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Quadratic modules of groupoids and related structures

Elis Soylu Yilmaz[a]

[a]*Department of Mathematics and Computer Science, Science Faculty, Eskişehir Osmangazi University, Eskişehir, Turkey*

Abstract. In this paper, we define functors between quadratic modules of groupoids and related structures. The focus of this paper is on the study of functors between quadratic modules of groupoids and related structures. Specifically, we explore the existence of functors between nil(2)-modules, quadratic modules, and 2-quadratic modules over groupoids. We also define functors that relate these different types of modules over groupoids, and investigate the categorical equivalence between them. We show that functors exist between the categories of 2-quadratic modules of groups, nil(2)-modules, and quadratic modules over group(oid)s.

1. Introduction

Whitehead defined the concept "crossed module" to describe algebraic models for homotopy 2-types [23]. The crossed modules are also equivalent to Cat^1-group [16] and to G-groupoids [7] which are also called group-groupoids [9] enables providing additional examples of crossed modules. Simplicial groups with a Moore complex of length 1 are equivalent to crossed modules. Conduché defined "2-crossed modules" or "crossed modules of length 2" as an algebraic model for homotopy 3-types by using simplicial groups with Moore complexes of length 2 as a result of this equivalence[12]. Quadratic modules are another homotopy 3-type algebraic model. The structure of a quadratic module is a 2-crossed module with additional nilpotent conditions. Quadratic modules and 2-crossed modules are related algebraic structures that have been extensively studied in algebraic topology and related fields [2]. Quadratic modules and 2-crossed modules have many applications in algebraic topology, including the study of homotopy groups of spaces, higher-dimensional knot theory, and the classification of topological phases of matter in condensed matter physics. They are also closely related to other algebraic structures, such as Lie algebras, categorical algebras, Kac-Moody algebras, and quantum groups. For more details, see [22], [17], [20], [1], [19], [14], [18].

Introducing an alternate model by extending the notion of quadratic modules, 2-quadratic modules were defined by Atik and Ulualan[3]. 2-quadratic modules represent algebraic models for homotopy 4-types. Atik and Ulualan also use the image of $F_{\alpha\beta}$ functions to give relations between 2-quadratic modules via simplicial groups. Brandt was the first to introduce groupoids [5] in 1926. Brown's survey [8] provides prospects for many threads of groupoids' usage. According to Brown, Eilenberg and Mac Lane were influenced by Brandt's axioms for groupoids when defining a category[13]. Since every small category

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Communicated by Ljubiša D. R. Kočinac
Email address: esoylu@ogu.edu.tr (Elis Soylu Yilmaz)
ORCID iD: https://orcid.org/0000-0002-0869-310X (Elis Soylu Yilmaz)

having invertible elements forms a groupoid, interest in groupoids has grown following Whitehead's papers in category theory. Kamps and Porter were the first to mention 2-crossed modules of groupoids [15]. In this work, based on the quadratic module definition defined by Baues on groups, we adapt quadratic module definition on groupoids. For more details about the theory of groupoids, see [6], [10], [24], [11]. The purpose of this paper is to demonstrate certain connections between quadratic modules of groupoids and related structures. As a result, the main points of this study can be given as:

i) To obtain a groupoid structure from any nil(2)-module

ii) To obtain a nil(2)-module of groupoid from any quadratic module

iii) To obtain a quadratic module from any 2-quadratic module

We also remark possible functors from quadratic module of groupoids to nil(2)-module of groupoids, from groupoids to nil(2)-module of groupoids, from nil(2)-modules to quadratic modules and from quadratic modules to 2-quadratic modules with examples. Our goal is to provide an understanding of the categorical relationships among them from an introductory and combinatorial perspective, not only to reprove some (known) results in a simple approach.

2. Preliminaries

Note that a group homomorphism $\partial : M \to N$ is *a pre-crossed module* with an action of N on M, written ${}^{n}m$ for $n \in N$ and $m \in M$, which satisfy $\partial({}^{n}m) = n\partial(m)n^{-1}$ for all $n \in N$ and $m \in M$.

Definition 2.1. ([4]) Let M be a group then the Peiffer commutator $\langle x_1, x_2, x_3\rangle$ of length 3 generates the subgroup P of M. A pre-crossed module $\partial : M \to N$ with the additional $P_3(\partial) = 1$ "nilpotency"condition is called a nil(2)-module. For $x, y \in M$

$$\langle x, y\rangle =^{\partial_1 x} (y)xy^{-1}x^{-1}$$

is the Peiffer commutator of the pre-crossed module $\partial : M \to N$.

A morphism of nil(2)-modules

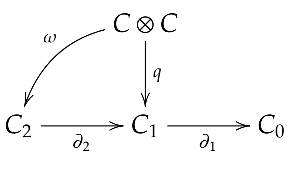
$$(g, f) : \left(M \xrightarrow{\partial} N\right) \to \left(M' \xrightarrow{\partial'} N'\right)$$

consists of groups homomorphisms $f : N \to N'$ and $g : M \to M'$ preserving the action of M on N such that $f\partial = \partial' g$. Throughout this study, we will refer to the category of *nil(2)-modules* as **Nil**(2).

Example 2.2. $\{e_G\} \to G$ is a nil(2)-module for a group G, where e_G is the identity element of G, and $\partial : \{e_G\} \to G$ is defined by $\partial(e_G) = e_G$.

Since $\{e_G\}$ only contains the identity element, we have that y, x, and $y^{-1}x^{-1}$ are all equal to e_G. Thus, we have $<x, y> = e_G$, which satisfies the Peiffer commutator condition. Finally, we need to show that $P_3(\partial) = \{e_G\}$. Since e_G is a trivial group, we have $P_3(\partial) = \{x \in e_G | <x, e_G, e_G> = e_G\} = \{e_G\}$, which is a trivial subgroup of $\{e_G\}$. Therefore, $P_3(\partial) = \{e_G\}$, that is $\partial : \{e_G\} \to G$ is a nil(2)-module.

Definition 2.3. ([4]) A diagram



of group homomorphisms which satisfy:
QM1) $\partial_1 : C_1 \to C_0$ is a nil(2)-module. For the abelianization of the associated crossed module $C_1^{cr} \to C_0$, $C = (C_1^{cr})^{ab}$, the quotient map $C_1 \twoheadrightarrow C = (C_1^{cr})^{ab}$ is defined as $x \mapsto \{x\}$, where the class represented by $x \in C_1$ is denoted by $\{x\} \in C$.
QM2) $\partial_1\partial_2 = 1$ and $\partial_2\omega = q$.
QM3) C_2 is a C_0-group and all ∂_2 and ∂_1 preserves the action of C_0. Additionally, for $a \in C_2, x \in C_1$:

$$^{\partial_1 x}aa^{-1} = \omega((\{\partial_2 a\} \otimes \{x\})(\{x\} \otimes \{\partial_2 a\})).$$

QM4) For $a, b \in C_2$

$$\omega(\{\partial_2 a\} \otimes \{\partial_2 b\}) = [b, a]$$

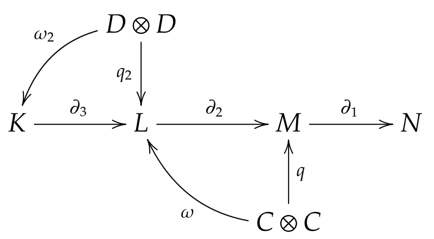
is called a quadratic module.

A quadratic modules morphism, $\varphi = (f_2, f_1, f_0) : (\omega, \partial_2, \partial_1) \to (\omega', \partial_2', \partial_1')$ is a commutative diagram,

$$\begin{array}{ccccccc} C \otimes C & \xrightarrow{\omega} & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 \\ \downarrow{\scriptstyle \varphi_* \otimes \varphi_*} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ C' \otimes C' & \xrightarrow[\omega']{} & C_2' & \xrightarrow[\partial_2']{} & C_1' & \xrightarrow[\partial_1']{} & C_0' \end{array}$$

where (f_1, f_0) is a nil(2)-module morphism which induces $\varphi_* : C \to C'$ and f_2 preserves the action of C_0 on C_2. We will refer to the category of quadratic modules as **QM**.

Definition 2.4. ([3]) A 2-quadratic module is a diagram of groups homomorphism

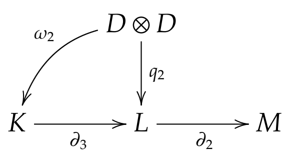


with an action of L on K, an action of M on L, K and an action of N on M, L, K. There are also 2-quadratic maps

$$\Phi_0 : D \otimes C \to K, \;\; \Phi_1 : C \otimes D \to K, \;\; \Phi_2 : C \otimes D \to K,$$
$$\omega : C \otimes C \to L, \;\; q : C \otimes C \to M, \;\; q_2 : D \otimes D \to L,$$
$$\omega_0 : D \otimes D \to K, \; \omega_1 : D \otimes D \to K, \;\; \omega_2 : D \otimes D \to K$$

where $C = (M^{cr})^{ab}$ and $D = (L^{cr})^{ab}$. The quotient maps $M \twoheadrightarrow C = (M^{cr})^{ab}$ and $L \twoheadrightarrow D = (L^{cr})^{ab}$ are given by $x \mapsto \{x\}$. The following axioms must be met by this data.

2QM1)



is a quadratic module,

2QM2) $\partial_1 : M \to N$ is a nil(2)-module and

$$\partial_2\omega(\{m\} \otimes \{m'\}) = q(\{m\} \otimes \{m'\}) = [m, m'],$$

2QM3) $\Phi_2^{-1}\Phi_1(\{m\} \otimes \{\partial_3 k\}) = \Phi_0(\{\partial_3 k\} \otimes \{m\})k^{\partial_2 m}(k)^{-1}$,
2QM4) $\Phi_0(\{\partial_3 k\} \otimes \{m\}) ={}^m k(k)^{-1}$,
2QM5) $\Phi_0(\{l\} \otimes \{\partial_2 l'\}) = \omega_2(\{l\} \otimes \{l'\})^{-1}\omega_1(\{l\} \otimes \{l'\})$,
2QM6) ${\Phi_1}^{-1}\Phi_2(\{\partial_2 l\} \otimes \{l'\}) = ({}^{[l',l]}\omega_2(\{l\} \otimes \{l'\}))\omega_1(\{l\} \otimes \{l'\})$,
2QM7) $\partial_3\omega_1(\{l\} \otimes \{l'\}) = [l, l']\omega(\{\partial_2 l\} \otimes \{\partial_2 l'\})$,
2QM8) $(i)\omega(\{\partial_2 l\} \otimes \{m\}) = l^m(l)^{-1}\partial_3(\Phi_0(\{l\} \otimes \{m\}))$ and
$(ii)\omega(\{m\} \otimes \{\partial_2 l\}) ={}^m l^{\partial_1 m}(l)^{-1}\partial_3({\Phi_1}^{-1}\Phi_2(\{m\} \otimes \{l\}))$,
2QM9) $\omega_1(\{\partial_3 k\} \otimes \{l\}\{l\} \otimes \{\partial_3 k\}) = \omega_0(\{\partial_3 k\} \otimes \{l\}) = 1$,
2QM10) $\omega_1(\{\partial_3 k\} \otimes \{\partial_3 k'\}) = [k, k']$,
2QM11) $\Phi_0(\{\partial_3 k\} \otimes \{\partial_2 l\})\Phi_1\Phi_2(\{\partial_2 l\} \otimes \{\partial_3 k\}) = 1$
for $k, k' \in K$, $l, l' \in L$ and $m, m' \in M$.

A morphism of the 2-quadratic modules $(f_0, f_1, f_2, f_3) : (K, L, M, N, \omega_2, q) \to (K', L', M', N', \omega_2', q')$ is a commutative diagram

$$\begin{array}{ccccccccc}
D \otimes D & \longrightarrow & K & \xrightarrow{\partial_3} & L & \xrightarrow{\partial_2} & M & \xrightarrow{\partial_1} & N \\
\downarrow{\scriptstyle \varphi_* \otimes \varphi_*} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\
D' \otimes D' & \longrightarrow & K' & \xrightarrow[\partial_3']{} & L' & \xrightarrow[\partial_2']{} & M' & \xrightarrow[\partial_1']{} & N'
\end{array}$$

such that
(a) For $m \in M$ and $n \in N$,

$$f_1({}^n m) ={}^{f_0(n)} f_1(m), f_2({}^n l) ={}^{f_0(n)} f_2(l), f_3({}^n k) ={}^{f_0(n)} f_3(k).$$

(b) For $a, b \in L$ $(i = 0, 1, 2)$

$$\omega_i'(\{f_2(a)\} \otimes \{f_2(b)\}) = f_3\omega_i(\{a\} \otimes \{b\}).$$

(c) For $a \in M, b \in L$ $(i = 1, 2)$

$$\varphi_i'(\{f_1(a)\} \otimes \{f_2(b)\}) = f_3\varphi_i(\{a\} \otimes \{b\}).$$

(d) For $b \in M$ and $a \in L$

$$\varphi_0'(\{f_2(a)\} \otimes \{f_1(b)\}) = f_3\varphi_0(\{a\} \otimes \{b\}).$$

(e) For $a, b \in M$,

$$\omega'(\{f_1(a)\} \otimes \{f_1(b)\}) = f_2\omega(\{a\} \otimes \{b\}).$$

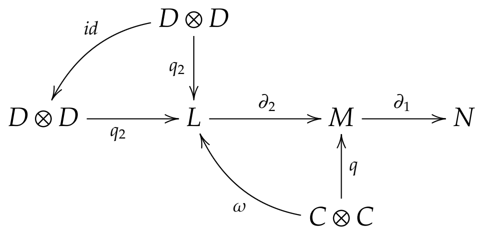
We will refer to the category of 2-quadratic modules as **2QM**.

Example 2.5. ([3]) Let

$$\begin{array}{ccccc}
 & & C \otimes C & & \\
 & \swarrow{\scriptstyle \omega} & \downarrow{\scriptstyle q} & & \\
L & \xrightarrow[\partial_2]{} & M & \xrightarrow[\partial_1]{} & N
\end{array}$$

be a quadratic module. Then we can define a 2-quadratic module as follows:

$q_2 : D \otimes D \to L$ defined as $q_2(\{l\} \otimes \{l'\}) =< l, l' >$ for $l, l' \in L$ where $D = (L^{cr})^{ab}$, then the diagram



is a 2-quadratic module. Thus, the functor $G_2 : QM \to 2QM$ is obtained.

3. Quadratic module of groupoids

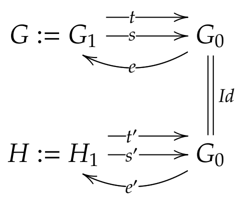
A groupoid is a small category that contains all invertible arrows, consisting of a set of objects C_0 (referred as the base), a set of arrows C_1, source and target maps $s; t : C_1 \to C_0$, composition $\circ : C_1 \times C_1 \longrightarrow C_1$, and identity $e : C_0 \longrightarrow C_1$.

We will denote such a groupoid with $(C_1, C_0, s, t, e, \circ)$ with C_0 as the set of objects and C_1 as the set of arrows. $C_1(x, y)$ represents an arrow from x to y. For $a \in C_1(y, z)$ and $b \in C_1(x, y)$, $b \circ a \in C_1(x, z)$ represents the composition of a and b. For $a \in C_1$, there exist $a^{-1} \in C_1$ such that $a \circ a^{-1} = e_{t(a)}$ and $a^{-1} \circ a = e_{s(a)}$.

Let G be a groupoid and $x, y \in G_0$. The groupoid G is called totally disconnected if $G_1(x, y)$ is empty for every $x \neq y$.

In [21] the definition of right action for groupoids is given. In the following, we intend to provide the definition of left groupoid action.

Definition 3.1. Let **G** be a totally disconnected groupoid and **H** be any groupoid over the same object set of **G**.



The groupoid (left) action of **H** on **G** is a map

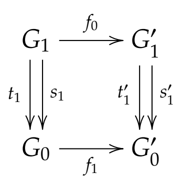
$$\begin{array}{ccc} H_1 \times G_1 & \longrightarrow & G_1 \\ (h, g) & \longmapsto & {}^h g \end{array}$$

satisfying the following conditions;

1. ${}^h g$ is defined if and only if $t(h) = s(g)$, and then $s(h) = s({}^h g)$,
2. ${}^{h_1 \circ h_2} g = {}^{h_2}({}^{h_1} g)$ and ${}^{e_x} g_1 = g_1$,
3. ${}^h(g_1 \circ g_2) = {}^h g_1 \circ {}^h g_2$ and ${}^{h_1} e_y = e_x$

for $g, g_1, g_2 \in G(x, x)$ and $h, h_1 \in H(x, y)$, $h_2 \in H(y, z)$.

A groupoid morphism is a commutative diagram



compatible with source and target maps that is:

$$\begin{aligned} s_1' f_0 &= f_1 s_1 \\ t_1' f_0 &= f_1 t_1 \end{aligned}$$

We will refer to the category of groupoids as **Grpd**.

Quadratic module of groups is given by Baues [4]. For the groupoid structure existence of $C \otimes C$ can be seen from [19]. Because quadratic modules are a special case of 2-crossed modules.

Definition 3.2. Let

$$\begin{array}{ccc} \partial_1 : G_1 & \longrightarrow & G_0 \\ \downarrow\downarrow & & \downarrow\downarrow \\ O & = & O \end{array}$$

be a pre-crossed module of groupoids where G_1 is a totally disconnect and G_0 is a normal groupoid. There exists a groupoid action (as given in Def 3.1) of G_0 on G_1. The Peiffer product in totally disconnected groupoid $(G_1, \circ)$ is given as

$$< g_1, g_2 >=^{\partial(g_1)} g_2 \circ g_1 \circ {g_2}^{-1} \circ {g_1}^{-1}$$

for $g_1, g_2 \in G_1$. $P_2(\partial)$ generated with $< g_1, g_2 >$ is the normal subgroupoid of the totally disconnected groupoid. For the subgroupoid $P_3(\partial)$ generated with the elements of the form

$$<< g_1, g_2 >, g_3 > \text{ and } < g_1, < g_2, g_3 >>$$

if $P_3(\partial) = \{1\}$ then the pre-crossed module $\partial : G_1 \to C_0$ is called nil(2)-module of groupoids.

Morphisms of Nil(2)-module of groupoids are defined similarly to morphisms of crossed module of groupoids. We will refer to the category of nil(2)-module of groupoids as $\mathbf{Nil(2)}_{Grpd}$.

Let

$$\begin{array}{ccccc} & \overset{\omega}{\swarrow} & G \otimes G & & \\ & & \downarrow q & & \\ G_2 & \xrightarrow[\partial_2]{} & G_1 & \xrightarrow[\partial_1]{} & G_0 \end{array}$$

be a quadratic module. To adapt this notion to groupoids G_2, G_1, G_0 should be groupoids over same object set O where if $x, y \in O (x \neq y), G_2(x, y) = G_1(x, y) = G$. That is $G_2 \rightrightarrows G_0$ and $G_1 \rightrightarrows G_0$ are totally disconnected groupoids. The groupoid action of G_0 on G_2 can be seen as: if $g_0 \in G_0(x, y)$ and $g_2 \in G_2(x, x)$ then ${}^{g_0}g_2 \in G_2(y, y)$. Similar for $g_1 \in G_1(x, x)$ and for conjugate action $g_0' \in G_0(x, x)$. All of the formulas still make sense after this adjustment. With this manner, we will refer to the category of quadratic module of groupoids as $\mathbf{QM}_{Grpd}$.

Next with propositions and examples, we define functors between quadratic modules of groupoids and related structures.

Proposition 3.3. *If $\partial : M \to N$ is a nil(2)-module, then (G_1, G_0, s, t, e) is a groupoid with source, target, and identity mappings defined as $s(m, n) = n$, $t(m, n) = \partial(m)n$, and $e(n) = (1_M, n)$ for $n \in N, m \in M$, and the composition*

$$(m, n) \circ (m', n') = (m'm, n).$$

Proof. Let $G_0 = N$ and $G_1 = M \ltimes N$. For $(m', n'), (m, n) \in M \ltimes N$ we have

$$\begin{aligned} s[(m', n') \circ (m, n)] &= s(m'm, n) \\ &= n \\ &= s(m, n) \end{aligned}$$

$$
\begin{aligned}
t[(m',n')\circ(m,n)] &= t(m'm,n)\\
&= \partial(m'm)n\\
&= \partial(m')\partial(m)n\\
&= \partial(m')n' \qquad (\text{since } \circ \text{ is defined } \partial(m)n = n')\\
&= t(m',n')
\end{aligned} \tag{1}
$$

and for $n \in N$ we have

$$
\begin{aligned}
se(n) &= s(1,n) = n = Id(n)\\
te(n) &= t(1,n) = \partial(1)n = n = Id(n)
\end{aligned}
$$

The inverse of $(m,n) \in G_1$ can be defined as

$$(m^{-1}, \partial(m)n) : \partial(m)n \to n$$

□

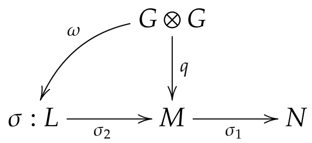
That is, there exists a functor

$$F_1 : \textbf{Nil(2)} \to \textbf{Grpd}$$

from the category of nil(2)-modules to that of groupoids.

Proposition 3.4. *A quadratic module gives a nil(2)-module.*

Proof. Let



be a quadratic module. Since N acts on M and L we define semi-direct products $M \ltimes N$ and $L \ltimes N$. As given in proposition 3.3, $M \ltimes N$ is a groupoid over N with source, target and identity maps $s_1(m,n) = n, t_1(m,n) = \sigma_1(m)n$ and $e_1(m_1) = (1_M, m_1)$ for $(m,n) \in M \ltimes N$ and $n_1 \in N$ respectively. Similarly $L \ltimes N$ is a groupoid with $s_2(l,n) = n, t_2(l,n) = \sigma_1(\sigma_2(l))n$ and $e_2(n) = (1_L, n)$. Since

$$L \xrightarrow{\sigma_2} M \xrightarrow{\sigma_1} N$$

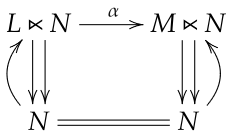
is a complex of groups $\sigma_1(\sigma_2(l))n = n$ makes $L \ltimes N$ a totally disconnected groupoid over N. The groupoid action of $M \ltimes N$ on $L \ltimes N$ can be defined as

$$^{(m,n)}(l,n) = (^{m}l, n) = (\omega(\sigma_2(l)\otimes m)l, n)$$

for $(m,n) \in M \ltimes N, (l,n) \in L \ltimes N$. Define $\alpha : L \ltimes N \to M \ltimes N, \alpha(l,n) = (\sigma_2(l), n)$. Then for $(m,n) \in M \ltimes N$ and $(l,n),(l',n) \in L \ltimes N$ we have

$$
\begin{aligned}
\alpha(^{(m,n)}(l,n)) &= \alpha(^{m}l, n)\\
&= (\sigma_2(^{m}l), n)\\
&= (\sigma_2\omega(\sigma_2(l)\otimes m), n)\\
&= (<\sigma_2(l), m> \sigma_2(l), n)\\
&= (^{\sigma_1\sigma_2(l)}m\sigma_2(l)m^{-1}\sigma_2(l)^{-1}\sigma_2(l), n)\\
&= (m,n)\circ\alpha(l,n)\circ(m,n)^{-1}
\end{aligned}
$$

with $P_3(\alpha) = 1$.



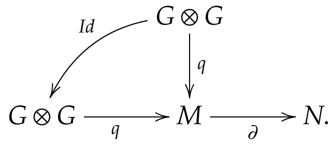
is a nil(2)-module of groupoids. □

That is there exists a functor

$$F_2 : \mathbf{QM} \rightarrow \mathbf{Nil(2)}_{Grpd}$$

from the category of quadratic modules to that of nil(2)-module of groupoids.

Example 3.5. ([4]) A quadratic module



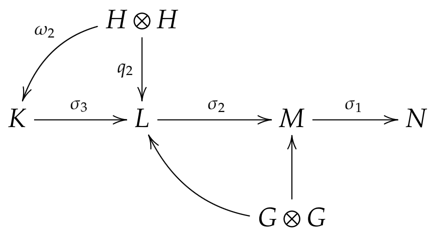
arises from any nil(2)-module, $\partial : M \rightarrow N$. That is we get a functor

$$G_1 : Nil(2) \rightarrow QM$$

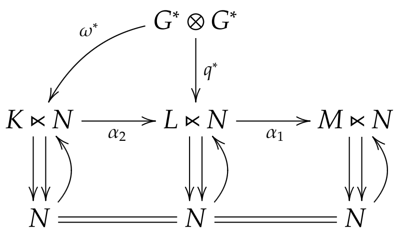
from the category of nil(2)-modules to that of quadratic modules.

Proposition 3.6. *A 2-quadratic module provides a quadratic module.*

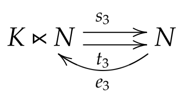
Proof. Let



be a 2-quadratic module. We claim that



is a quadratic module of groupoids where $\alpha_1(l, n) = (\sigma_2(l), n), \alpha_2(k, n) = (\sigma_3(k), n)$ and $G^* = ((L \ltimes N)^{cr})^{ab}$. In proposition 3.4, we obtain groupoid structures on $L \ltimes N$ and $M \ltimes N$. Also



with $s(k, n) = n$, $t(k, n) = \sigma_1\sigma_2\sigma_3(k)n = n$ is a totally disconnected groupoid. We define

$$\begin{array}{rcl} \omega^* : G^* \otimes G^* & \rightarrow & K \ltimes N \\ (\{l, n\} \otimes \{l', n\}) & \mapsto & (\omega_2\{l \otimes l'\}, n) \end{array}$$

by using $\omega_2 : H \otimes H \rightarrow K$ in 2-quadratic module.

QM1) In proposition 3.4, we show that

$$\begin{array}{ccc} L \ltimes N & \longrightarrow & M \ltimes N \\ \uparrow\downarrow\downarrow & & \downarrow\downarrow\uparrow \\ N & = & N \end{array}$$

is a nil(2)-module of groupoids.

QM2) For $(k, n) \in K \ltimes N$

$$\begin{array}{rcl} \alpha_1\alpha_2(k, n) & = & \alpha_1(\sigma_3(k), n) \\ & = & \sigma_2\sigma_3(k)n \\ & = & n \end{array}$$

and for $\{l_1, n\} \otimes \{l_2, n\} \in G^* \otimes G^*$.

$$\begin{array}{rcl} \alpha_2\omega^*(\{l_1, n\} \otimes \{l_2, n\}) & = & \alpha_2(\{l_1 \otimes l_2\}, n) \\ & = & \sigma_3(\omega_2\{l_1 \otimes l_2\}, n) \\ & = & ({}^{\partial_1(l_1)}l_2 \circ l_1 \circ l_2^{-1} \circ l_1^{-1}, n) \\ & = & ({}^{\partial_1(l_1)}l_2, n) \circ (l_1, n) \circ (l_2^{-1}, n) \circ (l_1^{-1}, n) \end{array}$$

QM3) For $(k, n) \in K \ltimes N$ and $(l, n) \in L \ltimes N$

$$\begin{array}{rcl} \omega^*[(\{\alpha_2(k, n)\} \otimes \{l, n\})(\{l, n\} \otimes \{\alpha_2(k, n)\})] & = & [(\omega_2(\{\alpha_2(k, n)\} \otimes \{l, n\}))\omega_2(\{l, n\} \otimes \{\alpha_2(k, n)\})] \\ & = & [(\omega_2(\{\alpha_2(k)\} \otimes \{l\}))(\omega_2\{l \otimes \alpha_2(k)\}, n)] \\ & = & (\omega_2(\{\alpha_2(k) \otimes l\}\{l \otimes \alpha_2(k)\}), n) \\ & = & ({}^{\alpha_1(l)}k \circ k^{-1}, n) \\ & = & {}^{\alpha_1(l,n)}(k, n) \circ (k^{-1}, n) \\ & = & {}^{\alpha_1(l,n)}(k, n) \circ (k, n)^{-1} \end{array}$$

QM4) For $(k, n), (k', n) \in K \ltimes N$

$$\begin{array}{rcl} \omega^*(\{\alpha_2(k, n)\} \otimes \{\alpha_2(k', n)\}) & = & \omega_2(\{\alpha_2(k, n) \otimes \alpha_2(k', n)\}) \\ & = & [(k', n), (k, n)] \\ & = & (k', n) \circ (k, n) \circ (k', n)^{-1} \circ (k, n)^{-1}. \end{array}$$

□

That is there exists a functor

$$F_3 : \mathbf{2QM} \rightarrow \mathbf{QM}_{Grpd}$$

from the category of 2-quadratic modules to that quadratic module of groupoids.

Example 3.7.

$$G_2 \longrightarrow G_1 \underset{t}{\overset{s}{\rightrightarrows}} G_0$$

is a nil(2)-module of groupoids. We know that $\{e_M\}$ is a totally disconnected groupoid over G_0, then we obtain the following sequence of groupoids

$$\{e_M\} \xrightarrow{i} G_2 \xrightarrow{\partial} G_1 \rightrightarrows G_0.$$

In this construction, if we define $\mathbf{C}$ as $((G_2)^{cr})^{ab}$ where

$$(G_2)^{cr} = G_2/_{P_2(\partial)}$$

and

$$P_2(\partial) = \{< g_1, g_2 >: g_1, g_2 \in G_2\}$$

$< g_1, g_2 >$ is the Peiffer commutators of morphisms in the groupoid

$$G_1 \rightrightarrows G_0$$

.

$$(G_1)^{cr} \rightrightarrows G_0$$

is a quotient groupoid. That is there exists a functor $G_3 : Grpd \to Nil(2)_{Grpd}$ from groupoids to nil(2)-module of groupoids.

Example 3.8. A quadratic module of groupoids can be obtained from a given nil(2)-module of groupoids by setting the example 3.5 for groupoids. For this if $\partial : M \to N$ is a nil(2)-module of groupoids, in the group case, the group M satisfying the nil(2) module conditions is sufficient, while in the groupoid case M must also be extremely disconnected groupoid. We need to take $L = 1_M$ and the resulting sequence

$$1_M \xrightarrow{q} M \xrightarrow{\partial} N$$

with the groupoid actions is a quadratic module of groupoids since 1_M is extremely disconnected groupoid. That is

$$\begin{array}{ccccc} & \overset{\omega}{\swarrow} & G \otimes G & & \\ & & \downarrow q & & \\ 1_M & \xrightarrow[q]{} & M & \xrightarrow[\partial]{} & N. \end{array}$$

is a quadratic module of groupoids where ω is identity morphism. Then we get a functor $G_4 : Nil(2)_{Grpd} \to QM_{Grpd}$.

4. Gray groupoids from 2-quadratic modules

In this section we will show that a functor can be defined from 2-quadratic modules to Gray groupoids.

Definition 4.1. A Gray groupoid [15] $\mathfrak{G}$ consists of G_0 class of objects, G_1 set of morphisms, G_2 set of 2-morphisms and G_3 set of 3-morphisms with

i) $s_n, t_n : G_i \to G_n$ are the n-source and n-target functions for $0 \le n < i \le 3$.

ii) $\#_n : G_{n+1} \times G_{n+1} \to G_{n+1}$ is a vertical composition for $0 \leq n < 3$.

iii) $\#_n : G_i \times G_{n+1} \to G_i$ and $\#_n : G_{n+1} \times G_i \to G_i$ are the whiskering functions for $0 \leq n \leq 1, n+1 < i \leq 3$.

iv) $\circ_h : G_2 \times G_2 \to G_3$ is a horizontal composition.

v) $id_{\_} : G_i \to G_{i+1}$ is an identity function for $0 \leq i \leq 2$.

such that

G1) $\mathfrak{G}$ is a globular set.

G2) $G(g, g')$ is a 2-category with g 0-source, g' 0-target of composition $\#_{n+1}$ and $id_{\_}$ map for $g, g' \in G_0$.

G3) For $\phi : g' \to g''$ 1-morphism and for all $g, g''' \in G_0$

$$\_\#_0\phi : G(g'', g''') \to G(g', g''')$$

is a 2-functor. In a similar way,

$$\phi\#_0\_ : G(g, g') \to G(g, g'')$$

is also a 2-functor.

G4) For $g, g', g'' \in G_0$, $\_\#_0 id_{g'}$ is equivalent to identity functor on $G(g', g'')$. $id_{g'}\#_0\_$ is equivalent to identity functor on $G(g, g')$.

G5) For all $\gamma, \delta \in G_2$, $\gamma : f \implies f'$ and $\delta : \phi \implies \phi'$ that hold the $t_0(\gamma) = s_0(\delta)$ equality is

$$\begin{aligned} s_2(\delta\#_0\gamma) &= (\delta\#_0 f')\#_1(\phi\#_0\gamma) \\ t_2(\delta\#_0\gamma) &= (\phi'\#_0\gamma)\#_1(\delta\#_0 f) \end{aligned}$$

for $f, f', \phi, \phi' \in G_1$.

G6) Let

$$\nu : \gamma \implies \gamma' : f \implies f' : g \to g'$$

be a 3-morphism and

$$\delta : \phi \implies \phi' : g' \to g''$$

be a 2-morphism. For $\gamma : f \implies f' : g \to g'$ 2-morphism, $\nu : \delta \implies \delta' : \phi \implies \phi' : g' \to g''$ 3-morphism and

$$((\phi'\#_0\nu)\#_1(\delta\#_0 f))\#_2(\delta\#_0\gamma) = (\delta\#_0\gamma')\#_2((\delta\#_0 f')\#_1(\phi\#_0\nu))$$

the following property is hold.

$$(\delta'\#_0\gamma)\#_2((\nu\#_0 f')\#_1(\phi\#_0\gamma)) = ((\phi'\#_0\gamma)\#_1(\nu\#_0 f))\#_2(\delta\#_0\gamma)$$

G7) Let $\gamma : f \implies f' : g \to g', \gamma' : f' \implies f'' : g \to g'$ and $\delta : \phi \implies \phi' : g' \to g''$ be 2-morphisms. The following equality is hold.

$$\delta\#_0(\gamma'\#_1\gamma) = ((\phi'\#_0\gamma')\#_1(\delta\#_0\gamma))\#_2((\delta\#_0\gamma')\#_1(\phi\#_0\gamma))$$

Let $\gamma : f \implies f' : g \to g'$ and $\delta : \phi \implies \phi' : g' \to g'', \delta' : \phi' \implies \phi'' : g' \to g''$ be 2-morphisms. So below equality is obtained.

$$(\delta'\#_1\delta)\#_0\gamma = ((\delta'\#_0\gamma)\#_1(\delta\#_0 f))\#_2((\delta'\#_0 f')\#_1(\delta\#_0 f))$$

$G8)$ Let $f : g \to g'$ be a 1-morphism and $\delta : \phi \implies \phi' : g' \to g''$ be a 2-morphism so,

$$\delta \#_0 id_f = id_{\delta \#_0 f}$$

must be correct. Let $\gamma : f \implies f' : g \to g'$ be a 2-morphism and $\phi : g' \to g''$ be a 1-morphism then,

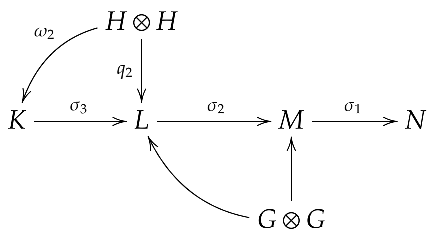
$$id_\phi \#_0 \gamma = id_{\phi \#_0 \gamma}$$

$G9)$ For $a \in G(g, g')_p, a' \in G(g', g'')_q, a'' \in G(g'', g''')_r$ and $p + q + r \le 2$,

$$(a'' \#_0 a') \#_0 a = a'' \#_0 (a' \#_0 a).$$

$\mathfrak{G} = (G_3, G_2, G_1, G_0)$ is called a Gray groupoid.

Let



be a 2-quadratic module. From proposition 3.3 ($G_1 = M \ltimes N, G_0 = N, s, t, e$) is a groupoid. From the definition of quadratic module L is a N-group. Furthermore from proposition 3.4 ($L \ltimes N, N, s, t, e$) is a groupoid and there exists a groupoid action of ($G_1 = M \ltimes N, G_0 = N, s_1, t_1, e_1$) on ($L \ltimes N, N, s_2, t_2, e_2$). That is we can define

$$G_2 = (L \ltimes N) \ltimes (M \ltimes N)$$

Similiar way using the action of N on K we can define

$$G_3 = (K \ltimes N) \ltimes (L \ltimes N) \ltimes (M \ltimes N).$$

Then $\mathfrak{G} = ((K \ltimes N) \ltimes (L \ltimes N) \ltimes (M \ltimes N), (L \ltimes N) \ltimes (M \ltimes N), M \ltimes N, N)$ satifies the conditions of a Gray groupoid. As a result we obtain a functor

$$\delta : \mathbf{2QM} \to \mathbf{Gray_{Grpd}}$$

from the category of 2-quadratic modules to that of Gray groupoids.

5. Conclusion

In proposition 3.3, we obtain a groupoid structure from a given nil(2)-module. In proposition 3.4, we obtain a nil(2)-module of groupoid from any quadratic module. In proposition 3.6, we obtain a quadratic module of groupoid from any 2-quadratic module. Thus there exists functors

$$\begin{array}{lll} F_1 : \mathbf{Nil(2)} & \to & \mathbf{Grpd}, \\ F_2 : \mathbf{QM} & \to & \mathbf{Nil(2)}_{Grpd}, \\ F_3 : \mathbf{2QM} & \to & \mathbf{QM}_{Grpd}. \end{array}$$

Also examples 2.5, 3.5, 3.7 and 3.8 induce functors

$$\begin{array}{lcl} G_1 : \mathbf{Nil(2)} & \rightarrow & \mathbf{QM}, \\ G_2 : \mathbf{QM} & \rightarrow & \mathbf{2QM}, \\ G_3 : \mathbf{Grpd} & \rightarrow & \mathbf{Nil(2)}_{Grpd}, \\ G_4 : \mathbf{Nil(2)}_{Grpd} & \rightarrow & \mathbf{QM}_{Grpd} \end{array}$$

As a result, the relations we examine in this work can be summarized with the following diagram,

$$\begin{array}{ccccc} \mathbf{Nil(2)} & \xrightarrow{G_1} & \mathbf{QM} & \xrightarrow{G_2} & \mathbf{2QM} \\ \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle F_2} & & \downarrow{\scriptstyle F_3} \\ \mathbf{Grpd} & \xrightarrow[G_3]{} & \mathbf{Nil(2)}_{Grpd} & \xrightarrow[G_4]{} & \mathbf{QM}_{Grpd} \end{array}$$

The functors that have been provided throughout the paper can be used to derive examples of these categories.

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