Filomat 39:3 (2025), 861–869
https://doi.org/10.2298/FIL2503861W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Extension of the pseudo Drazin inverse in Banach algebras

Weifeng Wang[a], Kai Yan[a,*]

[a] *School of Mathematics and Statistics, Fuzhou University, Fuzhou 350108, P.R. China*

Abstract. In this paper, we study a new class of generalized inverses in Banach algebras, which is termed extended pseudo Drazin inverse. We derive several characterizations and elementary properties of the extended pseudo Drazin inverse. Moreover, we obtain the generalized versions of Jacobson's lemma for extended Drazin inverse, extended generalized Drazin inverse and extended pseudo Drazin inverse.

1. Introduction

Throughout this paper, $\mathcal{A}$ denotes a complex unital Banach algebra. The notations $\mathcal{A}^{-1}$, $\mathcal{A}^{nil}$ and $\mathcal{A}^{qnil}$ denote the sets of all *invertible*, *nilpotent* and *quasi-nilpotent* elements of $\mathcal{A}$, respectively. For any element $a \in \mathcal{A}$, the *commutant* and the *double commutant* of a are defined by

$$\mathrm{comm}(a) = \{x \in \mathcal{A} : ax = xa\}$$

and

$$\mathrm{comm}^2(a) = \{x \in \mathcal{A} : xy = yx \text{ for all } y \in \mathrm{comm}(a)\},$$

respectively. Recall the *Jacobson radical* [12] in $\mathcal{A}$, which is defined by

$$J(\mathcal{A}) = \left\{a \in \mathcal{A} : 1 + ax \in \mathcal{A}^{-1} \text{ for any } x \in \mathcal{A}\right\}.$$

Let $\sqrt{J(\mathcal{A})}$ denote root of the Jacobson radical, which is given as

$$\sqrt{J(\mathcal{A})} = \{a \in \mathcal{A} : a^n \in J(\mathcal{A}) \text{ for some } n \geq 1\}.$$

In 1958, Drazin [9] pioneered the concept of Drazin inverse in associative rings and semigroups. Let $\mathcal{A}$ denote a Banach algebra. An element $a \in \mathcal{A}$ is *Drazin invertible* if there exists an element $x \in \mathcal{R}$ such that

$$ax = xa, \ xax = x \text{ and } a - axa \in \mathcal{A}^{nil}.$$

2020 *Mathematics Subject Classification*. Primary 15A09; Secondary 15A27, 16H99.
Keywords. extended pseudo Drazin inverse, Jacobson's lemma, Banach algebra.
Received: 20 May 2024; Revised: 04 November 2024; Accepted: 15 November 2024
Communicated by Dijana Mosić
Research supported by the Natural Science Foundation of Fujian Province, China (Grant No. 2022J01104).
* Corresponding author: Kai Yan
Email addresses: weifengwangzzz@163.com (Weifeng Wang), yklolxj@163.com (Kai Yan)
ORCID iDs: https://orcid.org/0009-0005-3824-7513 (Weifeng Wang), https://orcid.org/0000-0003-3557-7945 (Kai Yan)

Such x, if it exists, is unique and will be denoted by a^D. The nilpotency index of $a - axa$ is called the index of a, and is denoted by ind(a). If ind(a)=1, then a is called *group invertible* and the group inverse of a is denoted by $a^\#$. Let $\mathcal{A}^D$ and $\mathcal{A}^\#$ denote the sets of all Drazin invertible and group invertible elements of $\mathcal{A}$, respectively. In 1996, Koliha [11] extended this theory by introducing the concept of generalized Drazin inverse (called g-Drazin inverse) in Banach algebras. An element $a \in \mathcal{A}$ is *g-Drazin invertible* if there exists an element $x \in \mathcal{A}$ such that

$$ax = xa,\ xax = x \text{ and } a - axa \in \mathcal{A}^{qnil}.$$

Such x, if it exists, is unique and will be denoted by a^d. Let $\mathcal{A}^d$ denote the set of all g-Drazin invertible elements of $\mathcal{A}$. In 2012, Wang and Chen [18] introduced the concept of pseudo Drazin inverse (called p-Drazin inverse) in rings and Banach algebras. An element $a \in \mathcal{A}$ is *p-Drazin invertible* if there exists an element $x \in \mathcal{A}$ such that

$$ax = xa,\ xax = x \text{ and } a - axa \in \sqrt{J(\mathcal{A})}.$$

Such x, if it exists, is unique and will be denoted by a^{pD}. Let $\mathcal{A}^{pD}$ denote the set of all p-Drazin invertible elements of $\mathcal{A}$. More interesting results on pseudo Drazin inverse can be found in [3, 23].

Cline's formula and Jacobson's lemma for generalized inverse play significant roles in mathematics, particularly in ring theory and matrix theory. Let $a, b, c, d \in \mathcal{A}$. In 1965, Cline [5] demonstrated that ab is Drazin invertible if and only if ba is Drazin invertible. Moreover, $(ba)^D = b((ab)^D)^2a$, which is known as Cline's formula. Liao, Chen and Cui [13] extended Cline's formula for g-Drazin inverse in rings. Later, Zeng, Wu and Wen [21] established Cline's formula for ac and bd when $acd = dbd$ and $dba = aca$. Further intriguing findings related to Cline's formula are detailed in references [2, 14].

On the other hand, Jacobson's lemma states that $1 - ab$ is invertible if and only if $1 - ba$ is invertible, see [10]. In 2010, Castro-González and Cvetković-Ilic, along with colleagues, extended Jacobson's lemma for Drazin inverse in [1, 8]. In 2012, Zhuang, Chen and Cui [22] extended Jacobson's lemma for g-Drazin inverse. Further, in 2020, Yan, Zeng and Zhu [19] established Jacobson's lemma for ac and bd in cases when $acd = dbd$ and $dba = aca$. This provides an affirmative answer to Mosić's question in [17]. For more intriguing results related to Jacobson's lemma, one can refer to studies found in [6].

In 2020, Mosić [15] introduced the notions of extended Drazin inverse (called e-Drazin inverse) and extended generalized Drazin inverse (called eg-Drazin inverse) in Banach algebras. An element $a \in \mathcal{A}$ is *e-Drazin invertible* if there exists an element $x \in \mathcal{A}$ such that

$$ax = xa,\ xax = x \text{ and } a - axa \in \mathcal{A}^D.$$

And an element $a \in \mathcal{A}$ is *eg-Drazin invertible* if there exists an element $x \in \mathcal{A}$ such that

$$ax = xa,\ xax = x \text{ and } a - axa \in \mathcal{A}^d.$$

Let $\mathcal{A}^{eD}$ and $\mathcal{A}^{ed}$ denote the sets of all e-Drazin invertible and eg-Drazin invertible elements of $\mathcal{A}$, respectively. The e-Drazin invertibility and eg-Drazin invertibility provide new characterizations of Drazin invertibility and g-Drazin invertibility, respectively. It is not hard to see that e-Drazin inverse and eg-Drazin inverse are not unique, so we can determine the existence of g-Drazin inverse more effectively by utilizing the existence of eg-Drazin inverse. Although the existence of eg-Drazin inverse can characterize the existence of g-Drazin inverse, the eg-Drazin inverse can not yet be fully expressed by the g-Drazin inverse. Therefore, it is meaningful to investigate certain properties such as Cline's formula and Jacobson's lemma for the eg-Drazin inverse and e-Drazin inverse.

This article is organized as follows. In Section 2, based on the concept of p-Drazin inverse proposed by Wang [18] in Banach algebras, we introduce the notion of ep-Drazin inverse in Banach algebras and investigate its elementary properties. In section 3, drawing on Yan's approach [19] to Jacobson's lemma for Drazin inverse and g-Drazin inverse, we supplement Mosić's work [15] by proving the Jacobson's lemma for e-Drazin inverse, eg-Drazin inverse and ep-Drazin inverse.

2. Extended pseudo Drazin inverse

In this section, we firstly define the extended pseudo Drazin inverse (called ep-Drazin inverse) in Banach algebras by replacing the condition $a - axa \in \sqrt{J(\mathcal{A})}$ in the definition of p-Drazin inverse with $a - axa \in \mathcal{A}^{pD}$.

Definition 2.1. *An element $a \in \mathcal{A}$ is extended pseudo Drazin invertible (or ep-Drazin invertible for short) if there exists an element $x \in \mathcal{A}$ such that*

$$ax = xa, \ xax = x \text{ and } a - axa \in \mathcal{A}^{pD}.$$

In this case, x is an extended pseudo Drazin inverse of a.

In the following theorem, we offer several characterizations for ep-Drazin invertible elements by utilizing corresponding idempotents.

Theorem 2.2. *Let $a \in \mathcal{A}$. The following statements are equivalent:*
(i) *a is ep-Drazin invertible.*
(ii) *There exists an idempotent $p \in \mathcal{A}$ such that $ap = pa$, $ap \in (p\mathcal{A}p)^{-1}$ and $a(1-p) \in \mathcal{A}^{pD}$.*
(iii) *There exists an idempotent $p \in \mathcal{A}$ such that $ap = pa$, $ap \in \mathcal{A}^{\#}$ and $a(1-p) \in \mathcal{A}^{pD}$.*

Proof. (i) ⇒ (ii) Suppose that $x \in \mathcal{A}$ is an ep-Drazin inverse of a. Put $p = ax$, we observe that $p^2 = axax = ax = p$, $pa = axa = ap$ and $a(1-p) = a(1-ax) = a - axa \in \mathcal{A}^{pD}$. From

$$apx = axp = p^2 = p = p^2 = axp = xap,$$

it follows that $ap \in (p\mathcal{A}p)^{-1}$ and $x = (ap)^{-1}_{p\mathcal{A}p}$.

(ii) ⇒ (iii) Assume that there exists an idempotent p satisfying (ii). In order to prove that this p satisfies (iii), we only need to verify that $ap \in \mathcal{A}^{\#}$. Since $(ap)^{-1}_{p\mathcal{A}p} \in p\mathcal{A}p$, there exists an element $x \in \mathcal{A}$ such that $(ap)^{-1}_{p\mathcal{A}p} = pxp$. Thus,

$$p(ap)^{-1}_{p\mathcal{A}p} = p^2xp = pxp = (ap)^{-1}_{p\mathcal{A}p}.$$

From

$$ap(ap)^{-1}_{p\mathcal{A}p} = p = (ap)^{-1}_{p\mathcal{A}p}ap, \ ap(ap)^{-1}_{p\mathcal{A}p}ap = pap = ap^2 = ap$$

and

$$(ap)^{-1}_{p\mathcal{A}p}ap(ap)^{-1}_{p\mathcal{A}p} = p(ap)^{-1}_{p\mathcal{A}p} = (ap)^{-1}_{p\mathcal{A}p},$$

it follows that $ap \in \mathcal{A}^{\#}$ and $(ap)^{\#} = (ap)^{-1}_{p\mathcal{A}p}$.

(iii) ⇒ (i) Assume that there exists an idempotent p satisfying (iii). Put $x = (ap)^{\#}$, by double commutativity of the group inverse, we have $ax = xa$ and $px = xp$. From $x = (ap)^{\#} = ((ap)^{\#})^2ap = ((ap)^{\#})^2ap^2 = (ap)^{\#}p = xp$, it follows that $xax = xapx = x$ and $a - axa = a - apxpa = a - ap = a(1-p) \in \mathcal{A}^{pD}$. Thus, we can obtain that a is ep-Drazin invertible. □

For any element $a \in \sqrt{J(\mathcal{A})}$, we can infer $a^{pD} = 0$, which implies $a \in \mathcal{A}^{pD}$. Hence, $\sqrt{J(\mathcal{A})} \subseteq \mathcal{A}^{pD}$. Likewise, we have $\mathcal{A}^{nil} \subseteq \mathcal{A}^{D}$ and $\mathcal{A}^{qnil} \subseteq \mathcal{A}^{d}$. In the following theorem, we reveal the relationship between the p-Drazin inverse and the ep-Drazin inverse.

Theorem 2.3. *Let $a \in \mathcal{A}$. The following statements are equivalent:*
(i) *a is p-Drazin invertible.*
(ii) *a is ep-Drazin invertible.*
(iii) *There exists an idempotent $p \in \mathcal{A}$ commuting with a such that*

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p,$$

where $a_1 \in (p\mathcal{A}p)^{-1}$ and $a_2 \in \mathcal{A}^{pD}$.

(iv) *There exists an idempotent $p \in \mathcal{A}$ commuting with a such that*

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p,$$

where $a_1 \in (p\mathcal{A}p)^{-1}$ and $a_2 \in ((1-p)\mathcal{A}(1-p))^{pD}$.

Proof. (i) ⇒ (ii) Suppose that $x \in \mathcal{A}$ is the p-Drazin inverse of a. Then $ax = xa$, $xax = x$ and $a - axa \in \sqrt{J(\mathcal{A})}$. Since $\sqrt{J(\mathcal{A})} \subseteq \mathcal{A}^{pD}$, we can obtain that a is ep-Drazin invertible, and x is an ep-Drazin inverse of a.

(ii) ⇒ (iii) Since a is ep-Drazin invertible, there exists an idempotent $p \in \mathcal{A}$ commuting with a such that $ap \in (p\mathcal{A}p)^{-1}$ and $a(1-p) \in \mathcal{A}^{pD}$ by Theorem 2.2. In addition, we can represent element $a \in \mathcal{A}$ as

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_p,$$

where $a_{11} = pap$, $a_{12} = pa(1-p)$, $a_{21} = (1-p)ap$, $a_{22} = (1-p)a(1-p)$.
Hence, $a_1 = a_{11} = pap = ap^2 = ap \in (p\mathcal{A}p)^{-1}$, $a_{12} = pa(1-p) = ap(1-p) = 0$, $a_{21} = (1-p)ap = (1-p)pa = 0$, $a_2 = a_{22} = (1-p)a(1-p) = a(1-p)^2 = a(1-p) \in \mathcal{A}^{pD}$.

(iii) ⇒ (iv) [23, Lemma 2.7].

(iv) ⇒ (i) [23, Theorem 3.2]. □

We use the notation $\mathcal{A}^{epD}$ to denote the set of all ep-Drazin invertible elements of $\mathcal{A}$. Due to the equivalence relation between the p-Drazin invertibility and the ep-Drazin invertibility, it follows that $\mathcal{A}^{eD} \subseteq \mathcal{A}^{epD} \subseteq \mathcal{A}^{ed}$. In addition, if $a \in \mathcal{A}$ is p-Drazin invertible, then it is evident that both 0 and a^{pD} are ep-Drazin inverses of a. Thus, ep-Drazin inverse is not unique in general.

Lemma 2.4. *Let $a \in \mathcal{A}$. Then the following statements are equivalent:*

(i) *$a \in \mathcal{A}^{epD}$.*

(ii) *$a^n \in \mathcal{A}^{epD}$ for any integer $n \geq 1$.*

(iii) *$a^n \in \mathcal{A}^{epD}$ for some integer $n \geq 1$.*

Proof. It is obvious by [23, Lemma 2.8] and Theorem 2.3. □

In [[24]], some additive properties of p-Drazin inverse were presented. Next, we examine the additive property of ep-Drazin inverse.

Theorem 2.5. *If $a, b \in \mathcal{A}^{epD}$ with $ab = ba = 0$, $x \in \mathcal{A}$ is an ep-Drazin inverse of a, and $y \in \mathcal{A}$ is an ep-Drazin inverse of b, then $a + b \in \mathcal{A}^{epD}$ and $x + y$ is an ep-Drazin inverse of $a + b$.*

Proof. Suppose that $ab = ba = 0$. Then we have $ay = aby^2 = 0$, $ya = y^2ba = 0$, $bx = bax^2 = 0$ and $xb = x^2ab = 0$. Thus, $(a+b)(x+y) = (x+y)(a+b)$. Note that $xax = x$ and $yby = y$. We have

$$(x+y)(a+b)(x+y) = xax + yby = x + y.$$

According to $a - axa \in \mathcal{A}^{pd}$, $b - byb \in \mathcal{A}^{pd}$ and $(a - axa)(b - byb) = 0$, we obtain that

$$(a+b) - (a+b)(x+y)(a+b) = (a - axa) + (b - byb) \in \mathcal{A}^{pD}$$

by [24, Theorem 2.5]. Hence, $a + b \in \mathcal{A}^{epD}$ and $x + y$ is an ep-Drazin inverse of $a + b$. □

Corollary 2.6. *If $a_1, a_2, \cdots, a_n \in \mathcal{A}^{epD}$ with $a_ia_j = 0$ $(i, j = 1, \cdots, n; i \neq j)$, and $x_i \in \mathcal{A}$ is an ep-Drazin inverse of a_i for $i = 1, \cdots, n$, respectively. Then $a_1 + \cdots + a_n \in \mathcal{A}^{epD}$ and $x_1 + \cdots + x_n$ is an ep-Drazin inverse of $a_1 + \cdots + a_n$.*

The following theorem demonstrates certain decomposition properties of the ep-Drazin invertible element in Banach algebras.

Theorem 2.7. *Let $a \in \mathcal{A}$. The following statements are equivalent:*
(i) *a is ep-Drazin invertible.*
(ii) *$a = x + y$, where $xy = yx = 0$, $x \in \mathcal{A}^{\#}$ and $y \in \mathcal{A}^{pD}$.*
(iii) *$a = x + y$, where $xy = yx = 0$, $x \in \mathcal{A}^{D}$ and $y \in \mathcal{A}^{pD}$.*
(iv) *$a = x + y$, where $xy = yx = 0$, $x \in \mathcal{A}^{pD}$ and $y \in \mathcal{A}^{pD}$.*

Proof. (i) ⇒ (ii): It is obvious by Theorem 2.2.
(ii) ⇒ (iii): This part is clear.
(iii) ⇒ (iv): This part is clear.
(iv) ⇒ (i): It is obvious by Theorem 2.3 and Theorem 2.5. □

Theorem 2.8. *If $a \in \mathcal{A}^{epD}$ and x is an ep-Drazin inverse of a, then*
(i) *a is an ep-Drazin inverse of a if and only if $a^3 = a$.*
(ii) *x^n is an ep-Drazin inverse of a^n for any positive integer n.*
(iii) *x is an ep-Drazin inverse of a^2x.*
(iv) *a^2x is an ep-Drazin inverse of x.*

Proof. (i) Assume that a is an ep-Drazin inverse of a. Then $a^3 = aaa = a$. Conversely, if $a^3 = a$, then for $b = a$, we have $ab = ba$, $bab = b$ and $a - aba = 0 \in \mathcal{A}^{pD}$. Thus, a is ep-Drazin invertible, and a is an ep-Drazin inverse of a.

(ii) For any integer n, if x is an ep-Drazin inverse of a, then $a^n x^n = x^n a^n$, $x^n a^n x^n = (xax)^n = x^n$ and $a^n - a^n x^n a^n = (a - axa)^n \in \mathcal{A}^{pD}$ by [23, Lemma 2.8]. Thus, a^n is ep-Drazin invertible, and x^n is an ep-Drazin inverse of a^n.

(iii) If x is an ep-Drazin inverse of a, then $(a^2x)x = x(a^2x)$, $x(a^2x)x = x$ and $a^2x - (a^2x)x(a^2x) = 0 \in \mathcal{A}^{pD}$. Thus, a^2x is ep-Drazin invertible, and x is an ep-Drazin inverse of a^2x.

(iv) Since x is an ep-Drazin inverse of a, we have $x(a^2x) = (a^2x)x$, $(a^2x)x(a^2x) = a^2x$ and $x - x(a^2x)x = 0 \in \mathcal{A}^{pD}$. Thus, x is ep-Drazin invertible, and a^2x is an ep-Drazin inverse of x. □

3. Jacobson's lemma and Cline's formula for the ep-Drazin inverse

At the beginning of this section, we prove the Jacobson's lemma for ep-Drazin inverse under the conditions that $acd = dbd$ and $dba = aca$.

Theorem 3.1. *Let $a, b, c, d \in \mathcal{A}$ satisfy $acd = dbd$ and $dba = aca$. Then*

$$\alpha = 1 - bd \in \mathcal{A}^{epD} \Leftrightarrow \beta = 1 - ac \in \mathcal{A}^{epD}.$$

In this case, if $x \in \mathcal{A}$ is an ep-Drazin inverse of α, then

$$(1 - ds^{\pi}\alpha^{\pi}[1 - s^{\pi}\alpha^{\pi}\alpha(1 + bd)]^{-1}bac)(1 + ac) + dxbac + d\alpha^{\pi}s^{pD}bac$$

is an ep-Drazin inverse of β. Conversely, if $y \in \mathcal{A}$ is an ep-Drazin inverse of β, then

$$(1 - bact^{\pi}\beta^{\pi}[1 - t^{\pi}\beta^{\pi}\beta(1 + ac)]^{-1}d)(1 + bd) + bacyd + bac\beta^{\pi}t^{pD}d$$

is an ep-Drazin inverse of α. In both directions, the notations $\alpha^{\pi} = 1 - \alpha x$, $s = \alpha - \alpha x\alpha$, $s^{\pi} = 1 - ss^{pD}$, $\beta^{\pi} = 1 - \beta y$, $t = \beta - \beta y\beta$, $t^{\pi} = 1 - tt^{pD}$.

Proof. Suppose that $\alpha = 1 - bd$ is ep-Drazin invertible, and x is an ep-Drazin inverse of α. Since $s = \alpha - \alpha x\alpha \in \mathcal{A}^{pD}$, we obtain

$$s^{\pi}\alpha^{\pi}\alpha = s^{\pi}s = s - ss^{pD}s \in \sqrt{J(\mathcal{A})} \subseteq \mathcal{A}^{qnil}$$

by [7]. From $\alpha = 1 - bd \in \mathrm{comm}(x)$, it follows that $1 + bd = 2 - (1 - bd) = 2 - \alpha \in \mathrm{comm}(x)$. Note that pseudo Drazin inverse is generalized Drazin inverse. Then $s^{pD} \in \mathrm{comm}^2(s)$, we can deduce that $1 + bd \in \mathrm{comm}(s^{\pi}\alpha^{\pi}\alpha)$. Thus, $1 - s^{\pi}\alpha^{\pi}\alpha(1 + bd)$ is invertible. Set

$$r = (1 - ds^{\pi}\alpha^{\pi}[1 - s^{\pi}\alpha^{\pi}\alpha(1 + bd)]^{-1}bac)(1 + ac) + dxbac + d\alpha^{\pi}s^{pD}bac.$$

In order to prove that r is an ep-Drazin inverse of β, we need to show that

$$\text{(i) } r\beta = \beta r; \text{ (ii) } r\beta r = r; \text{ (iii) } \beta - \beta r\beta \in \mathcal{A}^{pD}.$$

(i) It is obvious that α^π, s^π are idempotent. Since bd commutes with α, we get $(bd)\alpha^\pi = \alpha^\pi(bd)$. Note that $s^{pD} \in \text{comm}^2(s)$, which implies that bd commutes with s^{pD} and s^π. Hence, $\alpha, \alpha^\pi, s^{pD}, s^\pi$ and $[1-s^\pi\alpha^\pi\alpha(1+bd)]^{-1}$ commute with each other. So

$$\begin{aligned}
r\beta &= 1-(ac)^2 - ds^\pi\alpha^\pi[1-s^\pi\alpha^\pi\alpha(1+bd)]^{-1}bac(1+ac)(1-ac) \\
&\quad + dxbac(1-ac) + d\alpha^\pi s^{pD}bac(1-ac) \\
&= 1-[dbac - dxbac(1-ac)] + d\alpha^\pi s^{pD}\alpha bac \\
&\quad - ds^\pi\alpha^\pi[1-s^\pi\alpha^\pi\alpha(1+bd)]^{-1}(1-bd)(1+bd)bac \\
&= 1-(dbac - dx\alpha bac) + d\alpha^\pi s^{pD}\alpha bac \\
&\quad - ds^\pi\alpha^\pi[1-s^\pi\alpha^\pi\alpha(1+bd)]^{-1}(1-bd)(1+bd)bac \\
&= 1-d\alpha^\pi bac + d(\alpha^\pi)^2 s^{pD}\alpha bac - d(s^\pi\alpha^\pi)^2[1-s^\pi\alpha^\pi\alpha(1+bd)]^{-1}\alpha(1+bd)bac \\
&= 1-d\alpha^\pi(1-s^{pD}\alpha^\pi\alpha)bac - ds^\pi\alpha^\pi[1-s^\pi\alpha^\pi\alpha(1+bd)]^{-1}s^\pi\alpha^\pi\alpha(1+bd)bac \\
&= 1-d\alpha^\pi s^\pi bac - ds^\pi\alpha^\pi[1-s^\pi\alpha^\pi\alpha(1+bd)]^{-1}(1-[1-s^\pi\alpha^\pi\alpha(1+bd)])bac \\
&= 1-ds^\pi\alpha^\pi bac - ds^\pi\alpha^\pi[1-s^\pi\alpha^\pi\alpha(1+bd)]^{-1}bac + ds^\pi\alpha^\pi bac \\
&= 1-ds^\pi\alpha^\pi[1-s^\pi\alpha^\pi\alpha(1+bd)]^{-1}bac
\end{aligned}$$

and

$$\begin{aligned}
\beta r &= 1-(ac)^2 - (1-ac)ds^\pi\alpha^\pi[1-s^\pi\alpha^\pi\alpha(1+bd)]^{-1}bac(1+ac) \\
&\quad + (1-ac)dxbac + (1-ac)d\alpha^\pi s^{pD}bac \\
&= 1-[dbac - (1-ac)dxbac] + d\alpha\alpha^\pi s^{pD}bac \\
&\quad - d\alpha s^\pi\alpha^\pi[1-s^\pi\alpha^\pi\alpha(1+bd)]^{-1}(1+bd)bac \\
&= 1-(dbac - d\alpha xbac) + d\alpha^\pi s^{pD}\alpha bac \\
&\quad - ds^\pi\alpha^\pi[1-s^\pi\alpha^\pi\alpha(1+bd)]^{-1}\alpha(1+bd)bac \\
&= 1-d\alpha^\pi bac + d(\alpha^\pi)^2 s^{pD}\alpha bac - d(s^\pi\alpha^\pi)^2[1-s^\pi\alpha^\pi\alpha(1+bd)]^{-1}\alpha(1+bd)bac \\
&= 1-d\alpha^\pi(1-s^{pD}\alpha^\pi\alpha)bac - ds^\pi\alpha^\pi[1-s^\pi\alpha^\pi\alpha(1+bd)]^{-1}s^\pi\alpha^\pi\alpha(1+bd)bac \\
&= 1-d\alpha^\pi s^\pi bac - ds^\pi\alpha^\pi[1-s^\pi\alpha^\pi\alpha(1+bd)]^{-1}(1-[1-s^\pi\alpha^\pi\alpha(1+bd)])bac \\
&= 1-ds^\pi\alpha^\pi bac - ds^\pi a^\pi[1-s^\pi\alpha^\pi\alpha(1+bd)]^{-1}bac + ds^\pi\alpha^\pi bac \\
&= 1-ds^\pi\alpha^\pi[1-s^\pi\alpha^\pi\alpha(1+bd)]^{-1}bac.
\end{aligned}$$

Therefore, r commutes with β.

(ii) From

$$s^\pi\alpha^\pi[1-s^\pi\alpha^\pi\alpha(1+bd)] = s^\pi\alpha^\pi[1-\alpha(1+bd)] = s^\pi\alpha^\pi(bd)^2,$$

it follows that

$$s^\pi a^\pi[1-s^\pi a^\pi\alpha(1+bd)]^{-1}bacd = s^\pi a^\pi(bd)^2[1-s^\pi a^\pi\alpha(1+bd)]^{-1} = s^\pi\alpha^\pi. \tag{1}$$

In addition, it is obvious that $\alpha^\pi x = s^\pi s^{pD} = 0$. By the equality (1), we have

$$\begin{aligned}
r\beta r &= r - ds^\pi\alpha^\pi[1-s^\pi\alpha^\pi\alpha(1+bd)]^{-1}bacr \\
&= r - (ds^\pi\alpha^\pi[1-s^\pi\alpha^\pi\alpha(1+bd)]^{-1}bac - ds^\pi\alpha^\pi[1-s^\pi\alpha^\pi\alpha(1+bd)]^{-1}bac)(1+ac) \\
&\quad - ds^\pi\alpha^\pi xbac - ds^\pi\alpha^\pi\alpha^\pi s^{pD}bac \\
&= r.
\end{aligned}$$

(iii) Set $A = \alpha s^{\pi}\alpha^{\pi}[1 - s^{\pi}\alpha^{\pi}\alpha(1 + bd)]^{-1}bac$ and $B = d$, we have

$$\begin{aligned} AB &= \alpha s^{\pi}\alpha^{\pi}[1 - s^{\pi}\alpha^{\pi}\alpha(1 + bd)]^{-1}bacd \\ &= s^{\pi}\alpha^{\pi}\alpha = s - ss^{pD}s \in \sqrt{J(\mathcal{A})} \subseteq \mathcal{A}^{pD}. \end{aligned}$$

By the Cline's formula for the pseudo Drazin inverse from [21, Theorem 2.8], we can obtain that

$$\begin{aligned} \beta - \beta r\beta &= \beta(1 - r\beta) \\ &= \beta d s^{\pi}\alpha^{\pi}[1 - s^{\pi}\alpha^{\pi}\alpha(1 + bd)]^{-1}bac \\ &= d\alpha s^{\pi}\alpha^{\pi}[1 - s^{\pi}\alpha^{\pi}\alpha(1 + bd)]^{-1}bac \\ &= BA \in \mathcal{A}^{pD}. \end{aligned}$$

The proof of the converse implication follows by a similar approach to that previously discussed. □

In the following two theorems, we provide Jacobson's lemma for eg-Drazin inverse and e-Drazin inverse.

Theorem 3.2. *Let $a, b, c, d \in \mathcal{A}$ satisfy $acd = dbd$ and $dba = aca$. Then*

$$\alpha = 1 - bd \in \mathcal{A}^{ed} \Leftrightarrow \beta = 1 - ac \in \mathcal{A}^{ed}.$$

In this case, if $x \in \mathcal{A}$ is an eg-Drazin inverse of α, then

$$(1 - ds^{\pi}\alpha^{\pi}[1 - s^{\pi}\alpha^{\pi}\alpha(1 + bd)]^{-1}bac)(1 + ac) + dxbac + d\alpha^{\pi}s^{d}bac$$

is an eg-Drazin inverse of β. Conversely, if $y \in \mathcal{A}$ is an eg-Drazin inverse of β, then

$$(1 - bact^{\pi}\beta^{\pi}[1 - t^{\pi}\beta^{\pi}\beta(1 + ac)]^{-1}d)(1 + bd) + bacyd + bac\beta^{\pi}t^{d}d$$

is an eg-Drazin inverse of β. In both directions, the notations $\alpha^{\pi} = 1 - \alpha x$, $s = \alpha - \alpha x\alpha$, $s^{\pi} = 1 - ss^{d}$, $\beta^{\pi} = 1 - \beta y$, $t = \beta - \beta y\beta$, $t^{\pi} = 1 - tt^{d}$.

Proof. Suppose that $\alpha = 1 - bd$ is eg-Drazin invertible, and x is an eg-Drazin inverse of α. Since $s = \alpha - \alpha x\alpha \in \mathcal{A}^{d}$, we get $s^{\pi}\alpha^{\pi}\alpha = s^{\pi}s = s - ss^{d}s \in \mathcal{A}^{qnil}$. Thus, $1 - s^{\pi}\alpha^{\pi}\alpha(1 + bd)$ is invertible by Theorem 3.1. Set

$$r = (1 - ds^{\pi}\alpha^{\pi}[1 - s^{\pi}\alpha^{\pi}\alpha(1 + bd)]^{-1}bac)(1 + ac) + dxbac + d\alpha^{\pi}s^{d}bac.$$

In order to prove that r is an eg Drazin inverse of β, we need to show that

$$\text{(i) } r\beta = \beta r; \text{ (ii) } r\beta r = r; \text{ (iii) } \beta - \beta r\beta \in \mathcal{A}^{d}.$$

Obviously, the proofs of (i) and (ii) are completely analogous to Theorem 3.1. For (iii), set $A = \alpha s^{\pi}\alpha^{\pi}[1 - s^{\pi}\alpha^{\pi}\alpha(1 + bd)]^{-1}bac$ and $B = d$, we have

$$\begin{aligned} AB &= \alpha s^{\pi}\alpha^{\pi}[1 - s^{\pi}\alpha^{\pi}\alpha(1 + bd)]^{-1}bacd \\ &= \alpha s^{\pi}\alpha^{\pi}[1 - s^{\pi}\alpha^{\pi}\alpha(1 + bd)]^{-1}(bd)^{2} \\ &= s^{\pi}\alpha^{\pi}\alpha = s - ss^{d}s \in \mathcal{A}^{qnil} \subseteq \mathcal{A}^{d}. \end{aligned}$$

By the Cline's formula for the generalized Drazin inverse from [13, Theorem 2.2], we can deduce that

$$\begin{aligned} \beta - \beta r\beta &= \beta(1 - r\beta) \\ &= \beta d s^{\pi}\alpha^{\pi}[1 - s^{\pi}\alpha^{\pi}\alpha(1 + bd)]^{-1}bac \\ &= d\alpha s^{\pi}\alpha^{\pi}[1 - s^{\pi}\alpha^{\pi}\alpha(1 + bd)]^{-1}bac \\ &= BA \in \mathcal{A}^{d}. \end{aligned}$$

The proof of the converse implication follows a similar approach to the one previously discussed. □

Theorem 3.3. *Let $a, b, c, d \in \mathcal{A}$ satisfy $acd = dbd$ and $dba = aca$. Then*

$$\alpha = 1 - bd \in \mathcal{A}^{eD} \Leftrightarrow \beta = 1 - ac \in \mathcal{A}^{eD}.$$

In this case, if $x \in \mathcal{A}$ is an e-Drazin inverse of α, then

$$(1 - ds^{\pi}\alpha^{\pi}[1 - s^{\pi}\alpha^{\pi}\alpha(1 + bd)]^{-1}bac)(1 + ac) + dxbac + d\alpha^{\pi}s^{D}bac$$

is an e-Drazin inverse of β. Conversely, if $y \in \mathcal{A}$ is an e-Drazin inverse of β, then

$$(1 - bact^{\pi}\beta^{\pi}[1 - t^{\pi}\beta^{\pi}\beta(1 + ac)]^{-1}d)(1 + bd) + bacyd + bac\beta^{\pi}t^{D}d$$

is an e-Drazin inverse of α. In both directions, the notations $\alpha^{\pi} = 1 - \alpha x$, $s = \alpha - \alpha x\alpha$, $s^{\pi} = 1 - ss^{D}$, $\beta^{\pi} = 1 - \beta y$, $t = \beta - \beta y\beta$, $t^{\pi} = 1 - tt^{D}$.

Proof. Using the similar computation of Theorem 3.2 and applying the Cline's formula for Drazin inverse, one can complete the proof similarly. □

In [15], Mosić has already established the Cline's formula for eg-Drazin inverse and e-Drazin inverse under the conditions $acd = dbd$ and $dba = aca$. Next, we establish the Cline's formula for ep-Drazin inverse under the same conditions.

Theorem 3.4. *Let $a, b, c, d \in \mathcal{A}$ satisfy $acd = dbd$ and $dba = aca$. Then*

$$ac \in \mathcal{A}^{epD} \Leftrightarrow bd \in \mathcal{A}^{epD}.$$

In this case, if $x \in \mathcal{A}$ is an ep-Drazin inverse of ac, then bx^2d is an ep-Drazin inverse of bd. Conversely, if $y \in \mathcal{A}$ is an ep-Drazin inverse of bd, then dy^3bac is an ep-Drazin inverse of ac. In addition, we have

$$(b(1 - acx)d)^{pD} = b[(ac(1 - acx))^{pD}]^2 d$$

and

$$((1 - dyb)ac)^{pD} = d(1 - bdy)[(bd(1 - bdy))^{pD}]^3 bac.$$

Proof. Suppose that ac is ep-Drazin invertible, and x is an ep-Drazin inverse of ac. Set $s = bx^2d$, in order to prove that s is an ep-Drazin inverse of bd, we need to show that

$$\text{(i) } bds = sbd;\ \text{(ii) } sbds = s;\ \text{(iii) } bd - bdsbd \in \mathcal{A}^{pD}.$$

(i) Since

$$bds = bdbx^2d = bdbacx^3d = bacacx^3d = bxd$$

and

$$sbd = bx^2dbd = bx^2acd = bxd,$$

we get $bds = sbd$.

(ii) Now, we have

$$sbds = bx^2dbdbx^2d = bx^2dbdbacx^3d = bx^2acacacx^3d = bx^2d = s.$$

(iii) Given that

$$bd - bdsbd = (1 - bds)bd = (1 - bxd)bd = bd - bxacd = b(1 - acx)d. \tag{2}$$

Set $A = ac$, $C = 1 - acx$, $B = b$ and $D = (1 - acx)d$. We observe that

$$ACA = ac(1 - acx)ac = (1 - acx)acac = (1 - acx)dbac = DBA$$

and

$$\begin{aligned} DBD &= (1 - acx)db(1 - acx)d \\ &= dbd - dbacxd - acxdbd + acxdbacxd \\ &= acd - acacxd - acxacd + acxacacxd \\ &= (1 - acx)ac(1 - acx)d \\ &= ac(1 - acx)(1 - acx)d \\ &= ACD. \end{aligned}$$

Since $ac \in \mathcal{A}^{epD}$, we have $AC = ac(1 - acx) \in \mathcal{A}^{pD}$. By [21, Theorem 2.8] and the equality (2), we can deduce that

$$bd - bdsbd = b(1 - acx)d = BD \in \mathcal{A}^{pD}.$$

Moreover, it is not hard to see that $1 - acx$ is idempotent, so we have

$$\begin{aligned} (b(1 - acx)d)^{pD} = (BD)^{pD} &= B((AC)^{pD})^2 D \\ &= b[(ac(1 - acx))^{pD}]^2(1 - acx)d \\ &= b[(ac(1 - acx))^{pD}]^3 ac(1 - acx)^2 d \\ &= b[(ac(1 - acx))^{pD}]^3 ac(1 - acx)d \\ &= b[(ac(1 - acx))^{pD}]^2 d. \end{aligned}$$

The proof of the converse implication follows a similar approach to the one previously discussed. □

Acknowledgement. The authors thank the reviewers for their careful reading and valuable comments.

References

[1] N. Castro-González, C. Mendes-Araújo, P. Patricio, Generalized inverses of a sum in rings, Bull. Aust. Math. Soc. 82(2010), 156-164.
[2] H. Chen, M.S. Abdolyousefi, Generalized Cline's formula for the generalized Drazin inverse in rings, Filomat 37(2023), 3021-3028.
[3] J. Chen, X. Chen, H. Zguitti, On the weighted pseudo Drazin invertible elements in associative rings and Banach algebras, Filomat 33(2019), 6359-6367.
[4] J. Chen, H. Zguitti. On the pseudo Drazin inverse of the sum of two elements in a Banach algebra, Filomat 31(2017), 2011-2022.
[5] R.E. Cline, An application of representation for the generalized inverse of a matrix, MRC Technical Report 592, 1965.
[6] G. Corach, B. Duggal, R. Harte, Extensions of Jacobson's lemma, Comm. Algebra 41(2013), 520-531.
[7] J. Cui. Quasinilpotents in rings and their applications, Turk. J. Math. 42(2018), 2854-2862.
[8] D. Cvetković-Ilic, R. Harte, On Jacobson's lemma and Drazin invertibility, Appl. Math. Lett. 23(2010), 417-420.
[9] M.P. Drazin, Pseudo-inverses in associative rings and semigroups, Amer. Math. Monthly 65(1958), 506-514.
[10] P.R. Halmos, Does mathematics have elements? Math. Intelligencer 3(1981), 147-153.
[11] J.J. Koliha, A generalized Drazin inverse, Glasgow Math. J. 38(1996), 367-381.
[12] T.Y. Lam, A First Course in Noncommutative Rings, Graduate Texts in Math-ematics, Vol. 131, 2nd edn. Springer-Verlag, New York, 2001.
[13] Y. Liao, J. Chen, J. Cui, Cline's formula for the generalized Drazin inverse, Bull. Malays. Math. Sci. Soc. 37(2014), 37-42.
[14] D. Mosić, A note on Cline's formula for the generalized Drazin inverse, Linear Multilinear Algebra 63(6) (2015), 1106-1110.
[15] D. Mosić, Extended g-Drazin inverse in a Banach algebra, Bull. Malays. Math. Sci. Soc. 43(2020), 879-892.
[16] D. Mosić, Extension of the generalized n-strong Drazin inverse, Filomat 37(2023), 7781-7790.
[17] D. Mosić, Extensions of Jacobson's lemma for Drazin inverses, Aequ. Math. 91(2017), 419-428.
[18] Z. Wang, J. Chen, Pseudo Drazin inverses in associative rings and Banach algebras, Linear Algebra Appl. 437(2012), 1332-1345.
[19] K. Yan, Q. Zeng, Y. Zhu, Generalized Jacobson's lemma for Drazin inverses and its applications, Linear Multilinear Algebra 68(2020), 81-93.
[20] K. Yan, Q. Zeng, Y. Zhu, On Drazin spectral equation for the operator products, Complex Anal. Oper. Theory 14(2020), 1-15.
[21] Q. Zeng, Z. Wu, Y. Wen, New extensions of Cline's formula for generalized inverses, Filomat 31(2017), 1973-1980.
[22] G. Zhuang, J. Chen, J. Cui, Jacobson's lemma for the generalized Drazin inverse, Linear Algebra Appl. 436(2012), 742-746.
[23] H. Zou, J. Chen, On the pseudo Drazin inverse of the sum of two elements in a Banach algebra, Filomat 31(2017), 2011-2022.
[24] H. Zhu, J. Chen, Additive property of pseudo Drazin inverse of elements in Banach algebras, Filomat 28(2014), 1773-1781.