{A}_1 \cup \mathcal{A}_2 \to \mathcal{A}_1 \cup \mathcal{A}_2$*, which is cyclic (noncyclic), is said to be a* $(\mathbb{P}, \mathbb{V}, \mathbb{U})$*-contractive operator such that for any pair of convex, nonempty, proximal, closed, bounded and T-invariant subsets* $(\mathtt{M}_1, \mathtt{M}_2)$ *such that dist* $(\mathtt{M}_1, \mathtt{M}_2) = dist(\mathcal{A}_1, \mathcal{A}_2)$*, we have,*

$$\begin{aligned}&\mathbb{P}\left[\mathcal{M}\left(T(\mathtt{M}_1) \cup T(\mathtt{M}_2)\right), \mathbb{V}\left(\mathcal{M}\left(T(\mathtt{M}_1) \cup T(\mathtt{M}_2)\right)\right)\right] \\ &\le \mathbb{U}\left[\mathbb{P}\left\{\mathcal{M}(\mathtt{M}_1 \cup \mathtt{M}_2), \mathbb{V}\left(\mathcal{M}(\mathtt{M}_1 \cup \mathtt{M}_2)\right)\right\}\right] - \mathbb{U}\left[\mathbb{P}\left\{\mathcal{M}\left(T(\mathtt{M}_1) \cup T(\mathtt{M}_2)\right), \mathbb{V}\left(\mathcal{M}\left(T(\mathtt{M}_1) \cup T(\mathtt{M}_2)\right)\right)\right\}\right]. \qquad (1)\end{aligned}$$

Theorem 3.3. *Consider a relatively non expansive, cyclic and* $(\mathbb{P}, \mathbb{V}, \mathbb{U})$*-contractive operator* $T : \mathcal{A}_1 \cup \mathcal{A}_2 \to \mathcal{A}_1 \cup \mathcal{A}_2$*. Then* T *has a* $\mathcal{BPP}$*, if* $\mathcal{A}_0 \ne \emptyset$*.*

Proof. Since $\mathcal{A}_0 \ne \emptyset$, $(\mathcal{A}_0, \mathcal{B}_0) \ne \emptyset$. By the given conditions on T, clearly $(\mathcal{A}_0, \mathcal{B}_0)$ is a closed, convex, proximinal and T-invariant pair. For each $\psi \in \mathcal{A}_0$, there is a $\theta \in \mathcal{B}_0$ satisfying $\|\psi - \theta\| = dist(\mathcal{A}_1, \mathcal{A}_2)$. Since T is relatively non expansive, so we get $\|T\psi - T\theta\| \le \|\psi - \theta\| = dist(\mathcal{A}_1, \mathcal{A}_2)$, which implies $T\psi \in \mathcal{B}_0$, that is, $T(\mathcal{A}_0) \subseteq \mathcal{B}_0$. Similarly, $T(\mathcal{B}_0) \subseteq \mathcal{A}_0$. Hence we get T is cyclic on $\mathcal{A}_0 \cup \mathcal{B}_0$.

Let us assume that $\mathtt{I}_0 = \mathcal{A}_0$ and $\mathsf{L}_0 = \mathcal{B}_0$ and $\left\{(\mathtt{I}_n, \mathsf{L}_n)\right\}$ be a sequence of pairs with $\mathtt{I}_n = \overline{conv}(T(\mathtt{I}_{n-1}))$ and $\mathsf{L}_n = \overline{conv}(T(\mathsf{L}_{n-1}))$, for all $n \in \mathbb{N}$. Now our claim is, $\mathtt{I}_{n+1} \subseteq \mathsf{L}_n$ and $\mathsf{L}_n \subseteq \mathtt{I}_{n-1}$ for all $n \in \mathbb{N}$. In fact $\mathsf{L}_1 = \overline{conv}(T(\mathsf{L}_0)) = \overline{conv}(T(\mathcal{B}_0)) \subseteq \overline{conv}(\mathcal{A}_0) = \mathcal{A}_0 = \mathtt{I}_0$. Hence we can write,

$$T(\mathsf{L}_1) \subseteq T(\mathtt{I}_0)\, and\, \mathsf{L}_2 = \overline{conv}(T(\mathsf{L}_1)) \subseteq \overline{conv}(T(\mathtt{I}_0)) = \mathtt{I}_1.$$

With the similar argument, we get by using induction that $\mathsf{L}_n \subseteq \mathtt{I}_{n-1}$. Similarly, we get $\mathtt{I}_{n+1} \subseteq \mathsf{L}_n$, for all $n \in \mathbb{N}$. Hence, we can write $\mathtt{I}_{n+2} \subseteq \mathsf{L}_{n+1} \subseteq \mathtt{I}_n \subseteq \mathsf{L}_{n-1}$, for all $n \in \mathbb{N}$. So, in $\mathcal{A}_0 \times \mathcal{B}_0$, the decreasing sequence of NBCC pairs is $\left\{(\mathtt{I}_{2n}, \mathsf{L}_{2n})\right\}$. Moreover,

$$T(\mathsf{L}_{2n}) \subseteq T(\mathtt{I}_{2n-1}) \subseteq \overline{conv}(T(\mathtt{I}_{2n-1})) = \mathtt{I}_{2n}, \qquad (2)$$

$$T(\mathtt{I}_{2n}) \subseteq T(\mathsf{L}_{2n-1}) \subseteq \overline{conv}(T(\mathsf{L}_{2n-1})) = \mathsf{L}_{2n}. \qquad (3)$$

Hence, we get $(\mathtt{I}_{2n}, \mathsf{L}_{2n})$ is a $T-invariant$ pair, for all n $\in \mathbb{N}$. Now, if $(\mathsf{a}, \mathsf{b}) \in \mathcal{A}_0 \times \mathcal{B}_0$ which is proximinal, we have,

$$dist(\mathtt{I}_{2n}, \mathsf{L}_{2n}) \le \|T^{2n}\mathsf{a} - T^{2n}\mathsf{b}\| \le \|\mathsf{a} - \mathsf{b}\| = dist(\mathcal{A}_1, \mathcal{A}_2).$$

Now, we are to show that, for all $n \in \mathbb{N}$, the pair $(\mathtt{I}_n, \mathsf{L}_n)$ is proximinal. For n=0, we have $(\mathtt{I}_0, \mathsf{L}_0)$ is a proximinal pair. Let us assume that $(\mathtt{I}_k, \mathsf{L}_k)$ is proximinal and there is an arbitrary μ such that $\mu \in \mathtt{I}_{k+1} = \overline{conv}(T(\mathtt{I}_k))$. So $\mu = \sum_{j=1}^{m} f_j T\left(g_j\right)$ with $g_j \in \mathtt{I}_k,\ m \in [1, \infty),\ f_j \ge 0$ and $\sum_{j=1}^{m} f_j = 1$. By assumption, we have

$(\mathtt{I}_k, \mathsf{L}_k)$ is a proximinal pair, so there exists $h_j \in \mathsf{L}_k$ $(1 \leq j \leq m)$ such that $\|g_j - h_j\| = dist(\mathtt{I}_k, \mathsf{L}_k) = dist(\mathcal{A}_1, \mathcal{A}_2)$. Consider $\epsilon = \sum_{j=1}^{m} f_j T\left(h_j\right)$. Then $\epsilon \in \overline{conv}(T(\mathsf{L}_k)) = \mathsf{L}_{k+1}$, and

$$\|\mu - \epsilon\| = \|\sum_{j=1}^{m} f_j T\left(g_j\right) - \sum_{j=1}^{m} f_j T\left(h_j\right)\| \leq \sum_{j=1}^{m} f_j \|g_j - h_j\| = dist(\mathcal{A}_1, \mathcal{A}_2). \tag{4}$$

Hence, $(\mathtt{I}_{k+1}, \mathsf{L}_{k+1})$ is a proximinal pair and by induction hypothesis our claim is proved.

Now, if $\max\left\{\mathcal{M}(\mathtt{I}_{2n_0}), \mathcal{M}(\mathsf{L}_{2n_0})\right\} = 0$ for some $n_0 \in [1, \infty) \cup \{0\}$, then we have $T : \mathtt{I}_{2n_0} \cup \mathsf{L}_{2n_0} \to \mathtt{I}_{2n_0} \cup \mathsf{L}_{2n_0}$ is compact. By corollary (2.4), T has a $\mathcal{BPP}$. Hence, we consider that $\max\left\{\mathcal{M}(\mathtt{I}_n), \mathcal{M}(\mathsf{L}_n)\right\} > 0$, for all $n \in [1, \infty)$. Since $\mathtt{I}_{2n+1} \subseteq T(\mathtt{I}_{2n})$ and $\mathsf{L}_{2n+1} \subseteq T(\mathsf{L}_{2n})$, we have,

$$
\begin{aligned}
&\mathbb{P}\left[\mathcal{M}(\mathtt{I}_{2n+1} \cup \mathsf{L}_{2n+1}), \mathbb{V}(\mathcal{M}(\mathtt{I}_{2n+1} \cup \mathsf{L}_{2n+1}))\right] \\
&= \mathbb{P}\left[\max\left\{\mathcal{M}(\mathtt{I}_{2n+1}), \mathcal{M}(\mathsf{L}_{2n+1})\right\}, \mathbb{V}\left(\max\left\{\mathcal{M}(\mathtt{I}_{2n+1}), \mathcal{M}(\mathsf{L}_{2n+1})\right\}\right)\right] \\
&= \mathbb{P}\Big[\max\Big\{\mathcal{M}(\overline{conv}(T(\mathtt{I}_{2n}))), \mathcal{M}(\overline{conv}(T(\mathsf{L}_{2n})))\Big\}, \\
&\mathbb{V}\Big(\max\Big\{\mathcal{M}(\overline{conv}(T(\mathtt{I}_{2n}))), \mathcal{M}(\overline{conv}(T(\mathsf{L}_{2n})))\Big\}\Big)\Big] \\
&= \mathbb{P}\left[\max\left\{\mathcal{M}(T(\mathtt{I}_{2n})), \mathcal{M}(T(\mathsf{L}_{2n}))\right\}, \mathbb{V}\left(\max\left\{\mathcal{M}(T(\mathtt{I}_{2n})), \mathcal{M}(T(\mathsf{L}_{2n}))\right\}\right)\right] \\
&= \mathbb{P}\left[\mathcal{M}(T(\mathtt{I}_{2n}) \cup T(\mathsf{L}_{2n})), \mathbb{V}(\mathcal{M}(T(\mathtt{I}_{2n}) \cup T(\mathsf{L}_{2n})))\right] \\
&\leq \mathbb{U}\left[\mathbb{P}\left\{\mathcal{M}(\mathtt{I}_{2n} \cup \mathsf{L}_{2n}), \mathbb{V}(\mathcal{M}(\mathtt{I}_{2n} \cup \mathsf{L}_{2n}))\right\}\right] - \\
&- \mathbb{U}\left[\mathbb{P}\left\{\mathcal{M}(T(\mathtt{I}_{2n}) \cup T(\mathsf{L}_{2n})), \mathbb{V}(\mathcal{M}(T(\mathtt{I}_{2n}) \cup T(\mathsf{L}_{2n})))\right\}\right] \\
&\leq \mathbb{U}\left[\mathbb{P}\left\{\mathcal{M}(\mathtt{I}_{2n} \cup \mathsf{L}_{2n}), \mathbb{V}(\mathcal{M}(\mathtt{I}_{2n} \cup \mathsf{L}_{2n}))\right\}\right] - \\
&- \mathbb{U}\left[\mathbb{P}\left\{\mathcal{M}(\mathtt{I}_{2n+1} \cup \mathsf{L}_{2n+1}), \mathbb{V}(\mathcal{M}(\mathtt{I}_{2n+1} \cup \mathsf{L}_{2n+1}))\right\}\right].
\end{aligned}
$$

If $\lim_{n\to\infty} \mathbb{P}\left[\mathcal{M}(\mathtt{I}_{2n+1} \cup \mathsf{L}_{2n+1}), \mathbb{V}(\mathcal{M}(\mathtt{I}_{2n+1} \cup \mathsf{L}_{2n+1}))\right] = \mathsf{N}$
Then $0 \leq \mathsf{N} \leq \mathbb{U}(\mathsf{N}) - \mathbb{U}(\mathsf{N}) = 0$.
ie. $\mathsf{N} = 0$.

Thus $\mathbb{P}\left[\mathcal{M}(\mathtt{I}_{2n+1} \cup \mathsf{L}_{2n+1}), \mathbb{V}(\mathcal{M}(\mathtt{I}_{2n+1} \cup \mathsf{L}_{2n+1}))\right] \to 0$ as $n \to \infty$ and $\lim_{n\to 0} \mathcal{M}(\mathtt{I}_{2n} \cup \mathsf{L}_{2n}) = \lim_{n\to 0} \mathbb{V}(\mathcal{M}(\mathtt{I}_{2n} \cup \mathsf{L}_{2n})) = 0$. Also,

$$\lim_{n\to\infty} \mathcal{M}(\mathtt{I}_{2n} \cup \mathsf{L}_{2n}) = \max\left\{\lim_{n\to\infty} \mathcal{M}(\mathtt{I}_{2n}), \lim_{n\to\infty} \mathcal{M}(\mathsf{L}_{2n})\right\} = 0.$$

Let $\mathtt{I}_\infty = \cap_{n=0}^{\infty} \mathtt{I}_{2n}$ and $\mathsf{L}_\infty = \cap_{n=0}^{\infty} \mathsf{L}_{2n}$, so, we get a nonempty, compact, convex pair $(\mathtt{I}_\infty, \mathsf{L}_\infty)$ which is T-invariant with dist$(\mathtt{I}_\infty, \mathsf{L}_\infty)$ =dist$(\mathcal{A}_1, \mathcal{A}_2)$. Hence from corollary 2.4, T has a $\mathcal{BPP}$. □

Theorem 3.4. *Consider a relatively nonexpansive, noncyclic and* $(\mathbb{P}, \mathbb{V}, \mathbb{U})$*-contractive operator* $T : \mathcal{A}_1 \cup \mathcal{A}_2 \to \mathcal{A}_1 \cup \mathcal{A}_2$ *on a strictly convex Banach space Z. Then T has a best proximity pair, if* $\mathcal{A}_0 \neq \emptyset$.

Proof. Following the proof of the theorem (3.3), define a pair $(\mathtt{I}_n, \mathsf{L}_n)$ as $\mathtt{I}_n = \overline{conv}(T(\mathtt{I}_{n-1}))$ and $\mathsf{L}_n = \overline{conv}(T(\mathsf{L}_{n-1}))$, $n \in \mathbb{N}$ with $\mathtt{I}_0 = \mathcal{A}_0$ and $\mathsf{L}_0 = \mathcal{B}_0$, we get NBCC and decreasing sequence of pairs $\left\{(\mathtt{I}_n, \mathsf{L}_n)\right\}$

in $\mathcal{A}_0 \times \mathcal{B}_0$. Also,

$$T(\mathtt{I}_n) \subseteq T(\mathtt{I}_{n-1}) \subseteq \overline{conv}(T(\mathtt{I}_{n-1})) = \mathtt{I}_n, \tag{5}$$
$$T(\mathsf{L}_n) \subseteq T(\mathsf{L}_{n-1}) \subseteq \overline{conv}(T(\mathsf{L}_{n-1})) = \mathsf{L}_n. \tag{6}$$

Thus, the pair $(\mathtt{I}_n, \mathsf{L}_n)$ is T-invariant, for all n ≥ 1. Following the proof of the theorem (3.3), we obtain a proximinal pair $(\mathtt{I}_n, \mathsf{L}_n)$, for all non-negative integer n such that $\mathrm{dist}(\mathtt{I}_n, \mathsf{L}_n) = \mathrm{dist}(\mathcal{A}_1, \mathcal{A}_2)$. Now, if $\max\left\{\mathcal{M}(\mathtt{I}_{n_0}), \mathcal{M}(\mathsf{L}_{n_0})\right\} = 0$, for some positive integer n_0, then $T : \mathtt{I}_{n_0} \cup \mathsf{L}_{n_0} \to \mathtt{I}_{n_0} \cup \mathsf{L}_{n_0}$ is compact. Hence, from corollary (2.5) we get the desired result. Thus, we consider that $\max\left\{\mathcal{M}(\mathtt{I}_n), \mathcal{M}(\mathsf{L}_n)\right\} > 0$. Since $\mathtt{I}_{n+1} \subseteq T(\mathtt{I}_n)$ and $\mathsf{L}_{n+1} \subseteq T(\mathsf{L}_n)$, we have,

$$\begin{aligned}
&\mathbb{P}[\mathcal{M}(\mathtt{I}_{n+1} \cup \mathsf{L}_{n+1}), \mathbb{V}(\mathcal{M}(\mathtt{I}_{n+1} \cup \mathsf{L}_{n+1}))] \\
&= \mathbb{P}\left[\max\left\{\mathcal{M}(\mathtt{I}_{n+1}), \mathcal{M}(\mathsf{L}_{n+1})\right\}, \mathbb{V}\left(\max\left\{\mathcal{M}(\mathtt{I}_{n+1}), \mathcal{M}(\mathsf{L}_{n+1})\right\}\right)\right] \\
&= \mathbb{P}\Big[\max\left\{\mathcal{M}(\overline{conv}(T(\mathtt{I}_n))), \mathcal{M}(\overline{conv}(T(\mathsf{L}_n)))\right\}, \\
&\mathbb{V}\left(\max\left\{\mathcal{M}(\overline{conv}(T(\mathtt{I}_n))), \mathcal{M}(\overline{conv}(T(\mathsf{L}_n)))\right\}\right)\Big] \\
&= \mathbb{P}\left[\max\left\{\mathcal{M}(T(\mathtt{I}_n)), \mathcal{M}(T(\mathsf{L}_n))\right\}, \mathbb{V}\left(\max\left\{\mathcal{M}(T(\mathtt{I}_n)), \mathcal{M}(T(\mathsf{L}_n))\right\}\right)\right] \\
&= \mathbb{P}[\mathcal{M}(T(\mathtt{I}_n) \cup T(\mathsf{L}_n)), \mathbb{V}(\mathcal{M}(T(\mathtt{I}_n) \cup T(\mathsf{L}_n)))] \\
&\leq \mathbb{U}\left[\mathbb{P}\left\{\mathcal{M}(\mathtt{I}_n \cup \mathsf{L}_n), \mathbb{V}(\mathcal{M}(\mathtt{I}_n \cup \mathsf{L}_n))\right\}\right] - \\
&- \mathbb{U}\left[\mathbb{P}\left\{\mathcal{M}(T(\mathtt{I}_n) \cup T(\mathsf{L}_n)), \mathbb{V}(\mathcal{M}(T(\mathtt{I}_n) \cup T(\mathsf{L}_n)))\right\}\right] \\
&\leq \mathbb{U}\left[\mathbb{P}\left\{\mathcal{M}(\mathtt{I}_n \cup \mathsf{L}_n), \mathbb{V}(\mathcal{M}(\mathtt{I}_n \cup \mathsf{L}_n))\right\}\right] - \\
&- \mathbb{U}\left[\mathbb{P}\left\{\mathcal{M}(\mathtt{I}_{n+1} \cup \mathsf{L}_{n+1}), \mathbb{V}(\mathcal{M}(\mathtt{I}_{n+1} \cup \mathsf{L}_{n+1}))\right\}\right].
\end{aligned}$$

If $\lim_{n\to\infty} \mathbb{P}[\mathcal{M}(\mathtt{I}_{n+1} \cup \mathsf{L}_{n+1}), \mathbb{V}(\mathcal{M}(\mathtt{I}_{n+1} \cup \mathsf{L}_{n+1}))] = \mathsf{N}$,
then $0 \leq \mathsf{N} \leq \mathbb{U}(\mathsf{N}) - \mathbb{U}(\mathsf{N}) = 0$.
ie. $\mathsf{N} = 0$.

Thus, $\mathbb{P}[\mathcal{M}(\mathtt{I}_{n+1} \cup \mathsf{L}_{n+1}), \mathbb{V}(\mathcal{M}(\mathtt{I}_{n+1} \cup \mathsf{L}_{n+1}))] \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} \mathcal{M}(\mathtt{I}_n \cup \mathsf{L}_n) = \lim_{n\to\infty} \mathbb{V}(\mathcal{M}(\mathtt{I}_n \cup \mathsf{L}_n)) = 0$. Also,

$$\lim_{n\to\infty} \mathcal{M}(\mathtt{I}_n \cup \mathsf{L}_n) = \max\left\{\lim_{n\to\infty} \mathcal{M}(\mathtt{I}_n), \lim_{n\to\infty} \mathcal{M}(\mathsf{L}_n)\right\} = 0.$$

Let $\mathtt{I}_\infty = \cap_{n=0}^{\infty} \mathtt{I}_n$ and $\mathsf{L}_\infty = \cap_{n=0}^{\infty} \mathsf{L}_n$, so, we get a nonempty, compact, convex pair $(\mathtt{I}_\infty, \mathsf{L}_\infty)$ which is T-invariant with $\mathrm{dist}(\mathtt{I}_\infty, \mathsf{L}_\infty) = \mathrm{dist}(\mathcal{A}_1, \mathcal{A}_2)$. Hence from corollary 2.5, T has a best proximity pair. □

Corollary 3.5. *Consider a relatively nonexpansive, cyclic operator* $T : \mathcal{A}_1 \cup \mathcal{A}_2 \to \mathcal{A}_1 \cup \mathcal{A}_2$ *such that,*

$$\begin{aligned}
&\mathcal{M}(T(\zeta_1) \cup T(\zeta_2)) + \mathbb{V}(\mathcal{M}(T(\zeta_1) \cup T(\zeta_2))) \\
&\leq \mathbb{U}[\mathcal{M}(\zeta_1 \cup \zeta_2) + \mathbb{V}(\mathcal{M}(\zeta_1 \cup \zeta_2))] - \mathbb{U}[\mathcal{M}(T(\zeta_1) \cup T(\zeta_2)) + \mathbb{V}(\mathcal{M}(T(\zeta_1) \cup T(\zeta_2)))],
\end{aligned} \tag{7}$$

for $(\zeta_1, \zeta_2) \in (\mathcal{A}_1, \mathcal{A}_2)$. *Then T has a* $\mathcal{BPP}$, *if* $\mathcal{A}_0 \neq \emptyset$.

Proof. Putting $\mathbb{P}(\mu, \nu) = \mu + \nu$ in equation (1) of definition (3.2) and using theorem 3.3, we obtain the result shown above. □

Corollary 3.6. *Consider a relatively nonexpansive, cyclic operator $T : \mathcal{A}_1 \cup \mathcal{A}_2 \to \mathcal{A}_1 \cup \mathcal{A}_2$ such that,*

$$\mathcal{M}(T(\zeta_1) \cup T(\zeta_2)) \leq \mathbb{U}[\mathcal{M}(\zeta_1 \cup \zeta_2)] - \mathbb{U}[\mathcal{M}(T(\zeta_1) \cup T(\zeta_2))], \tag{8}$$

for $(\zeta_1, \zeta_2) \in (\mathcal{A}_1, \mathcal{A}_2)$. Then T has a $\mathcal{BPP}$, if $\mathcal{A}_0 \neq \emptyset$.

Proof. Putting $\mathbb{V}(w) = 0$ in equation (7) of corollary (3.5), we obtain the result shown above. □

Corollary 3.7. *Consider a relatively nonexpansive, cyclic operator $T : \mathcal{A}_1 \cup \mathcal{A}_2 \to \mathcal{A}_1 \cup \mathcal{A}_2$ such that,*

$$\mathcal{M}(T(\zeta_1) \cup T(\zeta_2)) \leq \mathcal{M}(\zeta_1 \cup \zeta_2), \tag{9}$$

for $(\zeta_1, \zeta_2) \in (\mathcal{A}_1, \mathcal{A}_2)$. Then T has a $\mathcal{BPP}$, if $\mathcal{A}_0 \neq \emptyset$.

Proof. Putting $\mathbb{U}(w) = 2w$, in equation (8) of corollary (3.6), we obtain the result shown above. □

4. Applications

We apply our findings to investigate the optimum solution the following system of integro differential equations:

$$\begin{cases} \Omega(t) = a_0 - \Delta(t) \int_0^\delta \Omega(s)\, ds + \dfrac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} \left\{ \Lambda_1(s, \Omega(s)) + \int_0^s \Theta(\mathrm{z}, \Omega(\mathrm{z}))\, d\mathrm{z} \right\} ds, \\ \Psi(t) = b_0 - \Delta(t) \int_0^\delta \Psi(s)\, ds + \dfrac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} \left\{ \Lambda_2(s, \Psi(s)) + \int_0^s \Theta(\mathrm{z}, \Psi(\mathrm{z}))\, d\mathrm{z} \right\} ds, \end{cases} \tag{10}$$

for $t, \delta, \mathrm{z}, s, \theta \in [0,1] = \mathtt{J}$ and $a_0, b_0 \in \mathbb{R}$. Also, assume that $(\mathbb{R}, \|.\|)$ be a Banach space and two closed ball $\mathsf{K}_1 = \mathcal{S}(a_0, \kappa)$ and $\mathsf{K}_2 = \mathcal{S}(b_0, \kappa)$ in $\mathbb{R}$ with $\kappa \in \mathbb{R}$ and $\|\Lambda_1(., \Omega(.))\| \leq \mathtt{D}_1$, $\|\Lambda_2(., \Psi(.))\| \leq \mathtt{D}_2$, $\|\Theta\| \leq \mathcal{P}$, where Λ_1, Λ_2, Δ and Θ are continuous functions. Consider a standard Banach space $\mathbf{R} = C(J, \mathbb{R})$ of continuous function with supremum norm for $J = [0,1) \subseteq \mathtt{J}$. Also $\Gamma(,)$ denotes Euler's gamma function. Let:
$\mathbf{R}_1 = C(J, \mathsf{K}_1) = \{\Omega : J \to \mathsf{K}_1 : \Omega \in \mathbf{R}\}$,
$\mathbf{R}_2 = C(J, \mathsf{K}_2) = \{\Psi : J \to \mathsf{K}_2 : \Psi \in \mathbf{R}\}$.
Then $(\mathbf{R}_1, \mathbf{R}_2)$ is an NBCC pair in $\mathbf{R}$. Now, for every $\Omega \in \mathbf{R}_1$ and $\Psi \in \mathbf{R}_2$,
$\|\Omega - \Psi\| = \sup_{t \in J} \|\Omega(t) - \Psi(t)\| \geq \|a_0 - b_0\|$.

Thus, $\mathrm{dist}(\mathbf{R}_1, \mathbf{R}_2) = \|a_0 - b_0\|$.

Now, we define $T : \mathbf{R}_1 \cup \mathbf{R}_2 \to \mathbf{R}$ such that

$$T(\Omega(t)) = \begin{cases} b_0 - \Delta(t) \int_0^\delta \Omega(s)\, ds + \dfrac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} \left\{ \Lambda_2(s, \Omega(s)) + \right. \\ \left. + \int_0^s \Theta(\mathrm{z}, \Omega(\mathrm{z}))\, d\mathrm{z} \right\} ds, \quad \Omega \in \mathbf{R}_1, \\ a_0 - \Delta(t) \int_0^\delta \Omega(s)\, ds + \dfrac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} \left\{ \Lambda_1(s, \Omega(s)) + \right. \\ \left. + \int_0^s \Theta(\mathrm{z}, \Omega(\mathrm{z}))\, d\mathrm{z} \right\} ds, \quad \Omega \in \mathbf{R}_2. \end{cases}$$

Clearly, T is cyclic, and if $\|s - T(s)\| = dist(\mathbf{R}_1, \mathbf{R}_2)$, for $s \in \mathbf{R}_1 \cup \mathbf{R}_2$, then s is a $\mathcal{BPP}$ for the operator T and it is equivalent to that s is an optimum solution of the system (10).

Theorem 4.1. *[9] Consider $f \in C[n_1, n_2]$ with $n_1 < n_2$. Let g is Lebesgue integrable on $[n_1, n_2]$ and g does not change its sign in $[n_1, n_2]$. Then the generalized mean value theorem of integral calculus gives,*

$$\int_{n_1}^{n_2} f(t)\, g(t)\, dt = f(\varsigma) \int_{n_1}^{n_2} g(t)\, dt, \tag{11}$$

for some $\varsigma \in (n_1, n_2)$.

Theorem 4.2. *Assume that* $\mathtt{H}_0$ *be a* MNC *on* $\mathbf{R}$ *with* $|\Psi - \Omega| \leq |b_0 - a_0|$, $|\Omega| \leq \nu$, $\left\{|\Delta(t)|\nu + \frac{1}{\Gamma(\theta+1)}\left|\mathtt{D}_2 + \frac{\mathcal{P}}{\theta+1}\right|\right\} \leq \mathcal{M}_1$, *and* $|b_0| + \mathcal{M}_1 \leq \mathcal{M}_2$ *for* $\mathcal{M}_1 > 0, \mathcal{M}_2 > 0$ *and* $\|\Theta(\mathrm{z}, \Omega(\mathrm{z})) - \Theta(\mathrm{z}, \Psi(\mathrm{z}))\| \leq \Gamma(\theta+2)\frac{\|\Omega - \Psi\|}{2}$, *where* $\Theta, \Omega, \Psi, \theta$ *and* z *are defined as above discussion. Then an optimal solution exists for the system of integro differential equations (10) if:*

1. *For any bounded pair* $(\zeta_1, \zeta_2) \subseteq (\mathbf{R}_1, \mathbf{R}_2)$, $\mathtt{H}_0\Big(\big\{\Lambda_2(J \times \zeta_1) + \Theta(J \times \zeta_1)\big\}\Big), \mathtt{H}_0\Big(\big\{\Lambda_1(J \times \zeta_2) + \Theta(J \times \zeta_2)\big\}\Big) > 0$ *implies that,*

$$\mathtt{H}_0\Bigg(\Big\{\Lambda_2(J \times \zeta_1) + \Theta(J \times \zeta_1)\Big\} \cup \Big\{\Lambda_1(J \times \zeta_2) + \Theta(J \times \zeta_2)\Big\}\Bigg) \leq \mathtt{H}_0(\zeta_1 \cup \zeta_2).$$

2. *For all* $\Omega \in \mathbf{R}_1$, $\Psi \in \mathbf{R}_2$ *and* $s \in [0, 1)$,

$$\left\|\Big(\Lambda_2(s, \Omega(s)) - \Lambda_1(s, \Psi(s))\Big)\right\| \leq \Gamma(\theta+1)\left\{\frac{\|\Omega - \Psi\|}{2} - |b_0 - a_0|\Big(1 + \Delta(t)\Big)\right\}$$

Proof. First we show that the operator T is cyclic.
For $\Omega \in \mathbf{R}_1$,

$$\begin{aligned}
\|T(\Omega(t)) - b_0\| &= \left\|b_0 - \Delta(t)\int_0^\delta \Omega(s)\,ds + \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}\left\{\Lambda_2(s, \Omega(s)) + \int_0^s \Theta(\mathrm{z}, \Omega(\mathrm{z}))\,d\mathrm{z}\right\}ds - b_0\right\| \\
&= \left\| - \Delta(t)\int_0^\delta \Omega(s)\,ds + \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}\left\{\Lambda_2(s, \Omega(s)) + \int_0^s \Theta(\mathrm{z}, \Omega(\mathrm{z}))\,d\mathrm{z}\right\}ds\right\| \\
&\leq \left|\Delta(t)\int_0^\delta \Omega(s)\,ds\right| + \frac{1}{\Gamma(\theta)}\left|\int_0^t (t-s)^{\theta-1}\left\{\mathtt{D}_2 + \int_0^s \mathcal{P}\,d\mathrm{z}\right\}ds\right| \\
&= \left|\Delta(t)\int_0^\delta \Omega(s)\,ds\right| + \frac{1}{\Gamma(\theta)}\left|\int_0^t (t-s)^{\theta-1}\left\{\mathtt{D}_2 + \mathcal{P}s\right\}ds\right| \\
&= \left|\Delta(t)\int_0^\delta \Omega(s)\,ds\right| + \frac{1}{\Gamma(\theta)}\left|\int_0^t (t-s)^{\theta-1}\,\mathtt{D}_2\,ds + \mathcal{P}\int_0^t (t-s)^{\theta-1}\,s\,ds\right| \\
&\leq |\Delta(t)|\nu\delta + \frac{1}{\Gamma(\theta+1)}\left|(t)^\theta\,\mathtt{D}_2 + \mathcal{P}\left\{\Big(-s(t-s)^\theta\Big)_0^t + \int_0^t (t-s)^\theta\,ds\right\}\right| \\
&\leq |\Delta(t)|\nu\delta + \frac{1}{\Gamma(\theta+1)}\left|(t)^\theta\,\mathtt{D}_2 + \mathcal{P}\frac{t^{\theta+1}}{\theta+1}\right| \\
&\leq |\Delta(t)|\nu + \frac{1}{\Gamma(\theta+1)}\left|\mathtt{D}_2 + \frac{\mathcal{P}}{\theta+1}\right| \\
&\leq \mathcal{M}_1.
\end{aligned}$$

Hence, $T(\Omega(t)) \in \mathbf{R}_2$.

Using a similar method, we can show that $T(\Omega(t)) \in \mathbf{R}_1$ for $\Omega \in \mathbf{R}_2$. Thus, T is cyclic. We now show that $T(\mathbf{R}_1)$ is an equicontinuous and bounded subset of $\mathbf{R}_2$. Consider $\Omega \in \mathbf{R}_1$ and $t \in \mathtt{J}$. We have,

$$\begin{aligned}
\|T(\Omega(t))\| &= \left\| b_0 - \Delta(t) \int_0^\delta \Omega(s)\, ds + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} \left\{ \Lambda_2(s, \Omega(s)) + \int_0^s \Theta(\mathrm{z}, \Omega(\mathrm{z}))\, d\mathrm{z} \right\} ds \right\| \\
&\leq \left| b_0 - \Delta(t) \int_0^\delta \Omega(s)\, ds + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} \left\{ \mathtt{D}_2 + \mathcal{P}s \right\} ds \right| \\
&\leq |b_0| + |\Delta(t)|\nu\delta + \frac{1}{\Gamma(\theta+1)} \left| (t)^\theta\, \mathtt{D}_2 + \mathcal{P} \left\{ \Big(- s(t-s)^\theta \Big)_0^t + \int_0^t (t-s)^\theta\, ds \right\} \right| \\
&\leq |b_0| + |\Delta(t)|\nu\delta + \frac{1}{\Gamma(\theta+1)} \left| (t)^\theta\, \mathtt{D}_2 + \mathcal{P} \frac{t^{\theta+1}}{\theta+1} \right| \\
&\leq |b_0| + |\Delta(t)|\nu + \frac{1}{\Gamma(\theta+1)} \left| \mathtt{D}_2 + \frac{\mathcal{P}}{\theta+1} \right| \\
&\leq |b_0| + \mathcal{M}_1 \\
&\leq \mathcal{M}_2.
\end{aligned}$$

Thus, $T(\mathbf{R}_1)$ is bounded. Suppose that $t, \bar{t} \in \mathtt{J}$, $t > \bar{t}$ and $\Omega \in \mathbf{R}_1$. Then

$$\begin{aligned}
&\left\| T(\Omega(t)) - T(\Omega(\bar{t})) \right\| \\
&= \left\| b_0 - \Delta(t) \int_0^\delta \Omega(s)\, ds + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} \left\{ \Lambda_2(s, \Omega(s)) + \int_0^s \Theta(\mathrm{z}, \Omega(\mathrm{z}))\, d\mathrm{z} \right\} ds - b_0 + \Delta(\bar{t}) \int_0^\delta \Omega(s)\, ds - \frac{1}{\Gamma(\theta)} \int_0^{\bar{t}} (\bar{t}-s)^{\theta-1} \left\{ \Lambda_2(s, \Omega(s)) + \int_0^s \Theta(\mathrm{z}, \Omega(\mathrm{z}))\, d\mathrm{z} \right\} ds \right\| \\
&\leq \left| - \Delta(t) \int_0^\delta \Omega(s)\, ds + \Delta(\bar{t}) \int_0^\delta \Omega(s)\, ds \right| + \left| \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} \left\{ \Lambda_2(s, \Omega(s)) + \int_0^s \Theta(\mathrm{z}, \Omega(\mathrm{z}))\, d\mathrm{z} \right\} ds - \frac{1}{\Gamma(\theta)} \int_0^{\bar{t}} (\bar{t}-s)^{\theta-1} \left\{ \Lambda_2(s, \Omega(s)) + \int_0^s \Theta(\mathrm{z}, \Omega(\mathrm{z}))\, d\mathrm{z} \right\} ds \right| \\
&= \left| \Big(\Delta(\bar{t}) - \Delta(t) \Big) \int_0^\delta \Omega(s)\, ds \right| + \left| \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} \left\{ \Lambda_2(s, \Omega(s)) + \int_0^s \Theta(\mathrm{z}, \Omega(\mathrm{z}))\, d\mathrm{z} \right\} ds - \frac{1}{\Gamma(\theta)} \int_0^t (\bar{t}-s)^{\theta-1} \left\{ \Lambda_2(s, \Omega(s)) + \int_0^s \Theta(\mathrm{z}, \Omega(\mathrm{z}))\, d\mathrm{z} \right\} ds + \frac{1}{\Gamma(\theta)} \int_0^t (\bar{t}-s)^{\theta-1} \left\{ \Lambda_2(s, \Omega(s)) + \right. \right. \\
&\quad \left. \left. + \int_0^s \Theta(\mathrm{z}, \Omega(\mathrm{z}))\, d\mathrm{z} \right\} ds - \frac{1}{\Gamma(\theta)} \int_0^{\bar{t}} (\bar{t}-s)^{\theta-1} \left\{ \Lambda_2(s, \Omega(s)) + \int_0^s \Theta(\mathrm{z}, \Omega(\mathrm{z}))\, d\mathrm{z} \right\} ds \right| \\
&= \left| \Big(\Delta(\bar{t}) - \Delta(t) \Big) \nu\delta \right| + \left| \frac{1}{\Gamma(\theta)} \int_0^t \Big((t-s)^{\theta-1} - (\bar{t}-s)^{\theta-1} \Big) \left\{ \Lambda_2(s, \Omega(s)) + \int_0^s \Theta(\mathrm{z}, \Omega(\mathrm{z}))\, d\mathrm{z} \right\} ds + \frac{1}{\Gamma(\theta)} \int_{\bar{t}}^t (\bar{t}-s)^{\theta-1} \left\{ \Lambda_2(s, \Omega(s)) + \int_0^s \Theta(\mathrm{z}, \Omega(\mathrm{z}))\, d\mathrm{z} \right\} ds \right| \\
&= \left| \Big(\Delta(\bar{t}) - \Delta(t) \Big) \right| \nu + \left| \frac{1}{\Gamma(\theta)} \int_0^t \Big((t-s)^{\theta-1} - (\bar{t}-s)^{\theta-1} \Big) \left\{ \mathtt{D}_2 + \mathcal{P}s \right\} ds \right| + \left| \frac{1}{\Gamma(\theta)} \int_{\bar{t}}^t (\bar{t}-s)^{\theta-1} \left\{ \mathtt{D}_2 + \mathcal{P}s \right\} ds \right| \\
&= \left| \Big(\Delta(\bar{t}) - \Delta(t) \Big) \right| \nu + \left| \frac{\mathtt{D}_2}{\Gamma(\theta)} \int_0^t \Big((t-s)^{\theta-1} - (\bar{t}-s)^{\theta-1} \Big) ds + \frac{\mathcal{P}}{\Gamma(\theta)} \int_0^t s \Big((t-s)^{\theta-1} - (\bar{t}-s)^{\theta-1} \Big) ds \right| + \left| \frac{\mathtt{D}_2}{\Gamma(\theta)} \int_{\bar{t}}^t (\bar{t}-s)^{\theta-1}\, ds + \frac{\mathcal{P}}{\Gamma(\theta)} \int_{\bar{t}}^t s(\bar{t}-s)^{\theta-1}\, ds \right| \\
&= \left| \Big(\Delta(\bar{t}) - \Delta(t) \Big) \right| \nu + \left| \frac{\mathtt{D}_2}{\Gamma(\theta+1)} \Big(- (t-s)^\theta + (\bar{t}-s)^\theta \Big)_0^t + \frac{\mathcal{P}}{\Gamma(\theta)} \int_0^t s(t-s)^{\theta-1}\, ds - \frac{\mathcal{P}}{\Gamma(\theta)} \int_0^t s(\bar{t}-s)^{\theta-1}\, ds \right| + \left| \frac{\mathtt{D}_2}{\Gamma(\theta+1)} \Big(- (\bar{t}-s)^\theta \Big)_{\bar{t}}^t + \frac{\mathcal{P}}{\Gamma(\theta)} \int_{\bar{t}}^t s(\bar{t}-s)^{\theta-1}\, ds \right| \\
&= \left| \Big(\Delta(\bar{t}) - \Delta(t) \Big) \right| \nu + \left| \frac{\mathtt{D}_2}{\Gamma(\theta+1)} \Big((\bar{t}-t)^\theta + t^\theta - \bar{t}^\theta \Big) + \frac{\mathcal{P}}{\Gamma(\theta+2)} (t)^{\theta+1} - \frac{\mathcal{P}}{\Gamma(\theta+2)} (\bar{t})^{\theta+1} \right| + \left| - \frac{\mathtt{D}_2}{\Gamma(\theta+1)} (\bar{t}-t)^\theta - \frac{\mathcal{P}}{\Gamma(\theta+1)} t(\bar{t}-t)^\theta - \frac{\mathcal{P}}{\Gamma(\theta+2)} (\bar{t}-t)^{\theta+1} \right|
\end{aligned}$$

As $\bar{t} \to t$, we have, $\|T(\Omega(t)) - T(\Omega(\bar{t}))\| \to 0$,
that is, $T(\mathbf{R}_1)$ is equicontinuous. We can show that $T(\mathbf{R}_2)$ is equicontinuous and bounded with the similar manner. Thus, by Arzela-Ascoli theorem we conclude that $(\mathbf{R}_1, \mathbf{R}_2)$ is a relatively compact pair. Now, for each $(\Omega, \Psi) \in \mathbf{R}_1 \times \mathbf{R}_2$, we have,

$$\begin{aligned}
&\left\| T(\Omega(t)) - T(\Psi(t)) \right\| \\
&= \left\| b_0 - \Delta(t) \int_0^\delta \Omega(s)\, ds + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} \left\{ \Lambda_2(s, \Omega(s)) + \int_0^s \Theta(\mathrm{z}, \Omega(\mathrm{z}))\, d\mathrm{z} \right\} ds - a_0 + \Delta(t) \int_0^\delta \Psi(s)\, ds - \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} \left\{ \Lambda_1(s, \Psi(s)) + \int_0^s \Theta(\mathrm{z}, \Psi(\mathrm{z}))\, d\mathrm{z} \right\} ds \right\| \\
&\leq \left| b_0 - a_0 \right| + \left| \Delta(t) \int_0^\delta \Psi(s)\, ds - \Delta(t) \int_0^\delta \Omega(s)\, ds \right| + \left| \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} \left\{ \Lambda_2(s, \Omega(s)) + \int_0^s \Theta(\mathrm{z}, \Omega(\mathrm{z}))\, d\mathrm{z} \right\} ds - \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} \left\{ \Lambda_1(s, \Psi(s)) + \int_0^s \Theta(\mathrm{z}, \Psi(\mathrm{z}))\, d\mathrm{z} \right\} ds \right| \\
&\leq \left| b_0 - a_0 \right| + \left| \Delta(t) \int_0^\delta \Big(\Psi - \Omega \Big) ds \right| + \left| \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} \left\{ \Big(\Lambda_2(s, \Omega(s)) - \Lambda_1(s, \Psi(s)) \Big) + \int_0^s \Big(\Theta(\mathrm{z}, \Omega(\mathrm{z})) - \Theta(\mathrm{z}, \Psi(\mathrm{z})) \Big) d\mathrm{z} \right\} ds \right| \\
&\leq \left| b_0 - a_0 \right| + \Delta(t) \left| b_0 - a_0 \right| \delta + \left| \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} \left\{ \left(\Gamma(\theta+1) \left\{ \frac{\|\Omega - \Psi\|}{2} - |b_0 - a_0| \Big(1 + \Delta(t) \Big) \right\} \right) + \int_0^s \left(\Gamma(\theta+2) \frac{\|\Omega - \Psi\|}{2} \right) d\mathrm{z} \right\} ds \right| \\
&\leq \left| b_0 - a_0 \right| \Big(1 + \Delta(t) \Big) + \frac{1}{\Gamma(\theta+1)} \Gamma(\theta+1) \left\{ \frac{\|\Omega - \Psi\|}{2} - |b_0 - a_0| \Big(1 + \Delta(t) \Big) \right\} (t)^\theta + \left(\Gamma(\theta+2) \frac{\|\Omega - \Psi\|}{2} \right) \frac{1}{\Gamma(\theta)} \left| \int_0^t (t-s)^{\theta-1} s\, ds \right| \\
&\leq \left| b_0 - a_0 \right| \Big(1 + \Delta(t) \Big) + \left\{ \frac{\|\Omega - \Psi\|}{2} - |b_0 - a_0| \Big(1 + \Delta(t) \Big) \right\} + \left(\Gamma(\theta+2) \frac{\|\Omega - \Psi\|}{2} \right) \frac{1}{\Gamma(\theta+2)} (t)^{\theta+1} \\
&\leq |b_0 - a_0| (1 + \Delta(t)) + \frac{\|\Omega - \Psi\|}{2} - |b_0 - a_0| \Big(1 + \Delta(t) \Big) + \frac{\|\Omega - \Psi\|}{2} \\
&= \|\Omega - \Psi\|.
\end{aligned}$$

Thus T is relatively nonexpansive.
Now, we assume that the pair $(\zeta_1, \zeta_2) \subseteq (\mathbf{R}_1, \mathbf{R}_2)$ is a NBCC, T-invariant, proximinal pair and $dist(\zeta_1, \zeta_2) = dist(\mathbf{R}_1, \mathbf{R}_2)$. Using assumption (1) of theorem 4.2 and theorem 4.1, we get,

$$
\begin{aligned}
&\mathtt{H}_0(T(\zeta_1) \cup T(\zeta_2)) \\
&= \max\{\mathtt{H}_0(T(\zeta_1)),\ \mathtt{H}_0(T(\zeta_2))\} \\
&\leq \max\{\sup_{t\in J}\{\mathtt{H}_0(\{T\Omega(t) : \Omega \in \zeta_1\})\},\ \sup_{t\in J}\{\mathtt{H}_0(\{T\Psi(t) : \Psi \in \zeta_2\})\}\} \\
&= \max\Big\{\sup_{t\in J}\Big\{\mathtt{H}_0\Big(\Big\{b_0 - \Delta(t)\int_0^\delta \Omega(s)\, ds + \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}\Big\{\Lambda_2(s, \Omega(s)) + \int_0^s \Theta(\mathrm{z}, \Omega(\mathrm{z}))\, d\mathrm{z}\Big\} ds\Big\}\Big)\Big\}, \\
&\sup_{t\in J}\Big\{\mathtt{H}_0\Big(\Big\{a_0 - \Delta(t)\int_0^\delta \Psi(s)\, ds + \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}\Big\{\Lambda_1(s, \Psi(s)) + \int_0^s \Theta(\mathrm{z}, \Psi(\mathrm{z}))\, d\mathrm{z}\Big\} ds\Big\}\Big)\Big\}\Big\} \\
&\leq \max\Big\{\sup_{t\in J}\Big\{\mathtt{H}_0\Big(\Big\{\frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}\Big\{\Lambda_2(s, \Omega(s)) + \int_0^s \Theta(\mathrm{z}, \Omega(\mathrm{z}))\, d\mathrm{z}\Big\} ds\Big\}\Big)\Big\},\ \sup_{t\in J}\Big\{\mathtt{H}_0\Big(\Big\{\frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}\Big\{\Lambda_1(s, \Psi(s)) + \int_0^s \Theta(\mathrm{z}, \Psi(\mathrm{z}))\, d\mathrm{z}\Big\} ds\Big\}\Big)\Big\}\Big\} \\
&\leq \max\Big\{\sup_{t\in J}\Big\{\mathtt{H}_0\Big(\Big\{\frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}\Big\{\Lambda_2(s, \Omega(s)) + \Theta(\varsigma, \Omega(\varsigma))\, s\Big\} ds : \varsigma \in (0,t)\Big\}\Big)\Big\},\ \sup_{t\in J}\Big\{\mathtt{H}_0\Big(\Big\{\frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}\Big\{\Lambda_1(s, \Psi(s)) + \Theta(\varsigma, \Psi(\varsigma))\, s\Big\} ds : \varsigma \in (0,t)\Big\}\Big)\Big\}\Big\} \\
&\leq \max\Big\{\sup_{t\in J}\Big\{\mathtt{H}_0\Big(\Big\{\frac{1}{\Gamma(\theta)}\Big\{\Lambda_2(\bar\xi, \Omega(\bar\xi)) + \Theta(\varsigma, \Omega(\varsigma))\, \bar\xi\Big\}\int_0^t (t-s)^{\theta-1}\, ds : \varsigma, \bar\xi \in (0,t)\Big\}\Big)\Big\},\ \sup_{t\in J}\Big\{\mathtt{H}_0\Big(\Big\{\frac{1}{\Gamma(\theta)}\Big\{\Lambda_1(\bar\xi, \Psi(\bar\xi)) + \Theta(\varsigma, \Psi(\varsigma))\, \bar\xi\Big\}\int_0^t (t-s)^{\theta-1}\, ds : \varsigma, \bar\xi \in (0,t),\Big\}\Big)\Big\}\Big\} \\
&= \max\Big\{\sup_{t\in J}\Big\{\mathtt{H}_0\Big(\Big\{\frac{1}{\Gamma(\theta+1)}\Big\{\Lambda_2(\bar\xi, \Omega(\bar\xi)) + \Theta(\varsigma, \Omega(\varsigma))\, \bar\xi\Big\}(t)^\theta : \varsigma, \bar\xi \in (0,t)\Big\}\Big)\Big\},\ \sup_{t\in J}\Big\{\mathtt{H}_0\Big(\Big\{\frac{1}{\Gamma(\theta+1)}\Big\{\Lambda_1(\bar\xi, \Psi(\bar\xi)) + \Theta(\varsigma, \Psi(\varsigma))\, \bar\xi\Big\}(t)^\theta : \varsigma, \bar\xi \in (0,t)\Big\}\Big)\Big\}\Big\} \\
&\leq \max\Big\{\sup_{t\in J}\Big\{\mathtt{H}_0\Big(\Big\{\Big\{\Lambda_2(\bar\xi, \Omega(\bar\xi)) + \Theta(\varsigma, \Omega(\varsigma))\Big\} : \varsigma, \bar\xi \in (0,t)\Big\}\Big)\Big\},\ \sup_{t\in J}\Big\{\mathtt{H}_0\Big(\Big\{\Big\{\Lambda_1(\bar\xi, \Psi(\bar\xi)) + \Theta(\varsigma, \Psi(\varsigma))\Big\} : \varsigma, \bar\xi \in (0,t)\Big\}\Big)\Big\}\Big\} \\
&\leq \mathtt{H}_0\Big(\Big\{\Lambda_2(J\times\zeta_1) + \Theta(J\times\zeta_1)\Big\} \cup \Big\{\Lambda_1(J\times\zeta_2) + \Theta(J\times\zeta_2)\Big\}\Big) \\
&\leq \mathtt{H}_0(\zeta_1 \cup \zeta_2).
\end{aligned}
$$

Thus, by corollary 3.7, T has a $\mathcal{BPP}$. Hence, we conclude that the system of equations (10) has $s \in \mathbf{R}_1 \cup \mathbf{R}_2$ as an optimal solution. □

Example 4.3. *Assume the following system of integro differential equations with* $\|\Lambda_1(s, \Omega(s))\| \leq 2$, $\|\Lambda_2(s, \Psi(s))\| \leq 3$, *for* $t \in J = [0, 1)$ *as*

$$
\begin{aligned}
\Omega(t) &= \int_0^\delta \Omega(s)\, ds + \frac{1}{\Gamma(1)}\int_0^t (t-s)^{\theta-1}\Big\{\frac{\Omega}{2} + \int_0^s \Omega\, d\mathrm{z}\Big\} ds, \\
\Psi(t) &= 1 + \int_0^\delta \Psi(s)\, ds + \frac{1}{\Gamma(1)}\int_0^t (t-s)^{\theta-1}\Big\{\log\Big(\frac{\Psi}{2}\Big) + \int_0^s \Psi\, d\mathrm{z}\Big\} ds.
\end{aligned}
$$

Consider $\mathbf{R}_1 = \{t\}$ *and* $\mathbf{R}_2 = \{t+1\}$ *on* $J = (0, 1)$.
Define an operator $T : \mathbf{R}_1 \cup \mathbf{R}_2 \to \mathbf{R}$ *such that,*

$$
T(\Omega(t)) = \begin{cases} 1 + \int_0^\delta \Omega(s)\, ds + \frac{1}{\Gamma(1)}\int_0^t (t-s)^{\theta-1}\Big\{\log\Big(\frac{\Omega}{2}\Big) + \int_0^s \Omega\, d\mathrm{z}\Big\} ds, & \Omega \in \mathbf{R}_1, \\ \int_0^\delta \Omega(s)\, ds + \frac{1}{\Gamma(1)}\int_0^t (t-s)^{\theta-1}\Big\{\frac{\Omega}{2} + \int_0^s \Omega\, d\mathrm{z}\Big\} ds, & \Omega \in \mathbf{R}_2. \end{cases} \tag{12}
$$

Here $\Lambda_1 : J \times \zeta_2 \longrightarrow \zeta_2$, and $\Lambda_2 : J \times \zeta_1 \longrightarrow \zeta_1$ with $\zeta_1 = \{t\}$, $\zeta_2 = \{t+1\}$ and $(\zeta_1, \zeta_2) \subseteq (\mathbf{R}_1, \mathbf{R}_2)$ on $J = [0, 1)$, $\mathtt{H}_0\Big(\Big\{\Lambda_2(J\times\zeta_1) + \Theta(J\times\zeta_1)\Big\}\Big) > 0$, $\mathtt{H}_0\Big(\Big\{\Lambda_1(J\times\zeta_2) + \Theta(J\times\zeta_2)\Big\}\Big) > 0$.

Now

$$
\begin{aligned}
&\mathtt{H}_0\Big(\Big\{\Lambda_2(J\times\zeta_1)+\Theta\,(J\times\zeta_1)\Big\}\cup\Big\{\Lambda_1(J\times\zeta_2)+\Theta\,(J\times\zeta_2)\Big\}\Big)\\
&= max\Big\{\mathtt{H}_0\left(\left\{\log\left(\frac{\Omega}{2}\right)+\Omega\right\}\right),\mathtt{H}_0\left(\frac{\Psi}{2}+\Psi\right)\Big\}\\
&\le max\{\mathtt{H}_0\left(\{\zeta_1\}\right),\mathtt{H}_0\left(\{\zeta_2\}\right)\}\\
&\le \mathtt{H}_0(\zeta_1\cup\zeta_2).
\end{aligned}
$$

Now, for all $\Omega\in\mathbf{R}_1$, $\Psi\in\mathbf{R}_2$, we have,

$$
\begin{aligned}
\left\|\Big(\Lambda_2(s,\Omega\,(s))-\Lambda_1(s,\Psi\,(s))\Big)\right\| &= \left\|\log\left(\frac{\Omega}{2}\right)-\frac{\Psi}{2}\right\|\\
&\le \left\|\frac{\Omega}{2}-\frac{\Psi}{2}\right\|\\
&\le \Gamma(2)\frac{\|\Omega-\Psi\|}{2}.
\end{aligned}
$$

Since the above system of equation satisfies all the conditions of theorem (4.2). Therefore $s\in\{0,1\}=\mathbf{R}_1\cup\mathbf{R}_2$ is the optimal solution for the above system of equation at $t=0$ as $\|s-T\,(s)\,\|=1=dist\,(\mathbf{R}_1,\mathbf{R}_2)$.

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