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On Fréchet-Urysohn gyrogroups

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Abstract. In this article, it is shown that if G is a strongly topological gyrogroup, H is a closed strong subgyrogroup of G and H is inner neutral, then the quotient space G/H is a sequential α_4-space if and only if it is a strongly Fréchet-Urysohn space, which deduces that the quotient space G/H is a weakly first-countable space if and only if it is metrizable. Then, it is shown that every Fréchet-Urysohn Hausdorff paratopological gyrogroup having the property $(**)$ is a strong α_4-space, which deduces that every Fréchet-Urysohn Hausdorff paratopological gyrogroup having the property $(**)$ is a strongly Fréchet-Urysohn space. Moreover, it is shown that if a Hausdorff paratopological gyrogroup having the property $(**)$ is a sequential α_4-space, then it is a strongly Fréchet-Urysohn space. Finally, we investigate the Fréchet-Urysohn Hausdorff paratopological gyrogroup with an ω^ω-base and show that every Fréchet-Urysohn Hausdorff paratopological gyrogroup having the property $(**)$ with an ω^ω-base is first-countable.

1. Introduction

A gyrogroup is a generalization of a group such that the associative law has been replaced by a weaker one. The concept of gyrogroups was introduced by A.A. Ungar when he study the c-ball of relativistically admissible velocities with Einstein velocity addition in [38], where Einstein velocity addition is the standard velocity addition of relativistically admissible velocities that Einstein introduced in [20] that founded the special theory of relativity in 1905. In 2017, W. Atiponrat [4] introduced the concept of topological gyrogroups. A topological gyrogroup is a gyrogroup G endowed with a topology such that the binary operation $\oplus : G \times G \to G$ is jointly continuous and the inverse mapping $\ominus(\cdot) : G \to G$, i.e. $x \to \ominus x$, is also continuous. By further study on the classical Möbius gyrogroups, Bao and Lin [8] introduced the concept of strongly topological gyrogroups. A topological gyrogroup G is called a strongly topological gyrogroup if there exists a neighborhood base $\mathscr{U}$ of 0 such that, for every $U \in \mathscr{U}$, $\mathrm{gyr}[x, y](U) = U$ for any $x, y \in G$. Clearly, every topological group is a strongly topological gyrogroup. A series of results on (strongly) topological gyrogroups have been obtained in [5, 8–11, 13, 15–17, 41–43].

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In [37], T. Suksumran and K. Wiboonton introduced the notion of L-subgyrogroups and showed that if H is an L-subgyrogroup of a gyrogroup G, then the set $\{a \oplus H : a \in G\}$ forms a disjoint partition of G. It is natural to research the quotient spaces of gyrogroups. In particular, Bao and Xu [12] constructed a subgyrogroup H in a strongly topological gyrogroup G such that $gyr[x, y](H) = H$ for all $x, y \in G$, hence they introduced the concept of strong subgyrogroups in a topological gyrogroup. Clearly, for a topological gyrogroup G, every strong subgyrogroup of G is an L-subgyrogroup.

The concept of ω^ω-base was introduced by Ferrando et al. in [23] in the framework of locally convex spaces for studying (DF)-spaces and C(X)-spaces. Then Gabriyelyan, Kakol and Leiderman investigated topological groups with an ω^ω-base and showed that a topological group G is metrizable if and only if it is Fréchet-Urysohn and has an ω^ω-base. In 2022, Bao and Xu [12] improved the result and showed that all Fréchet-Urysohn rectifiable spaces with countable cs^*-character are metrizable, which deduces that all Fréchet-Urysohn topological gyrogroups with countable cs^*-character are metrizable. Moreover, they extended the result that a topological group G is metrizable if and only if it is Fréchet-Urysohn and has an ω^ω-base to quotient spaces of strongly topological gyrogroups. They showed that if G is a strongly topological gyrogroup, H is a closed strong subgyrogroup of G and H is inner neutral, then G/H is first-countable if and only if G/H is Fréchet-Urysohn with an ω^ω-base. Therefore, it is natural to extend some important results of topological groups to quotient spaces of strongly topological gyrogroups. It is well-known that every weakly first-countable topological group is metrizable, then the following question is posed.

Question 1.1. *Let G be a strongly topological gyrogroup, H a closed inner neutral strong subgyrogroup of G. If the quotient space G/H is weakly first-countable, is it metrizable?*

In 2020, W. Atiponrat and R. Maungchang posed the concept of paratopological gyrogroups and showed that every regular micro-associative paratopological gyrogroup is completely regular, see [6, 27]. Then P. Li and R. Shen [29] proved that every T_0 compact paratopological gyrogroup is a Hausdorff topological gyrogroup. Since every paratopological group is a paratopological gyrogroup, we would like to extend some important results of paratopological groups to paratopological gyrogroups. In particular, Cai et al. [18] investigated paratopological groups with an ω^ω-base and proved that every Fréchet-Urysohn Hausdorff paratopological group having the property $(**)$ with an ω^ω-base is first-countable, hence submetrizable, where a paratopological group G has the property $(**)$ if there exists a non-trivial sequence $\{x_n\}_{n\in\mathbb{N}}$ in G such that both $\{x_n\}_{n\in\mathbb{N}}$ and $\{x_n^{-1}\}_{n\in\mathbb{N}}$ converge to the identity of G. It is natural to consider the following question.

Question 1.2. *Is every Fréchet-Urysohn Hausdorff paratopological gyrogroup G with an ω^ω-base first-countable? What if G has the property $(**)$?*

In this paper, we show that if G is a strongly topological gyrogroup, H is a closed strong subgyrogroup of G and H is inner neutral, then the quotient space G/H is a sequential α_4-space if and only if it is a strongly Fréchet-Urysohn space, which deduces that if the quotient space G/H is a weakly first-countable space if and only if it is metrizable, which gives an affirmative answer to Question 1.1. Then, it is shown that every Fréchet-Urysohn Hausdorff paratopological gyrogroup having the property $(**)$ is a strong α_4-space, which deduces that every Fréchet-Urysohn Hausdorff paratopological gyrogroup having the property $(**)$ is a strongly Fréchet-Urysohn space. Moreover, it is shown that if a Hausdorff paratopological gyrogroup having the property $(**)$ is a sequential α_4-space, then it is a strongly Fréchet-Urysohn space. Finally, we investigate the Fréchet-Urysohn Hausdorff paratopological gyrogroup with an ω^ω-base and show that every Fréchet-Urysohn Hausdorff paratopological gyrogroup having the property $(**)$ with an ω^ω-base is first-countable, which gives an affirmative answer to Question 1.2 when a paratopological gyrogroup G has the property $(**)$.

2. Preliminaries

Throughout this paper, if not specified, we assume that all topological spaces are T_1 spaces. Moreover, the set of all positive integers denoted by $\mathbb{N}$ and the first infinite ordinal denoted by ω. The reader may consult [3, 21, 31, 40] for notation and terminology not explicitly given here.

Definition 2.1. ([4]) Let G be a nonempty set, and let $\oplus : G \times G \to G$ be a binary operation on G. Then the pair $(G, \oplus)$ is called a *groupoid* or *magma*. A function f from a groupoid $(G_1, \oplus_1)$ to a groupoid $(G_2, \oplus_2)$ is called a *groupoid homomorphism* if $f(x \oplus_1 y) = f(x) \oplus_2 f(y)$ for any elements $x, y \in G_1$. Furthermore, a bijective groupoid homomorphism from a groupoid $(G, \oplus)$ to itself will be called a *groupoid automorphism*. We write $\mathrm{Aut}(G, \oplus)$ for the set of all automorphisms of a groupoid $(G, \oplus)$.

Definition 2.2. ([40]) Let $(G, \oplus)$ be a groupoid. The system $(G, \oplus)$ is called a *gyrogroup*, if its binary operation satisfies the following conditions:

$(G1)$ There exists a unique identity element $0 \in G$ such that $0 \oplus a = a = a \oplus 0$ for all $a \in G$.

$(G2)$ For each $x \in G$, there exists a unique inverse element $\ominus x \in G$ such that $\ominus x \oplus x = 0 = x \oplus (\ominus x)$.

$(G3)$ For all $x, y \in G$, there exists $\mathrm{gyr}[x, y] \in \mathrm{Aut}(G, \oplus)$ with the property that $x \oplus (y \oplus z) = (x \oplus y) \oplus \mathrm{gyr}[x, y](z)$ for all $z \in G$.

$(G4)$ For any $x, y \in G$, $\mathrm{gyr}[x \oplus y, y] = \mathrm{gyr}[x, y]$.

Notice that a group is a gyrogroup $(G, \oplus)$ such that $\mathrm{gyr}[x, y]$ is the identity function for all $x, y \in G$. The definition of a subgyrogroup is as follows.

Definition 2.3. ([37]) Let H be a nonempty subset of a gyrogroup G. We call H a *subgyrogroup* provided with H forming a gyrogroup under the operation inherited from G and $\mathrm{gyr}[x, y]|_H$ being an automorphism of H for each $x, y \in H$, denoted by $H \leq G$.

Furthermore, a subgyrogroup H of G is said to be an *L-subgyrogroup*, denoted by $H \leq_L G$, if $\mathrm{gyr}[a, h](H) = H$ for all $a \in G$ and $h \in H$.

Proposition 2.4. ([37]) *Let $(G, \oplus)$ be a gyrogroup, and let H be a nonempty subset of G. Then H is a subgyrogroup if and only if the following statements are true:*

1. *For any $x \in H$, $\ominus x \in H$.*
2. *For any $x, y \in H$, $x \oplus y \in H$.*

Lemma 2.5. ([40]). *Let $(G, \oplus)$ be a gyrogroup. Then for any $x, y, z \in G$, we obtain the following:*

1. $(\ominus x) \oplus (x \oplus y) = y$.
2. $(x \oplus (\ominus y)) \oplus gyr[x, \ominus y](y) = x$.
3. $(x \oplus gyr[x, y](\ominus y)) \oplus y = x$.
4. $gyr[x, y](z) = \ominus(x \oplus y) \oplus (x \oplus (y \oplus z))$.

Definition 2.6. ([39]) Let $(G, \oplus)$ be a gyrogroup, and $x \in G$. We define the left gyrotranslation by x to be the function $L_x : G \to G$ such that $L_x(y) = x \oplus y$ for any $y \in G$; the right gyrotranslation by x is defined to be the function $R_x : G \to G$ such that $R_x(y) = y \oplus x$ for any $y \in G$.

Definition 2.7. ([4]) A triple $(G, \tau, \oplus)$ is called a *topological gyrogroup* if the following statements hold:

1. (G, τ) is a topological space.
2. $(G, \oplus)$ is a gyrogroup.
3. The binary operation $\oplus : G \times G \to G$ is jointly continuous while $G \times G$ is endowed with the product topology, and the operation of taking the inverse $\ominus(\cdot) : G \to G$, i.e. $x \to \ominus x$, is also continuous.

If a triple $(G, \tau, \oplus)$ satisfies the first two conditions and its binary operation is continuous, it is called *a paratopological gyrogroup* [6]. In particular, if the binary operation and the topology are clear, we just say that G is a topological gyrogroup (paratopological gyrogroup).

Example 2.8. ([4]) The Einstein gyrogroup with the standard topology is a topological gyrogroup but not a topological group.

Let $\mathbb{R}^3_{\mathbf{c}} = \{\mathbf{v} \in \mathbb{R}^3 : \|\mathbf{v}\| < \mathbf{c}\}$, where $\mathbf{c}$ is the vacuum speed of light, and $\|\mathbf{v}\|$ is the Euclidean norm of a vector $\mathbf{v} \in \mathbb{R}^3$. The Einstein velocity addition $\oplus_E : \mathbb{R}^3_{\mathbf{c}} \times \mathbb{R}^3_{\mathbf{c}} \to \mathbb{R}^3_{\mathbf{c}}$ is given as follows:

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1 + \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{c}^2}} (\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}} \mathbf{v} + \frac{1}{\mathbf{c}^2} \frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}} (\mathbf{u} \cdot \mathbf{v}) \mathbf{u}),$$

for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$, $\mathbf{u} \cdot \mathbf{v}$ is the usual dot product of vectors in $\mathbb{R}^3$, and $\gamma_{\mathbf{u}}$ is the gamma factor which is given by

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \frac{\mathbf{u}\cdot\mathbf{u}}{\mathbf{c}^2}}}.$$

It was proved in [40] that $(\mathbb{R}^3_c, \oplus_E)$ is a gyrogroup but not a group. Moreover, with the standard topology inherited from $\mathbb{R}^3$, it is clear that $\oplus_E$ is continuous. Finally, $-\mathbf{u}$ is the inverse of $\mathbf{u} \in \mathbb{R}^3$ and the operation of taking the inverse is also continuous. Therefore, the Einstein gyrogroup $(\mathbb{R}^3_c, \oplus_E)$ with the standard topology inherited from $\mathbb{R}^3$ is a topological gyrogroup but not a topological group.

The following concepts are important in our researches.

Definition 2.9. Let X be a topological space.

(1) X is called a *weakly first-countable space* or *gf-countable space* [2] if for each point $x \in X$ it is possible to assign a sequence $\{B(n, x) : n \in \mathbb{N}\}$ of subsets of X containing x in such a way that $B(n+1, x) \subseteq B(n, x)$ and so that a set U is open if, and only if, for each $x \in U$ there exists $n \in \mathbb{N}$ such that $B(n, x) \subseteq U$.

(2) X is called a *sequential space* [24] if for each non-closed subset $A \subseteq X$, there are a point $x \in X \setminus A$ and a sequence in A converging to x in X.

(3) X is called a *Fréchet-Urysohn space* [24] if for any subset $A \subseteq X$ and $x \in \overline{A}$, there is a sequence in A converging to x in X.

(4) X is called a *strongly Fréchet-Urysohn space* [36] if the following condition is satisfied:

(SFU) For each $x \in X$ and every sequence $\xi = \{A_n : n \in \mathbb{N}\}$ of subsets of X such that $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$, there exists a sequence $\eta = \{b_n : n \in \mathbb{N}\}$ in X converging to x and intersecting infinitely many members of ξ.

(5) X is called an *α_4-space* [35], if for every point $x \in X$ and each sheaf $\{S_n : n \in \mathbb{N}\}$ with the vertex x, there exists a sequence converging to x which meets infinitely many sequences S_n.

Definition 2.10. ([3]) Let ζ be a family of non-empty subsets of a topological space X.

(1) ζ is called a *prefilter* on X if whenever P_1 and P_2 are in ζ, there exists $P \in \zeta$ such that $P \subseteq P_1 \cap P_2$.

(2) A prefilter ζ on X is said to *converge to a point* $x \in X$ if every open neighbourhood of x contains an element of ζ.

(3) A prefilter ζ on X is said to *accumulate to a point* $x \in X$ if x belongs to the closure of each element of ζ.

(4) Two prefilters ζ and η on X are said to be *synchronous* if, for any $P \in \eta$ and $Q \in \eta$, $P \cap Q \neq \emptyset$.

(5) X is called a *bisequential space* if, for every prefilter ζ on X accumulating to a point $x \in X$, there exists a countable prefilter η on X converging to the same point x such that ζ and η are synchronous.

The following diagram shows basic relationships between the classes of spaces defined in Definitions 2.9 and 2.10.

$$\begin{tikzcd}
 & \text{bisequential space} \arrow[r] & \text{strongly Fréchet-Urysohn space} \arrow[dd] \arrow[r] & \text{Fréchet-Urysohn space} \arrow[dd] \\
\text{first-countable space} \arrow[ur] \arrow[dr] & & & \\
 & \text{weakly first-countable space} \arrow[r] \arrow[rr, bend right] & \alpha_4\text{-space} & \text{sequential space}
\end{tikzcd}$$

The relationships between the classes of spaces defined in Definitions 2.9 and 2.10

3. Quotient spaces with inner neutral strong subgyrogroups

In this section, it is shown that if G is a strongly topological gyrogroup, H is a closed strong subgyrogroup of G and H is inner neutral, then the quotient space G/H is a sequential α_4-space if and only if it is a strongly Fréchet-Urysohn space, which deduces that if the quotient space G/H is a weakly first-countable space if and only if it is metrizable, which gives an affirmative answer to Question 1.1.

First, we recall the following concept of the coset space of a topological gyrogroup.

Let $(G, \tau, \oplus)$ be a topological gyrogroup and H an L-subgyrogroup of G. It follows from [37, Theorem 20] that $G/H = \{a \oplus H : a \in G\}$ is a coset space which defines a partition of G. We denote by π the mapping $a \mapsto a \oplus H$ from G onto G/H. Clearly, for each $a \in G$, we have $\pi^{-1}(\pi(a)) = a \oplus H$. Indeed, for any $a \in G$ and $h \in H$,

$$\begin{aligned}(a \oplus h) \oplus H &= a \oplus (h \oplus \mathrm{gyr}[h, a](H)) \\ &= a \oplus (h \oplus \mathrm{gyr}^{-1}[a, h](H)) \\ &= a \oplus (h \oplus H) \\ &= a \oplus H\end{aligned}$$

Denote by $\tau(G)$ the topology of G, the quotient topology on G/H is as follows:

$$\tau(G/H) = \{O \subseteq G/H : \pi^{-1}(O) \in \tau(G)\}.$$

Definition 3.1. ([12]) A subgyrogroup H of a topological gyrogroup G is called *strong subgyrogroup* if for any $x, y \in G$, we have $gyr[x, y](H) = H$.

Obviously, for a topological gyrogroup G, every strong subgyrogroup of G is an L-subgyrogroup. Moreover, the authors claimed that every strongly topological gyrogroup G contains some strong subgyrogroups which are union-generated from open neighborhoods of the identity element by construction, see [12, Proposition 3.11].

A topological space X is called a coset space if X is homeomorphic to G/H, for some closed subgroup H of a topological group G. It is well-known that every first-countable topological group is metrizable by the Birkhoff-Kakutani theorem. However, the following example shows that it does not hold in coset spaces.

Example 3.2. ([22]) The two arrows space is a compact coset space which is first-countable, but not submetrizable.

Since every topological group is a strongly topological gyrogroup and each subgroup is a strong subgyrogroup, the Example 3.2 shows that the axioms of first-countability is not equivalent with metrizability in the quotient space G/H, where G is a strongly topological gyrogroup and H is a closed strong subgyrogroup of G.

Definition 3.3. ([12]) A subgyrogroup H of a topological gyrogroup G is called *inner (outer) neutral* if for every open neighborhood U of 0 in G, there exists an open neighborhood V of 0 such that $H \oplus V \subseteq U \oplus H$ ($V \oplus H \subseteq H \oplus U$). H is called *neutral* if it is not only inner neutral but also outer neutral.

Remark 3.4. According to [12, Proposition 3.16], it was proved that if G is a strongly topological gyrogroup, then every compact strong subgyrogroup H of G is outer neutral, that is, for every open neighborhood U of 0 in G, there exists an open neighborhood V of 0 such that $V \oplus H \subseteq H \oplus U$. It is not difficult to see that every compact strong subgyrogroup H of G is inner neutral by the similar proof. Therefore, every compact strong subgyrogroup of a strongly topological gyrogroup is neutral.

Lemma 3.5. ([12]) *Let G be a strongly topological gyrogroup and H a closed strong subgyrogroup of G. Then the family $\{\pi(x \oplus V) : V \in \tau, 0 \in U\}$ is a local base of the space G/H at the point $x \oplus H \in G/H$, and G/H is a homogeneous T_1-space.*

Theorem 3.6. *Suppose that G is a strongly topological gyrogroup, H is a closed strong subgyrogroup of G and H is inner neutral, then the followings are equivalent.*

(1) *G/H is a sequential α_4-space;*
(2) *G/H is Fréchet-Urysohn;*
(3) *G/H is strongly Fréchet-Urysohn.*

Proof. It is well-known that a topological space X is a strongly Fréchet-Urysohn space if and only if it is Fréchet-Urysohn and strong α_4-space. According to [12, Theorem 3.19], it was proved that if G is a strongly topological gyrogroup, H is an inner neutral closed strong subgyrogroup of G and G/H is Fréchet-Urysohn, then G/H is a strong α_4-space. Therefore, it suffices to prove that (1) ⇒ (2). Suppose further that the space G/H is non-discrete.

(1) ⇒ (2). For $A \subseteq G/H$, we write $[A]$ the set of all limit points of sequences in A. Suppose on the contrary that G/H is not Fréchet-Urysohn. There is a subset B of G/H such that $[B] \neq \overline{B}$. If $[B]$ is closed in G/H, then $\overline{B} \subseteq \overline{[B]} = [B] \subseteq \overline{B}$, which is a contradiction. Hence, $[B]$ is not closed in G. By the hypothesis, G/H is sequential, so $[B]$ is not sequentially closed, that is $[[B]] \neq [B]$. Thus there is $b \in [[B]] \setminus [B]$. We may assume $b = \pi(0)$ without loss of generality, since G/H is homogeneous by Lemma 3.5.

Let $\{b_n : n \in \mathbb{N}\}$ be a sequence of points of $[B]$ converging to $\pi(0)$. For each $n \in \mathbb{N}$, fix a point $x_n \in \pi^{-1}(b_n)$. For each b_n, let $\{b_n(j) : j \in \mathbb{N}\}$ be a sequence of points of B converging to b_n. For each $j \in \mathbb{N}$, fix a point $x_n(j) \in \pi^{-1}(b_n(j))$. We claim $\lim_{j\to\infty} \pi((\ominus x_n) \oplus x_n(j)) = \pi(0)$.

Indeed, let O be an open neighborhood of $\pi(0)$ in G/H, then there is an open neighborhood U of 0 in G such that $\pi(U) \subseteq O$. Since $\lim_{j\to\infty} \pi(x_n(j)) = \lim_{j\to\infty} b_n(j) = b_n = \pi(x_n)$, there is $m \in \mathbb{N}$ such that $\pi(x_n(j)) \in \pi(x_n \oplus U)$ for $j \geq m$. So $\pi((\ominus x_n) \oplus x_n(j)) \in \pi((\ominus x_n) \oplus (x_n \oplus U)) = \pi(U) \subseteq O$ for $j \geq m$. Hence $\lim_{j\to\infty} \pi((\ominus x_n) \oplus x_n(j)) = \pi(0)$.

Since G/H is an α_4-space, it is possible to pick n_k, j_k for each $k \in \mathbb{N}$ such that $\{\pi((\ominus x_{n_k}) \oplus x_{n_k}(j_k)) : k \in \mathbb{N}\}$ converges to $\pi(0)$ and $n_k < n_{k+1}$ for each $k \in \mathbb{N}$. It follows from [12, Lemma 3.17] that $\lim_{k\to\infty} b_{n_k}(j_k) = \lim_{k\to\infty} \pi(x_{n_k}(j_k)) = \lim_{k\to\infty} \pi(x_{n_k} \oplus ((\ominus x_{n_k}) \oplus x_{n_k}(j_k))) = \pi(0)$, this contradicts the assumption that $\pi(0) \notin [B]$. Therefore, G/H is Fréchet-Urysohn. □

Note. It was claimed in [34, Theorem 4.4] that for a strongly topological gyrogroup G, if H is a closed neutral strong subgyrogroup of G, then G/H is metrizable if and only if G/H is first-countable. Moreover, it follows from [32, Corollary 1.3.10(2)] that every weakly first-countable Fréchet-Urysohn space is first-countable. Therefore, we conclude the following result.

Corollary 3.7. *Suppose that G is a strongly topological gyrogroup, H is a closed strong subgyrogroup of G and H is inner neutral, then the quotient space G/H is a weakly first-countable space if and only if G/H is metrizable.*

4. On Fréchet-Urysohn paratopological gyrogroups

In this section, we show that every Fréchet-Urysohn Hausdorff paratopological gyrogroup having the property (∗∗) is a strong α_4-space, which deduces that every Fréchet-Urysohn Hausdorff paratopological gyrogroup having the property (∗∗) is a strongly Fréchet-Urysohn space. Moreover, it is shown that if a Hausdorff paratopological gyrogroup having the property (∗∗) is a sequential α_4-space, then it is a strongly Fréchet-Urysohn space. Finally, we show that every Fréchet-Urysohn Hausdorff paratopological gyrogroup having the property (∗∗) with an ω^ω-base is first-countable, which gives an affirmative answer to Question 1.2 when a paratopological gyrogroup G has the property (∗∗).

Proposition 4.1. *Let G be a paratopological gyrogroup and x be any element of G. Then the left gyrotranslation L_x of G by x is a homeomorphism of the space G onto itself.*

Proof. It is clear that L_x is a continuous bijection. For any $y \in G$, $L_x(y) = x \oplus y$. Then $L_{\ominus x}(L_x(y)) = L_{\ominus x}(x \oplus y) = \ominus x \oplus (x \oplus y) = y$. Therefore, $L_{\ominus x} \circ L_x$ is the identity mapping, it follows that the inverse of L_x is also continuous. Hence, L_x is a homeomorphism of the space G onto itself. □

Proposition 4.2. *Every paratopological gyrogroup is a homogeneous space.*

Proof. Let G be a paratopological gyrogroup. For arbitrary $x, y \in G$, put $z = y \oplus gyr[y, x](\ominus x)$. Then $L_z(x) = z \oplus x = (y \oplus gyr[y, x](\ominus x)) \oplus x = y$ by Lemma 2.5. Moreover, L_z is homeomorphic by Proposition 4.1, so G is homogeneous. □

Proposition 4.3. *Let G be a paratopological gyrogroup, U a neighborhood of the identity element* 0*. Then there exists an open neighborhood V of* 0 *such that $V \subseteq U$ and $V \oplus V \subseteq U$.*

Proof. Since the mapping $\oplus : G \times G \to G$ is continuous, $(\oplus)^{-1}(Int(U))$ is an open set containing $(0, 0)$. We can find open neighborhoods V_1 and V_2 of 0 such that $0 \in V_1 \oplus V_2 \subseteq Int(U) \subseteq U$. Put $V = V_1 \cap V_2$. Then V is an open neighborhood of 0 such that $V \subseteq U$ and $V \oplus V \subseteq U$. □

Definition 4.4. A paratopological gyrogroup is said to *have the property* (∗∗) if there exists a non-trivial sequence $\{x_n\}_{n\in\mathbb{N}}$ in G such that both $\{x_n\}_{n\in\mathbb{N}}$ and $\{\ominus x_n\}_{n\in\mathbb{N}}$ converge to the identity element 0 of G.

Definition 4.5. ([18]) A topological space X is called a *strong α_4-space* if for any subset $\{x_{m,n} : m, n \in \mathbb{N}\} \subseteq X$ with $lim_{n\to\infty} x_{m,n} = x \in X$ for each $m \in \mathbb{N}$, there are strictly increasing sequences of natural numbers $\{i_k\}_{k\in\mathbb{N}}$ and $\{j_k\}_{k\in\mathbb{N}}$ such that $lim_{k\to\infty} x_{i_k, j_k} = x$.

The idea of the following proof is originated from [19, Lemma 1.3], then Lin, Cai and Ling used the same method to investigate rectifiable spaces, paratopological groups and coset spaces of topological groups, respectively, see [30, Lemma 5.1], [18, Theorem 2.3] and [33, Theorem 3.7].

Theorem 4.6. *Every Fréchet-Urysohn Hausdorff paratopological gyrogroup having the property* (∗∗) *is a strong α_4-space.*

Proof. Let G be a Fréchet-Urysohn Hausdorff paratological gyrogroup having the property (∗∗) and $\{x_{m,n} : m, n \in \mathbb{N}\} \subseteq G$ be such that $lim_{n\to\infty} x_{m,n} = 0$ for each $m \in \mathbb{N}$. Since G is homogeneous by Proposition 4.2, it suffices to show that there exist strictly increasing sequences of natural numbers $\{i_k\}_{k\in\mathbb{N}}$ and $\{j_k\}_{k\in\mathbb{N}}$ such that $lim_{k\to\infty} x_{i_k, j_k} = 0$.

Since G has the property (∗∗), there exists a non-trivial sequence $\{a_m\}_{m\in\mathbb{N}}$ in G with $lim_{m\to\infty} a_m = 0$ and $lim_{m\to\infty} \ominus a_m = 0$. For every $m, l \in \mathbb{N}$, put

$$y_{m,l} = \begin{cases} a_m \oplus x_{m,l+m} & , \quad a_m \oplus x_{m,l+m} \neq 0; \\ a_m & , \quad \textit{otherwise}. \end{cases}$$

Set $M = \{y_{m,l} : (m, l) \in \mathbb{N} \times \mathbb{N}\}$. Then $0 \notin M$, we show that $0 \in \overline{M}$.

Indeed, for each open neighborhood U of 0 in G, there exists a neighborhood V of 0 such that $V \oplus V \subseteq U$ by Proposition 4.3. Since $lim_{m\to\infty} a_m = 0$, we can find $m \in \mathbb{N}$ such that $a_m \in V$. It follows from $lim_{n\to\infty} x_{m,n} = 0$ that $x_{m,l+m} \in U$ for some $l \in \mathbb{N}$. Hence $y_{m,l} \in V \oplus V \subseteq U$, which deduces that $M \cap U \neq \emptyset$ and $0 \in \overline{M}$.

Since G is Fréchet-Urysohn, we can find a sequence $\{y_{m_k, l_k}\}_{k\in\mathbb{N}}$ converging to 0 in G.

Case 1. The sequence $\{l_k\}_{k\in\mathbb{N}}$ is bounded.

Choose a subsequence if necessary, then assume that $l_k = r, k = 1, 2, \cdots$, for some natural number r. Since $lim_{k\to\infty} y_{m_k, r} = lim_{k\to\infty} y_{m_k, l_k} = 0$ and $y_{m_k, r} \neq 0$, we know that $lim_{k\to\infty} m_k = \infty$. Choosing once more a subsequence, we assume further that $m_k < m_{k+1}$ for each $k \in \mathbb{N}$.

Subcase 1.1. The set $N_1 = \{k \in \mathbb{N} : y_{m_k, r} = a_{m_k}\}$ is infinite.

Denote $N_1 = \{p_1, p_2, \cdots\}$ with $p_k < p_{k+1}$ for each $k \in \mathbb{N}$. Then $a_{m_{p_k}} \oplus x_{m_{p_k}, r+m_{p_k}} = 0$ for each $k \in \mathbb{N}$. As $lim_{k\to\infty} \ominus a_{m_{p_k}} = 0$, we have $lim_{k\to\infty} x_{m_{p_k}, r+m_{p_k}} = lim_{k\to\infty}[(\ominus a_{m_{p_k}}) \oplus (a_{m_{p_k}} \oplus x_{m_{p_k}, r+m_{p_k}})] = 0$. For each $k \in \mathbb{N}$, put $i_k = m_{p_k}$ and $j_k = r + m_{p_k}$, then $\{i_k\}_{k\in\mathbb{N}}$ and $\{j_k\}_{k\in\mathbb{N}}$ are strictly increasing sequences such that $lim_{k\to\infty} x_{i_k, j_k} = 0$.

Subcase 1.2. The set $N_1 = \{k \in \mathbb{N} : y_{m_k, r} = a_{m_k}\}$ is finite.

Then put $N_2 = \{k \in \mathbb{N} : y_{m_k, r} \neq a_{m_k}\}$, so N_2 is infinite. Denote $N_2 = \{q_1, q_2, \cdots\}$ with $q_k < q_{k+1}$ for each $k \in \mathbb{N}$. Then $y_{m_{q_k}, l_{q_k}} = a_{m_{q_k}} \oplus x_{m_{q_k}, r+m_{q_k}}$ for each $k \in \mathbb{N}$. Since $lim_{k\to\infty} \ominus a_{m_{q_k}} = 0$, we have that

$lim_{k\to\infty}x_{m_{q_k},r+m_{q_k}} = lim_{k\to\infty}[(\ominus a_{m_{q_k}}) \oplus (a_{m_{q_k}} \oplus x_{m_{q_k},r+m_{q_k}})] = 0$. For each $k \in \mathbb{N}$, put $i_k = m_{q_k}$ and $j_k = r + m_{q_k}$, then $\{i_k\}_{k\in\mathbb{N}}$ and $\{j_k\}_{k\in\mathbb{N}}$ are also strictly increasing sequences such that $lim_{k\to\infty}x_{i_k,j_k} = 0$.

Case 2. The sequence $\{l_k\}_{k\in\mathbb{N}}$ is not bounded.

Assume that $\{l_k\}_{k\in\mathbb{N}}$ is strictly increasing, then $lim_{k\to\infty}m_k = \infty$. Otherwise, taking a subsequence if necessary, there exists $s \in \mathbb{N}$ such that $m_k = s$ for each $k \in \mathbb{N}$. Since $\{l_k\}_{k\in\mathbb{N}}$ is strictly increasing, $lim_{k\to\infty}x_{s,s+l_k} = 0$. From $lim_{k\to\infty}y_{s,l_k} = 0$, it follows that $a_{m_k} = a_s = 0$, which is a contradiction with the choice of $\{a_m\}_{m\in\mathbb{N}}$. Thus $lim_{k\to\infty}m_k = \infty$. Then there exists a strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that $m_{n_1} < m_{n_2} < \cdots$. As $lim_{k\to\infty}a_{m_{n_k}} \oplus x_{m_{n_k},l_{n_k}+m_{n_k}} = lim_{k\to\infty}y_{m_{n_k},l_{n_k}} = 0$ and $lim_{k\to\infty} \ominus a_{m_{n_k}} = 0$, we have that $lim_{k\to\infty}x_{m_{n_k},l_{n_k}+m_{n_k}} = lim_{k\to\infty}[(\ominus a_{m_{n_k}}) \oplus (a_{m_{n_k}} \oplus x_{m_{n_k},l_{n_k}+m_{n_k}})] = 0$. For each $k \in \mathbb{N}$, put $i_k = m_{n_k}$ and $j_k = l_{n_k} + m_{n_k}$, so $\{i_k\}_{k\in\mathbb{N}}$ and $\{j_k\}_{k\in\mathbb{N}}$ are strictly increasing sequences such that $lim_{k\to\infty}x_{i_k,j_k} = 0$. We conclude that G is a strong α_4-space. □

It is well-known that a topological space X is a strongly Fréchet-Urysohn space if and only if it is Fréchet-Urysohn and strong α_4-space. Then the following result is obtained.

Corollary 4.7. *Every Fréchet-Urysohn Hausdorff paratological gyrogroup having the property* (∗∗) *is a strongly Fréchet-Urysohn space.*

Theorem 4.8. *Let G be a Hausdorff paratopological gyrogroup having the property* (∗∗)*. If G is a sequential α_4-space, then it is a strongly Fréchet-Urysohn space.*

Proof. For $A \subseteq G$, we write $[A]$ the set of all limit points of sequences in A. First, we show that G is a Fréchet-Urysohn space. Suppose on the contrary that G is not Fréchet-Urysohn. There is a subset B of G such that $[B] \neq \overline{B}$. If $[B]$ is closed in G, then $\overline{B} \subseteq \overline{[B]} = [B] \subseteq \overline{B}$, which is a contradiction. Hence, $[B]$ is not closed in G. By the hypothesis, G is sequential, so $[B]$ is not sequentially closed, that is $[[B]] \neq [B]$. Thus there is $b \in [[B]] \setminus [B]$. We may assume $b = 0$ without loss of generality, since G is homogeneous by Proposition 4.2.

Since $0 \in [[B]]$, we can find a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points of $[B]$ such that $\{x_n\}_{n\in\mathbb{N}}$ converges to 0. For each $n \in \mathbb{N}$, there exists a sequence $\{x_{n,j}\}_{j\in\mathbb{N}} \subseteq B$ such that the sequence $\{x_{n,j}\}_{j\in\mathbb{N}}$ converges to x_n. Then $\lim_{j\to\infty}\{(\ominus x_n)\oplus x_{n,j}\} = 0$. By the hypothesis, G is a α_4-space, it is possible to pick n_k, j_k for each $k \in \mathbb{N}$ such that $\lim_{k\to\infty}\{(\ominus x_{n_k})\oplus x_{n_k,j_k}\} = 0$ and $n_k < n_{k+1}$ for each $k \in \mathbb{N}$. Then $\lim_{k\to\infty} x_{n_k,j_k} = \lim_{k\to\infty}(x_{n_k} \oplus \{(\ominus x_{n_k})\oplus x_{n_k,j_k}\}) = 0$, this contradicts the assumption that $0 \notin [B]$. Therefore, G is Fréchet-Urysohn.

Finally, it follows from Corollary 4.7 that G is a strongly Fréchet-Urysohn space. □

Definition 4.9. ([25]) A point x of a topological space X is said to have a *neighborhood ω^ω-base* or a *local $\mathfrak{G}$-base* if there exists a base of neighborhoods at x of the form $\{U_\alpha(x) : \alpha \in \mathbb{N}^\mathbb{N}\}$ such that $U_\beta(x) \subseteq U_\alpha(x)$ for all elements $\alpha \leq \beta$ in $\mathbb{N}^\mathbb{N}$, where $\mathbb{N}^\mathbb{N}$ consisting of all functions from $\mathbb{N}$ to $\mathbb{N}$ is endowed with the natural partial order, ie., $f \leq g$ if and only if $f(n) \leq g(n)$ for all $n \in \mathbb{N}$. The space X is said to have an *ω^ω-base* or a *$\mathfrak{G}$-base* if it has a neighborhood ω^ω-base or a local $\mathfrak{G}$-base at every point $x \in X$.

Suppose that G is a paratopological gyrogroup and G has an ω^ω-base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. Set

$$I_k(\alpha) = \{\beta \in \mathbb{N}^\mathbb{N} : \beta_i = \alpha_i \text{ for } i = 1, \cdots, k\}, \text{ and } D_k(\alpha) = \bigcap_{\beta\in I_k(\alpha)} U_\beta,$$

where $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$ and $k \in \mathbb{N}$. It is clear that $\{D_k(\alpha)\}_{k\in\mathbb{N}}$ is an increasing sequence of subsets of G and contains the identity element 0.

Lemma 4.10. ([26]) *Let $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$ and $\beta_k = (\beta^k_i)_{i\in\mathbb{N}} \in I_k(\alpha)$ for every $k \in \mathbb{N}$. Then there is $\gamma \in \mathbb{N}^\mathbb{N}$ such that $\alpha \leq \gamma$ and $\beta_k \leq \gamma$ for every $k \in \mathbb{N}$.*

Theorem 4.11. *Every Fréchet-Urysohn Hausdorff paratopological gyrogroup having the property* (∗∗) *with an ω^ω-base is first-countable.*

Proof. Let G be a Fréchet-Urysohn Hausdorff paratopological gyrogroup having the property $(**)$ with an ω^ω-base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. First, we show that for each $\alpha \in \mathbb{N}^\mathbb{N}$, there is $k \in \mathbb{N}$ such that $D_k(\alpha)$ is a neighborhood of the identity element 0. Suppose on the contrary, we can find $\alpha \in \mathbb{N}^\mathbb{N}$ such that $D_k(\alpha)$ is not a neighborhood of 0, for any $k \in \mathbb{N}$. It means that $0 \in \overline{G \setminus D_k(\alpha)}$ for any $k \in \mathbb{N}$. Since G is Fréchet-Urysohn, we can find a sequence $\{x_{n,k}\}_{n\in\mathbb{N}}$ in $G \setminus D_k(\alpha)$ which converges to 0. It follows from Theorem 4.6 that G is a strong α_4-space, so there are strictly increasing sequences $(n_i)_{i\in\mathbb{N}}$ and $(k_i)_{i\in\mathbb{N}}$ of natural numbers such that $lim_{i\to\infty} x_{n_i,k_i} = 0$. For every $i \in \mathbb{N}$, there exists $\beta_{k_i} \in I_{k_i}(\alpha)$ such that $x_{n_i,k_i} \notin U_{\beta_{k_i}}$. It follows from Lemma 4.10 that there is $\gamma \in \mathbb{N}^\mathbb{N}$ such that $\beta_{k_i} \le \gamma$ for every $i \in \mathbb{N}$. Therefore, for any $i \in \mathbb{N}$, $x_{n_i,k_i} \notin U_\gamma$. We conclude that the sequence $\{x_{n_i,k_i}\}_{i\in\mathbb{N}}$ does not converge to 0 and this is a contradiction.

Therefore, for each $\alpha \in \mathbb{N}^\mathbb{N}$, we can find a minimal natural number k_α such that $D_{k_\alpha}(\alpha)$ is a neighborhood of the identity element 0. It is clear that $D_{k_\alpha}(\alpha) \subseteq U_\alpha$. Moreover, for $i \in \mathbb{N}$, fix $\alpha^{(i)} = (i, \alpha_2, \alpha_3, \cdots) \in \mathbb{N}^\mathbb{N}$. Then for any $\beta = (\beta_1, \beta_2, \cdots) \in I_1(\alpha^{(i)})$, $D_1(\beta) = D_1(\alpha^i)$. Therefore, $\{D_1(\alpha) : \alpha \in \mathbb{N}^\mathbb{N}\} = \{D_1(\alpha^{(i)}) : i \in \mathbb{N}\}$ is countable. So, $\{D_k(\alpha) : k \in \mathbb{N}, \alpha \in \mathbb{N}^\mathbb{N}\}$ is countable. Furthermore, $\{D_{k_\alpha}(\alpha) : \alpha \in \mathbb{N}^\mathbb{N}\} \subseteq \{D_k(\alpha) : k \in \mathbb{N}, \alpha \in \mathbb{N}^\mathbb{N}\}$. Thus, we have that the family $\{D_{k_\alpha}(\alpha) : \alpha \in \mathbb{N}^\mathbb{N}\}$ is countable. In conclusion, the family $\{int(D_{k_\alpha}(\alpha)) : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a countable base of open neighborhoods at 0 in G. It follows from Proposition 4.2 that G is first-countable. □

It follows from Proposition 4.2 that if a paratopological gyrogroup G is first-countable, then it has an ω^ω-base. Therefore, by Theorem 4.11, the following result is clear.

Corollary 4.12. *Let G be a Hausdorff paratopological gyrogroup having the property $(**)$. Then G is first-countable if and only if G is Fréchet-Urysohn with an ω^ω-base.*

A topological space X is *hemicompact* if $X = \bigcup_{n\in\mathbb{N}} X_n$, where X_n is compact for every $n \in \mathbb{N}$ and for every compact $K \subseteq X$, there is $n \in \mathbb{N}$ such that $K \subseteq X_n$.

Proposition 4.13. *Every Fréchet-Urysohn hemicompact topological gyrogroup is locally compact and metrizable.*

Proof. Let G be a Fréchet-Urysohn hemicompact topological gyrogroup and $G = \bigcup_{n\in\mathbb{N}} K_n$, where $\{K_n\}_n$ is an increasing sequence of compact subsets of K containing the identity element 0 such that every compact set in G is contained in some K_n. Then we can find $n \in \mathbb{N}$ such that K_n is a neighborhood of 0. Suppose on the contrary that K_n is not a neighborhood of 0 for any $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$ and each neighborhood U of 0, there exists $x_{U,n} \in U \setminus K_n$. For each $n \in \mathbb{N}$, set $B_n = \{x_{U,n} : U \text{ is an open neighborhood of } 0\}$. Then $0 \in \overline{B_n}$. Since G is Fréchet-Urysohn, for each $n \in \mathbb{N}$, we can find an open neighborhood sequence $\{U_n(k)\}_k$ of 0 such that $x_{U_n(k),n} \to 0$ at $k \to \infty$. Since every Fréchet-Urysohn topological gyrogroup is a strong α_4-space by [14, Lemma 3.3], there exists strictly increasing sequences $\{k_p\}_{p\in\mathbb{N}}$ and $\{n_p\}_{p\in\mathbb{N}}$ such that $x_{U_{n_p}(k_p),n_p} \to 0$ at $p \to \infty$. Since the set $B = \{x_{U_{n_p}(k_p),n_p} : p \in \mathbb{N}\} \cup \{0\}$ is compact in G, we can find $m \in \mathbb{N}$ such that $B \subseteq K_m$, which is a contradiction. Taking into account that G is homogeneous by Proposition 4.2, it follows that G is locally compact.

Since every locally compact topological gyrogroup is feathered and every topological gyrogroup is a rectifiable space, it follows from [1, Theorem 3.4] that every feathered topological gyrogroup with countable tightness is first-countable. Hence, G is metrizable by [1, Theorem 3.2]. □

Revising the proof of Proposition 4.13, we show that every Fréchet-Urysohn Hausdorff paratopological gyrogroup having the property $(**)$ is locally compact.

Proposition 4.14. *Every Fréchet-Urysohn Hausdorff hemicompact paratopological gyrogroup having the property $(**)$ is locally compact.*

Proof. Let G be a Fréchet-Urysohn Hausdorff hemicompact paratopological gyrogroup having the property $(**)$ and $G = \bigcup_{n\in\mathbb{N}} K_n$, where K_n is compact for every $n \in \mathbb{N}$ and for every compact $K \subseteq G$ there exists $n \in \mathbb{N}$ such that $K \subseteq K_n$. Then we show that there exists $n \in \mathbb{N}$ such that K_n is a neighborhood of 0. Suppose on the contrary that K_n is not a neighborhood of 0 for any $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$ and each neighborhood U of 0, there exists $x_{U,n} \in U \setminus K_n$. For each $n \in \mathbb{N}$, set $B_n = \{x_{U,n} : U \text{ is an open neighborhood of } 0\}$. Then

$0 \in \overline{B_n}$. Since G is Fréchet-Urysohn, for each $n \in \mathbb{N}$, we can find an open neighborhood sequence $\{U_n(k)\}_k$ of 0 such that $x_{U_n(k),n} \to 0$ at $k \to \infty$. Since every Fréchet-Urysohn Hausdorff paratological gyrogroup having the property (∗∗) is a strong α_4-space by Theorem 4.6, there exists strictly increasing sequences $\{k_p\}_{p\in\mathbb{N}}$ and $\{n_p\}_{p\in\mathbb{N}}$ such that $x_{U_{n_p}(k_p),n_p} \to 0$ at $p \to \infty$. Since the set $B = \{x_{U_{n_p}(k_p),n_p} : p \in \mathbb{N}\} \cup \{0\}$ is compact in G, we can find $m \in \mathbb{N}$ such that $B \subseteq K_m$, which is a contradiction. Therefore, we obtain that G is locally compact. □

In [28], Jin and Xie introduced the concept of strongly paratopological gyrogroups, that is, G is called a *strongly paratopological gyrogroup* if G is a paratopological gyrogroup and there exists a neighborhood base $\mathscr{U}$ of 0 in G such that for every $U \in \mathscr{U}$, $\mathrm{gyr}[x, y](U) = U$ for any $x, y \in G$. They showed that every locally compact Hausdorff strongly paratopological gyrogroup is a topological gyrogroup. Then we obtain the following result by Propositions 4.13 and 4.14.

Corollary 4.15. *Every Fréchet-Urysohn Hausdorff hemicompact strongly paratopological gyrogroup having the property* (∗∗) *is locally compact and metrizable.*

A space X is *submetrizable* if there exists a continuous one-to-one mapping of X onto a metrizable space. In [7], Banakh and Ravsky showed that every first-countable Hausdorff paratopological group is submetrizable. Then the following question is natural.

Question 4.16. *Is every first-countable Hausdorff paratopological gyrogroup submetrizable?*

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