Filomat 39:3 (2025), 871–881
https://doi.org/10.2298/FIL2503871B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: `http://www.pmf.ni.ac.rs/filomat`

Separation and compactness in topological categories

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Abstract. In previous papers, several extensions of the notions of closedness, separation properties, and compactness in a set-based topological category were introduced. In this paper, we develop further these extensions in a much larger non set-based topological categories. Moreover, we show that the categories $\mathbf{KT_2ConFCO}$ (the category of KT_2 constant filter convergence spaces and continuous functions) and **Chy** (the category of Cauchy spaces and Cauchy maps) are isomorphic. Also, we characterize strongly compact constant filter convergence spaces and investigate some invariance properties of them. Finally, we compare our results and give some applications.

1. Introduction

The fundamental objects of a topological space are its open (closed) sets. The notion of closedness is used to define, for example, the $T_i, i = 1, 2, 3, 4$ separation axioms and compactness. Compact Hausdorff spaces are one of the most important classes of topological spaces to deal with can be formulated in terms of closed sets. This formulation is used by several authors to study these spaces in categorical setting. For example, the notions of compactness and Hausdorffness with respect to a factorization structure were defined in [12, p.167] and [16, p.350] for a general category, with respect to closure operators were done in [10, p.14] for abstract categories, and with respect to initial lifts, final lifts, products, pushouts, discreteness were defined in [6, p.225] for set-based topological categories.

In view of this, it will be useful to be able to not only extend these notions to an arbitrary topological category but also to have the characterization of each of them and present important theorems in general topology such as the Tietze Extension Theorem, the Tychonoff Theorem, the Baire Theorem, the Urysohn Lemma among others in certain topological categories of interest.

In this paper, we develop the extensions of each of the notions of strong closedness, compactness, the T_0 and T_2 separation axioms in non set-based topological categories in order to open the way to the investigation of these concepts.

In Section 3, we show that the categories $\mathbf{KT_2ConFCO}$ and **Chy** are isomorphic and prove that if a constant filter convergence space (A, K) is finite, then there is a bijection from the set $KT_2(A)$ of all KT_2 constant filter convergence stuctures on A onto the set $Eq(A)$ of all equivalence relations on A.

In Section 4, we characterize strongly compact constant filter convergence spaces and investigate some invariance properties of them.

Finally, in Section 5, we compare our results and mention some applications.

2020 *Mathematics Subject Classification*. Primary 54A20; Secondary 54A05, 54B30, 54D10.
Keywords. Constant filter convergence spaces, strongly compact spaces, strong closedness, T_2 spaces, Cauchy spaces.
Received: 01 April 2024; Revised: 16 December 2024; Accepted: 27 December 2024
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2. Premilinaries

Let $\mathcal{B}$ be a category with finite products and cokernel pairs (i.e., given any morphism $f : A \to B$ in $\mathcal{B}$, the pushout of f with itself), 1 be a terminal object, x be the constant object, i.e., subterminal, and B be an object in $\mathcal{B}$. We denote by B^n, $n = 1, 2, ..., n$, the product of B with itself n times and by $\pi_j : B^n \to B$ the j th projection morphism, $j = 1, 2, ..., n$. If $A \in \mathcal{B}$ and $f_i : A \to B$ are morphisms in $\mathcal{B}$, then there exists a unique morphism $f = (f_1, f_2, ..., f_n) : A \to B^n$ such that $\pi_i \circ f = f_i$ for each $i = 1, 2, ..., n$.

The diagonal $\Delta : B \to B^2$ is given by $\Delta = (1_B, 1_B)$, where $1_B : B \to B$ is the identity morphism. Define $\pi_{jk} : B^2 \vee_{\triangle} B^2 \to B$ to be $\pi_j + \pi_k$ for $j, k = 1, 2$, where $B^2 \vee_{\triangle} B^2$ denotes the cokernel of Δ along itself. More precisely, if i_1 and $i_2 : B^2 \to B^2 \vee_{\triangle} B^2$ denote the inclusions of B^2 as the first and second factor, respectively, then $i_1 \circ \Delta = i_2 \circ \Delta$ is a pushout diagram.

$$\begin{tikzcd}
B \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & B^2 \arrow[d, "i_2"] \\
B^2 \arrow[r, "i_1"'] & B^2 \vee_{\triangle} B^2
\end{tikzcd}$$

Note that for morphisms $(\pi_1, \pi_1, \pi_2) : B^2 \to B^3$ and $(\pi_1, \pi_2, \pi_1) : B^2 \to B^3$, $(\pi_1, \pi_1, \pi_2) \circ \Delta = (1_B, 1_B, 1_B) = (\pi_1, \pi_2, \pi_1) \circ \Delta$, and consequently, $A = (\pi_{11}, \pi_{21}, \pi_{12}) : B^2 \vee_{\triangle} B^2 \to B^3$ is the unique morphism called the principal axis morphism for which $A \circ i_2 = (\pi_1, \pi_1, \pi_2)$ and $A \circ i_1 = (\pi_1, \pi_2, \pi_1)$, i.e.,

$$\begin{tikzcd}
B \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & B^2 \arrow[d, "i_2"] \arrow[ddr, bend left, "{(\pi_1,\pi_1,\pi_2)}"] & \\
B^2 \arrow[r, "i_1"'] \arrow[drr, bend right, "{(\pi_1,\pi_2,\pi_1)}"'] & B^2 \vee_{\triangle} B^2 \arrow[dr, dotted, "{\exists! A=(\pi_{11},\pi_{21},\pi_{12})}"] & \\
 & & B^2
\end{tikzcd}$$

Similarly, $(\pi_1, \pi_1, \pi_2) \circ \Delta = (1_B, 1_B, 1_B) = (\pi_1, \pi_2, \pi_2) \circ \Delta$ (resp. $1_{B^2} \circ \Delta = \Delta$) and consequently, there exists a unique morphism $S = (\pi_{11}, \pi_{12}, \pi_{22}) : B^2 \vee_{\triangle} B^2 \to B^3$ (resp. $\nabla = (\pi_{11}, \pi_{22}) : B^2 \vee_{\triangle} B^2 \to B^2$), called the skewed axis (resp. the fold) morphism.

Note that

$$\pi_1 \circ S = \pi_{11} = \pi_1 \circ A, \pi_2 \circ S = \pi_{21} = \pi_2 \circ A, \pi_3 \circ S = \pi_{22}, \pi_3 \circ A = \pi_{12},$$

$$S \circ i_1 = (\pi_1, \pi_2, \pi_2), S \circ i_2 = (\pi_1, \pi_1, \pi_2) = A \circ i_2, A \circ i_1 = (\pi_1, \pi_2, \pi_1),$$

and $\nabla \circ i_k = 1_{B^2}$ for $k = 1, 2$.

Let $f : B \to B$ be the constant morphism, i.e., there exists a morphism $x : 1 \to B$ such that $x \circ h = f$, where $h : B \to 1$ is the unique morphism from B to the terminal object 1.

Define $p_1, \nabla_x : B \bigvee_x B \to B$ to be $f + 1_B$ and $1_B + 1_B$ respectively, where $B \bigvee_x B$ denotes the cokernel of $x : 1 \to B$ along itself, i.e., if i_1 and $i_2 : B \to B \bigvee_x B$ denote the inclusions of B as the first and second factor, respectively, then $i_1 \circ x = i_2 \circ x$ is a pushout diagram.

$$\begin{tikzcd}
1 \arrow[r, "x"] \arrow[d, "x"'] & B \arrow[d, "i_2"] \\
B \arrow[r, "i_1"'] & B \bigvee_x B
\end{tikzcd}$$

Note that for morphisms $(f, 1_B) : B \to B^2$ and $(1_B, 1_B) :\to B^2$, one has $(f, 1_B) \circ x = (x, x) = (1_B, 1_B) \circ x$, and consequently, $S_x = (p_1, \nabla_x) : B \bigvee_x B \to B^2$ is the unique morphism called the skewed x-axis morphism for which $S_x \circ i_2 = (f, 1_B)$ and $S_x \circ i_1 = (1_B, 1_B)$, i.e.,

$$\begin{tikzcd}
1 \arrow[r, "x"] \arrow[d, "x"'] & B \arrow[d, "i_2"] \arrow[ddr, bend left, "{(1_B,1_B)}"] & \\
B \arrow[r, "i_1"'] \arrow[drr, bend right, "{(f,1_B)}"'] & B \bigvee_x B \arrow[dr, dotted, "{\exists! S_x=(p_1,\nabla_x)}"] & \\
 & & B^2
\end{tikzcd}$$

Note that $\pi_1 \circ S_x = p_1$ and $\pi_2 \circ S_x = \nabla_x$.

Similarly, $1_B \circ x = x$ and consequently, there exists a unique morphism $\nabla_x : B \bigvee_x B \to B)$, called the fold morphism at x and $\nabla_x i_k = 1_B$ for $k = 1, 2$.

If $\mathcal{B} = \mathbf{Set}$, then a point (a, b) in $B^2 \vee_\triangle B^2$ will be denoted by $(a, b)_1$ (resp. $(a, b)_2$) if (a, b) is in the first (resp. second) component of $B^2 \vee_\triangle B^2$, and $S(a, b)_1 = (a, b, b)$, $A(a, b)_2 = S(a, b)_2 = (a, a, b)$, $A(a, b)_1 = (a, b, a)$, and $\triangledown((a, b)_i) = (a, b)$ for $i = 1, 2$ [2].

A point a in $B \bigvee_x B$ will be denoted by a_1 (resp. a_2) if a is in the first (resp. second) component of $B \bigvee_x B$ and $S_x(a_1) = (a, a)$, $S_x(a_2) = (x, a)$, and $\nabla_x(a_i) = a$ for $i = 1, 2$ [2, p.334].

A functor $U : \mathcal{E} \to \mathcal{B}$ is said to be topological or $\mathcal{E}$ is a topological category over $\mathcal{B}$ if and only if the following conditions hold:

(1) U is concrete, i.e., faithful (U is mono on hom sets) and amnestic (if $U(f) = id$ and f is an isomorphism, then $f = id$) [17, p.278].

(2) U has small fibers, i.e., $U^{-1}(b)$ is a set for all b in $\mathcal{B}$.

(3) For every U-source, i.e., family $g_i : b \to U(X_i)$ of maps in **Set**, there exists a family $f_i : X \to X_i$ in $\mathcal{E}$ such that $U(f_i) = g_i$ and if $U(h_i : Y \to X_i) = g_i k : UY \to b \to U(X_i)$, then there exists a lift $\bar{k} : Y \to X$ of $k : UY \to UX$, i.e., $U(\bar{k}) = k$. This latter condition means that every U-source has an initial lift [1, p.333]. It is well known that the existence of initial lifts of arbitrary U-source is equivalent to the existence of final lifts (the dual of the initial lifts) for arbitrary U-sink [1, p.335].

A topological functor $U : \mathcal{E} \to \mathcal{B}$ is said to be normalized if the constant objects have a unique structure [7, p.592]. Z is called a subspace of X if there exists monomorphism $i : Z \to X$ that is an initial lift (i.e., an embedding) and we denote it by $Z \subset X$.

Note that a topological functor $U : \mathcal{E} \to \mathcal{B}$ has a left adjoint $D : \mathcal{B} \to \mathcal{E}$, where $D(e)$ is obtained as the final lift of the empty sink on e. An object of the form $e = DUe$ is called a discrete object in $\mathcal{E}$. An object e in $\mathcal{E}$ is discrete if and only if every morphism $U(e) \to U(c)$ lifts to a morphism $e \to c$ for each object c in $\mathcal{E}$ [1, p.336].

Let $\mathcal{B}$ be a category with finite products, a terminal object, and pushouts. Let $U : \mathcal{E} \to \mathcal{B}$ be a normalized topological functor, x be the subterminal, $X \in Ob(\mathcal{E})$ with $U(X) = B$, and $Z \subset X$.

Definition 2.1. (1) If the initial lift of the U-source $S_x : B \bigvee_x B \to U(X^2)$ and $\nabla_x : B \bigvee_x B \to UD(B)$ is discrete, then X is called T_1 at x.

(2) If X/Z is T_1 at 1, then Z is called a strongly closed subobject of X, where X/Z is the pushout of $i : Z \to X$ with $g : Z \to 1$.

For $\mathcal{B} = \mathbf{Set}$, Definition 2.1 is given in [2, p.336].

For $\mathcal{E} = \mathbf{Top}$ and $\mathcal{B} = \mathbf{Set}$, a topological space is T_1 at x if and only if for any distinct point y from x, there is a neighborhood of each missing the other. If a topological space is T_1, then strong closedness coincide with the usual closedness [2, p.337]. Note that X/Z is the final lift of the epi U-sink $Q : U(X) = B \to B/Z = (B\backslash Z) \cup 1$ identifying Z with 1 for $Z \neq \emptyset$, the initial object in **Set**. If $Z = \emptyset$, then $B/Z = B \coprod 1$.

Let $B \neq \emptyset$ and $F(B)$ be the set of filters on B. A filter α is said to be proper (resp., improper) if and only if $\emptyset \notin \alpha$ (resp. $\emptyset \in \alpha$). If $\alpha = [U] = \{V \subset B : U \subset V\}$, then α is said to be a principal filter. If $\alpha = [x]$ for $x \in B$, then α is called a point filter. An ultrafilter that is not a point filter is called a free filter.

If a function $K : B \to P(F(B))$ satisfies the following axioms:

(*i*) for each $x \in B$, $[x] \in K(x)$.

(*ii*) if $\alpha \in K(x)$ and $\beta \supset \alpha$, then $\beta \in K(x)$,

then (B, K) is called a filter convergence space; filter convergence spaces are referred to as convergence functions in [13, p.128] and generalized convergence spaces in [18, p.31]. If K is a constant function, then (B, K) is called a constant filter convergence space [19].

A function $f : (B, K) \to (C, L)$ between constant filter convergence spaces is said to be continuous if $\alpha \in K$, then $f(\alpha) \in L$, where $f(\alpha) = [\{f(V) : V \in \alpha\}]$.

The category of constant filter convergence spaces and continuous functions is denoted by **ConFCO** which is a bireflective subcategory of **FCO** filter convergence spaces [19, p.353]. For a filter convergence space (A, K), by Theorem 4 of [19, p.353], $1_A : (A, K) \to (A, L)$ is the bireflection, where $L(a) = \{K(x) : x \in A\}$.

Proposition 2.2. *Let $B \neq \emptyset$, $\{(B_i, K_i), i \in I\}$ be a class of constant filter convergemce spaces, and $\{f_i : B \to B_i, i \in I\}$ be a source in the category* **Set**. *A source $\{f_i : (B, K) \to (B_i, K_i), i \in I\}$ in* **ConFCO** *is an initial lift if and only if $\alpha \in K$ precisely when $f_i(\alpha) \in K_i$ for all $i \in I$.*

An epi sink $\{f_i : (B_i, K_i) \to (B, K), i \in I\}$ in **ConFCO** *is a final lift if and only if $\alpha \in K$ precisely when there exist $i \in I$ and $\alpha_i \in K_i$ with $f_i(\alpha_i) \subset \alpha$.*

These are special cases of Theorem 4 of [19, p.353]

The category **Chy**, which is a cartesian closed topological category [11, p.12], of Cauchy spaces has as objects $(A, \mathbf{S})$, where A is a set and $\mathbf{S} \subset F(A)$ is subject to the following axioms:

(1) $[x] \in \mathbf{S}$ for each $x \in A$.

(2) $\alpha \in \mathbf{S}$ and $\alpha \subset \beta$ implies $\beta \in \mathbf{S}$ for any filter β on A.

(3) For $\beta, \alpha \in \mathbf{S}$ if $\beta \cup \alpha$ is proper, then $\beta \cap \alpha \in \mathbf{S}$.

Let $f : (A, \mathbf{S})$ and $(B, \mathbf{T})$ be Cauchy spaces. A morphism $f : (A, \mathbf{S}) \to (B, \mathbf{T})$ is a function such that $f(\alpha) \in \mathbf{T}$ if $\alpha \in \mathbf{S}$, i.e., f is a Cauchy map.

If $\mathbf{S}$ satisfies axioms **(1)** and **(2)**, then $(A, \mathbf{S})$ is called a filter space and the category of filter spaces and Cauchy maps is denoted by **Fil** [18], p. 32.

Theorem 2.3. ([4], p.391), ([14], p.18) *Let $(B, K) \in$* **ConFCO** *(resp. $(B, \mathbf{S}) \in$* **Chy***) and $\emptyset \neq Z \subset B$. Then, Z is strongly closed if and only if $\alpha \not\subset [a]$ or $\alpha \cup [Z]$ is improper for every $a \in B$ with $a \notin Z$ and every $\alpha \in K$ (resp. $\alpha \in \mathbf{S}$).*

Example 2.4. (1) By Theorem 2.3, all subsets of the discrete constant filter convergence space $(B, K = \{[x], [\emptyset] : x \in B\})$ are strongly closed.

(2) Let $(B, F(B))$ be the indiscrete constant filter convergence space with *cardB* ≥ 2. By Theorem 2.3, the only strongly closed subsets of B are $\emptyset$ and B.

(3) Let $B = \{x, y, z, w\}$ and define constant filter convergence structures K and L on B as follow:
$K = \{[x], [y], [z], [w], [x] \cap [y], [y] \cap [z], [z] \cap [w], [x] \cap [z], [x] \cap [y] \cap [z], [\emptyset]\}$.
$L = \{[x], [y], [z], [w], [x] \cap [y], [\emptyset]\}$.

By Theorem 2.3, the only strongly closed subsets of (B, K) (resp. (B, L)) are B and $\emptyset$ (resp. $\{z\}$, $\{w\}$, $\{x, y\}$, $\{z, w\}$, $\{x, y, z\}$, $\{x, y, w\}$, B, and $\emptyset$).

Let $(B, K) \in$ **ConFCO** and $Z \subset B$. The strong closure $Scl_B(Z)$ of Z is the intersection of all strongly closed subsets of B containing Z.

Theorem 2.5. (1) *The categories* **ConFCO** *and* **Fil** *are isomorphic.*

(2) *Let* $(B, K) \in$ **ConFCO**. *If* $Z_i \subset B$ *is strongly closed for each* $i \in I$, *then* $\bigcap_{i \in I} Z_i$ *is strongly closed.*

(3) $Scl_B(Z)$ *is strongly closed.*

Proof. (1) For $(B, K) \in$ **ConFCO**, define $\mathbf{S_K}=\{\alpha \in F(B) : \alpha \in K\}$. Then $(B, \mathbf{S_K}) \in$ **Fil**. For $(B, \mathbf{S}) \in$ **Fil**, define $K_{\mathbf{S}}(x) = K_{\mathbf{S}}(y) = \{\alpha \in F(B) : \alpha \in \mathbf{S}\}$ for every $x, y \in B$, i.e., $K_{\mathbf{S}} : B \to P(F(B))$ is a constant function. Then $(B, K_{\mathbf{S}}) \in$ **ConFCO**. Note that $\mathbf{S}_{K_{\mathbf{S}}} = \mathbf{S}$ for each filter structure $\mathbf{S}$ and $K_{\mathbf{S}_K} = K$ for each constant filter convergence structure K. Finally, if $f : (B, K) \to (C, L)$ is continuous between constant filter convergence spaces, then clearly, $f : (B, \mathbf{S_K}) \to (\mathbf{C}, \mathbf{S_L})$ is a Cauchy map. If $f : (A, \mathbf{S}) \to (B, \mathbf{T})$ is a Cauchy map between filter spaces, then $f : (B, K_{\mathbf{S}}) \to (C, L_{\mathbf{T}})$ is continuous. Hence, **ConFCO** and **Fil** are isomorphic.

(2) Suppose $a \in B$, $a \notin \bigcap_{i \in I} Z_i$ and $\alpha \in K$. There exists $k \in I$ such that $a \notin Z_k$. Since Z_k is strongly closed, by Theorem 2.3, $\alpha \cup [Z_k]$ is improper or $\alpha \not\subset [a]$. If $\alpha \cup [Z_k]$ is improper, then $\alpha \cup [\bigcap_{i \in I} Z_i]$ is improper since $\alpha \cup [Z_k] \subset \alpha \cup [\bigcap_{i \in I} Z_i]$. Consequently, by Theorem 2.3, $\bigcap_{i \in I} Z_i$ is strongly closed.

(3) By Part(2), $Scl_B(Z)$ is strongly closed. □

In Cauchy spaces, the concept of completeness is meaningful, whereas in (constant) filter convergence spaces, this concept is unavailable. By Theorem 2.5, the category **ConFCO** is a link between the categories **Fil** and **FCO**. Cauchy spaces proved to be extremely useful in the completion theory of convergence vector spaces. The reader is referred to [11, 18] for more details concerning Cauchy spaces and convergence spaces.

Theorem 2.6. ([3], p.100)

(*i*) *Let* σ *be a filter on* $B^2 \vee_\Delta B^2$. *If* $\sigma_0 = \bigcup \pi_{ij}^{-1} \pi_{ij} \sigma$, $j, i = 1, 2$, *then* $\sigma_0 \subset \sigma$ *and* $\pi_{ij}\sigma = \pi_{ij}\sigma_0$ *for all* $j, i = 1, 2$, *where* π_{ij} *is defined above. Let* α_{ij}, $i, j = 1, 2$ *be proper filters on* B.

(*ii*) $\sigma = \bigcup_{j,i=1}^{2} \pi_{ij}^{-1} \alpha_{ij}$ *is proper if and only if either (a)* $(\alpha_{11} \cup \alpha_{12})$ *and* $(\alpha_{21} \cup \alpha_{22})$ *are proper or (b)* $(\alpha_{21} \cup \alpha_{11})$ *and* $(\alpha_{22} \cup \alpha_{12})$ *are proper.*

(*iii*) *There exists a proper filter* σ *on* $B^2 \vee_\Delta B^2$ *such that* $\pi_{ij}\sigma = \alpha_{ij}$ *for all* $j, i = 1, 2$ *if and only if*

(1) *If* $(\alpha_{11} \cup \alpha_{12})$ *is improper or* $(\alpha_{21} \cup \alpha_{22})$ *is improper, then* $\alpha_{11} = \alpha_{21}$ *and* $\alpha_{22} = \alpha_{12}$.

(2) *If* $(\alpha_{11} \cup \alpha_{21})$ *is improper or* $(\alpha_{12} \cup \alpha_{22})$ *is improper, then* $\alpha_{11} = \alpha_{12}$ *and* $\alpha_{21} = \alpha_{22}$.

(3) *If both (a) and (b) hold, then* $\alpha_{11} \cap \alpha_{22} = \alpha_{12} \cap \alpha_{21}$.

3. T_2 constant filter convergence spaces

We show that the categories **KT₂ConFCO** and **Chy** are isomorphic. Moreover, we prove that if a constant filter convergence space (A, K) is finite, then there is a bijection from the set $KT_2(A)$ of all KT_2 constant filter convergence stuctures on A onto the set $Eq(A)$ of all equivalence relations on A.

Let $\mathcal{B}$ be a category with finite products and pushouts, $U : \mathcal{E} \to \mathcal{B}$ be topological, and $X \in Ob(\mathcal{E})$ with $U(X) = B$.

Let S_B (resp. A_B) be the initial lift of the U-source S (resp. A) : $B^2 \bigvee_\Delta B^2 \to U(X^3)$ and $W_{(B^2 \bigvee_\Delta B^2)}$ be the final lift of the U-sink $\{q \circ i_1, q \circ i_2 : U(X^2) \to B^2 \vee_\Delta B^2\}$, where $i_k : B^2 \to B^2 \coprod B^2, k = 1, 2$ are the canonical injection maps and $q : B^2 \coprod B^2 \to B^2 \bigvee_\Delta B^2$ is the quotient map.

Definition 3.1. (1) If $S_B = A_B$, then X is said to be a $Pre\overline{T}_2$ object.

(2) If the initial lift of the U-source $\nabla : B^2 \vee_\Delta B^2 \to U(D(B^2))$ and $id : B^2 \vee_\Delta B^2 \to U(B^2 \vee_\Delta B^2, W_{(B^2 \vee_\Delta B^2)})$ is discrete, then X is said to be a T_0' object, where D is the discrete functor.

(3) If the diagonal Δ is strongly closed in X^2, then X is said to be a ST_2 object.

(4) If X is T_0' and $Pre\overline{T}_2$, then X is said to be a KT_2 object.

Remark 3.2. (1) For $\mathcal{E}$ = **Top** and $\mathcal{B}$ = **Set**, T_0' (resp. KT_2 or ST_2) reduce to the usual T_0 (resp. T_2) axiom [5, p.43]. A topological space is $Pre\overline{T}_2$ or pre-Hausdorff if and only if for any two distinct points, if there is a neighborhood of one missing the other, then the points have disjoint neighborhoods [2, p.338]. There is no implication between $Pre\overline{T}_2$ and each of T_0 and T_1. Take the integers set Z with indiscrete and cofinite topologies. In the realm of $Pre\overline{T}_2$ topological spaces, by the Theorem 4.3 of [8], all T_0, T_1, and T_2 spaces are equivalent.

(2) For $\mathcal{B}$ = **Set**, Definition 3.1 is given in [2, p.337] and if $\mathcal{B}$ is a category with finite limits and colimit, then (1) of Definition 3.1 is given in [20, p.18].

Theorem 3.3. *A constant filter convergence space (A, K) is KT_2 if and only if it is a Cauchy space.*

Proof. Suppose (A, K) is KT_2 and $\alpha \cup \delta$ is proper for any proper filters $\alpha, \delta \in K$. In Theorem 2.6, let

$$\alpha_{11} = \alpha, \alpha_{21} = \alpha \cup \delta, \alpha_{12} = \alpha \cap \delta, \alpha_{12} = \delta.$$

Note that

$$\alpha_{12} \cup \alpha_{11} = \alpha, \alpha_{21} \cup \alpha_{22} = \alpha \cup \delta, \alpha_{21} \cup \alpha_{11} = \alpha \cup \delta, \alpha_{22} \cup \alpha_{12} = \delta$$

are proper and $\alpha_{11} \cap \alpha_{22} = \alpha \cap \delta = \alpha_{12} \cap \alpha_{21}$. By Theorem 2.6, there exists a proper filter β on the wedge with

$$\pi_1 A\beta = \alpha_{11} = \pi_1 S\beta, \pi_2 A\beta = \alpha_{21} = \pi_2 S\beta, \pi_3 A\beta = \alpha_{12}, \pi_3 S\beta = \alpha_{22}.$$

Since (A, K) is $pre\overline{T}_2$ and $\pi_1 A\beta = \pi_1 S\beta, \pi_1 A\beta = \pi_2 S\beta, \pi_3 S\beta \in K$, we get $\pi_3 A\beta = \alpha \cap \delta \in K$ and hence, (A, K) is a Cauchy space.

Suppose (A, K) is a Cauchy space. First, we show that (A, K) is $pre\overline{T}_2$, i.e., for any filter β on wedge, $\pi_1 A\beta$, $\pi_2 A\beta$, $\pi_3 A\beta \in K$ if and only if $\pi_1 S\beta$, $\pi_2 S\beta$, $\pi_3 S\beta \in K$. Note that $\pi_1 A\beta = \pi_1 S\beta$ and $\pi_2 A\beta = \pi_2 S\beta$. We need to show that $\pi_3 A\beta \in K$ if and only if $\pi_3 S\beta \in K$.

If $\beta = [\emptyset]$, then nothing to show. If $\beta \neq [\emptyset]$, then let

$$\beta_0 = \pi_1^{-1}(\pi_1 A\beta) \cup \pi_2^{-1}(\pi_2 A\beta) \cup \pi_3^{-1}(\pi_3 A\beta) \cup \pi_3^{-1}(\pi_3 S\beta).$$

By Theorem 2.6, $\beta_0 \subset \beta$, $\pi_i A\beta_0 = \pi_i A\beta$, and $\pi_i S\beta_0 = \pi_i S\beta$ for each $i = 1, 2, 3$.

We apply Theorem 2.6 with

$$\alpha_{11} = \pi_1 A\beta = \pi_1 S\beta,$$

$$\alpha_{21} = \pi_2 A\beta = \pi_2 S\beta,$$

$$\alpha_{12} = \pi_3 A\beta,$$

and

$$\alpha_{22} = \pi_3 S\beta.$$

If (1) of Theorem 2.6 (iii) holds, then $\alpha_{22} = \alpha_{12}$ and consequently $\alpha_{22} \in K$ if and only if $\alpha_{12} \in K$. If (2) of Theorem 2.6 (ii) holds, then $\alpha_{11} = \alpha_{12}$, $\alpha_{22} = \alpha_{21}$, and consequently $\alpha_{12} \in K$ and $\alpha_{22} \in K$. If (3) of Theorem 2.6 (iii) holds, then $\alpha_{11} \cap \alpha_{22} = \alpha_{12} \cap \alpha_{21}$. Let $\alpha_{21} = \alpha, \alpha_{12} = \delta, \alpha_{11} = \alpha \cup \delta$ and note that $\alpha \cup \delta$ is proper. By assumption, we get $\alpha \cap \delta \in K$. But $\alpha \cap \delta \subset \alpha_{22} = \pi_3 S\beta$ and $\alpha \cap \delta \subset \alpha_{12} = \pi_3 A\beta$ imply $\pi_3 S\beta \in K$ and $\pi_3 A\beta \in K$. Hence, by Definition 3.1, (A, K) is $pre\overline{T}_2$.

Next, we show that (A,K) is T_0'. Suppose α is any filter on the wedge with $\alpha \supset (q \circ i_k)(\beta)$ for some β in K^2, where K^2 is the product constant filter convergence structure on A^2, $k = 1$ or 2 and $\nabla(\alpha) = [\emptyset]$ or $[(t,s)]$ for some $(t,s) \in A^2$. Then it follows easily that

$$\alpha = [\emptyset], [(t,s)_2], [(t,s)_1]$$

or

$$\alpha \supset [(t,s)_2] \cap [(t,s)_1].$$

If $t = s$, then $\alpha = [(t,t)]$. Suppose $t \neq s$. If $\alpha = [(t,s)_2] \cap [(t,s)_1]$, then $\alpha \supset (q \circ i_k)(\beta)$ for some β in K^2 and $k = 1$ or 2, a contradiction since $t \neq s$.

If $[\emptyset] \neq \alpha \neq [(t,s)_2] \cap [(t,s)_1]$, then $\alpha \supset [(t,s)_2] \cap [(t,s)_1]$ if and only if $\alpha = [(t,s)_2]$ or $[(t,s)_1]$.

If $\alpha = [(t,s)_2]$ or $[(t,s)_1]$, then $\alpha \supset [(t,s)_2] \cap [(t,s)_1]$. If $\alpha \supset [(t,s)_2] \cap [(t,s)_1]$ and $[\emptyset] \neq \alpha \neq [(t,s)_2] \cap [(t,s)_1]$, then $\exists U \in \alpha$ with $V = \{(t,s)_2, (t,s)_1\} \not\subset U$. Since α is a filter and $U, V \in \alpha$, $U \cap V = \{(t,s)_2\}$ or $\{(t,s)_1\}$ belongs α and consequently, $\alpha = [(t,s)_2]$ or $[(t,s)_1]$. Hence, (A,K) is T_0' and thus, (A,K) is KT_2. □

Theorem 3.4. *The categories* $\mathbf{KT_2ConFCO}$ *and* **Chy** *are isomorphic.*

Proof. Combine Theorems 2.5 and 3.3. □

Theorem 3.5. *If A is finite, then there is a bijection from the set $KT_2(A)$ of all KT_2 constant filter convergence stuctures on A onto the set $Eq(A)$ of all equivalence relations on A.*

Proof. Let (A,K) be a finite constant filter convergence space. Define functions $f : Eq(A) \to KT_2(A)$ and $g : KT_2(A) \to Eq(A)$ as follow.

If $K \in KT_2(A)$, then let $g(K) = R_K$ be given by aR_Kb if and only if $[a] \cap [b] \in K$ for all $a, b \in A$. R_K is reflexive and symmetric. Suppose aR_Kb and bR_Kc for $a,b,c \in A$. Since $([a] \cap [b]) \cup ([b] \cap [c]) = [b]$ is proper and (A,K) is KT_2, by Theorem 3.3, $([a] \cap [b]) \cap ([b] \cap [c]) \in K$ and thus, $[a] \cap [c] \in K$ ((A,K) is a constant filter convergence space), i.e., aR_Kc, i.e., R_K is transitive. Hence, $R_K \in Eq(A)$.

If $R \in Eq(A)$, then let $f(R) = K_R$ be given by $K_R(a) = \{\alpha : [a_R] \subset \alpha\} \cup \{[\emptyset]\}$ for all $a \in A$, where $a_R = \{b \in B : aRb\}$, the equivalence class of a. We show that (A, K_R) is a KT_2 constant filter convergence space. Since R is reflexive and $[a_R] \subset [a]$, $[a] \in K_R$ for each $a \in A$. If $\alpha \in K_R$ and $\beta \supset \alpha$ for any filter β on A, then $[a_R] \subset \alpha$ and hence, $[a_R] \subset \beta \in K_R$.

If $\alpha, \beta \in K_R$, then $[a_R] \subset \alpha$ and $[b_R] \subset \beta$ for some $a, b \in A$. If $\alpha \cup \beta$ is proper, then $[a_R] \cup [b_R]$ is proper. Thus, $a_R \cap b_R \neq \emptyset$ and since R is the equivalence relation on A, we have $a_R = b_R$ and thus, $\alpha \cap \beta \in K_R$. By Theorem 3.3, (A, K_R) is a KT_2 constant filter convergence space.

Finally, we need to verify that $gof = id_{Eq(A)}$ and $fog = id_{KT_2(A)}$. Let $R \in Eq(A)$ and $gof(R) = g(K_R) = S_{K_R}$. Then for each $a, b \in A$, $aS_{K_R}b$ if and only if $[a] \cap [b] \in K_R$ and only if $[a_R] \subset [a] \cap [b]$ if and only if aRb which shows that $gof = id_{Eq(A)}$.

Let $K \in KT_2(A)$ and $fog(K) = f(R_K) = L_{R_K}$. Then a proper filter $\alpha \in L_{R_K}$ if and only if $[a_{R_K}] \subset \alpha$ for some $a \in A$. Since (A,K) is finite, $\alpha = [D]$, where $D = \{a_1, a_2, ..., a_n\} \subset a_{R_K}$ and aR_Ka_i, $i = 1, 2, ...n$. Note that $[a_i] \in K$ and $[a_1] \cup \bigcap_{i=2}^{n} [a_i]$ is proper. Since (A,K) is a KT_2 constant filter convergence space, by Theorem 3.3, we get $\alpha = [D] = \bigcap_{i=1}^{n} [a_i] \in K$. Hence, $fog(K) = f(R_K) = L_{R_K} = K$, i.e., $gof = id_{KT_2(A)}$. □

Example 3.6. (1) By Theoren 3.3, both indiscrete and discret constant filter convergence spaces are KT_2 constant filter convergence spaces.

(2) Let $A = \{a,b\}$, $K = \{[\emptyset], [a], [b], [c]\}$, and $L = \{[\emptyset], [a], [b], [a] \cap [b] = [A]\} = F(A)$. By Theoren 3.3, both (A,K) and (A,L) are KT_2 constant filter convergence spaces.

(3) Let $A = \{a,b,c\}$, $K = \{[\emptyset], [a], [b], [c], [a] \cap [b], [a] \cap [c], [b] \cap [c]\}$, and $L = \{[\emptyset], [a], [b], [c], [a] \cap [b]\}$. By Theoren 3.3, (A,K) is not KT_2 constant filter convergence space since $([a] \cap [b]) \cup ([a] \cap [c]) = [a]$ is proper but $([a] \cap [b]) \cap ([a] \cap [c]) = [A] \notin K$. On the other hand, (A,L) is a KT_2 constant filter convergent space.

(4) By Theorem 3.5, if A is finite, then every equivalence relation on A induces KT_2 constant filter convergent structure on A.

(5) By Theorem 4 of [19, p.353], **ConFCO** is a bireflective subcategory of **FCO** and for a filter convergence space (A, K), $1_A : (A, K) \to (A, L)$ is the bireflection, where for $a \in A$, $L(a) = \{K(x) : x \in A\}$ is a constant function. (A, L) is constant filter convergence space. By Theorem 4.10 of [7, p.598] and Corollary 3.14 of [4, p.392], T_2' and ST_2 filter convergence spaces are KT_2 constant filter convergence spaces.

(6) Every topological space (A, τ) induces a filter convergence space. Indeed, for $a \in A$, let $\eta_a = \{U \subset A : \exists V \in \tau \text{ such that } a \in V \subset U\}$ be a neighborhood filter at the point a and define $K(a) = \{\alpha : \alpha \supset \eta_a\}$. Then (A, K) is a filter convergence space and by Part (5), the bireflection (A, L) is constant filter convergence space. Moreover, if (A, τ) is T_2 topological space, then by Theoren 3.3 and Part (5), (A, L) is KT_2 constant filter convergence space. In particular, all metric spaces induce KT_2 constant filter convergence spaces.

We denote by $\mathbf{T}\mathcal{E}$ the subcategory of $\mathcal{E}$ whose objects are the T-spaces, where $T = Pre\overline{T}_2, KT_2, T_0'$ and $\mathcal{E} =$ **ConFCO** or **Chy**.

Remark 3.7. **(A)** In **Top**, there is no implication between $Pre\overline{T}_2$ and T_0'. Take the integers set Z with indiscrete and cofinite topologies. In the realm of $Pre\overline{T}_2$ topological spaces, by the Theorem 4.3 of [8], T_0', ST_2, and KT_2 are equivalent.

(B) In **ConFCO**, by Theorem 3.3, $\mathbf{KT_2ConFCO} = \mathbf{Pre\overline{T}_2ConFCO} \subset \mathbf{T_0'ConFCO} = \mathbf{ConFCO}$. In particular, every $Pre\overline{T}_2$ constant filter convergence space is T_0'. However, $(B = \{x, y, z, w\}, K)$ in Example 2.4 is T_0' but it is not $Pre\overline{T}_2$ since $([y] \cap [z]) \cup ([z] \cap [w]) = [z]$ is proper and $([z] \cap [y]) \cap ([z] \cap [w]) = [\{z, y, w\}] \notin K$.

In the realm of $Pre\overline{T}_2$ constant filter convergence spaces, by Theorem 3.3, T_0' and KT_2 are equivalent.

(C) By Theorems 4.1-4.4 of [14], $\mathbf{ST_2Chy} \subset \mathbf{KT_2Chy} = \mathbf{T_0'Chy} = \mathbf{Pre\overline{T}_2Chy} = \mathbf{Chy}$. In particular, all KT_2, $Pre\overline{T}_2$, and T_0' Cauchy spaces are equal.

4. Strongly compact constant filter convergence spaces

In this section, we characterize strongly compact constant filter convergence spaces and investigate some invariance properties of them.

Definition 4.1. Let $U : \mathcal{E} \to \mathcal{B}$ be a normalized topological functor, $A, B \in Ob(\mathcal{E})$, and $f : A \longrightarrow B$ be a morphism in $\mathcal{E}$ that has epi-mono factorization, where $\mathcal{B}$ is a category with finite products, a terminal object, and pushouts.

(1) If the image of each strongly closed subobject of A is a strongly closed subobject of B, then f is said to be strongly closed.

(2) If the projection $\pi_2 : A \times B \longrightarrow B$ is strongly closed for each object B in $\mathcal{E}$, then A is said to be strongly compact.

If $\mathcal{B} =$ **Set**, then Definition 4.1 reduces to the one that was given in [6, p.225] and if $\mathcal{E} =$ **Top**, $\mathcal{B} =$ **Set**, and a topological space is T_1, then the notion of strong compactness (resp. closedness) reduces to usual one.

Theorem 4.2. *A constant filter convergence space (A, K) is strongly compact if and only if every ultra filter in A converges.*

Proof. Suppose (A, K) is a strongly compact space and α is a non covergent ultrafilter on A. Let B be the set obtained by adjoining a new element to A, i.e., $B = A \cup \{\infty\}$. Define a convergence structure L on B by

$$L = \{[\emptyset], [x] : x \in A\} \cup \{\beta \in F(B) : \alpha = \beta \cup [A] \quad \text{or} \quad \beta = [\infty]\}$$

and let $\triangle = \{[(x, y) \in A \times B : x = y\} \subset A \times B$. Note that (B, L) is a constant filter convergence space. Let $\sigma = \pi_1^{-1}([x]) \cup \pi_2^{-1}(\beta)$. Since $\pi_1(\sigma) = [x] \in K$ and $\pi_2(\sigma) = \beta \in L$, we have $\sigma \in S$, where S is the product structure on $A \times B$. By Theorem 2.5, the strong closure $Scl_{A\times B}(\triangle)$ of $\triangle$ is a strongly closed subset of $A \times B$ and $Scl_{B\times A}(\triangle) \subset A \times A \subset A \times B$. Consequently, $\pi_2(Scl_{A\times B}(\triangle)) = A$ which is not a strongly closed subset of A since $\infty \in B$, $\infty \notin \pi_2(\triangle) = A$, $\alpha \not\subset [\infty]$ and $\alpha \cup [\pi_2(\triangle)] = \alpha \cup [A]$ is proper for $\alpha \in L$. Thus, by Theorem 2.3, $\pi_2(\triangle) \subset B$ is not strongly closed, a contradiction since (B, L) is strongly compact.

Suppose every ultrafilter in A converges. We need to show that for each constant filter convergence space (B, L), the projection map $\pi_2 : (A \times B, S) \longrightarrow (B, L)$ is strongly closed, where S is the product structure on $A \times B$. Suppose $M \subset A \times B$ is strongly closed and $\pi_2(M)$ is not strongly closed. By Theorem 2.3, $\alpha \subset [a]$ and $\alpha \cup [\pi_2(M)]$ is proper for some $a \in B$ with $a \notin \pi_2(M)$ and some $\alpha \in L$. Let $\sigma = [M] \cup \pi_2^{-1}(\alpha)$. Note that σ is proper, $\pi_1(\sigma)$ is a proper filter on A, and there exists a ultrafilter $\beta \in K$ (by assumption) with $\pi_1(\sigma) \subset \beta$. Let $\theta = \pi_1^{-1}(\beta) \cup \pi_2^{-1}(\alpha)$. Note that $\pi_1(\theta) = \beta \in K$, $\pi_2(\theta) = \alpha \in L$ and by Proposition 2.1, $\theta \in S$. Since $\pi_1(\sigma) \subset \beta$, $\theta \cup [M]$ is proper. Also, $a \notin \pi_2(M)$ implies $(x, a) \notin M$ for $x \in A$ and $\theta \subset [(x, a)]$ (since $\pi_1(\sigma) \subset \beta = \pi_1(\theta) \in K$) and $\alpha \subset [a]$. Thus, by Theorem 2.3, M is not strongly closed, a contradiction. Hence, $\pi_2(M)$ has to be strongly closed subset of B and by Definition 4.1, (A, K) is strongly compact. □

Example 4.3. By Theorem 4.2,

1. every indiscrete constant filter convergence space is strongly compact.
2. every finite constant filter convergence space is strongly compact.
3. the discrete constant filter convergence space is strongly compact if and only if it is finite. Therefore, the infinite discrete constant filter convergence space is not strongly compact.
4. Let R be the reals and define K and L as $K = \{[\emptyset], [x] : x \in R\} \cup \{\alpha : \alpha \text{ is a free filter}\}$ and $L = \{\alpha \in F(R) :$ there is some ultrafilter β on R and some $x \in R$ with $\alpha \supset \beta \cap [x]\} \cup \{\alpha \in F(R) : \alpha \supset \{U \subset R : R \setminus U$ is finite$\}\}$. By Theorem 4.2, (R, K) and (R, L) are strongly compact.

Theorem 4.4. (1) *If Z is strongly closed subset of a strongly compact constant filter convergence space (A, K), then Z is strongly compact.*

(2) *A strongly compact subset of KT_2 constant filter convergence space need not be strongly closed.*

Proof. (1) Suppose Z is strongly closed subset of a strongly compact space (A, K) and α is any ultrafilter on Z. Let K_Z be the initial structure on Z induced by the inclusion map $i : Z \to (A, K)$. Note that $i(\alpha)$ is an ultrafilter on A with $i(\alpha) \in K$ since (A, K) is strongly compact and by Proposition 2.1, $\alpha \in K_Z$. Hence, by Theorem 4.2, Z is strongly compact.

(2) Let R be the reals. Then, by Theorem 3.3, the indiscrete space $(R, F(R))$ is KT_2 and by Theorem 2.3, the subset $[0, 3]$ of R is not strongly closed but by Theorem 4.2, $[0, 3]$ is strongly compact. □

Theorem 4.5. *Let $f : (A, K) \longrightarrow (B, L)$ be continuous.*

(1) *If (A, K) is strongly compact, then $f(A)$ is strongly compact.*

(2) *If (B, L) is KT_2 and (A, K) is strongly compact, then f need not be strongly closed.*

Proof. (1) Let α be a ultrafilter on $f(A)$ and $L_{f(A)}$ be the initial structure on $f(A)$ induced by the inclusion map $i : f(A) \to B$. Then, $f^{-1}(i(\alpha)) = [\{f^{-1}(V) : V \in i(\alpha)\}] \in F(A)$ and there is a ultrafilter β on A with $\beta \supset f^{-1}(i(\alpha))$. Since (A, K) is strongly compact, by Theorem 4.2, $\beta \in K$. Note that $f(\beta) \supset f(f^{-1}(i(\alpha)) \supset i(\alpha)$ and thus, $i(\alpha) = f(\beta)$ ($i(\alpha)$ is a ultrafilter on B). Since $f : (A, K) \longrightarrow (B, L)$ is continuous and $\beta \in K$, then $i(\alpha) = f(\beta) \in L$, and so, $\alpha \in L_{f(A)}$. Hence, by Theorem 4.2, $(f(A), L_{f(A)})$ is strongly compact.

(2) Let (R, K) be as in Example 4.3 (4) and $(R, F(R))$ be the indiscrete constant filter convergence space. The identity function $id : (R, K) \longrightarrow (R, F(R))$ is continuous and by Theorem 4.2, (R, K) is strongly compact and by Theorem 3.3, $(R, F(R))$ is KT_2. By Theorem 2.3, $\{3\}$ is strongly closed subset of (R, K). Indeed, for

every $a \in R$ with $a \notin \{3\}$ and every $\alpha \in K$, we have $\alpha = [\emptyset], [a]$ for $a \in R$ or α is a free filter. If $\alpha = [3]$, then $\alpha \not\subset [a]$ for every $a \in R$ with $a \notin \{3\}$ and if $\alpha \neq [3]$, then $\alpha \cup [3]$ is improper. If α is a free filter, then $\alpha \not\subset [a]$ for every $a \in R$ with $a \notin \{3\}$ and $\alpha \cup [3]$ is improper (if $\alpha \cup [3]$ were proper, then $\alpha = [3]$, a contradiction since α is a free filter). However, $id(\{3\}) = \{3\}$ is not strongly closed subset of $(R, F(R))$. □

Theorem 4.6. *An arbitrary product of strongly compact constant filter convergence spaces is strongly compact.*

Proof. Let (A_i, K_i) be strongly compact constant filter convergence spaces for every $i \in I$ and α be any ultrafilter on the product space $(A = \prod_{i \in I} A_i, K)$. Since $\pi_i(\alpha)$ is an ultrafilter on A_i and each (A_i, K_i) is strongly compact, by Theorem 4.2, $\pi_i(\alpha) \in K_i$ for each $i \in I$ and by Proposition 2.1, $\alpha \in K$. Hence, by Theorem 4.2, (A, K) is strongly compact. □

5. Comments

For $\mathcal{B} = \mathbf{Set}$, Definitions 2.1, 3.1, and 4.1 reduce to the ones that were given in [2, p.337], [6, p.225] and if $\mathcal{B}$ is a category with finite limits and colimit, then (1) of Definition 3.1 reduces to that was given in [20, p.18]. In general, by Remark 2.8 of [5] and Remark 3.6, there is no implication between $Pre\overline{T}_2$ (resp. KT_2) and T_0' (resp. ST_2). In the realm of $Pre\overline{T}_2$ topological spaces, by the Theorem 4.3 of [8], T_0', ST_2, and KT_2 are equivalent. Does this result hold in general?

Note that a notion of closedness at the level of set-based topological categories was defined in [4]. In **Top**, if a space is T_1, then the notions of closedness and strong closedness coincide. By Theorem 3.1, 3.2, 3.9, and 3.10 of [4], these notions are independent of each other in a topological category, in general. When do these notions coincide?

Definitions 2.1, 3.1, and 4.1 open the way to the investigation of separation properties, compactness, disconnectedness, and connectedness in a much larger non set-based topological categories. Therefore, it will be useful to have the characterization of each of them and present important theorems in general topology.

If $U : \mathcal{E} \to \mathcal{B}$ is topological and $\mathcal{D}$ is a full subcategory of $\mathcal{E}$ such that the restriction $U_1 = U|_{\mathcal{D}} : \mathcal{D} \to \mathcal{B}$ is still topological, then for an object $X \in \mathcal{D}$ we have two notions of strong closedness, KT_2, $Pre\overline{T}_2$, and T_0' objects one with respect to U and one with respect to U_1. One may expect that the two notions may differ. Take $\mathcal{E} = \mathbf{ConFCO}$ and $\mathcal{D} = \mathbf{Top}$. Then by Remark 3.6 and Theorem 3.3,

$$\mathbf{T_0'ConFCO} = \mathbf{ConFCO}, \qquad \mathbf{KT_2ConFCO} = \mathbf{Chy} = \mathbf{Pre\overline{T}_2ConFCO}$$

$$\mathbf{T_0'Top} = \mathbf{T_0Top}, \qquad \mathbf{KT_2Top} = \mathbf{ST_2Top} = \mathbf{T_2Top} \subset \mathbf{Pre\overline{T}_2Top}.$$

In **Top**, by Remark 3.6, there is no implication between $Pre\overline{T}_2$ and T_0' and in the realm of $Pre\overline{T}_2$ topological spaces, by the Theorem 4.3 of [8], T_0', ST_2, and KT_2 are equivalent. Every zero-dimensional topological space is $Pre\overline{T}_2$. If B is a $Pre\overline{T}_2$ topological space and an arbitrary intersection of open subsets of B is open, then B is zero-dimensional [20, p.84].

By Theorem 3.3 and Remark 3.6, all KT_2, $Pre\overline{T}_2$, and T_0' Cauchy spaces are equal and every ST_2 Cauchy space is KT_2.

By Theorem 2.5, the category **ConFCO** is a link between the categories **Fil** and **FCO**. In constant filter convergence spaces the concept of completeness is not available but in KT_2 constant filter convergence spaces, this concept is available since by Theorem 3.3, KT_2 constant filter convergence structure induces the associated Cauchy structure.

In **ConFCO**, if (A, K) is a finite constant filter convergence space, then

(1) by Theorem 3.5, the partitions of A are in one-to-one correspondence with the distinct KT_2 constant filter convergence structures on A.

(2) by Theorem 3.3 and Remark 3.6, every $Pre\overline{T}_2$ constant filter convergence space is T_0' and in the realm of $Pre\overline{T}_2$ constant filter convergence spaces, T_0' and KT_2 are equivalent.

T'_0 and $Pre\overline{T}_2$ were used to define T_2, T_3, and T_4 objects in topological categories [2, p.340].

The equivalence relations can be characterized in terms of KT_2 reflexive spaces [8]. The equivalence (rep. partial, equals) relations can be characterized in terms of $Pre\overline{T}_2$ (resp. T'_0, KT_2) preordered spaces [8].

If an extended pseudo-quasi-semi metric space (A, d) is KT_2, then A has a partition consisting of strongly closed subsets [9, p.4759] and [15, p.709].

Acknowledgment

I would like to thank the referee for his/her useful suggestions

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